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Topological Uniform Descent, Quasi-Fredholmness and Operators Originated from Semi-B-Fredholm Theory

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Abstract

In this paper we study operators originated from semi-B-Fredholm theory and as a consequence we get some results regarding boundaries and connected hulls of the corresponding spectra. In particular, we prove that a bounded linear operator T acting on a Banach space, having topological uniform descent, is a **BR** operator if and only if 0 is not an accumulation point of the associated spectrum $\sigma_{\mathbf{R}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathbf{R}\}$, where **R** denote any of the following classes: upper semi-Weyl operators, Weyl operators, upper semi-Fredholm operators, Fredholm operators, operators with finite (essential) descent and **BR** the B-regularity associated to **R** as in Berkani (Studia Mathematica 140(2):163–174, 2000). Under the stronger hypothesis of quasi-Fredholmness of T, we obtain a similar characterisation for T being a **BR** operator for much larger families of sets **R**.

Keywords Semi-B-Fredholm operators · Topological uniform descent · Quasi-Fredholm operators · Boundary · Connected hull

Mathematics Subject Classification 47A53 · 47A10

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1 Introduction

Let $\mathbb{N}$ ($\mathbb{N}_0$) denote the set of all positive (non-negative) integers, and let $\mathbb{C}$ denote the set of all complex numbers. We use $L(X)$ to denote the Banach algebra of bounded linear operators acting on an infinite dimensional complex Banach space X. The group of all invertible operators is denoted by $L(X)^{-1}$. Let $\mathcal{I}(X)$ denote the set of all bounded below operators and let $\mathcal{S}(X)$ denote the set of all surjective operators. For $T \in L(X)$, denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ its spectrum, point spectrum, approximate point spectrum and surjective spectrum, respectively. Also, write $N(T)$ for its null-space, $R(T)$ for its range, $\alpha(T)$ for its nullity and $\beta(T)$ for its defect. *The compression spectrum* of $T \in L(X)$, denoted by $\sigma_{cp}(T)$, is the set of all complex λ such that $T-\lambda I$ does not have dense range.

An operator $T \in L(X)$ is *upper semi-Fredholm* if $\alpha(T) < \infty$ and $R(T)$ is closed, while T is *lower semi-Fredholm* if $\beta(T) < \infty$. In the sequel $\Phi_+(X)$ (resp. $\Phi_-(X)$) will denote the set of upper (resp. lower) semi-Fredholm operators. If T is upper or lower semi-Fredholm, then T is called *semi-Fredholm*. The set of semi-Fredholm operators is denoted by $\Phi_\pm(X)$. For semi-Fredholm operators the index is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. The set of Fredholm operators is defined as $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$. The sets of upper semi-Weyl, lower semi-Weyl and Weyl operators are defined as $\mathcal{W}_+(X) = \{T \in \Phi_+(X) : \text{ind}(T) \leq 0\}$, $\mathcal{W}_-(X) = \{T \in \Phi_-(X) : \text{ind}(T) \geq 0\}$ and $\mathcal{W}(X) = \{T \in \Phi(X) : \text{ind}(T) = 0\}$, respectively.

For $T \in L(X)$, the *upper semi-Fredholm spectrum*, the *lower semi-Fredholm spectrum*, the *semi-Fredholm spectrum*, the *Fredholm spectrum*, the *upper semi-Weyl spectrum*, the *lower semi-Weyl spectrum* and the *Weyl spectrum* are defined, respectively, by:

$$\begin{aligned}
\sigma_{\Phi_+}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_+(X)\}, \\
\sigma_{\Phi_-}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_-(X)\}, \\
\sigma_{\Phi_\pm}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_\pm(X)\}, \\
\sigma_{\Phi}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi(X)\}, \\
\sigma_{\mathcal{W}_+}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{W}_+(X)\}, \\
\sigma_{\mathcal{W}_-}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{W}_+(X)\}, \\
\sigma_{\mathcal{W}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{W}(X)\}.
\end{aligned}$$

For $n \in \mathbb{N}_0$ we set $c_n(T) = \dim R(T^n)/R(T^{n+1})$ and $c'_n(T) = \dim N(T^{n+1})/N(T^n)$. From [21, Lemmas 3.1 and 3.2] it follows that $c_n(T) = \text{codim}\,(R(T) + N(T^n))$ and $c'_n(T) = \dim(N(T) \cap R(T^n))$. Obviously, the sequences $(c_n(T))_n$ and $(c'_n(T))_n$ are decreasing. For each $n \in \mathbb{N}_0$, T induced a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to the space $R(T^{n+1})/R(T^{n+2})$ and let $k_n(T)$ denote the dimension of the null space of the induced map. From [14, Lemma 2.3] it follows that

$$\begin{aligned}
k_n(T) &= \dim(R(T^n) \cap N(T))/(R(T^{n+1}) \cap N(T)) \\
&= \dim(R(T) + N(T^{n+1}))/(R(T) + N(T^n)).
\end{aligned}$$

From this it is easily seen that $k_n(T) = c'_n(T) - c'_{n+1}(T)$ if $c'_{n+1}(T) < \infty$ and $k_n(T) = c_n(T) - c_{n+1}(T)$ if $c_{n+1}(T) < \infty$.

The *descent* $\delta(T)$ and the *ascent* $a(T)$ of T are defined by $\delta(T) = \inf\{n \in \mathbb{N}_0 : c_n(T) = 0\} = \inf\{n \in \mathbb{N}_0 : R(T^n) = R(T^{n+1})\}$ and $a(T) = \inf\{n \in \mathbb{N}_0 : c'_n(T) = 0\} = \inf\{n \in \mathbb{N}_0 : N(T^n) = N(T^{n+1})\}$. We set formally $\inf \emptyset = \infty$.

The *essential descent* $\delta_e(T)$ and the *essential ascent* $a_e(T)$ of T are defined by $\delta_e(T) = \inf\{n \in \mathbb{N}_0 : c_n(T) < \infty\}$ and $a_e(T) = \inf\{n \in \mathbb{N}_0 : c'_n(T) < \infty\}$.

The sets of upper semi-Browder, lower semi-Browder and Browder operators are defined as $\mathcal{B}_+(X) = \{T \in \Phi_+(X) : a(T) < \infty\}$, $\mathcal{B}_-(X) = \{T \in \Phi_-(X) : \delta(T) < \infty\}$ and $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$, respectively. For $T \in L(X)$, the *upper semi-Browder spectrum*, the *lower semi-Browder spectrum* and the *Browder spectrum* are defined, respectively, by:

$$
\begin{aligned}
\sigma_{\mathcal{B}_+}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}_+(X)\},\\
\sigma_{\mathcal{B}_-}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}_-(X)\},\\
\sigma_{\mathcal{B}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathcal{B}(X)\}.
\end{aligned}
$$

Sets of *left and right Drazin invertible* operators, respectively, are defined as $LD(X) = \{T \in L(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed}\}$ and $RD(X) = \{T \in L(X) : \delta(T) < \infty \text{ and } R(T^{\delta(T)}) \text{ is closed}\}$. If $a(T) < \infty$ and $\delta(T) < \infty$, then T is called *Drazin invertible* [3,4]. By $D(X)$ we denote the set of Drazin invertible operators.

An operator $T \in L(X)$ is a *left essentially Drazin invertible* operator if $a_e(T) < \infty$ and $R(T^{a_e(T)+1})$ is closed. If $\delta_e(T) < \infty$ and $R(T^{\delta_e(T)})$ is closed, then T is called *right essentially Drazin invertible*. In the sequel $LD^e(X)$ (resp. $RD^e(X)$) will denote the set of left (resp. right) essentially Drazin invertible operators.

For $T \in L(X)$, the *left Drazin spectrum*, the *right Drazin spectrum*, the *Drazin spectrum*, the *left essentially Drazin spectrum*, the *right essentially Drazin spectrum*, the *descent spectrum* and the *essential descent spectrum* are defined, respectively, by:

$$
\begin{aligned}
\sigma_{LD}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(X)\},\\
\sigma_{RD}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin RD(X)\},\\
\sigma_{D}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin D(X)\},\\
\sigma^e_{LD}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin LD^e(X)\},\\
\sigma^e_{RD}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \notin RD^e(X)\},\\
\sigma_{dsc}(T) &= \{\lambda \in \mathbb{C} : \delta(T - \lambda I) = \infty\},\\
\sigma^e_{dsc}(T) &= \{\lambda \in \mathbb{C} : \delta_e(T - \lambda I) = \infty\}.
\end{aligned}
$$

An operator $T \in L(X)$ is said to be *quasi-Fredholm* if there is $d \in \mathbb{N}_0$ such that $k_n(T) = 0$ for all $n \geq d$ and $R(T^{d+1})$ is closed. The set of quasi-Fredholm operators includes many sets of operators such as left (right) Drazin invertible operators, left (right) essentially Drazin invertible operators, upper (lower) semi-B-Weyl operators (see [7]).

For $T \in L(X)$ we say that it is *Kato* if $R(T)$ is closed and $N(T) \subset R(T^n)$ for every $n \in \mathbb{N}$. An operator $T \in L(X)$ is *nilpotent* when $T^n = 0$ for some $n \in \mathbb{N}$. An operator $T \in L(X)$ is said to be of *Kato type* if there exist closed subspaces X_1, X_2 such that $X = X_1 \oplus X_2$, $T(X_i) \subset X_i$, $i = 1, 2$, $T_{|X_1}$ is nilpotent and $T_{|X_2}$ is Kato. Every operator of Kato type is a quasi-Fredholm operator. In the case of Hilbert spaces, the set of quasi-Fredholm operators coincides with the set of Kato type operators.

For $T \in L(X)$ and every $d \in \mathbb{N}_0$, the operator range topology on $R(T^d)$ is defined by the norm $\|\cdot\|_d$ such that for every $y \in R(T^d)$,

$$\|y\|_d = \inf\{\|x\| : x \in X,\ y = T^d x\}.$$

Operators which have eventual topological uniform descent were introduced by Grabiner in [14]:

Definition 1.1 Let $T \in L(X)$. If there is $d \in \mathbb{N}_0$ for which $k_n(T) = 0$ for $n \geq d$, then T is said to have uniform descent for $n \geq d$. If in addition, $R(T^n)$ is closed in the operator range topology of $R(T^d)$ for $n \geq d$, then we say that T has *eventual topological uniform descent* and, more precisely, that T has *topological uniform descent for* (TUD for brevity) $n \geq d$.

It is easily seen that if T has finite nullity, defect, ascent or essential ascent, then it has uniform descent. If T has finite descent or essential descent, then T has TUD. Also, the set of operators which have TUD contains the set of quasi-Fredholm operators [7].

For $T \in L(X)$, the *Kato type spectrum*, the *quasi-Fredholm spectrum* and the *topological uniform descent spectrum* are defined, respectively, by:

$$\begin{aligned}
\sigma_{Kt}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not of Kato type}\},\\
\sigma_{q\Phi}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not quasi-Fredholm}\},\\
\sigma_{TUD}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ does not have TUD}\}.
\end{aligned}$$

We use the following notation [7,23]:

$$\begin{array}{ccc}
\mathbf{R_1} = \mathcal{S}(X) & \mathbf{R_2} = \mathcal{B}_-(X) & \mathbf{R_3} = RD(X)\\
\mathbf{R_4} = \Phi_-(X) & \mathbf{R_5} = RD^e(X) & \\
\mathbf{R_6} = \mathcal{I}(X) & \mathbf{R_7} = \mathcal{B}_+(X) & \mathbf{R_8} = LD(X)\\
\mathbf{R_9} = \Phi_+(X) & \mathbf{R_{10}} = LD^e(X) &
\end{array}$$

and

$$\begin{aligned}
\mathbf{R_4^a} &= \{T \in L(X) : \delta(T) < \infty\},\\
\mathbf{R_5^a} &= \{T \in L(X) : \delta^e(T) < \infty\}.
\end{aligned}$$

For a bounded linear operator T and $n \in \mathbb{N}_0$ define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular, $T_0 = T$). If $T \in L(X)$ and if there exist an integer n for which the range space $R(T^n)$ is closed and T_n belongs to the class $\mathbf{R}$, we will say that T belongs to the class $\mathbf{BR}$, where

$\mathbf{R} \in \{\mathbf{R}_i : i = 1, \ldots, 10\} \cup \{\mathbf{R}_4^a, \mathbf{R}_5^a\} \cup \{\Phi(X), \mathcal{B}(X), \mathcal{W}_+(X), \mathcal{W}_-(X), \mathcal{W}(X)\}$. For $T \in L(X)$ let $\sigma_{\mathbf{R}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathbf{R}\}$ and $\sigma_{\mathbf{BR}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathbf{BR}\}$.

More details, if for an integer n the range space $R(T^n)$ is closed and T_n is Fredholm (resp. upper semi-Fredholm, lower semi-Fredholm, Browder, upper semi-Browder, lower semi-Browder), then T is called a *B-Fredholm* (resp. *upper semi-B-Fredholm, lower semi-B-Fredholm*, *B-Browder*, *upper semi-B-Browder, lower semi-B-Browder*) operator. If $T \in L(X)$ is upper or lower semi-B-Fredholm, then T is called *semi-B-Fredholm*. The index $\mathrm{ind}(T)$ of a semi-B-Fredholm operator T is defined as the index of the semi-Fredholm operator T_n. By [6, Proposition 2.1] the definition of the index is independent of the integer n. An operator $T \in L(X)$ is *B-Weyl* (resp. *upper semi-B-Weyl, lower semi-B-Weyl*) if T is B-Fredholm and ind(T) = 0 (resp. T is upper semi-B-Fredholm and $\mathrm{ind}(T) \leq 0$, T is lower semi-B-Fredholm and $\mathrm{ind}(T) \geq 0$).

For $T \in L(X)$, the *upper semi-B-Fredholm spectrum*, the *lower semi-B-Fredholm spectrum*, the *B-Fredholm spectrum*, the *upper semi-B-Weyl spectrum*, the *lower semi-B-Weyl spectrum*, the *B-Weyl spectrum*, the *upper semi-B-Browder spectrum*, the *lower semi-B-Browder spectrum* and the *B-Browder spectrum* are defined, respectively, by:

$$\begin{aligned}
\sigma_{B\Phi_+}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-B-Fredholm}\},\\
\sigma_{B\Phi_-}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-B-Fredholm}\},\\
\sigma_{B\Phi}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Fredholm}\},\\
\sigma_{B\mathcal{W}_+}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-B-Weyl}\},\\
\sigma_{B\mathcal{W}_-}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-B-Weyl}\},\\
\sigma_{B\mathcal{W}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Weyl}\},\\
\sigma_{B\mathcal{B}_+}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-B-Browder}\},\\
\sigma_{B\mathcal{B}_-}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-B-Browder}\},\\
\sigma_{B\mathcal{B}}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Browder}\}.
\end{aligned}$$

We recall that the set of Drazin invertible operators (resp, $LD(X)$, $RD(X)$) coincides with the set of B-Browder (resp. upper semi-B-Browder, lower semi-B-Browder) operators, while the set of left (right) essentially Drazin invertible operator coincides with the set of upper (lower) semi-B-Fredholm operators [7, Theorem 3.6], [3,4]. Therefore, for any $T \in L(X)$ it holds:

$$\sigma_D(T) = \sigma_{B\mathcal{B}}(T), \quad \sigma_{LD}(T) = \sigma_{B\mathcal{B}_+}(T), \quad \sigma_{RD}(T) = \sigma_{B\mathcal{B}_-}(T),$$

and

$$\sigma_{LD}^e(T) = \sigma_{B\Phi_+}(T), \quad \sigma_{RD}^e(T) = \sigma_{B\Phi_-}(T).$$

An operator $T \in L(X)$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0 for breviety) if for every open disc $\mathcal{D}_{\lambda_0}$ centerd at λ_0 the only analitic function $f : \mathcal{D}_{\lambda_0} \to X$ satisfying $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in \mathcal{D}_{\lambda_0}$ is the function $f \equiv 0$.

If $K \subset \mathbb{C}$, then ∂K is the boundary of K, acc K is the set of accumulation points of K, int K is the set of interior points of K and iso K is the set of isolated points of K. For a compact set $K \subset \mathbb{C}$, ηK denotes its connected hull.

The aim of this paper is to give characterization of the **BR** classes through properties such as topological uniform descent or quasi-Fredholmness, and properties of the appropriate spectra $\sigma_{\mathbf{R}}$, as well as to get some results regarding boundaries and connected hulls of **BR**-spectra.

Jiang et al. in [19, Theorem 3.2] characterize the set of left Drazin invertible operators proving that if $T - \lambda I$ has TUD, then $T - \lambda I$ is left Drazin invertible if and only if $\sigma_{ap}(T)$ does not cluster at λ, and also, if and only if λ is not an interior point of $\sigma_{ap}(T)$. M. Berkani, N. Castro and S.V. Djordjević proved in [11, Theorem 2.5] that, under the same condition that $T - \lambda I$ has TUD, $\sigma_p(T)$ does not cluster at λ if and only if $a(T - \lambda I) < \infty$. Further Jiang et al. in [19, Theorem 3.4] proved that if $T - \lambda I$ has TUD, then $\delta(T - \lambda I) < \infty$ if and only if $\sigma_{su}(T)$ does not cluster at λ, and also, if and only if λ is not an interior point of $\sigma_{su}(T)$.

In this paper we characterize the sets of upper and lower semi-B-Weyl operators, as well as the sets of left and right essentially Drazin invertible operators. We also give further characterisations of left and right Drazin invertible operators. By using Grabiner's punctured neighborhood theorem [14, Theorem 4.7], [7, Thorem 4.5] we prove that

$$\begin{aligned} T \in \mathbf{BR} &\Longleftrightarrow T \text{ is quasi-Fredholm and } 0 \notin \operatorname{acc} \sigma_{\mathbf{R}}(T) \\ &\Longleftrightarrow T \text{ is quasi-Fredholm and } 0 \notin \operatorname{int} \sigma_{\mathbf{R}}(T), \end{aligned} \tag{1.1}$$

for $\mathbf{R} \in \{\mathbf{R_2}, \mathbf{R_3}, \mathbf{R_4}, \mathbf{R_5}, \mathcal{W}_-(X)\}$. By an example we show that the condition that T is quasi-Fredholm in the previous formulas can not be replaced by a weaker condition that T has topological uniform descent.

Further we prove that

$$\begin{aligned} T \in \mathbf{BR} &\Longleftrightarrow T \text{ has TUD and } 0 \notin \operatorname{acc} \sigma_{\mathbf{R}}(T) \\ &\Longleftrightarrow T \text{ has TUD and } 0 \notin \operatorname{int} \sigma_{\mathbf{R}}(T), \end{aligned} \tag{1.2}$$

for $\mathbf{R} \in \{\mathbf{R_7}, \mathbf{R_8}, \mathbf{R_9}, \mathbf{R_{10}}, \mathbf{R_4^a}, \mathbf{R_5^a}, \mathcal{W}_+(X), \mathcal{W}(X), \Phi(X), \mathcal{B}(X)\}$.

The condition that T has TUD in the previous equivalences (1.2) cannot be omitted and it is demonstrated by an example. Also, the condition that T is quasi-Fredholm in the equivalences (1.1) cannot be omitted which is also demonstrated by an example.

As a consequence of these characterizations, for $\mathbf{R} \in \{\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_4}, \mathbf{R_6}, \mathbf{R_7}, \mathbf{R_9}\} \cup \{\mathcal{W}_+(X), \mathcal{W}_-(X), \mathcal{W}(X), \Phi(X)\}$ we obtain that $\operatorname{int} \sigma_{\mathbf{R}}(T) = \operatorname{int} \sigma_{\mathbf{BR}}(T)$, $\partial \sigma_{\mathbf{BR}}(T) \subset \partial \sigma_{\mathbf{R}}(T)$ and the set $\sigma_{\mathbf{R}}(T) \setminus \sigma_{\mathbf{BR}}(T)$ consists of at most countably many isolated points. Also we obtain that the boundary of $\sigma_{\mathbf{BR}}(T)$, for $\mathbf{R} \in \{\mathbf{R_6}, \mathbf{R_7}, \mathbf{R_8}, \mathbf{R_9}, \mathbf{R_{10}}, \mathbf{R_4^a}, \mathbf{R_5^a}, \mathcal{W}_+(X), \mathcal{W}(X), \Phi(X), \mathcal{B}(X)\}$ is contained in $\sigma_{TUD}(T)$, while the boundary of $\sigma_{\mathbf{BR}}(T)$, where $\mathbf{R} \in \{\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}, \mathbf{R_4}, \mathbf{R_5}, \mathcal{W}_-(X)\}$, is contained in $\sigma_{q\Phi}(T)$, and by an example it is shown that it is not contained in the TUD spectrum.

Boundaries of spectra originated from Fredholm theory were investigated by Miličić and Veselić in [25, Theorem 7]. They proved the following inclusions:

$$
\begin{array}{ccccc}
 & & & \partial\sigma_{\Phi_+}(T) & \\
 & & \subset & & \subset \\
\partial\sigma_{\mathcal{B}}(T) \subset \partial\sigma_{\mathcal{W}}(T) \subset \partial\sigma_{\Phi}(T) & & & & \partial\sigma_{\Phi_\pm}(T). \\
 & & \subset & & \subset \\
 & & & \partial\sigma_{\Phi_-}(T) &
\end{array}
$$

V. Rakočević proved (see [27, Theorem 1]) that $\partial\sigma_{\mathcal{W}}(T) \subset \sigma_{\mathcal{W}_+}(T)$ and hence there is the inclusion $\partial\sigma_{\mathcal{W}}(T) \subset \partial\sigma_{\mathcal{W}_+}(T)$. In [28, Corollary 2.5] it is proved that $\partial\sigma_{\mathcal{B}}(T) \subset \partial\sigma_{\mathcal{B}_+}(T) \subset \partial\sigma_{\mathcal{W}_+}(T)$, as well as that $\eta\sigma_{\mathcal{B}}(T) = \eta\sigma_{\mathcal{B}_+}(T) = \eta\sigma_{W_+}(T)$. The following inclusions are known:

$$
\begin{array}{ccccccccc}
 & & \partial\sigma_{\mathcal{B}_+}(T) & \subset & \partial\sigma_{\mathcal{W}_+}(T) & \subset & \partial\sigma_{\Phi_+}(T) & & \\
 & \subset & & \subset & & \subset & & \subset & \\
\partial\sigma_{\mathcal{B}}(T) & \subset & \partial\sigma_{\mathcal{W}}(T) & \subset & \partial\sigma_{\Phi}(T) & \subset & & & \partial\sigma_{\Phi_\pm}(T). \\
 & \subset & & \subset & & \subset & & \subset & \\
 & & \partial\sigma_{\mathcal{B}_-}(T) & \subset & \partial\sigma_{\mathcal{W}_-}(T) & \subset & \partial\sigma_{\Phi_-}(T) & &
\end{array}
$$

We generalize these results to the case of spectra originated from semi-B-Fredholm theory and prove the following inclusions:

$$
\begin{array}{ccccccccc}
 & & \partial\sigma_{B\mathcal{B}_+}(T) & \subset & \partial\sigma_{B\mathcal{W}_+}(T) & \subset & \partial\sigma_{B\Phi_+}(T) & & \\
 & \subset & & \subset & & \subset & & \subset & \\
\partial\sigma_{B\mathcal{B}}(T) & \subset & \partial\sigma_{B\mathcal{W}}(T) & \subset & \partial\sigma_{B\Phi}(T) & \subset & & & \partial\sigma_{q\Phi}(T), \\
 & \subset & & \subset & & \subset & & \subset & \\
 & & \partial\sigma_{B\mathcal{B}_-}(T) & \subset & \partial\sigma_{B\mathcal{W}_-}(T) & \subset & \partial\sigma_{B\Phi_-}(T) & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & \partial\sigma_{B\mathcal{B}_+}(T) & \subset & \partial\sigma_{B\mathcal{W}_+}(T) & & & & \\
 & \subset & & \subset & & \subset & & & \\
\partial\sigma_{B\mathcal{B}}(T) & \subset & \partial\sigma_{B\mathcal{W}}(T) & \subset & \partial\sigma_{B\Phi}(T) & \subset & \partial\sigma_{B\Phi_+}(T) & \subset & \partial\sigma_{TUD}(T), \\
 & \subset & & & & \subset & & \subset & \\
 & & & \partial\sigma_{dsc}(T) & & \subset & \partial\sigma^e_{dsc}(T) & &
\end{array}
$$

as well as that the connected hulls of all spectra mentioned in the previous inclusions are mutually equal and also coincide with the connected hull of Kato type spectrum.

As an application we get that a bounded linear operator T is meromorphic, that is its non-zero spectral points are poles of its resolvent, if and only if $\sigma_{B\Phi}(T) \subset \{0\}$ and this is exactly when $\sigma_{TUD}(T) \subset \{0\}$. This result was obtained earlier (see [10] and [20]). Jiang et al. in [20, Corollary 3.3] proved it by using the local constancy of the mappings $\lambda \mapsto K(\lambda I - T) + H_0(\lambda I - T)$ and $\lambda \mapsto \overline{K(\lambda I - T) \cap H_0(\lambda I - T)}$ [20, Theorem 2.6] and results about SVEP established in [19], but our method of proof is rather different and more direct. Jiang et al. also obtained that if $\rho_{TUD}(T)$ has only one component, then $\sigma_D(T) = \sigma_{TUD}(T)$ [20, Theorem 3.1] and hence, if $\sigma(T)$ is countable or contained in a line segment, then $\sigma_D(T) = \sigma_{TUD}(T)$ [20, p. 1156]. We give here an alternative proof of these results and get more than this: if $\sigma(T)$ is contained in a line, then $\sigma_D(T) = \sigma_{TUD}(T)$, and moreover, if $\sigma_{\mathbf{R}}(T)$ is contained in a line for

$\mathbf{R} \in \{\mathbf{R_6}, \mathbf{R_7}, \mathbf{R_8}, \mathbf{R_9}, \mathbf{R_{10}}, \mathbf{R_4^a}, \mathbf{R_5^a}, \mathcal{W}_+(X), \mathcal{W}(X), \Phi(X), \mathcal{B}(X)\}$, then $\sigma_{\mathbf{BR}}(T) = \sigma_{TUD}(T)$. On the other side if $\mathbf{R} \in \{\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}, \mathbf{R_4}, \mathbf{R_5}, \mathcal{W}_-(X)\}$ and $\sigma_{\mathbf{R}}(T)$ is contained in a line, then $\sigma_{\mathbf{BR}}(T) = \sigma_{qF}(T)$. We also prove that if $\sigma_*(T)$ is contained in a line for $\sigma_* \in \{\sigma_{B\Phi}, \sigma_{B\mathcal{W}}, \sigma_{LD}^e, \sigma_{B\mathcal{W}_+}, \sigma_{LD}, \sigma_{dsc}^e, \sigma_{dsc}\}$, then $\sigma_*(T) = \sigma_{TUD}(T)$, while if $\sigma_*(T)$ is contained in a line for $\sigma_* \in \{\sigma_{RD}^e, \sigma_{B\mathcal{W}_-}, \sigma_{RD}\}$, then $\sigma_*(T) = \sigma_{qF}(T)$. In particular, if $\sigma_p(T)$ ($\sigma_{cp}(T)$) is countable or contained in a line, then $\sigma_{LD}(T) = \sigma_{TUD}(T)$ ($\sigma_{RD}(T) = \sigma_{q\Phi}(T)$ and $\sigma_{dsc}(T) = \sigma_{TUD}(T)$). Furthermore, by using connected hulls we show that if $\mathbb{C}\setminus\sigma_*(T)$ has only one component where σ_* is one of $\sigma_{q\Phi}, \sigma_{Kt}, \sigma_{B\Phi}, \sigma_{B\mathcal{W}}, \sigma_{LD}^e, \sigma_{B\mathcal{W}_+}, \sigma_{LD}, \sigma_{RD}^e, \sigma_{B\mathcal{W}_-}, \sigma_{RD}, \sigma_{dsc}^e, \sigma_{dsc}$, then $\sigma_*(T) = \sigma_D(T)$. Also we give an alternative proof of Theorem 2.10 in [10]. As a consequence we get that if $\sigma_*(T) = \partial\sigma_*(T) = \operatorname{acc}\sigma_*(T)$, then $\sigma_*(T) = \sigma_{TUD}(T)$ for $\sigma_* \in \{\sigma_{\mathcal{W}_+}, \sigma_{\mathcal{W}_-}, \sigma_{\mathcal{W}}, \sigma_{B\mathcal{W}_-}, \sigma_{\Phi_+}, \sigma_{\Phi_-}, \sigma_{\Phi}, \sigma_{RD}^e, \sigma_{ap}, \sigma_{su}, \sigma_{\mathcal{B}_+}, \sigma_{\mathcal{B}_-}, \sigma_{\mathcal{B}}, \sigma_{RD}, \sigma\}$. In particular, if $\sigma_{ap}(T) = \partial\sigma(T)$ ($\sigma_{su}(T) = \partial\sigma(T)$) and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$, then $\sigma_{TUD}(T) = \sigma_{ap}(T)$ ($\sigma_{TUD}(T) = \sigma_{su}(T)$). It improves the corresponding results of P. Aiena and E. Rosas [2, Theorem 2.10, Corollary 2.11]. These results are then used to find the TUD spectrum of arbitrary non-invertible isometry. We also use them to find the TUD spectrum and B-spectra of the forward and backward unilateral shifts on $c_0(\mathbb{N})$, $c(\mathbb{N})$, $\ell_\infty(\mathbb{N})$ or $\ell_p(\mathbb{N})$, $p \geq 1$, and also of Cesáro operator.

2 Semi-B-Weyl and Semi-B-Fredholm Operators

We start with the following auxiliary assertions.

Lemma 2.1 *Let $T \in L(X)$ have TUD for $n \geq d$ and finite essential ascent. Then $R(T^n)$ is closed in X for each integer $n \geq d$.*

Proof Since T has finite essential ascent and TUD for $n \geq d$, we have that

$$\dim(N(T) \cap R(T^n)) < \infty \text{ for all } n \geq d.$$

It means that $\alpha(T_n) < \infty$ for $T_n : R(T^n) \to R(T^n)$ and hence $\alpha(T_n^d) \leq d \cdot \alpha(T_n) < \infty$. So we have that

$$\dim(N(T^d) \cap R(T^n)) < \infty \text{ for all } n \geq d. \tag{2.1}$$

From [14, Theorem 3.2] it follows that $N(T^d) + R(T^n)$ is closed in X for every $n \geq 0$. According to (2.1), $N(T^d) \cap R(T^n)$ is closed for every $n \geq d$ and then by [26, Lemma 20.3] we obtain that $R(T^n)$ is closed for every $n \geq d$. □

Lemma 2.2 *Let $T \in L(X)$. Then:*

(1) *T has TUD and $a_e(T) < \infty \iff T$ is left essentially Drazin invertible.*
(2) *T has TUD and $a(T) < \infty \iff T$ is left Drazin invertible.*

Proof (1) Suppose that T has TUD for $n \geq d$ and that $a_e(T) < \infty$. From Lemma 2.1 it follows that there exists $n \geq a_e(T) + 1$ such that $R(T^n)$ is closed. According to [23, Lemma 7] it follows that $R(T^{a_e(T)+1})$ is closed and hence T is left essentially Drazin invertible.

The opposite inclusion is clear (see [7, p. 166 and 172]).

(2) can be proved similarly. □

In the following two theorems we characterize upper and lower semi-B-Weyl operators.

Theorem 2.3 *Let $\lambda \in \mathbb{C}$, $T \in L(X)$ and let $T - \lambda I$ have TUD for $n \geq d$. Then the following statements are equivalent:*

(1) *$\sigma_{\mathcal{W}_+}(T)$ does not cluster at λ;*
(2) *λ is not an interior point of $\sigma_{\mathcal{W}_+}(T)$;*
(3) *$\sigma_{B\mathcal{W}_+}(T)$ does not cluster at λ;*
(4) *λ is not an interior point of $\sigma_{B\mathcal{W}_+}(T)$;*
(5) *$T - \lambda I$ is an upper semi-B-Weyl operator.*

Proof (1)$\Longrightarrow$(2), (3)$\Longrightarrow$(4) Clear.

(1)$\Longrightarrow$(3), (2)$\Longrightarrow$(4) It follows from the inclusion $\sigma_{B\mathcal{W}_+}(T) \subset \sigma_{\mathcal{W}_+}(T)$.

(4)$\Longrightarrow$(5) Since $T - \lambda I$ has TUD for $n \geq d$, from [14, Theorem 4.7] it follows that there exists an $\epsilon > 0$ such that for every $\mu \in \mathbb{C}$ the following implication holds:

$$\begin{aligned}&0 < |\mu - \lambda| < \epsilon \Longrightarrow\\ &c_n(T - \mu I) = c_d(T - \lambda I) \text{ and } c'_n(T - \mu I) = c'_d(T - \lambda I) \text{ for all } n \geq 0.\end{aligned} \tag{2.2}$$

Suppose that λ is not an interior point of $\sigma_{B\mathcal{W}_+}(T)$. Then there exists $\mu \in \mathbb{C}$ such that $0 < |\mu - \lambda| < \epsilon$ and $T - \mu I$ is an upper semi-B-Weyl operator. Therefore, $c'_n(T - \mu I) = \dim(N(T - \mu I) \cap R((T - \mu I)^n)) < \infty$ for n large enough and according to (2.2) we obtain that $c'_d(T - \lambda I) < \infty$, and so $a_e(T - \lambda I) \leq d$.

From Lemma 2.1 it follows that $R((T - \lambda I)^d)$ and $R((T - \lambda I)^{d+1})$ are closed. As $\dim(N(T - \lambda I) \cap R((T - \lambda I)^d)) = c'_d(T - \lambda I) < \infty$, we have that the restriction of $T - \lambda I$ to $R((T - \lambda I)^d)$ is an upper semi-Fredholm operator. Consequently, $T - \lambda I$ is an upper semi-B-Fredholm operator and since

$$\begin{aligned}\operatorname{ind}(T - \lambda I) &= \dim(N(T - \lambda I) \cap R((T - \lambda I)^d)\\ &\quad - \dim R((T - \lambda I)^d)/R((T - \lambda I)^{d+1})\\ &= c'_d(T - \lambda I) - c_d(T - \lambda I) = c'_n(T - \mu I) - c_n(T - \mu I)\\ &= \operatorname{ind}(T - \mu I) \leq 0,\end{aligned}$$

it follows that $T - \lambda I$ is an upper semi-B-Weyl operator.

(5)$\Longrightarrow$(1) Suppose that $T - \lambda I$ is an upper semi-B-Weyl operator. Then there exists $d \in \mathbb{N}_0$ such that $T - \lambda I$ has TUD for $n \geq d$, and $c'_d(T - \lambda I) = \dim(N(T - \lambda I) \cap R((T - \lambda I)^d)) < \infty$ and $\operatorname{ind}(T - \lambda I) = c'_d(T - \lambda I) - c_d(T - \lambda I) \leq 0$. For arbitrary $\mu \in \mathbb{C}$ such that $0 < |\mu - \lambda| < \epsilon$, according to (2.2) we obtain that $\alpha(T - \mu I) = c'_0(T - \mu I) = c'_d(T - \lambda I) < \infty$ and since $R(T - \mu I)$ is closed by [14, Theorem 4.7], we conclude that $T-\mu I$ is upper semi-Fredholm with $\mathrm{ind}(T-\mu I)=c_0'(T-\mu I)-c_0(T-\mu I)=c_d'(T-\lambda I)-c_d(T-\lambda I)\le 0$, that is $T-\mu I$ is upper semi-Weyl. Therefore, λ is not an accumulation point of $\sigma_{\mathcal{W}_+}(T)$. □

Theorem 2.4 *Let $\lambda\in\mathbb{C}$, $T\in L(X)$ and let $T-\lambda I$ have TUD for $n\ge d$. Then the following statements are equivalent:*

(1) *$\sigma_{\mathcal{W}_-}(T)$ does not cluster at λ;*
(2) *λ is not an interior point of $\sigma_{\mathcal{W}_-}(T)$;*
(3) *$\sigma_{B\mathcal{W}_-}(T)$ does not cluster at λ;*
(4) *λ is not an interior point of $\sigma_{B\mathcal{W}_-}(T)$.*
In particular, if $T-\lambda I$ is quasi-Fredholm, then the statements (1)–(4) are equivalent to the following satement:
(5) *$T-\lambda I$ is a lower semi-B-Weyl operator.*

Proof (1)$\Longrightarrow$(2), (3)$\Longrightarrow$(4) Clear.

(1)$\Longrightarrow$(3), (2)$\Longrightarrow$(4) It follows from the inclusions $\sigma_{B\mathcal{W}_-}(T)\subset\sigma_{\mathcal{W}_-}(T)$.

(4)$\Longrightarrow$(1) Suppose that λ is not an interior point of $\sigma_{B\mathcal{W}_-}(T)$. Since $T-\lambda I$ has TUD for $n\ge d$, according to [14, Theorem 4.7] there exists an $\epsilon>0$ such that for every $\mu\in\mathbb{C}$ the implication (2.2) holds. From $\lambda\notin\mathrm{int}\,\sigma_{B\mathcal{W}_-}(T)$ it follows that there exists $\mu_0\in\mathbb{C}$ such that $0<|\mu_0-\lambda|<\epsilon$ i $T-\mu_0 I$ is a lower semi-B-Weyl operator. Hence there exists $n\in\mathbb{N}_0$ such that $c_n(T-\mu_0 I)=\dim R((T-\mu_0 I)^n)/R((T-\mu_0 I)^{n+1})<\infty$ and $\mathrm{ind}(T-\mu_0 I)=c_n'(T-\mu_0 I)-c_n(T-\mu_0 I)\ge 0$, which according to (2.2) implies that $c_d(T-\lambda I)<\infty$ and $c_d'(T-\lambda I)-c_d(T-\lambda I)\ge 0$. Using (2.2) again we get that for every $\mu\in\mathbb{C}$ such that $0<|\mu-\lambda|<\epsilon$ we have that $\beta(T-\mu I)=c_0(T-\mu I)=c_d(T-\lambda I)<\infty$ and hence $T-\mu I$ is lower semi-Fredholm with $\mathrm{ind}(T-\mu I)=c_0'(T-\mu I)-c_0(T-\mu I)=c_d'(T-\lambda I)-c_d(T-\lambda I)\ge 0$. This means that λ is not an accumulated point of $\sigma_{\mathcal{W}_-}(T)$.

(4)$\Longrightarrow$(5) Suppose that $T-\lambda I$ is quasi-Fredholm. Then there exists $d\in\mathbb{N}_0$ such that $R(T-\lambda I)+N((T-\lambda I)^n)=R(T-\lambda I)+N((T-\lambda I)^d)$ for all $n\ge d$ and $R((T-\lambda I)^{d+1})$ is closed. So $T-\lambda I$ has TUD for $n\ge d$. From [14, Theorem 4.7] it follows that there exists an $\epsilon>0$ such that for every $\mu\in\mathbb{C}$ the implication (2.2) holds.

Further, suppose that $\lambda\notin\mathrm{int}\,\sigma_{B\mathcal{W}_-}(T)$. Then there exists $\mu\in\mathbb{C}$ such that $0<|\mu-\lambda|<\epsilon$ and $T-\mu I$ is a lower semi-B-Weyl operator. Therefore,

$$c_n(T-\mu I)=\dim(R((T-\mu I)^n)/R((T-\mu I)^{n+1})<\infty \text{ for } n \text{ large enough},$$

and according to (2.2) we obtain that $c_d(T-\lambda I)<\infty$. As $R((T-\lambda I)^{d+1})$ is closed, from [23, Lemma 12], we conclude that $R((T-\lambda I)^d)$ is closed. Since

$$\dim(R((T-\lambda I)^d)/R((T-\lambda I)^{d+1})=c_d(T-\lambda I)<\infty,$$

we have that the restriction of $T-\lambda I$ to $R((T-\lambda I)^d)$ is a lower semi-Fredholm operator. Therefore, $T-\lambda I$ is a lower semi-B-Fredholm operator and, as in the proof of the implication (4)$\Longrightarrow$(5) in Theorem 2.3, we conclude that $\operatorname{ind}(T-\lambda I) = \operatorname{ind}(T-\mu I) \geq 0$. Consequently, $T-\lambda I$ is a lower semi-B-Weyl operator.

(5)$\Longrightarrow$(1) Suppose that $T-\lambda I$ is a lower semi-B-Weyl operator. Then there is $d \in \mathbb{N}_0$ such that $T-\lambda I$ has TUD for $n \geq d$ and hence there exists an $\epsilon > 0$ such that for every $\mu \in \mathbb{C}$ the implication (2.2) holds. Also we have that

$$c_d(T-\lambda I) = \dim(R((T-\lambda I)^d)/R((T-\lambda I)^{d+1}) < \infty$$

and

$$0 \leq \operatorname{ind}(T-\lambda I) = c'_d(T-\lambda I) - c_d(T-\lambda I).$$

For arbitrary $\mu \in \mathbb{C}$ such that $0 < |\mu-\lambda| < \epsilon$, according to (2.2), we obtain that $\beta(T-\mu I) = c_0(T-\mu I) = c_d(T-\lambda I) < \infty$ and $\operatorname{ind}(T-\mu I) = c'_0(T-\mu I) - c_0(T-\mu I) = c'_d(T-\lambda I) - c_d(T-\lambda I) \geq 0$, which implies that $T-\mu I$ is a lower semi-Weyl operator. Consequently, λ is not an accumulation point of $\sigma_{\mathcal{W}_-}(T)$. □

Theorem 2.5 *Let $\lambda \in \mathbb{C}$, $T \in L(X)$ and let $T-\lambda I$ have TUD for $n \geq d$. Then the following statements are equivalent:*

(1) *$\sigma_{\mathcal{W}}(T)$ does not cluster at λ;*
(2) *λ is not an interior point of $\sigma_{\mathcal{W}}(T)$;*
(3) *$\sigma_{B\mathcal{W}}(T)$ does not cluster at λ;*
(4) *λ is not an interior point of $\sigma_{B\mathcal{W}}(T)$;*
(5) *$T-\lambda I$ is a B-Weyl operator.*

Proof (4)$\Longrightarrow$(5) Suppose that $\lambda \notin \operatorname{int}\sigma_{B\mathcal{W}}(T)$ and that $T-\lambda I$ has TUD for $n \geq d$. According to [14, Theorem 4.7] there exists an $\epsilon > 0$ such that for every $\mu \in \mathbb{C}$ the implication (2.2) holds. From $\lambda \notin \operatorname{int}\sigma_{B\mathcal{W}}(T)$ it follows that there exists $\mu \in \mathbb{C}$ such that $0 < |\mu-\lambda| < \epsilon$ and $T-\mu I$ is a B-Weyl operator. Therefore, for n large enough we have that $c_n(T-\mu I) = \dim(R((T-\mu I)^n)/R((T-\mu I)^{n+1})) < \infty$, $c'_n(T-\mu I) = \dim(N(T-\mu I) \cap R((T-\mu I)^n)) < \infty$ and $0 = \operatorname{ind}(T-\mu I) = c'_n(T-\mu I) - c_n(T-\mu I)$. According to (2.2) we obtain that $c_d(T-\lambda I) = c'_d(T-\lambda I) < \infty$, that is

$$\dim(N(T-\lambda I) \cap R((T-\lambda I)^d)) = \dim(R((T-\lambda I)^d)/R((T-\lambda I)^{d+1})) < \infty.$$

It means that the restriction of $T-\lambda I$ to $R((T-\lambda I)^d)$ is a Weyl operator. Therefore, $T-\lambda I$ is a B-Weyl operator.

The implication (5)$\Longrightarrow$(1) follows from Theorems 2.3 and 2.4. □

We need the following well-known results (see [23], [9, Remark A (iii)], [6, Proposition 3.1], [12, Corollary 1.3], [13, Corollary 2.5], [14, Theorem 4.7 and Corollary 4.8], [1, Corollary 1.45]).

Proposition 2.6 *For $T \in L(X)$ the set $\sigma_*(T)$ is compact, where $\sigma_* \in \{\sigma_D, \sigma_{LD}, \sigma^e_{LD}, \sigma_{B\mathcal{W}}, \sigma_{B\Phi}, \sigma_{B\mathcal{W}_+}, \sigma_{dsc}, \sigma^e_{dsc}, \sigma_{RD}, \sigma^e_{RD}, \sigma_{B\mathcal{W}_-}, \sigma_{Kt}, \sigma_{q\Phi}, \sigma_{TUD}\}$.*

Corollary 2.7 *Let $T \in L(X)$. Then*

(1) $\sigma_{B\mathcal{W}_+}(T) = \sigma_{TUD}(T) \cup \operatorname{int}\sigma_*(T) = \sigma_{TUD}(T) \cup \operatorname{acc}\sigma_*(T)$, *for* $\sigma_* \in \{\sigma_{\mathcal{W}_+}, \sigma_{B\mathcal{W}_+}\}$;
(2) $\sigma_{B\mathcal{W}_-}(T) = \sigma_{q\Phi}(T) \cup \operatorname{int}\sigma_*(T) = \sigma_{q\Phi}(T) \cup \operatorname{acc}\sigma_*(T)$; *for* $\sigma_* \in \{\sigma_{\mathcal{W}_-}, \sigma_{B\mathcal{W}_-}\}$;
(3) $\sigma_{B\mathcal{W}}(T) = \sigma_{TUD}(T) \cup \operatorname{int}\sigma_*(T) = \sigma_{TUD}(T) \cup \operatorname{acc}\sigma_*(T)$; *for* $\sigma_* \in \{\sigma_{\mathcal{W}}, \sigma_{B\mathcal{W}}\}$.
(4) $\operatorname{int}\sigma_{\mathcal{W}_*}(T) = \operatorname{int}\sigma_{B\mathcal{W}_*}(T)$, *for* $\mathcal{W}_* \in \{\mathcal{W}_+, \mathcal{W}_-, \mathcal{W}\}$;
(5) $\partial\,\sigma_{B\mathcal{W}_*}(T) \subset \partial\,\sigma_{\mathcal{W}_*}(T)$, *for* $\mathcal{W}_* \in \{\mathcal{W}_+, \mathcal{W}_-, \mathcal{W}\}$;
(6) $\sigma_{\mathcal{W}_+}(T)\backslash\sigma_{B\mathcal{W}_+}(T) = (\operatorname{iso}\sigma_{\mathcal{W}_+}(T))\backslash\sigma_{TUD}(T)$,
$\sigma_{\mathcal{W}_-}(T)\backslash\sigma_{B\mathcal{W}_-}(T) = (\operatorname{iso}\sigma_{\mathcal{W}_-}(T))\backslash\sigma_{qF}(T)$,
$\sigma_{\mathcal{W}}(T)\backslash\sigma_{B\mathcal{W}}(T) = (\operatorname{iso}\sigma_{\mathcal{W}}(T))\backslash\sigma_{TUD}(T)$;
(7) $\sigma_{\mathcal{W}_*}(T)\backslash\sigma_{B\mathcal{W}_*}(T)$, *where* $\mathcal{W}_* \in \{\mathcal{W}_+, \mathcal{W}_-, \mathcal{W}\}$, *consists of at most countably many isolated points.*

Proof (1) Let $\sigma_* \in \{\sigma_{\mathcal{W}_+}, \sigma_{B\mathcal{W}_+}\}$. From Theorem 2.3 it follows that $T - \lambda I$ is upper semi-Weyl if and only if $T - \lambda I$ has TUD and λ is not an interior point of $\sigma_*(T)$, that is there is the following equality:

$$\sigma_{B\mathcal{W}_+}(T) = \sigma_{TUD}(T) \cup \operatorname{int}\sigma_*(T). \tag{2.3}$$

Also from Theorem 2.3 it follows that $T - \lambda I$ is upper semi-Weyl if and only if $T - \lambda I$ has TUD and λ is not an accumulation point of $\sigma_*(T)$, which implies that $\sigma_{B\mathcal{W}_+}(T) = \sigma_{TUD}(T) \cup \operatorname{acc}\sigma_*(T)$.

The equalities in (2) and (3) follow from Theorems 2.4 and 2.5, respectively.

(4) For $\mathcal{W}_* \in \{\mathcal{W}_+, \mathcal{W}_-, \mathcal{W}\}$, from (1), (2) and (3) it follows that $\operatorname{int}\sigma_{\mathcal{W}_*}(T) \subset \sigma_{B\mathcal{W}_*}(T)$ and hence, $\operatorname{int}\sigma_{\mathcal{W}_*}(T) \subset \operatorname{int}\sigma_{B\mathcal{W}_*}(T)$. The converse inclusion follows from the inclusion $\sigma_{B\mathcal{W}_*}(T) \subset \sigma_{\mathcal{W}_*}(T)$.

(5) Since $\sigma_{B\mathcal{W}_*}(T)$ is closed (Proposition 2.6), we have that $\partial\,\sigma_{B\mathcal{W}_*}(T) \subset \sigma_{B\mathcal{W}_*}(T)$. As $\sigma_{B\mathcal{W}_*}(T) \subset \sigma_{\mathcal{W}_*}(T)$ and $\sigma_{\mathcal{W}_*}(T) = \partial\,\sigma_{\mathcal{W}_*}(T) \cup \operatorname{int}\sigma_{\mathcal{W}_*}(T)$ since $\sigma_{\mathcal{W}_*}(T)$ is also closed, from (4) it follows that $\partial\,\sigma_{B\mathcal{W}_*}(T) \subset \partial\,\sigma_{\mathcal{W}_*}(T)$.

(6) Let $\lambda \in \sigma_{\mathcal{W}_+}(T)\backslash\sigma_{B\mathcal{W}_+}(T)$. From (1) we get that $\lambda \notin \operatorname{acc}\sigma_{\mathcal{W}_+}(T)$ and hence, $\lambda \in \operatorname{iso}\sigma_{\mathcal{W}_+}(T)$. As $\lambda \notin \sigma_{B\mathcal{W}_+}(T)$, it follows that $\lambda \notin \sigma_{TUD}(T)$ and so, $\lambda \in (\operatorname{iso}\sigma_{\mathcal{W}_+}(T))\backslash\sigma_{TUD}(T)$.

Suppose that $\lambda \in (\operatorname{iso}\sigma_{\mathcal{W}_+}(T))\backslash\sigma_{TUD}(T)$. Then $\lambda \in \sigma_{\mathcal{W}_+}(T)$, $\lambda \notin \operatorname{acc}\sigma_{\mathcal{W}_+}(T)$ and $T - \lambda I$ has TUD. According to Theorem 2.3 we get that $T - \lambda I$ is upper semi-B-Weyl and thus, $\lambda \in \sigma_{\mathcal{W}_+}(T)\backslash\sigma_{B\mathcal{W}_+}(T)$.

The rest of equalities can be proved similarly.

(7) follows from (6). □

In the following theorem we characterize left essentially Drazin invertible operators, that is, upper semi-B-Fredholm operators.

Theorem 2.8 *Let $\lambda \in \mathbb{C}$, $T \in L(X)$ and let $T - \lambda I$ have TUD for $n \geq d$. Then the following statements are equivalent:*

(1) $\sigma_{\Phi_+}(T)$ *does not cluster at* λ;

(2) *λ is not an interior point of $\sigma_{\Phi_+}(T)$;*
(3) *$\sigma^e_{LD}(T)$ does not cluster at λ;*
(4) *λ is not an interior point of $\sigma^e_{LD}(T)$;*
(5) *$a_e(T - \lambda I) < \infty$;*
(6) *$T - \lambda I$ is left essentially Drazin invertible.*

Proof (1)$\Longrightarrow$(2), (3)$\Longrightarrow$(4) Obvious.

(1)$\Longrightarrow$(3), (2)$\Longrightarrow$(4) It follows from the inclusions $\sigma^e_{LD}(T) \subset \sigma_{\Phi_+}(T)$.

(4)$\Longrightarrow$(5) Suppose that λ is not an interior point of $\sigma^e_{LD}(T)$. Since $T - \lambda I$ has TUD for $n \geq d$, according to [14, Theorem 4.7] there exists an $\epsilon > 0$ such that if $0 < |\lambda - \mu| < \epsilon$ we have that

$$c'_n(T - \mu I) = c'_d(T - \lambda I) \text{ for all } n \geq 0. \tag{2.4}$$

Since $\lambda \notin \operatorname{int} \sigma^e_{LD}(T)$, there is $\mu \in \mathbb{C}$ such that $0 < |\mu - \lambda| < \epsilon$ and $T - \mu I$ is left essentially Drazin invertible. Thus $a_e(T - \mu I) < \infty$, which implies that $c'_n(T - \mu I) < \infty$ for some $n \in \mathbb{N}_0$. According to (2.4) we conclude that $c'_d(T - \lambda I) < \infty$ and hence $a_e(T - \lambda I) \leq d$.

(5)$\Longrightarrow$(6) It follows from Lemma 2.2 (1).

(6)$\Longrightarrow$(5) It is obvious.

(5)$\Longrightarrow$(1) Let $a_e(T - \lambda I) < \infty$. Since $T - \lambda I$ has TUD, from [14, Corolary 4.8 (f)] we get that there is an $\epsilon > 0$ such that for every $\mu \in \mathbb{C}$, from $0 < |\lambda - \mu| < \epsilon$ it follows that $T - \mu I$ is upper semi-Fredholm. This means that λ is not an accumulation points of $\sigma_{\Phi_+}(T)$. □

We need the following result.

Proposition 2.9 [7, Proposition 3.4] *Let $T \in L(X)$. Then*

(1) *T is quasi-Fredholm and $\delta(T) < \infty \Longleftrightarrow T$ is right Drazin invertible.*
(2) *T is quasi-Fredholm and $\delta_e(T) < \infty \Longleftrightarrow T$ is right essentially Drazin invertible.*

Example 2.10 Let H be a Hilbert space with an orthonormal basis $\{e_{ij}\}_{i,j=1}^{\infty}$ and let the operator T defined by:

$$Te_{i,j} = \begin{cases} 0 & \text{if } j = 1, \\ \frac{1}{i} e_{i,1}, & \text{if } j = 2 \\ e_{i,j-1}, & \text{otherwise} \end{cases}$$

It is easily seen that $R(T) = R(T^2)$ and $R(T)$ is not closed. Hence $R(T^n)$ is not closed for all $n \geq 1$ and so T is neither a right Drazin invertible operator nor a right essentially Drazin invertible operator. However, since $R(T) = R(T^2)$, then T has uniform descent for $n \geq 1$ and $N(T) + R(T) = X$. Hence $N(T) + R(T)$ is closed and from [14, Theorem 3.2] it follows that T has TUD for $n \geq 1$. We remark that finite descent or finite essential descent of a bounded operator imply that it has TUD but does not imply closeness of ranges of its powers. So, T is an operator with $\delta(T) = \delta_e(T) < \infty$ which hence has TUD, but T is neither right Drazin invertible nor right essentially Drazin invertible and this shows that the condition that T is quasi-Fredholm in the assertions (1) and (2) in Proposition 2.9 can neither be omitted nor replaced by a weaker condition that T has TUD.

In the following theorem we give some characterizations of right essentially Drazin invertible, that is, lower semi-B-Fredholm operators.

Theorem 2.11 *Let $\lambda \in \mathbb{C}$, $T \in L(X)$ and let $T - \lambda I$ have TUD for $n \geq d$. Then the following statements are equivalent:*

(1) *$\sigma_{\Phi_-}(T)$ does not cluster at λ;*
(2) *λ is not an interior point of $\sigma_{\Phi_-}(T)$;*
(3) *$\sigma^e_{RD}(T)$ does not cluster at λ;*
(4) *λ is not an interior point of $\sigma^e_{RD}(T)$;*
(5) *$\sigma^e_{dsc}(T)$ does not cluster at λ;*
(6) *λ is not an interior point of $\sigma^e_{dsc}(T)$;*
(7) *$\delta_e(T - \lambda I) < \infty$.*
In particular, if $T - \lambda I$ is quasi-Fredholm then the statements (1)–(7) are equivalent to the following satement:
(8) *$T - \lambda I$ is right essentially Drazin invertible.*

Proof (1)$\Longrightarrow$(2), (3)$\Longrightarrow$(4), (5)$\Longrightarrow$(6) Obvious.

(1)$\Longrightarrow$(3)$\Longrightarrow$(5), (2)$\Longrightarrow$(4)$\Longrightarrow$(6) It follows from the inclusions $\sigma^e_{dsc}(T) \subset \sigma^e_{RD}(T) \subset \sigma_{\Phi_-}(T)$.

(6)$\Longrightarrow$(7) Suppose that λ is not an interior point of $\sigma^e_{dsc}(T)$. Since $T - \lambda I$ has TUD for $n \geq d$, by [14, Theorem 4.7] there exists an $\epsilon > 0$ such that for every $\mu \in \mathbb{C}$, from $0 < |\lambda - \mu| < \epsilon$ it follows that $c_n(T - \mu I) = c_d(T - \lambda I)$ for all $n \geq 0$. Since $\lambda \notin \operatorname{int} \sigma^e_{dsc}(T)$, there is $\mu \in \mathbb{C}$ such that $0 < |\mu - \lambda| < \epsilon$ and $\delta_e(T - \mu I) < \infty$. This implies that $c_d(T - \lambda I) < \infty$ and hence $\delta_e(T - \lambda I) \leq d$.

(7)$\Longrightarrow$(1) Let $\delta_e(T - \lambda I) < \infty$. Then $T - \lambda I$ has TUD and from [14, Corolary 4.8 (g)] it follows that there is an $\epsilon > 0$ such that if $0 < |\lambda - \mu| < \epsilon$ we have that $T - \mu I$ is lower semi-Fredholm. This means that λ is not an accumulation points of $\sigma_{\Phi_-}(T)$.

Under assumption that $T - \lambda I$ is quasi-Fredholm, the equivalence (7)$\Longleftrightarrow$(8) follows from Proposition 2.9 (2). □

Theorem 2.12 *Let $\lambda \in \mathbb{C}$, $T \in L(X)$ and let $T - \lambda I$ have TUD for $n \geq d$. Then the following statements are equivalent:*

(1) *$\sigma_{\Phi}(T)$ does not cluster at λ;*
(2) *λ is not an interior point of $\sigma_{\Phi}(T)$;*
(3) *$\sigma_{B\Phi}(T)$ does not cluster at λ;*
(4) *λ is not an interior point of $\sigma_{B\Phi}(T)$;*
(5) *$T - \lambda I$ is a B-Fredholm operator.*

Proof (4)$\Longrightarrow$(5): It can be proved similarly to the proof of the implication (4)$\Longrightarrow$(5) in Theorem 2.5.

(5)$\Longrightarrow$(1) It follows from Theorems 2.8 and 2.11. □

Corollary 2.13 *Let $T \in L(X)$. Then*

(1) $\sigma^e_{LD}(T) = \sigma_{TUD}(T) \cup \operatorname{int} \sigma_*(T) = \sigma_{TUD}(T) \cup \operatorname{acc} \sigma_*(T)$, *where* $\sigma_* \in \{\sigma_{\Phi_+}, \sigma^e_{LD}\}$;
(2) $\sigma^e_{dsc}(T) = \sigma_{TUD}(T) \cup \operatorname{int} \sigma_*(T) = \sigma_{TUD}(T) \cup \operatorname{acc} \sigma_*(T)$, *where* $\sigma_* \in \{\sigma_{\Phi_-}, \sigma^e_{RD}, \sigma^e_{dsc}\}$;
(3) $\sigma^e_{RD}(T) = \sigma_{q\Phi}(T) \cup \operatorname{int} \sigma_*(T) = \sigma_{q\Phi}(T) \cup \operatorname{acc} \sigma_*(T)$, *where* $\sigma_* \in \{\sigma_{\Phi_-}, \sigma^e_{RD}, \sigma^e_{dsc}\}$;
(4) $\sigma_{B\Phi}(T) = \sigma_{TUD}(T) \cup \operatorname{int} \sigma_*(T) = \sigma_{TUD}(T) \cup \operatorname{acc} \sigma_*(T)$, *where* $\sigma_* \in \{\sigma_{\Phi}, \sigma_{B\Phi}\}$;
(5) $\operatorname{int} \sigma_{\Phi_+}(T) = \operatorname{int} \sigma^e_{LD}(T)$, $\operatorname{int} \sigma_{\Phi_-}(T) = \operatorname{int} \sigma^e_{RD}(T) = \operatorname{int} \sigma^e_{dsc}(T)$, $\operatorname{int} \sigma_{\Phi}(T) = \operatorname{int} \sigma_{B\Phi}(T)$;
(6) $\partial\, \sigma^e_{LD}(T) \subset \partial\, \sigma_{\Phi_+}(T)$, $\partial\, \sigma^e_{dsc}(T) \subset \partial\, \sigma^e_{RD}(T) \subset \partial\, \sigma_{\Phi_-}(T)$, $\partial\, \sigma_{B\Phi}(T) \subset \partial\, \sigma_{\Phi}(T)$;
(7) $\sigma_{\Phi_+}(T) \backslash \sigma^e_{LD}(T) = (\operatorname{iso} \sigma_{\Phi_+}(T)) \backslash \sigma_{TUD}(T)$, $\sigma_{\Phi_-}(T) \backslash \sigma^e_{dsc}(T) = (\operatorname{iso} \sigma_{\Phi_-}(T)) \backslash \sigma_{TUD}(T)$, $\sigma_{\Phi_-}(T) \backslash \sigma^e_{RD}(T) = (\operatorname{iso} \sigma_{\Phi_-}(T)) \backslash \sigma_{qF}(T)$, $\sigma_{\Phi}(T) \backslash \sigma_{B\Phi}(T) = (\operatorname{iso} \sigma_{\Phi}(T)) \backslash \sigma_{TUD}(T)$;
(8) $\sigma_{\Phi_+}(T) \backslash \sigma^e_{LD}(T)$, $\sigma_{\Phi_-}(T) \backslash \sigma^e_{dsc}(T)$, $\sigma_{\Phi_-}(T) \backslash \sigma^e_{RD}(T)$, $\sigma_{\Phi}(T) \backslash \sigma_{B\Phi}(T)$ *are at most countable.*

Proof (1) follows from Theorem 2.8, (2) and (3) follow from Theorem 2.11 and (4) follows from Theorem 2.12. (5) and (7) follow from (1), (2), (3) and (4), while (6) follows from (5) and Proposition 2.6. (8) follows from (7). □

Further we focus to left and right Drazin invertible operators. Jiang et al. proved that if $\lambda \in \mathbb{C}$, $T \in L(X)$ and $T - \lambda I$ has TUD for $n \geq d$, then the following statements are equivalent (see [19, Theorem 3.2] and the proof of this theorem):

(1) $T - \lambda I$ is left Drazin invertible;
(2) $a(T - \lambda I) < \infty$;
(3) $\sigma_{ap}(T)$ does not cluster at λ;
(4) λ is not an interior point of $\sigma_{ap}(T)$,

while M. Berkani, N. Castro and S.V. Djordjević proved in [11, Theorem 2.5] that, under the same condition that $T - \lambda I$ has TUD, $\sigma_p(T)$ does not cluster at λ if and only if $a(T - \lambda I) < \infty$. In the following theorem we add some characterisations of left Drazin invertible operators.

Theorem 2.14 *Let $\lambda \in \mathbb{C}$, $T \in L(X)$ and let $T - \lambda I$ have TUD for $n \geq d$. Then the following statements are equivalent:*

(1) *λ is not an interior point of $\sigma_p(T)$;*
(2) *$\sigma_{\mathcal{B}_+}(T)$ does not cluster at λ;*
(3) *λ is not an interior point of $\sigma_{\mathcal{B}_+}(T)$;*
(4) *$\sigma_{LD}(T)$ does not cluster at λ;*
(5) *λ is not an interior point of $\sigma_{LD}(T)$;*
(6) *$T - \lambda I$ is left Drazin invertible.*

Proof (2)$\Longrightarrow$(3), (4)$\Longrightarrow$(5) It is obvious.

(2)$\Longrightarrow$(4), (3)$\Longrightarrow$(5) It follows from the inclusion $\sigma_{LD}(T) \subset \sigma_{\mathcal{B}_+}(T)$.

(1)$\Longrightarrow$(6) Suppose that λ is not an interior point of $\sigma_p(T)$. Since $T - \lambda I$ has TUD, from [14, Corolary 4.8 (d)] it follows that $a = a(T - \lambda I) < \infty$. Now from Lemma 2.2 (2) we get that $T - \lambda I$ is a left Drazin invertible operator.

(5)$\Longrightarrow$(6) Suppose that λ is not an interior point of $\sigma_{LD}(T)$. As $T - \lambda I$ has TUD, according to [14, Corolary 4.8, (a)] we conclude that $a(T - \lambda I) < \infty$ and by Lemma 2.2 (2) $T - \lambda I$ is left Drazin invertible.

(6)$\Longrightarrow$(2) It follows from the implication (6)$\Longrightarrow$(5) in [19, Theorem 3.2].

(6)$\Longrightarrow$(1) It follows from [11, Theorem 2.5]. □

Jiang et al. in [19, Theorem 3.4] proved that if $\lambda \in \mathbb{C}$, $T \in L(X)$ and $T - \lambda I$ has TUD for $n \geq d$, then the following statements are equivalent:

(1) $\sigma_{su}(T)$ does not cluster at λ;
(2) λ is not an interior point of $\sigma_{su}(T)$;
(3) $\delta(T - \lambda I) < \infty$.

In the following theorem we add some statements equivalent to those ones in [19, Theorem 3.4].

Theorem 2.15 *Let $\lambda \in \mathbb{C}$, $T \in L(X)$ and let $T - \lambda I$ have TUD for $n \geq d$. Then the following statements are equivalent:*

(1) *$\sigma_{cp}(T)$ does not cluster at λ;*
(2) *λ is not an interior point of $\sigma_{cp}(T)$;*
(3) *$\sigma_{\mathcal{B}_-}(T)$ does not cluster at λ;*
(4) *λ is not an interior point of $\sigma_{\mathcal{B}_-}(T)$;*
(5) *$\sigma_{RD}(T)$ does not cluster at λ;*
(6) *λ is not an interior point of $\sigma_{RD}(T)$;*
(7) *$\sigma_{dsc}(T)$ does not cluster at λ;*
(8) *λ is not an interior point of $\sigma_{dsc}(T)$;*
(9) *$\delta(T - \lambda I) < \infty$.*
In particular, if $T - \lambda I$ is quasi-Fredholm, then the statements (1)–(7) are equivalent to the following satement:
(10) *$T - \lambda I$ is right Drazin invertible.*

Proof (1)$\Longrightarrow$(2), (3)$\Longrightarrow$(4), (5)$\Longrightarrow$(6), (7)$\Longrightarrow$(8) Obvious.

(3)$\Longrightarrow$(5)$\Longrightarrow$(7), (4)$\Longrightarrow$(6)$\Longrightarrow$(8) It follows from the inclusions $\sigma_{dsc}(T) \subset \sigma_{RD}(T) \subset \sigma_{\mathcal{B}_-}(T)$.

(2)$\Longrightarrow$(1), (2)$\Longrightarrow$(3) Suppose that λ is not an interior point of $\sigma_{cp}(T)$. Since $T - \lambda I$ has TUD for $n \geq d$, from [14, Theorem 4.7] we have that there is an $\epsilon > 0$ such that if $0 < |\lambda - \mu| < \epsilon$ it follows that $R(T - \mu I)$ is closed and

$$c_n(T - \mu I) = c_d(T - \lambda I) \text{ for all } n \in \mathbb{N}_0. \tag{2.5}$$

From $\lambda \notin \operatorname{int} \sigma_{cp}(T)$ it follows that there exists $\mu_0 \in \mathbb{C}$ such that $0 < |\lambda - \mu_0| < \epsilon$ and $T - \mu_0 I$ has dense range. As $R(T - \mu_0 I)$ is closed, it implies that $T - \mu_0 I$ is onto and hence $c_n(T-\mu_0 I)=0$ for all $n\in\mathbb{N}_0$. Consequently, $c_d(T-\lambda I)=0$ and hence for all $\mu\in\mathbb{C}$ such that $0<|\lambda-\mu|<\epsilon$ we have that $\beta(T-\mu I)=c_0(T-\mu I)=0$, i.e. $T-\mu I$ is surjective, which means that $\lambda\notin\operatorname{acc}\sigma_{cp}(T)$ and $\lambda\notin\operatorname{acc}\sigma_{\mathcal{B}_-}(T)$.

(8)$\Longrightarrow$(9) Suppose that λ is not an interior point of $\sigma_{dsc}(T)$. Since $T-\lambda I$ has TUD for $n\geq d$, from [14, Theorem 4.7] we have that there is an $\epsilon>0$ such that if $0<|\lambda-\mu|<\epsilon$, then the equalities (2.5) hold. From $\lambda\notin\operatorname{int}\sigma_{dsc}(T)$ we have that there exists $\mu_0\in\mathbb{C}$ such that $0<|\lambda-\mu_0|<\epsilon$ and $\delta(T-\mu_0 I)<\infty$. So there is $n\in\mathbb{N}_0$ such that $c_n(T-\mu_0 I)=0$ and hence, according to (2.5), it follows that $c_d(T-\lambda I)=0$. Thus $\delta(T-\lambda I)<\infty$.

(9)$\Longrightarrow$(1) It follows from [14, Corollary 4.8 (c)].

Under assumption that $T-\lambda I$ is quasi-Fredholm, the equivalence (9)$\Longleftrightarrow$(10) follows from Proposition 2.9 (1). $\square$

Remark 2.16 Since the operator T in Example 2.10 has the finite descent, then according to [14, Theorem 4.7 and Corollary 4.8] there exists an $\epsilon>0$ such that for $\mu\in\mathbb{C}$ from $0<|\mu|<\epsilon$ it follows that $\delta(T-\mu I)=0$, i.e. $T-\mu I$ is surjective. This means that 0 is not an accumulation point of $\sigma_{su}(T)$, as well as $\sigma_{\Phi_-}(T)$, $\sigma_{\mathcal{W}_-}(T)$, $\sigma_{RD}(T)$, $\sigma_{B\mathcal{W}_-}(T)$ and $\sigma^e_{RD}(T)$. As for every $n\in\mathbb{N}$, $R(T^n)=R(T)$ is not closed, then T is neither a lower semi-Fredholm nor a lower semi-B-Weyl operator, and as we have already mentioned T is neither right Drazin invertible nor right essentially Drazin invertible. This means that the condition that $T-\lambda I$ is quasi-Fredholm in Theorems 2.4, 2.8 and 2.15 can not be replaced by a weaker condition that $T-\lambda I$ has TUD.

The next theorem follows immediately from [19, Theorems 3.2 and 3.4] and Theorems 2.14 and 2.15.

Theorem 2.17 *Let $\lambda\in\mathbb{C}$, $T\in L(X)$ and let $T-\lambda I$ have TUD for $n\geq d$. Then the following statements are equivalent:*

(1) *$\sigma(T)$ does not cluster at λ;*
(2) *λ is not an interior point of $\sigma(T)$;*
(3) *$\sigma_{\mathcal{B}}(T)$ does not cluster at λ;*
(4) *λ is not an interior point of $\sigma_{\mathcal{B}}(T)$;*
(5) *$\sigma_D(T)$ does not cluster at λ;*
(6) *λ is not an interior point of $\sigma_D(T)$;*
(7) *$T-\lambda I$ is Drazin invertible.*

Corollary 2.18 *Let $T\in L(X)$. Then*

(1) $\sigma_{LD}(T)=\sigma_{TUD}(T)\cup\operatorname{int}\sigma_*(T)=\sigma_{TUD}(T)\cup\operatorname{acc}\sigma_*(T)$, *where $\sigma_*\in\{\sigma_p,\sigma_{ap},\sigma_{\mathcal{B}_+},\sigma_{LD}\}$;*
(2) $\sigma_{dsc}(T)=\sigma_{TUD}(T)\cup\operatorname{int}\sigma_*(T)=\sigma_{TUD}(T)\cup\operatorname{acc}\sigma_*(T)$, *where $\sigma_*\in\{\sigma_{su},\sigma_{cp},\sigma_{\mathcal{B}_-},\sigma_{RD},\sigma_{dsc}\}$;*
(3) $\sigma_{RD}(T)=\sigma_{q\Phi}(T)\cup\operatorname{int}\sigma_*(T)=\sigma_{q\Phi}(T)\cup\operatorname{acc}\sigma_*(T)$, *where $\sigma_*\in\{\sigma_{su},\sigma_{cp},\sigma_{\mathcal{B}_-},\sigma_{RD},\sigma_{dsc}\}$;*
(4) $\sigma_D(T)=\sigma_{TUD}(T)\cup\operatorname{int}\sigma_*(T)=\sigma_{TUD}(T)\cup\operatorname{acc}\sigma_*(T)$, *where $\sigma_*\in\{\sigma,\sigma_{\mathcal{B}},\sigma_D\}$;*

(5) $\operatorname{int}\sigma_{ap}(T) = \operatorname{int}\sigma_{\mathcal{B}_+}(T) = \operatorname{int}\sigma_{LD}(T)$,
$\operatorname{int}\sigma_{su}(T) = \operatorname{int}\sigma_{\mathcal{B}_-}(T) = \operatorname{int}\sigma_{RD}(T) = \operatorname{int}\sigma_{dsc}(T)$,
$\operatorname{int}\sigma(T) = \operatorname{int}\sigma_{\mathcal{B}}(T) = \operatorname{int}\sigma_D(T)$;

(6) $\partial\,\sigma_{LD}(T) \subset \partial\,\sigma_{\mathcal{B}_+}(T) \subset \partial\,\sigma_{ap}(T)$,
$\partial\,\sigma_{dsc}(T) \subset \partial\,\sigma_{RD}(T) \subset \partial\,\sigma_{\mathcal{B}_-}(T) \subset \partial\,\sigma_{su}(T)$,
$\partial\,\sigma_D(T) \subset \partial\,\sigma_{\mathcal{B}}(T) \subset \partial\,\sigma(T)$;

(7) $\sigma_*(T)\backslash\sigma_{LD}(T) = (\operatorname{iso}\sigma_*(T))\backslash\sigma_{TUD}(T)$ *for* $\sigma_* \in \{\sigma_{ap}, \sigma_{\mathcal{B}_+}\}$,
$\sigma_*(T)\backslash\sigma_{dsc}(T) = (\operatorname{iso}\sigma_*(T))\backslash\sigma_{TUD}(T)$ *for* $\sigma_* \in \{\sigma_{su}, \sigma_{\mathcal{B}_-}, \sigma_{RD}\}$,
$\sigma_*(T)\backslash\sigma_{RD}(T) = (\operatorname{iso}\sigma_*(T))\backslash\sigma_{qF}(T)$ *for* $\sigma_* \in \{\sigma_{su}, \sigma_{\mathcal{B}_-}, \sigma_{dsc}\}$,
$\sigma_*(T)\backslash\sigma_D(T) = (\operatorname{iso}\sigma_*(T))\backslash\sigma_{TUD}(T)$ *for* $\sigma_* \in \{\sigma, \sigma_{\mathcal{B}}\}$.

Proof It follows from Theorems 2.14 and 2.15, [19, Theorem 3.2], [11, Theorem 2.5], Theorem 2.17 and Proposition 2.6, similarly to the proof of Corollary 2.7. ◻

We remark that from [7, Lemma 3.1] it follows that

$$\mathbf{BR_4^a} = \mathbf{R_4^a}, \quad \mathbf{BR_5^a} = \mathbf{R_5^a}. \tag{2.6}$$

Now we can formulate a general assertion:

Theorem 2.19 *Let* $T \in L(X)$.

(1) *If* $\mathbf{R} \in \{\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}, \mathbf{R_4}, \mathbf{R_5}, \mathcal{W}_-(X)\}$, *then*

$$\begin{aligned} T \in \mathbf{BR} &\Longleftrightarrow T \text{ is quasi} - \text{Fredholm} \;\;\wedge\;\; 0 \notin \operatorname{acc}\sigma_{\mathbf{R}}(T) \\ &\Longleftrightarrow T \text{ is quasi} - \text{Fredholm} \;\;\wedge\;\; 0 \notin \operatorname{int}\sigma_{\mathbf{R}}(T). \end{aligned}$$

If $\mathbf{R} \in \{\mathbf{R_6}, \mathbf{R_7}, \mathbf{R_8}, \mathbf{R_9}, \mathbf{R_{10}}, \mathbf{R_4^a}, \mathbf{R_5^a}, \mathcal{W}_+(X), \mathcal{W}(X), \Phi(X), \mathcal{B}(X), L(X)^{-1}\}$, *then*

$$\begin{aligned} T \in \mathbf{BR} &\Longleftrightarrow T \text{ has TUD} \;\;\wedge\;\; 0 \notin \operatorname{acc}\sigma_{\mathbf{R}}(T) \\ &\Longleftrightarrow T \text{ has TUD} \;\;\wedge\;\; 0 \notin \operatorname{int}\sigma_{\mathbf{R}}(T). \end{aligned}$$

(2) *If* $\mathbf{R} \in \{\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_4}, \mathbf{R_6}, \mathbf{R_7}, \mathbf{R_9}\} \cup \{\mathcal{W}_+(X), \mathcal{W}_-(X), \mathcal{W}(X), \Phi(X), \mathcal{B}(X), L(X)^{-1}\}$, *then*

$$\begin{aligned} &\operatorname{int}\sigma_{\mathbf{R}}(T) = \operatorname{int}\sigma_{\mathbf{BR}}(T), \\ &\partial\,\sigma_{\mathbf{BR}}(T) \subset \partial\,\sigma_{\mathbf{R}}(T) \end{aligned}$$

and $\sigma_{\mathbf{R}}(T)\backslash\sigma_{\mathbf{BR}}(T)$ *consists of at most countably many isolated points.*

3 Boundaries and Connected Hulls of Corresponding Spectra

The *connected hull* of a compact subset K of the complex plane $\mathbb{C}$, denoted by ηK, is the complement of the unbounded component of $\mathbb{C}\backslash K$ [16, Definition 7.10.1]. Given a compact subset K of the plane, a hole of K is a bounded component of $\mathbb{C}\setminus K$, and so a hole of K is a component of $\eta K\setminus K$.

We shall need the following well-known result (see [17, Theorems 1.2, 1.3], [16, Theorems 7.10.2, 7.10.3]).

Proposition 3.1 *Let $K, H \subset \mathbb{C}$ be compact and let*

$$\partial K \subset H \subset K.$$

Then

$$\partial K \subseteq \partial H \subseteq H \subseteq K \subseteq \eta K = \eta H.$$

If Ω is a component of $\mathbb{C}\setminus H$, then $\Omega \subset K$ or $\Omega \cap K = \emptyset$.
The set K can be obtained from H by filling in some holes of H.

Remark 3.2 If $K \subseteq \mathbb{C}$ is at most countable, then $\eta K = K$. Therefore, for compact subsets $H, K \subseteq \mathbb{C}$, if $\eta K = \eta H$, then H is finite (countable) if and only if K is finite (countable), and in that case $H = K$. Particularly, for compact subsets $H, K \subseteq \mathbb{C}$, if $\eta K = \eta H$, then K is empty if and only if H is empty.

Corollary 3.3 *Let $T \in L(X)$.*

(1) $\partial\sigma_*(T) \subset \partial\sigma_{TUD}(T)$, *where* $\sigma_* \in \{\sigma_{B\mathcal{W}_+}, \sigma_{B\mathcal{W}}, \sigma^e_{LD}, \sigma^e_{dsc}, \sigma_{B\Phi}, \sigma_{LD}, \sigma_{dsc}, \sigma_D\}$;
(2) $\partial\sigma_*(T) \subset \partial\sigma_{q\Phi}(T)$, *where* $\sigma_* \in \{\sigma_{B\mathcal{W}_-}, \sigma_{RD}, \sigma^e_{RD}\}$.

Proof Since $\sigma_{B\mathcal{W}_+}(T)$ is closed (Proposition 2.6), it follows that $\partial\sigma_{B\mathcal{W}_+}(T) \subset \sigma_{B\mathcal{W}_+}(T)$. Hence, by using Corollary 2.7 (1), we obtain that

$$\partial\sigma_{B\mathcal{W}_+}(T) = \partial\sigma_{B\mathcal{W}_+}(T) \cap \sigma_{B\mathcal{W}_+}(T) = \partial\sigma_{B\mathcal{W}_+}(T) \cap \sigma_{TUD}(T) \subset \sigma_{TUD}(T).$$

Now from $\partial\sigma_{B\mathcal{W}_+}(T) \subset \sigma_{TUD}(T) \subset \sigma_{B\mathcal{W}_+}(T)$, according to Proposition 3.1, it follows that $\partial\sigma_{B\mathcal{W}_+}(T) \subset \partial\sigma_{TUD}(T)$.

Similarly for the rest of inclusions. □

It is known that [1, Theorem 1.65 (i)]

$$\partial\sigma_\Phi(T) \cap \operatorname{acc}\sigma_\Phi(T) \subset \sigma_{Kt}(T).$$

We remark that it holds more than this: $\partial\sigma_\Phi(T) \cap \operatorname{acc}\sigma_\Phi(T) \subset \partial\sigma_{TUD}(T)$ (see Corollary 3.5 (5)).

Further we establish the inclusions of the similar type for other essential spectra.

Corollary 3.4 *Let $T \in L(X)$. Then*

(1) $\partial\sigma_{\mathcal{W}_+}(T) \cap \operatorname{acc}\sigma_{\mathcal{W}_+}(T) \subset \partial\sigma_{\mathcal{W}_+}(T) \cap \sigma_{B\mathcal{W}_+}(T) \subset \partial\sigma_{TUD}(T)$;
(2) $\partial\sigma_{\mathcal{W}_-}(T) \cap \operatorname{acc}\sigma_{\mathcal{W}_-}(T) \subset \partial\sigma_{TUD}(T)$;
(3) $\partial\sigma_{B\mathcal{W}_-}(T) \cap \operatorname{acc}\sigma_{B\mathcal{W}_-}(T) \subset \partial\sigma_{B\mathcal{W}_-}(T) \cap \operatorname{acc}\sigma_{\mathcal{W}_-}(T) \subset \partial\sigma_{TUD}(T)$;

(4) $\partial\sigma_{\mathcal{W}}(T)\cap \operatorname{acc}\sigma_{\mathcal{W}}(T)\subset \partial\sigma_{\mathcal{W}}(T)\cap\sigma_{B\mathcal{W}}(T)\subset\partial\sigma_{TUD}(T)$.

Proof (1) By using the first equality in Corollary 2.7 (1) we get

$$\begin{aligned}\partial\sigma_{\mathcal{W}_+}(T)\cap\sigma_{B\mathcal{W}_+}(T)&=\partial\sigma_{\mathcal{W}_+}(T)\cap(\sigma_{TUD}(T)\cup \operatorname{int}\sigma_{\mathcal{W}_+}(T))\\&=\partial\sigma_{\mathcal{W}_+}(T)\cap\sigma_{TUD}(T),\end{aligned}$$

and therefore

$$\partial\sigma_{\Phi_+}(T)\cap\sigma_{B\mathcal{W}_+}(T)\subset\sigma_{TUD}(T). \tag{3.1}$$

Let $\lambda\in\partial\sigma_{\mathcal{W}_+}(T)\cap\sigma_{B\mathcal{W}_+}(T)$. Then there exists a sequence (λ_n) which converges to λ and such that $T-\lambda_n$ is upper semi-Weyl for every $n\in\mathbb{N}$. As $T-\lambda_n$ is upper semi-Fredholm, then it has TUD and so $\lambda_n\notin\sigma_{TUD}(T)$, $n\in\mathbb{N}$. As (λ_n) converges to λ and since $\lambda\in\sigma_{TUD}(T)$ according to (3.1), we get that $\lambda\in\partial\sigma_{TUD}(T)$. Therefore, $\partial\sigma_{\mathcal{W}_+}(T)\cap\sigma_{B\mathcal{W}_+}(T)\subset\partial\sigma_{TUD}(T)$.

Further, from the second equality in Corollary 2.7 (1) it follows that

$$\partial\sigma_{\mathcal{W}_+}(T)\cap \operatorname{acc}\sigma_{\mathcal{W}_+}(T)\subset\partial\sigma_{\mathcal{W}_+}(T)\cap\sigma_{B\mathcal{W}_+}(T).$$

(2) Let $T-\lambda I$ have TUD and let $\lambda\in\partial\sigma_{\mathcal{W}_-}(T)$. Since $\lambda\notin \operatorname{int}\sigma_{\mathcal{W}_-}(T)$, according to Theorem 2.4 we conclude that $\lambda\notin \operatorname{acc}\sigma_{\mathcal{W}_-}(T)$. Therefore, $\partial\sigma_{\mathcal{W}_-}(T)\cap \operatorname{acc}\sigma_{\mathcal{W}_-}(T)\subset\sigma_{TUD}(T)$. Now proceeding as in the proof of (1) we get $\partial\sigma_{\mathcal{W}_-}(T)\cap \operatorname{acc}\sigma_{\mathcal{W}_-}(T)\subset\partial\sigma_{TUD}(T)$.

(3) Suppose that $T-\lambda I$ has TUD and $\lambda\in\partial\sigma_{B\mathcal{W}_-}(T)$. From Theorem 2.3 it follows that $\lambda\notin \operatorname{acc}\sigma_{\mathcal{W}_-}(T)$. Thus $\partial\sigma_{B\mathcal{W}_-}(T)\cap acc\,\sigma_{\mathcal{W}_-}(T)\subset\sigma_{TUD}(T)$. As in the proof of (1), we obtain that

$$\partial\sigma_{B\mathcal{W}_-}(T)\cap acc\,\sigma_{\mathcal{W}_-}(T)\subset\partial\sigma_{TUD}(T).$$

From $\sigma_{B\mathcal{W}_-}(T)\subset\sigma_{\mathcal{W}_-}(T)$ it follows that $\operatorname{acc}\sigma_{B\mathcal{W}_-}(T)\subset \operatorname{acc}\sigma_{\mathcal{W}_-}(T)$, which implies the first inclusion in (3).

(4) Similarly to the proof of (1) by using Corollary 2.7 (3). □

Corollary 3.5 *Let* $T\in L(X)$.

(1) $\partial\sigma_{\Phi_+}(T)\cap \operatorname{acc}\sigma_{\Phi_+}(T)\subset\partial\sigma_{\Phi_+}(T)\cap\sigma^e_{LD}(T)\subset\partial\sigma_{TUD}(T)$;
(2) $\partial\sigma_{\Phi_-}(T)\cap \operatorname{acc}\sigma_{\Phi_-}(T)\subset\partial\sigma_{\Phi_-}(T)\cap\sigma^e_{dsc}(T)\subset\partial\sigma_{TUD}(T)$;
(3) $\partial\sigma^e_{RD}(T)\cap \operatorname{acc}\sigma^e_{RD}(T)\subset\partial\sigma^e_{RD}(T)\cap \operatorname{acc}\sigma_{\Phi_-}(T)\subset\partial\sigma^e_{RD}(T)\cap\sigma^e_{dsc}(T)=\partial\sigma^e_{RD}(T)\cap\sigma_{TUD}(T)\subset\partial\sigma_{TUD}(T)$;
(4) $\partial\sigma_{\Phi_-}(T)\cap\sigma^e_{RD}(T)=\partial\sigma_{\Phi_-}(T)\cap\sigma_{q\Phi}(T)\subset\partial\sigma_{q\Phi}(T)$;
(5) $\partial\sigma_{\Phi}(T)\cap \operatorname{acc}\sigma_{\Phi}(T)\subset\partial\sigma_{\Phi}(T)\cap\sigma_{B\Phi}(T)\subset\partial\sigma_{TUD}(T)$.

Proof It follows from Corollary 2.13, similarly to the proof of Corollary 3.4. □

Corollary 3.6 *Let* $T\in L(X)$. *Then*

(1) $\partial\sigma_{ap}(T)\cap \operatorname{acc}\sigma_{ap}(T)\subset\partial\sigma_{ap}(T)\cap\sigma_{LD}(T)\subset\partial\sigma_{TUD}(T)$;
(2) $\partial\sigma_{\mathcal{B}_+}(T)\cap \operatorname{acc}\sigma_{\mathcal{B}_+}(T)\subset\partial\sigma_{\mathcal{B}_+}(T)\cap\sigma_{LD}(T)\subset\partial\sigma_{TUD}(T)$;

(3) $\partial\sigma_p(T) \cap \operatorname{acc}\sigma_p(T) \subset \partial\sigma_p(T) \cap \sigma_{LD}(T) \subset \sigma_{TUD}(T)$;
(4) $\partial\sigma_{su}(T) \cap \operatorname{acc}\sigma_{su}(T) \subset \partial\sigma_{su}(T) \cap \sigma_{dsc}(T) \subset \partial\sigma_{TUD}(T)$;
(5) $\partial\sigma_{cp}(T) \cap \operatorname{acc}\sigma_{cp}(T) \subset \partial\sigma_{cp}(T) \cap \sigma_{dsc}(T) \subset \sigma_{TUD}(T)$;
(6) $\partial\sigma_{\mathcal{B}_-}(T) \cap \operatorname{acc}\sigma_{\mathcal{B}_-}(T) \subset \partial\sigma_{\mathcal{B}_-}(T) \cap \sigma_{dsc}(T) \subset \partial\sigma_{TUD}(T)$;
(7) $\partial\sigma_{RD}(T) \cap \operatorname{acc}\sigma_{RD}(T) \subset \partial\sigma_{RD}(T) \cap \sigma_{dsc}(T) = \partial\sigma_{RD}(T) \cap \sigma_{TUD}(T) \subset \partial\sigma_{TUD}(T)$;
(8) $\partial\sigma_{su}(T) \cap \sigma_{RD}(T) \subset \partial\sigma_{q\Phi}(T)$;
(9) $\partial\sigma_{cp}(T) \cap \sigma_{RD}(T) = \partial\sigma_{cp}(T) \cap \sigma_{q\Phi}(T) \subset \sigma_{q\Phi}(T)$;
(10) $\partial\sigma(T) \cap \operatorname{acc}\sigma(T) \subset \partial\sigma(T) \cap \sigma_D(T) \subset \partial\sigma_{TUD}(T)$;
(11) $\partial\sigma_{\mathcal{B}}(T) \cap \operatorname{acc}\sigma_{\mathcal{B}}(T) \subset \partial\sigma_{\mathcal{B}}(T) \cap \sigma_D(T) \subset \partial\sigma_{TUD}(T)$.

Proof It follows from Corollary 2.18. □

Remark 3.7 For the operator T in Example 2.10, from Remark 2.16, we can conclude that $0 \in \partial\sigma_*(T)$ where $\sigma_* \in \{\sigma_{su}, \sigma_{\Phi_-}, \sigma_{\mathcal{W}_-}, \sigma_{RD}, \sigma_{B\mathcal{W}_-}, \sigma^e_{RD}\}$. As T has TUD, we have that $0 \notin \sigma_{TUD}(T)$. So, in the inclusions (2) in Corollary 3.3, as well as in the inclusion (4) in Corollary 3.5, and in the inclusion (8) in Corollary 3.6, $\sigma_{q\Phi}(T)$ can not be replaced by $\sigma_{TUD}(T)$.

In the proof of the next theorem we use the following inclusions:

$$\begin{array}{ccccccccccc}
 & & & & \sigma^e_{LD}(T) & \subset & \sigma_{B\mathcal{W}_+}(T) & \subset & \sigma_{LD}(T) & & \\
 & & & \subset & & \subset & & \subset & & \subset & \\
\sigma_{TUD}(T) & \subset & \sigma_{q\Phi}(T) & \subset & \sigma_{Kt}(T) & \subset & \sigma_{B\Phi}(T) & \subset & \sigma_{B\mathcal{W}}(T) & \subset & \sigma_D(T). \\
 & & & \subset & & \subset & & \subset & & \subset & \\
 & \subset & & & \sigma^e_{RD}(T) & \subset & \sigma_{B\mathcal{W}_-}(T) & \subset & \sigma_{RD}(T) & & \\
 & & & \subset & & & & \subset & & & \\
 & & \sigma^e_{dsc}(T) & & \subset & & \sigma_{dsc}(T) & & & &
\end{array}$$

Theorem 3.8 *Let $T \in L(X)$. Then*

1.

$$\begin{array}{ccccccccc}
 & & \partial\sigma_{LD}(T) & \subset & \partial\sigma_{B\mathcal{W}_+}(T) & \subset & \partial\sigma^e_{LD}(T) & & \\
 & \subset & & \subset & & \subset & & \subset & \\
\partial\sigma_D(T) & \subset & \partial\sigma_{B\mathcal{W}}(T) & \subset & \partial\sigma_{B\Phi}(T) & \subset & & & \partial\sigma_{TUD}(T), \\
 & \subset & & & & \subset & & \subset & \\
 & & & \partial\sigma_{dsc}(T) & & \subset & \partial\sigma^e_{dsc}(T) & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & \partial\sigma_{LD}(T) & \subset & \partial\sigma_{B\mathcal{W}_+}(T) & \subset & \partial\sigma^e_{LD}(T) & & \\
 & \subset & & \subset & & \subset & & \subset & \\
\partial\sigma_D(T) & \subset & \partial\sigma_{B\mathcal{W}}(T) & \subset & \partial\sigma_{B\Phi}(T) & \subset & & & \partial\sigma_{q\Phi}(T), \\
 & \subset & & \subset & & \subset & & \subset & \\
 & & \partial\sigma_{RD}(T) & \subset & \partial\sigma_{B\mathcal{W}_-}(T) & \subset & \partial\sigma^e_{RD}(T) & &
\end{array}$$

$$\partial\sigma_D(T) \subset \partial\sigma_{B\mathcal{W}}(T) \subset \partial\sigma_{B\Phi}(T) \subset \partial\sigma_{Kt}(T).$$

2.

$$\begin{aligned}\eta\sigma_{TUD}(T) &= \eta\sigma_{q\Phi}(T) = \eta\sigma_{Kt}(T) = \eta\sigma_{B\Phi}(T) = \eta\sigma_{B\mathcal{W}}(T) = \eta\sigma_D(T)\\ &= \eta\sigma^e_{LD}(T) = \eta\sigma_{B\mathcal{W}_+}(T) = \eta\sigma_{LD}(T)\\ &= \eta\sigma^e_{RD}(T) = \eta\sigma_{B\mathcal{W}_-}(T) = \eta\sigma_{RD}(T)\\ &= \eta\sigma^e_{dsc}(T) = \eta\sigma_{dsc}(T).\end{aligned}$$

3. *The set $\sigma_D(T)$ consists of $\sigma_*(T)$ and possibly some holes in $\sigma_*(T)$ where $\sigma_* \in \{\sigma_{q\Phi}, \sigma_{Kt}, \sigma_{B\Phi}, \sigma_{B\mathcal{W}}, \sigma^e_{LD}, \sigma_{B\mathcal{W}_+}, \sigma_{LD}, \sigma^e_{RD}, \sigma_{B\mathcal{W}_-}, \sigma_{RD}, \sigma^e_{dsc}, \sigma_{dsc}\}$.*

Proof It follows from Proposition 3.1, the previous inclusions, Proposition 2.6 and Corollary 3.3. □

From Theorem 3.8 and Remark 3.2 it follows that if one of $\sigma_{TUD}(T), \sigma_{q\Phi}(T), \sigma_{Kt}(T), \sigma_{B\Phi}(T), \sigma_{B\mathcal{W}}(T), \sigma_D(T), \sigma^e_{LD}(T), \sigma_{B\mathcal{W}_+}(T), \sigma_{LD}(T), \sigma^e_{RD}(T), \sigma_{B\mathcal{W}_-}(T), \sigma_{RD}(T)$, $\sigma^e_{dsc}(T)$ and $\sigma_{dsc}(T)$ is finite (countable), then all of them are equal and therefore finite (countable). This result is already obtained in [20, Corollary 3.4], but our method of proofing is different.

Example 3.9 Let $Q : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ be the operator defined by

$$Q(\xi_1, \xi_2, \xi_3, \dots) = \left(0, \xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \dots\right), \quad (\xi_1, \xi_2, \xi_3, \dots) \in \ell_2(\mathbb{N}).$$

From $\lim_{n\to\infty} ||Q^n||^{\frac{1}{n}} = \lim_{n\to\infty}(\frac{1}{n!})^{\frac{1}{n}} = 0$ we see that Q is quasinilpotent. It follows that 0 is not an accumulation point of $\sigma_{\mathbf{R}}(Q)$ for $\mathbf{R} \in \{\mathbf{R_i} : 1 \le i \le 10\} \cup \{\mathbf{R^a_4}, \mathbf{R^a_5}\} \cup \{\mathcal{W}_+(X), \mathcal{W}_-(X), \mathcal{W}(X), \Phi(X), \mathcal{B}(X), L(X)^{-1}\}$. As Q is the limit of finite rank operators, Q is compact. Since Q^n is compact and $R(Q^n)$ is infinite dimensional, we conclude that $R(Q^n)$ is not closed for every $n \in \mathbb{N}$. Therefore, Q is not quasi-Fredholm and so, $0 \in \sigma_{qF}(Q) \subset \sigma(Q) = \{0\}$. Thus $\sigma_{qF}(Q) = \{0\}$, which implies that $\sigma_{TUD}(Q) = \{0\}$, that is Q does not have TUD. It means that the condition that T has TUD can not be omitted in Theorems 2.3, 2.5, 2.8, 2.11, 2.12, 2.14, 2.15, 2.17, and 2.19. Also, the condition that T is quasi-Fredholm can not be omitted in Theorems 2.3, 2.8, 2.15, and 2.19.

Remark 3.10 Similar results can get for the Jeribi essential spectrum of operators defined on a Banach space which has no reflexive infinite dimensional subspace.

The Jeribi essential spectrum is defined for $T \in L(X)$ by

$$\sigma_j(T) = \bigcap_{K \in \mathcal{W}_*(X)} \sigma(T + K),$$

where $\mathcal{W}_*(X)$ denotes the set of all weakly compact operators on a Banach X (see the definition and some properties in [5,18]). Obviously, $\sigma_j(T)$ is compact and

$$\sigma_j(T) = \bigcap_{K \in \mathcal{W}_*(\mathcal{X})} \sigma(T + K) \subset \bigcap_{K \in K(X)} \sigma(T + K) = \sigma_{\mathcal{W}}(T). \tag{3.2}$$

According to [5, Theorem 3.3], if X is a Banach space which has no reflexive infinite dimensional subspace and $T \in L(X)$, then

$$\sigma_\Phi(T) \subset \sigma_j(T). \tag{3.3}$$

As $\partial\, \sigma_{\mathcal{W}}(T) \subset \sigma_\Phi(T)$, from (3.2) and (3.3) it follows that

$$\partial\, \sigma_{\mathcal{W}}(T) \subset \sigma_j(T) \subset \sigma_{\mathcal{W}}(T),$$

which according to Proposition 3.1 implies that

$$\partial\, \sigma_{\mathcal{W}}(T) \subset \partial\, \sigma_j(T), \quad \eta\sigma_j(T) = \eta\sigma_{\mathcal{W}}(T)$$

and the spectrum $\sigma_{\mathcal{W}}(T)$ can be obtained from $\sigma_j(T)$ by filling in some holes of $\sigma_j(T)$. Also, if $\sigma_j(T)$ is connected, $\sigma_{\mathcal{W}}(T)$ is connected.

4 Applications

An operator $T \in L(X)$ is *meromorphic* if its non-zero spectral points are poles of its resolvent. We say that T is *polinomially meromorphic* if there exists non-trivial polynomial p such that $p(T)$ is meromorphic.

In [10, Theorem 2.11] it is given a characterization of meromorphic operators in terms of B-Fredholm operators: an operator $T \in L(X)$ is meromorphic if and only if $\sigma_{B\Phi}(T) \subset \{0\}$. This result is extended by Jiang et al. in [20, Corollary 3.3] by including the characterization of meromorphic operators in terms of operators of topological uniform descent: $T \in L(X)$ is meromorphic if and only if $\sigma_{TUD}(T) \subset \{0\}$. Their proof is based on the local constancy of the mappings $\lambda \mapsto K(\lambda I - T) + H_0(\lambda I - T)$ and $\lambda \mapsto \overline{K(\lambda I - T) \cap H_0(\lambda I - T)}$ and results about SVEP established in [19]. We obtain the same assertion as a corollary of Theorem 3.8 and our method of proofing is rather different.

Theorem 4.1 *Let $T \in L(X)$. Then the following conditions are equivalent:*

1. *T is a meromorphic operator;*
2. $\sigma_{TUD}(T) \subset \{0\}$*;*
3. $\sigma_{B\Phi}(T) \subset \{0\}$.

Proof Since T is a meromorphic operator if and only if $\sigma_D(T) \subset \{0\}$, the assertion follows from Theorem 3.8 (see the comment after Theorem 3.8). □

For $T \in L(X)$ set $\rho_{TUD}(T) = \mathbb{C} \setminus \sigma_{TUD}(T)$.

Theorem 4.2 *Let $T \in L(X)$. If Ω is a component of $\rho_{TUD}(T)$, then $\Omega \subset \sigma_D(T)$ or $\Omega \setminus E \subset \rho(T)$, where $E = \{\lambda \in \mathbb{C} : \lambda$ is the pole of the resolvent od $T\}$.*

Proof Since $\partial\sigma_D(T) \subset \sigma_{TUD}(T)$, from Proposition 3.1 it follows that

$$\Omega \subset \sigma_D(T) \text{ or } \Omega \cap \sigma_D(T) = \emptyset. \tag{4.1}$$

If the second formula in (4.1) holds, then, as $\sigma_D(T) = \sigma(T)\setminus E$, we obtain that $(\Omega\setminus E) \cap \sigma(T) = \emptyset$, which implies $\Omega\setminus E \subset \rho(T)$. □

In [20, Corollary 2.12] Jiang et al. obtained the same result as Theorem 4.2 by using the constancy of the mappings $\lambda \mapsto \overline{K(\lambda I - T) + N(\lambda I - T)}$ and $\lambda \mapsto K(\lambda I - T) \cap H_0(\lambda I - T)$ on the components of $\rho_{TUD}(T)$, however our proof is rather different. They also obtained that if $\rho_{TUD}(T)$ has only one component, then $\sigma_{TUD}(T) = \sigma_D(T)$. We get this result in a different way and also obtain that analogous assertion holds for other spectra.

Theorem 4.3 *Let $T \in L(X)$ and let $\sigma_* \in \{\sigma_{TUD}, \sigma_{q\Phi}, \sigma_{Kt}, \sigma_{B\Phi}, \sigma_{B\mathcal{W}}, \sigma^e_{LD}, \sigma_{B\mathcal{W}_+}, \sigma_{LD}, \sigma^e_{RD}, \sigma_{B\mathcal{W}_-}, \sigma_{RD}, \sigma^e_{dsc}, \sigma_{dsc}\}$. Then there is implication*

$$\mathbb{C}\setminus\sigma_*(T) \text{ has only one component} \Longrightarrow \sigma_*(T) = \sigma_D(T).$$

Proof Since $\mathbb{C}\setminus\sigma_*(T)$ has only one component, it follows that $\sigma_*(T)$ has no holes and hence $\sigma_*(T) = \eta\sigma_*(T)$. According to Theorem 3.8 we conclude that $\sigma_D(T) \subset \eta\sigma_D(T) = \eta\sigma_*(T) = \sigma_*(T) \subset \sigma_D(T)$ and hence $\sigma_D(T) = \sigma_*(T)$. □

Let $F_0(X)$ denote set of finite rank operators on X. Now we can prove Theorem 2.10 in [10] in a different way.

Theorem 4.4 *Let $T \in L(X)$ and suppose that $\sigma_{B\mathcal{W}}(T)$ is simply connected. Then $T + F$ satisfies the generalized version II of the Weyl's theorem for every $F \in F_0(X)$.*

Proof From $F \in F_0(X)$ it follows that $\sigma_{B\mathcal{W}}(T) = \sigma_{B\mathcal{W}}(T + F)$ [9, Theorem 4.3] and $\sigma_{B\mathcal{W}}(T + F)$ is simply connected. According to Theorem 3.8, since there are no holes in $\sigma_{B\mathcal{W}}(T + F)$, we conclude that $\sigma_D(T + F) = \sigma_{B\mathcal{W}}(T + F)$, and so $T + F$ satisfies the generalized version II of the Weyl's theorem. □

Corollary 4.5 *Let $T \in L(X)$.*

(1) *If $\sigma_p(T) \subset \partial\sigma_p(T)$, then $\sigma_{LD}(T) = \sigma_{TUD}(T)$.*
(2) *If $\sigma_{cp}(T) \subset \partial\sigma_{cp}(T)$, then $\sigma_{dsc}(T) = \sigma_{TUD}(T)$ and $\sigma_{RD}(T) = \sigma_{q\Phi}(T)$.*
(3) *If $\sigma_*(T) = \partial\sigma_*(T)$, where $\sigma_* \in \{\sigma_D, \sigma_{LD}, \sigma^e_{LD}, \sigma_{B\Phi}, \sigma_{B\mathcal{W}_+}, \sigma_{B\mathcal{W}}\}$, then $\sigma_*(T) = \sigma_{TUD}(T)$.*
(4) *If $\sigma_{dsc}(T) = \partial\sigma_{dsc}(T)$, then $\sigma_{dsc}(T) = \sigma_{TUD}(T)$ and $\sigma_{RD}(T) = \sigma_{q\Phi}(T)$.*
(5) *If $\sigma^e_{dsc}(T) = \partial\sigma^e_{dsc}(T)$, then $\sigma^e_{dsc}(T) = \sigma_{TUD}(T)$ and $\sigma^e_{RD}(T) = \sigma_{q\Phi}(T)$.*
(6) *If $\sigma_{B\mathcal{W}_-}(T) = \partial\sigma_{B\mathcal{W}_-}(T)$, then $\sigma_{B\mathcal{W}_-}(T) = \sigma_{q\Phi}(T)$.*

Proof (1) From $\sigma_p(T) \subset \partial\sigma_p(T)$ it follows that $\operatorname{int}\sigma_p(T) = \emptyset$. Using Corollary 2.18 (1) we get $\sigma_{LD}(T) = \sigma_{TUD}(T)$.

The rest of assertions can be proved similarly by using Corollaries 2.18, 2.13 and 2.7. □

Remark 4.6 Jiang et al. concluded in [20, p. 1156] that if $\sigma(T)$ is contained in a line segment, then $\sigma_D(T) = \sigma_{TUD}(T)$. From Corollary 4.5 (3) we get that if $\sigma(T)$ is contained in a line, then $\sigma_D(T) = \sigma_{TUD}(T)$. If T is unitary operator on Hilbert space, then its spectrum is contained in a line and so, $\sigma_D(T) = \sigma_{TUD}(T)$.

From Corollary 4.5 it follows also that if $\sigma_*(T)$ is contained in a line for $\sigma_* \in \{\sigma_{LD}, \sigma_D, \sigma^e_{LD}, \sigma_{B\mathcal{W}_+}, \sigma_{B\mathcal{W}}, \sigma_{B\Phi}, \sigma^e_{dsc}, \sigma_{dsc}\}$, then $\sigma_*(T) = \sigma_{TUD}(T)$. Also, if $\sigma_*(T)$ is contained in a line for $\sigma_* \in \{\sigma^e_{RD}, \sigma_{B\mathcal{W}_-}, \sigma_{RD}\}$, then $\sigma_*(T) = \sigma_{qF}(T)$.

Therefore, if $\sigma_{\mathbf{R}}(T)$ is contained in a line for $\mathbf{R} \in \{\mathbf{R_6}, \mathbf{R_7}, \mathbf{R_8}, \mathbf{R_9}, \mathbf{R_{10}}, \mathbf{R^a_4}, \mathbf{R^a_5}, \mathcal{W}_+(X), \mathcal{W}(X), \Phi(X), \mathcal{B}(X), L(X)^{-1}\}$, then $\sigma_{\mathbf{BR}}(T) = \sigma_{TUD}(T)$. If $\mathbf{R} \in \{\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}, \mathbf{R_4}, \mathbf{R_5}, \mathcal{W}_-(X)\}$ and $\sigma_{\mathbf{R}}(T)$ is contained in a line, then $\sigma_{\mathbf{BR}}(T) = \sigma_{qF}(T)$ (see also Theorem 2.19).

Furthermore, if $\sigma^e_{dsc}(T)$ ($\sigma_{dsc}(T)$) is contained in a line, then $\sigma^e_{RD}(T) = \sigma_{qF}(T)$ ($\sigma_{RD}(T) = \sigma_{qF}(T)$). If $\sigma_p(T)$ ($\sigma_{cp}(T)$) is countable or contained in a line, then $\sigma_{LD}(T) = \sigma_{TUD}(T)$ ($\sigma_{RD}(T) = \sigma_{q\Phi}(T)$ and $\sigma_{dsc}(T) = \sigma_{TUD}(T)$).

Example 4.7 If X is one of $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geq 1$, then for the forward and backward bilateral shifts $W_1,\ W_2 \in L(X)$ there are equalities $\sigma(W_1) = \sigma(W_2) = \partial\mathbb{D}$, where $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. From $\operatorname{acc}\sigma(W_1) = \operatorname{acc}\sigma(W_2) = \partial\mathbb{D}$ we conclude that $\sigma_D(W_1) = \sigma_D(W_2) = \partial\mathbb{D}$ and since $\sigma(W_1)$ and $\sigma(W_2)$ are contained in a line, we obtain $\sigma_{TUD}(W_1) = \sigma_{TUD}(W_2) = \partial\mathbb{D}$.

Corollary 4.8 *Let $T \in L(X)$.*

If $\sigma_* \in \{\sigma_{\mathcal{W}_+}, \sigma_{\mathcal{W}_-}, \sigma_{\mathcal{W}}, \sigma_{B\mathcal{W}_-}, \sigma_{\Phi_+}, \sigma_{\Phi_-}, \sigma_{\Phi}, \sigma^e_{RD}, \sigma_{ap}, \sigma_{su}, \sigma_{\mathcal{B}_+}, \sigma_{\mathcal{B}_-}, \sigma_{\mathcal{B}}, \sigma_{RD}, \sigma\}$ and

$$\sigma_*(T) = \partial\sigma_*(T) = \operatorname{acc}\sigma_*(T), \tag{4.2}$$

then

$$\sigma_{TUD}(T) = \sigma_*(T). \tag{4.3}$$

Proof From Corollaries 3.4, 3.5 and 3.6 we have that $\partial\sigma_*(T) \cap \operatorname{acc}\sigma_*(T) \subset \sigma_{TUD}(T)$, which together with the equalities (4.2) gives the inclusion $\sigma_*(T) \subset \sigma_{TUD}(T)$. Since $\sigma_{TUD}(T) \subset \sigma_*(T)$, we get (4.3). □

We recall that if $K \subset \mathbb{C}$ is compact, then for $\lambda \in \partial K$ the following equivalence holds:

$$\lambda \in \operatorname{acc} K \iff \lambda \in \operatorname{acc} \partial K. \tag{4.4}$$

The following corollary is an improvement of Theorem 2.10 and Corollary 2.11 in [2].

Corollary 4.9 *Let $T \in L(X)$.*

1. *Let T be an operator for which $\sigma_{ap}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Then $\sigma_{ap}(T) = \sigma_{TUD}(T)$.*
2. *Let T be an operator for which $\sigma_{su}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Then $\sigma_{su}(T) = \sigma_{TUD}(T)$.*

Proof From $\sigma_{ap}(T) = \partial\sigma(T)$ and $\partial\sigma(T) \subset \partial\sigma_{ap}(T) \subset \sigma_{ap}(T)$ it follows that $\sigma_{ap}(T) = \partial\sigma_{ap}(T)$, while from (4.4) it follows that every $\lambda \in \partial\sigma(T)$ is not isolated in $\partial\sigma(T)$. Therefore, every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma_{ap}(T)$ and hence, $\sigma_{ap}(T) \subset \operatorname{acc}\sigma_{ap}(T)$. Thus $\sigma_{ap}(T) = \partial\sigma_{ap}(T) = \operatorname{acc}\sigma_{ap}(T)$ and from Corollary 4.8 it follows that $\sigma_{ap}(T) = \sigma_{TUD}(T)$.

The assertion (2) can be proved similarly. □

Example 4.10 For each $X \in \{c_0(\mathbb{N}), c(\mathbb{N}), \ell_\infty(\mathbb{N}), \ell_p(\mathbb{N})\}$, $p \geq 1$, and the forward and backward unilateral shifts $U, V \in L(X)$ there are equalities $\sigma(U) = \sigma(V) = \mathbb{D}$, $\sigma_D(U) = \sigma_D(V) = \mathbb{D}$ and $\sigma_{ap}(U) = \sigma_{su}(V) = \partial\mathbb{D}$. By using Corollary 4.8 (or Corollary 4.9) we obtain that $\sigma_{TUD}(U) = \sigma_{ap}(U) = \partial\mathbb{D}$ and $\sigma_{TUD}(V) = \sigma_{su}(V) = \partial\mathbb{D}$. It implies that

$$\sigma_{TUD}(U) = \sigma_{q\Phi}(U) = \sigma^e_{LD}(U) = \sigma_{B\mathcal{W}_+}(U) = \sigma_{LD}(U) = \sigma_{ap}(U) = \partial\mathbb{D}$$

and

$$\begin{aligned}\sigma_{TUD}(V) &= \sigma_{q\Phi}(V) = \sigma^e_{RD}(V) = \sigma_{B\mathcal{W}_-}(V) = \sigma_{RD}(V)\\ &= \sigma_{dsc}(V) = \sigma^e_{dsc}(V) = \sigma_{su}(V) = \partial\mathbb{D}.\end{aligned}$$

As $\sigma_{su}(U) = \mathbb{D}$, from Corollary 2.18 ((2), (3)) it follows that $\sigma_{dsc}(U) = \sigma_{RD}(U) = \mathbb{D}$. Since $\sigma_\Phi(U) = \sigma_\Phi(V) = \partial\mathbb{D}$ [29, Theorem 4.2], from Remark 4.6 we get that $\sigma_{B\Phi}(U) = \sigma_{TUD}(U) = \partial\mathbb{D}$ and similarly, $\sigma_{B\Phi}(V) = \partial\mathbb{D}$. From the inclusions $\sigma_{TUD}(U) \subset \sigma_{Kt}(U) \subset \sigma_{B\Phi}(U)$ we have that $\sigma_{Kt}(U) = \partial\mathbb{D}$ and similarly, $\sigma_{Kt}(V) = \partial\mathbb{D}$.

From $\partial\sigma_\Phi(U) \subset \sigma_{\Phi_-}(U) \subset \sigma_\Phi(U)$ it follows that $\sigma_{\Phi_-}(U) = \partial\mathbb{D}$, that is $\sigma_{\Phi_-}(U)$ is contained in the line and hence, by Remark 4.6 we get that $\sigma^e_{dsc}(U) = \sigma_{TUD}(U) = \partial\mathbb{D}$ and $\sigma^e_{RD}(U) = \sigma_{q\Phi}(U) = \partial\mathbb{D}$.

From $\partial\sigma_\Phi(V) \subset \sigma_{\Phi_+}(V) \subset \sigma_\Phi(V)$ we conclude that $\sigma_{\Phi_+}(V) = \partial\mathbb{D}$ which according to Remark 4.6 implies that $\sigma^e_{LD}(V) = \sigma_{TUD}(V) = \partial\mathbb{D}$. As $\sigma_{RD}(V) = \partial\mathbb{D}$ and $\sigma_D(V) = \mathbb{D}$, we get $\sigma_{LD}(V) = \mathbb{D}$. From $\sigma_\Phi(V) = \partial\mathbb{D}$, $\sigma_{ap}(V) = \mathbb{D}$ and $\sigma_{su}(V) = \partial\mathbb{D}$, we conclude that for $|\lambda| < 1$ it holds that $V - \lambda I$ is Fredholm with positive index and so, $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_{\mathcal{W}_+}(V) \subset \sigma_{\mathcal{W}}(V) \subset \mathbb{D}$, which implies that $\sigma_{\mathcal{W}_+}(V) = \sigma_{\mathcal{W}}(V) = \mathbb{D}$. From Corollary 2.7 (1), (3) it follows that $\sigma_{B\mathcal{W}_+}(V) = \mathbb{D}$ and $\sigma_{B\mathcal{W}}(V) = \mathbb{D}$. Similarly, from $\sigma_\Phi(U) = \partial\mathbb{D}$, $\sigma_{ap}(U) = \partial\mathbb{D}$ and $\sigma_{su}(U) = \mathbb{D}$ it follows that $\sigma_{\mathcal{W}_-}(U) = \sigma_{\mathcal{W}}(U) = \mathbb{D}$, which by Corollary 2.7 (2), (3) implies that $\sigma_{B\mathcal{W}_-}(U) = \sigma_{B\mathcal{W}}(U) = \mathbb{D}$.

Example 4.11 Every non-invertible isometry T has the property that $\sigma(T) = \mathbb{D}$ and $\sigma_{ap}(T) = \partial\mathbb{D}$ [2, p. 187]. Hence $\sigma_{ap}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Therefore, according to Corollary 4.9, for arbitrary non-invertible isometry T we get that $\sigma_{TUD}(T) = \sigma_{q\Phi}(T) = \sigma^e_{LD}(T) = \sigma_{B\mathcal{W}_+}(T) = \sigma_{LD}(T) = \sigma_{ap}(T) = \partial\mathbb{D}$.

Example 4.12 For the *Cesáro operator* C_p defined on the classical Hardy space $H_p(\mathbf{D})$, $\mathbf{D}$ the open unit disc and $1 < p < \infty$, by

$$(C_p f)(\lambda) = \frac{1}{\lambda}\int_0^\lambda \frac{f(\mu)}{1-\mu}\,d\mu, \quad \text{for all } f \in H_p(\mathbf{D}) \text{ and } \lambda \in \mathbf{D},$$

it is known that its spectrum is the closed disc Γ_p centered at $p/2$ with radius $p/2$, $\sigma_{Kt}(C_p) = \sigma_{ap}(C_p) = \partial\Gamma_p$ and also $\sigma_\Phi(C_p) = \partial\Gamma_p$ [2,24]. According to Corollary 4.8 or Corollary 4.9 we get that $\sigma_{TUD}(C_p) = \sigma_{q\Phi}(C_p) = \sigma^e_{LD}(C_p) =$

$\sigma_{B\mathcal{W}_+}(C_p) = \sigma_{LD}(C_p) = \sigma_{ap}(C_p) = \partial\Gamma_p$. Since $\sigma_\Phi(C_p)$ is contained in the line and hence also $\sigma_{\Phi_+}(C_p)$ and $\sigma_{\Phi_-}(C_p)$ are contained in the line, according to Remark 4.6 we conclude that $\sigma_{B\Phi}(C_p) = \sigma^e_{dsc}(C_p) = \sigma_{TUD}(C_p) = \partial\Gamma_p$ and $\sigma^e_{RD}(C_p) = \sigma_{qF}(C_p) = \partial\Gamma_p$. From $\sigma(C_p) = \Gamma_p$ and $\sigma_{ap}(C_p) = \partial\Gamma_p$ it follows that and $\sigma_{su}(C_p) = \Gamma_p$ which together with $\sigma_\Phi(C_p) = \partial\Gamma_p$ implies that $\sigma_{\mathcal{W}_-}(C_p) = \sigma_{\mathcal{W}}(C_p) = \Gamma_p$. Now from Corollary 2.7 (2), (3) we obtain that $\sigma_{B\mathcal{W}_-}(C_p) = \sigma_{B\mathcal{W}}(C_p) = \Gamma_p$. As $\sigma_D(C_p) = \Gamma_p$ and $\sigma_{LD}(C_p) = \partial\Gamma_p$, it follows that $\sigma_{RD}(C_p) = \Gamma_p$, which by Corollary 2.18 (2) implies that $\sigma_{dsc}(C_p) = \Gamma_p$.

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