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The Schwarz Type Lemmas and the Landau Type Theorem of Mappings Satisfying Poisson's Equations

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Abstract
For a given continuous function $g : \overline{\mathbb{D}} \to \mathbb{C}$ and a given continuous function $\psi : \mathbb{T} \to \mathbb{C}$, we establish some Schwarz type Lemmas for mappings f satisfying the PDE: $\Delta f = g$ in $\mathbb{D}$, and $f = \psi$ in $\mathbb{T}$, where $\mathbb{D}$ is the unit disk of the complex plane $\mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$ is the unit circle. Then we apply these results to obtain a Landau type theorem, which is a partial answer to the open problem in Chen and Ponnusamy (Bull Aust Math Soc 97: 80–87, 2018).

Keywords Schwarz's Lemma · Landau type theorem · Poisson's equation

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1 Preliminaries and Main Results

Let $\mathbb{C} \cong \mathbb{R}^2$ be the complex plane. For $a \in \mathbb{C}$ and $r > 0$, we let $\mathbb{D}(a, r) = \{z : |z - a| < r\}$ so that $\mathbb{D}_r := \mathbb{D}(0, r)$ and thus, $\mathbb{D} := \mathbb{D}_1$ denotes the open unit disk in the complex plane $\mathbb{C}$. Let $\mathbb{T} = \partial\mathbb{D}$ be the boundary of $\mathbb{D}$. We denote by $\mathcal{C}^m(\Omega)$ the set of all complex-valued m-times continuously differentiable functions from Ω into $\mathbb{C}$,

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where Ω is a subset of $\mathbb{C}$ and $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In particular, let $\mathcal{C}(\Omega) := \mathcal{C}^0(\Omega)$, the set of all continuous functions defined in Ω.

For a real 2×2 matrix A, we use the matrix norm $\|A\| = \sup\{|Az| : |z| = 1\}$ and the matrix function $\lambda(A) = \inf\{|Az| : |z| = 1\}$. For $z = x + iy \in \mathbb{C}$, the formal derivative of the complex-valued functions $f = u + iv$ is given by

$$D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

so that

$$\|D_f\| = |f_z| + |f_{\overline{z}}| \quad \text{and} \quad \lambda(D_f) = \big||f_z| - |f_{\overline{z}}|\big|,$$

where

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\big(f_x - if_y\big) \quad \text{and} \quad f_{\overline{z}} = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\big(f_x + if_y\big).$$

We use

$$J_f := \det D_f = |f_z|^2 - |f_{\overline{z}}|^2$$

to denote the *Jacobian* of f and

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4 f_{z\overline{z}}$$

is the *Laplacian* of f.

For $t \in \mathbb{R}$ and $z, w \in \mathbb{D}$ with $z \neq w$ and $|z| + |w| \neq 0$, let

$$G(z, w) = \log\left|\frac{1 - z\overline{w}}{z - w}\right| \quad \text{and} \quad P(z, e^{it}) = \frac{1 - |z|^2}{|1 - ze^{-it}|^2}$$

be the *Green function* and *Poisson kernel*, respectively.

Let $\psi : \mathbb{T} \to \mathbb{C}$ be a bounded integrable function and let $g \in \mathcal{C}(\mathbb{D})$. For $z \in \mathbb{D}$, the solution to the *Poisson's equation*

$$\Delta f(z) = g(z)$$

satisfying the boundary condition $f|_{\mathbb{T}} = \psi \in L^1(\mathbb{T})$ is given by

$$f(z) = \mathcal{P}_\psi(z) - \mathcal{G}_g(z), \tag{1.1}$$

where

$$\mathcal{G}_g(z) = \frac{1}{2\pi}\int_{\mathbb{D}} G(z, w)g(w)dA(w), \quad \mathcal{P}_\psi(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{it})\psi(e^{it})dt, \tag{1.2}$$

and $dA(w)$ denotes the Lebesgue measure in $\mathbb{D}$. It is well known that if ψ and g are continuous in $\mathbb{T}$ and in $\overline{\mathbb{D}}$, respectively, then $f = \mathcal{P}_{\psi} - \mathcal{G}_g$ has a continuous extension $\tilde{f}$ to the boundary, and $\tilde{f} = \psi$ in $\mathbb{T}$ (see [18, pp. 118–120] and [2,19,20,22]).

Heinz in his classical paper [17] proved the following result, which is called the *Schwarz Lemma* of complex-valued harmonic functions: If f is a complex-valued harmonic function from $\mathbb{D}$ into itself satisfying the condition $f(0) = 0$, then, for $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|. \tag{1.3}$$

Later, Pavlović [30, Theorem 3.6.1] removed the assumption $f(0) = 0$ and improved (1.3) into the following sharp form

$$\left| f(z) - \frac{1-|z|^2}{1+|z|^2} f(0) \right| \leq \frac{4}{\pi} \arctan |z|, \tag{1.4}$$

where f is a complex-valued harmonic function from $\mathbb{D}$ to itself.

The first aim of this paper is to extend (1.4) into mappings satisfying the Poisson's equation as follows.

Theorem 1 *For a given $g \in \mathcal{C}(\overline{\mathbb{D}})$ and a given $\psi \in \mathcal{C}(\mathbb{T})$, if a complex-valued function f satisfies $\Delta f = g$ in $\mathbb{D}$ and $f = \psi$ in $\mathbb{T}$, then, for $z \in \overline{\mathbb{D}}$,*

$$\left| f(z) - \frac{1-|z|^2}{1+|z|^2} \mathcal{P}_{\psi}(0) \right| \leq \frac{4\|\mathcal{P}_{\psi}\|_{\infty}}{\pi} \arctan |z| + \frac{\|g\|_{\infty}}{4}(1-|z|^2), \tag{1.5}$$

where

$$\mathcal{P}_{\psi}(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{it}) \psi(e^{it}) dt, \ \ \|\mathcal{P}_{\psi}\|_{\infty} = \sup_{z \in \mathbb{D}} |\mathcal{P}_{\psi}(z)| \text{ and } \|g\|_{\infty} = \sup_{z \in \mathbb{D}} |g(z)|.$$

If we take $g(z) = -4M$ and $f(z) = M(1 - |z|^2)$ for $z \in \overline{\mathbb{D}}$, where M is a positive constant, then the inequality (1.5) is sharp in $\overline{\mathbb{D}}$.

The following result is a classical Schwarz Lemma at the boundary.

Theorem A (see [15]) *Let f be a holomorphic function from $\mathbb{D}$ into itself. If f is holomorphic at $z = 1$ with $f(0) = 0$ and $f(1) = 1$, then $f'(1) \geq 1$. Moreover, the inequality is sharp.*

Theorem A has attracted much attention and has been generalized in various forms (See [6,23,26,27] for holomorphic functions, and see [21] for harmonic functions). In the following result, applying Theorem 1, we establish a Schwarz Lemma at the boundary for mappings satisfying the Poisson's equation, which is a generalization of Theorem A.

Theorem 2 *For a given $g \in \mathcal{C}(\overline{\mathbb{D}})$, let $f \in \mathcal{C}^2(\mathbb{D}) \cap \mathcal{C}(\mathbb{T})$ be a function of $\mathbb{D}$ into itself satisfying $\Delta f = g$, where $\|g\|_\infty < \frac{8}{3\pi}$. If $f(0) = 0$ and, for some $\zeta \in \mathbb{T}$, $\lim_{r\to 1^-} |f(r\zeta)| = 1$, then*

$$\liminf_{r\to 1^-} \frac{|f(\zeta) - f(r\zeta)|}{1-r} \geq \frac{2}{\pi} - \frac{3\|g\|_\infty}{4}. \tag{1.6}$$

In particular, if $\|g\|_\infty = 0$, then the estimate of (1.6) is sharp.

In [14], Colonna proved a sharp *Schwarz-Pick* type Lemma of complex-valued harmonic functions, which is as follows: If f is a complex-valued harmonic function from $\mathbb{D}$ into itself, then, for $z \in \mathbb{D}$,

$$\|D_f(z)\| \leq \frac{4}{\pi}\frac{1}{1-|z|^2}. \tag{1.7}$$

We extend (1.7) into the following form.

Theorem 3 *For a given $g \in \mathcal{C}(\overline{\mathbb{D}})$ and a given $\psi \in \mathcal{C}(\mathbb{T})$, if a complex-valued function f satisfies $\Delta f = g$ in $\mathbb{D}$ and $f = \psi$ in $\mathbb{T}$, then, for $z \in \mathbb{D}\backslash\{0\}$,*

$$\|D_f(z)\| \leq \frac{4\|\mathcal{P}_\psi\|_\infty}{\pi}\frac{1}{1-|z|^2} + 2\mu(|z|), \tag{1.8}$$

where

$$\frac{\|g\|_\infty}{4} \leq \mu(|z|) = \frac{\|g\|_\infty(1-|z|^2)}{8|z|^2}\left[\frac{1+|z|^2}{1-|z|^2} - \frac{(1-|z|^2)}{2|z|}\log\frac{1+|z|}{1-|z|}\right] \leq \frac{\|g\|_\infty}{3}$$

and $\mu(|z|)$ is decreasing on $|z| \in (0,1)$. In particular, if $z = 0$, then

$$\|D_f(0)\| \leq \lim_{|z|\to 0^+}\left(\frac{4\|\mathcal{P}_\psi\|_\infty}{\pi}\frac{1}{1-|z|^2} + 2\mu(|z|)\right) = \frac{4}{\pi}\|\mathcal{P}_\psi\|_\infty + \frac{2}{3}\|g\|_\infty. \tag{1.9}$$

Moreover, if $\|g\|_\infty = 0$, then the extremal functions

$$f(z) = \frac{2M\alpha}{\pi}\arg\left(\frac{1+\phi(z)}{1-\phi(z)}\right)$$

show that the estimate of (1.8) and (1.9) are sharp, where $|\alpha| = 1$ and $M > 0$ are constants, and ϕ is a conformal automorphism of $\mathbb{D}$.

We remark that if $\|g\|_\infty = 0$ and $\|\mathcal{P}_\psi\|_\infty = 1$ in Theorem 3, then (1.8) and (1.9) coincide with (1.7).

Let $\mathcal{A}$ denote the set of all analytic functions f defined in $\mathbb{D}$ satisfying the standard normalization: $f(0) = f'(0) - 1 = 0$. In the early 20th century, Landau [24] showed that there is a constant $r > 0$, independent of $f \in \mathcal{A}$, such that $f(\mathbb{D})$ contains a disk of radius r. Let L_f be the supremum of the set of positive numbers r such that $f(\mathbb{D})$ contains a disk of radius r, where $f \in \mathcal{A}$. Then we call $\inf_{f\in\mathcal{A}} L_f$ the Landau–Bloch constant. One of the long standing open problems in geometric function theory is to determine the precise value of the Landau–Bloch constant. It has attracted much attention, see [4,25,28,29,32] and references therein. The Landau theorem is an important tool in geometric function theory of one complex variable (cf. [5,33]). Unfortunately, for general class of functions, there is no Landau type theorem (see [7,32]). In order to obtain some analogs of the Landau type theorem for more general classes of functions, it is necessary to restrict the class of functions considered (cf. [1,3,7–11,13,16,32]). Let's recall some known results as follows.

Theorem B ([7, Theorem 2]) *Let f be a harmonic mapping in $\mathbb{D}$ such that $f(0) = J_f(0) - 1 = 0$ and $|f(z)| < M$ for $z \in \mathbb{D}$, where M is a positive constant. Then f is univalent in $\mathbb{D}_{\rho_0}$ with $\rho_0 = \pi^3/(64mM^2)$, and $f(\mathbb{D}_{\rho_0})$ contains a univalent disk $\mathbb{D}_{R_0}$ with*

$$R_0 = \frac{\pi}{8M}\rho_0 = \frac{\pi^4}{512mM^3},$$

where $m \approx 6.85$ is the minimum of the function $(3 - r^2)/[r(1 - r^2)]$ for $r \in (0, 1)$.

Theorem C ([1, Theorem 1]) *Let $f(z) = |z|^2G(z) + K(z)$ be a biharmonic mapping, that is $\Delta(\Delta f) = 0$, in $\mathbb{D}$ such that $f(0) = K(0) = J_f(0)-1 = 0$, where G and K are harmonic satisfying $|G(z)|,\ |K(z)| < M$ for $z \in \mathbb{D}$, where M is a positive constant. Then there is a constant $\rho_2 \in (0, 1)$ such that f is univalent in $\mathbb{D}_{\rho_2}$. Specifically, ρ_2 satisfies*

$$\frac{\pi}{4M} - 2\rho_2 M - 2M\left[\frac{\rho_2^2}{(1-\rho_2)^2} + \frac{1}{(1-\rho_2)^2} - 1\right] = 0$$

and $f(\mathbb{D}_{\rho_2})$ contains a disk $\mathbb{D}_{R_2}$, where

$$R_2 = \frac{\pi}{4M}\rho_2 - 2M\frac{\rho_2^3 + \rho_2^2}{1-\rho_2}.$$

For some $g \in \mathcal{C}(\overline{\mathbb{D}})$, let $\mathcal{F}_g(\overline{\mathbb{D}})$ denote the class of all complex-valued functions $f \in \mathcal{C}^2(\mathbb{D}) \cap \mathcal{C}(\mathbb{T})$ satisfying $\Delta f = g$ and $f(0) = J_f(0) - 1 = 0$. We extend Theorems B and C into the following from.

Theorem 4 *For a given $g \in \mathcal{C}(\overline{\mathbb{D}})$, let $f \in \mathcal{F}_g(\overline{\mathbb{D}})$ satisfying $\|g\|_\infty \le M_1$ and $\|f\|_\infty \le M_2$, where $M_1 \ge 0$ and $M_2 > 0$ are constants. Then f is univalent in $\mathbb{D}_{r_0}$, where r_0 satisfies the following equation*

$$\frac{1}{\frac{4}{\pi}M_2 + \frac{2}{3}M_1} - \frac{4M_2}{\pi}\frac{r_0(2-r_0)}{(1-r_0)^2} - 2M_1\big[\log 4(1+r_0) - \log r_0\big](2+r_0)r_0 = 0.$$

Moreover, $f(\mathbb{D}_{r_0})$ *contains an univalent disk* $\mathbb{D}_{R_0}$ *with*

$$R_0 \geq \frac{2M_2}{\pi}\frac{r_0^2(2-r_0)}{(1-r_0)^2}.$$

Remark 1.1 Theorem 4 gives an affirmative answer to the open problem of [13] for the *u-gradient mapping* $f \in \mathcal{C}^2(\mathbb{D})$. If g is harmonic, then all $f \in \mathcal{F}_g(\overline{\mathbb{D}})$ are biharmonic. Furthermore, if $\|g\|_\infty = 0$, then all $f \in \mathcal{F}_g(\overline{\mathbb{D}})$ are harmonic. Hence, Theorem 4 is also a generalization of a series of known results, such as [1, Theorem 2], [7, Theorems, 3, 4, 5 and 6], [8, Theorems 2 and 3].

In the following two Examples, we will show that there is no Landau type Theorem for $f \in \mathcal{F}_g(\overline{\mathbb{D}})$ without the boundedness hypothesis of $\|f\|_\infty$.

Example 1.10 For $g \equiv 1$ and $z = x + iy \in \mathbb{D}$, let $f_k(z) = kx + |z|^2/4 + i\frac{y}{k}$, where $k \in \{1, 2, \ldots\}$. Then, for all $k \in \{1, 2, \ldots\}$, f_k is univalent. For all $k \in \{1, 2, \ldots\}$, by simple calculations, we see that $J_{f_k}(0) - 1 = f_k(0) = 0$, and there is no an absolute constant $\rho_0 > 0$ such that $\mathbb{D}_{\rho_0}$ is contained in $f_k(\mathbb{D})$.

Example 1.11 For $\|g\|_\infty = 0$ and $z = x + iy \in \mathbb{D}$, let $f_k(z) = kx + i\frac{y}{k}$, where $k \in \{1, 2, \ldots\}$. For all $k \in \{1, 2, \ldots\}$, it is not difficult to see that f_k is univalent and $J_{f_k}(0) - 1 = f_k(0) = 0$. Moreover, for all $k \in \{1, 2, \ldots\}$, $f_k(\mathbb{D})$ contains no disk with radius bigger than $1/k$. Hence, for all $k \in \{1, 2, \ldots\}$, there is no an absolute constant $r_0 > 0$ such that $\mathbb{D}_{r_0}$ is contained in $f_k(\mathbb{D})$.

Corollary 1 *Under the same hypothesis of Theorem 4, there is a* $r_0 \in (0, 1)$ *such that* f *is bi-Lipschitz in* $\mathbb{D}_{r_0}$.

The proofs of Theorems 1, 2, 3, 4 and Corollary 1 will be presented in Sect. 2.

2 Proofs of the Main Results

Proof of Theorem 1 For a given $g \in \mathcal{C}(\mathbb{D})$, by (1.1), we have

$$f(z) = \mathcal{P}_\psi(z) - \mathcal{G}_g(z), \quad z \in \mathbb{D}, \tag{2.1}$$

where $\mathcal{P}_\psi$ and $\mathcal{G}_g$ are defined in (1.2). Since $\mathcal{P}_\psi$ is harmonic in $\mathbb{D}$, by (1.4), we see that, for $z \in \mathbb{D}$,

$$\left|\mathcal{P}_\psi(z) - \frac{1-|z|^2}{1+|z|^2}\mathcal{P}_\psi(0)\right| \leq \frac{4\|\mathcal{P}_\psi\|_\infty}{\pi} \arctan|z|. \tag{2.2}$$

On the other hand, for a fixed $z \in \mathbb{D}$, let

$$\zeta = \frac{z-w}{1-\overline{z}w},$$

which is equivalent to

$$w = \frac{z - \zeta}{1 - \overline{z}\zeta}.$$

Then

$$\begin{aligned}
\left|\mathcal{G}_g(z)\right| &= \left|\frac{1}{2\pi}\int_{\mathbb{D}}\left(\log\frac{1}{|\zeta|}\right) g\left(\frac{z-\zeta}{1-\overline{z}\zeta}\right)\frac{(1-|z|^2)^2}{|1-\overline{z}\zeta|^4}dA(\zeta)\right| \\
&\leq \frac{\|g\|_\infty}{2\pi}\left|\int_{\mathbb{D}}\left(\log\frac{1}{|\zeta|}\right)\frac{(1-|z|^2)^2}{|1-\overline{z}\zeta|^4}dA(\zeta)\right| \\
&= (1-|z|^2)^2\|g\|_\infty\int_0^1\left[\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{|1-\overline{z}re^{it}|^4}\right)r\log\frac{1}{r}\right]dr \\
&= (1-|z|^2)^2\|g\|_\infty\int_0^1\left[\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{|(1-\overline{z}re^{it})^2|^2}\right)r\log\frac{1}{r}\right]dr \\
&= (1-|z|^2)^2\|g\|_\infty\int_0^1\left[\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=0}^{\infty}(n+1)(r\overline{z})^n e^{int}\right|^2 dt\right)r\log\frac{1}{r}\right]dr \\
&= (1-|z|^2)^2\|g\|_\infty\int_0^1\left(r\log\frac{1}{r}\right)\sum_{n=0}^{\infty}(n+1)^2|z|^{2n}r^{2n}dr \\
&= (1-|z|^2)^2\|g\|_\infty\sum_{n=0}^{\infty}(n+1)^2|z|^{2n}\int_0^1 r^{2n+1}\left(\log\frac{1}{r}\right)dr \\
&= \frac{(1-|z|^2)^2\|g\|_\infty}{4}\sum_{n=0}^{\infty}|z|^{2n} \\
&= \frac{\|g\|_\infty}{4}(1-|z|^2). \qquad (2.3)
\end{aligned}$$

Hence, by (2.2) and (2.3), we conclude that

$$\begin{aligned}
\left|f(z) - \frac{1-|z|^2}{1+|z|^2}\mathcal{P}_\psi(0)\right| &\leq \left|\mathcal{P}_\psi(z) - \frac{1-|z|^2}{1+|z|^2}\mathcal{P}_\psi(0)\right| + \left|\mathcal{G}_g(z)\right| \\
&\leq \frac{4\|\mathcal{P}_\psi\|_\infty}{\pi}\arctan|z| + \frac{\|g\|_\infty}{4}(1-|z|^2).
\end{aligned}$$

Now we prove the sharpness part. For $z \in \overline{\mathbb{D}}$, let

$$g(z) = -4M \quad \text{and} \quad f(z) = M(1-|z|^2),$$

where M is a positive constant. Then $\psi \equiv 0$ in $\mathbb{T}$ and

$$\begin{aligned}\left|f(z)-\frac{1-|z|^2}{1+|z|^2}\mathcal{P}_\psi(0)\right| = |f(z)| &= \left|\frac{1}{2\pi}\int_{\mathbb{D}} G(z,w)g(w)dA(w)\right| \\ &= \frac{\|g\|_\infty}{4}(1-|z|^2),\end{aligned}$$

which shows (1.5) is sharp in $\overline{\mathbb{D}}$. The proof of this theorem is complete. □

Proof of Theorem 2 For a given $g \in \mathcal{C}(\overline{\mathbb{D}})$, by (1.1) with f in place of ψ, we have

$$f(z) = \mathcal{P}_f(z) - \mathcal{G}_g(z), \quad z \in \mathbb{D},$$

where $\mathcal{P}_f$ and $\mathcal{G}_g$ are defined in (1.2). Since $f(0) = 0$, we see that

$$\begin{aligned}|\mathcal{P}_f(0)| = |\mathcal{G}_g(0)| &= \left|\frac{1}{2\pi}\int_{\mathbb{D}} \log\frac{1}{|w|}g(w)dA(w)\right| \\ &\leq \frac{\|g\|_\infty}{2\pi}\int_0^{2\pi} dt \int_0^1 r\log\frac{1}{r}dr \\ &= \frac{\|g\|_\infty}{4}.\end{aligned} \tag{2.4}$$

Let $z = r\zeta \in \mathbb{D}$, where $\zeta \in \mathbb{T}$ is as in the statement of the theorem. Then, by (2.4) and Theorem 1, we have

$$\begin{aligned}|f(\zeta)-f(r\zeta)| &= \left|f(\zeta)+\mathcal{P}_f(0)\frac{1-|z|^2}{1+|z|^2}-\mathcal{G}_g(0)\frac{1-|z|^2}{1+|z|^2}-f(r\zeta)\right| \\ &\geq 1-\left|f(r\zeta)-\mathcal{P}_f(0)\frac{1-|z|^2}{1+|z|^2}\right| - |\mathcal{G}_g(0)|\frac{1-|z|^2}{1+|z|^2} \\ &\geq 1-\frac{4}{\pi}\arctan|z| - \frac{\|g\|_\infty}{4}(1-|z|^2) - |\mathcal{G}_g(0)|\frac{1-|z|^2}{1+|z|^2} \\ &\geq 1-\frac{4}{\pi}\arctan|z| - \frac{\|g\|_\infty}{4}(1-|z|^2) - \frac{\|g\|_\infty}{4}\frac{(1-|z|^2)}{1+|z|^2},\end{aligned}$$

which, together with L'Hospital's rule, gives that

$$\begin{aligned}\liminf_{r\to 1^-}\frac{|f(e^{i\theta})-f(re^{i\theta})|}{1-r} &\geq \lim_{r\to 1^-}\frac{1-\frac{4}{\pi}\arctan r-\frac{\|g\|_\infty}{4}(1-r^2)-\frac{\|g\|_\infty}{4}\frac{(1-r^2)}{1+r^2}}{1-r}\\ &=\lim_{r\to 1^-}\left[\frac{4}{\pi}\frac{1}{1+r^2}-\|g\|_\infty\frac{r}{2}-\|g\|_\infty\frac{r}{(1+r^2)^2}\right]\\ &=\frac{2}{\pi}-\frac{3\|g\|_\infty}{4}.\end{aligned}$$

Now we prove the sharpness part. For $z\in\mathbb{D}$, let

$$f(z)=\frac{2}{\pi}\arctan\frac{2\mathrm{Re}(z)}{1-|z|^2}.$$

Then f is harmonic in $\mathbb{D}$ with $f(0)=f(1)-1=0$, and

$$f(\rho)=\frac{4}{\pi}\arctan\rho,$$

where $\rho\in(-1,1)$. Elementary calculations show that

$$\liminf_{\rho\to 1^-}\frac{|f(1)-f(\rho)|}{1-\rho}=\frac{2}{\pi},$$

which implies that (1.6) is sharp for $\|g\|_\infty=0$. The proof of this theorem is complete. □

Theorem D ([31] or [19, Proposition 2.4]) *Suppose that X is an open subset of $\mathbb{R}$, and Ω a measure space. Suppose, further, that a function $F: X\times\Omega\to\mathbb{R}$ satisfies the following conditions:*

(1) *$F(x,w)$ is a measurable function of x and w jointly, and is integrable with respect to w for almost every $x\in X$.*
(2) *For almost every $w\in\Omega$, $F(x,w)$ is an absolutely continuous function with respect to x. [This guarantees that $\partial F(x,w)/\partial x$ exists almost everywhere.]*
(3) *$\partial F/\partial x$ is locally integrable, that is, for all compact intervals $[a,b]$ contained in X:*

$$\int_a^b\int_\Omega\left|\frac{\partial}{\partial x}F(x,w)\right|dwdx<\infty.$$

Then, $\int_\Omega F(x,w)dw$ is an absolutely continuous function with respect to x, and for almost every $x\in X$, its derivative exists, which is given by

$$\frac{d}{dx}\int_\Omega F(x,w)dw=\int_\Omega\frac{\partial}{\partial x}F(x,w)dw.$$

Proof of Theorem 3 For a given $g \in \mathcal{C}(\overline{\mathbb{D}})$, by (2.1), we have

$$f(z) = \mathcal{P}_{\psi}(z) - \mathcal{G}_g(z), \quad z \in \mathbb{D},$$

where $\mathcal{P}_{\psi}$ and $\mathcal{G}_g$ are the same as in (2.1). Applying [19, Lemma 2.3] and Theorem D, we have

$$\begin{aligned} \frac{\partial}{\partial z}\mathcal{G}_g(z) &= \frac{1}{2\pi}\int_{\mathbb{D}} \frac{\partial}{\partial z} G(z, w) g(w) dA(w) \\ &= \frac{1}{4\pi}\int_{\mathbb{D}} \frac{(1-|w|^2)}{(z-w)(z\overline{w}-1)} g(w) dA(w) \in \mathcal{C}(\mathbb{D}) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \frac{\partial}{\partial \overline{z}}\mathcal{G}_g(z) &= \frac{1}{2\pi}\int_{\mathbb{D}} \frac{\partial}{\partial \overline{z}} G(z, w) g(w) dA(w) \\ &= \frac{1}{4\pi}\int_{\mathbb{D}} \frac{(1-|w|^2)}{(\overline{z}-\overline{w})(w\overline{z}-1)} g(w) dA(w) \in \mathcal{C}(\mathbb{D}). \end{aligned}$$

For a fixed $z \in \mathbb{D}\backslash\{0\}$, let

$$\zeta = \frac{z-w}{1-\overline{z}w} \tag{2.6}$$

which implies that

$$w = \frac{z-\zeta}{1-\overline{z}\zeta}, \quad 1-\overline{z}w = \frac{1-|z|^2}{1-\overline{z}\zeta} \quad \text{and} \quad 1-|w|^2 = \frac{(1-|\zeta|^2)(1-|z|^2)}{|1-\overline{z}\zeta|^2}. \tag{2.7}$$

Then, by (2.5), (2.7) and the change of variables (2.6), we have

$$\begin{aligned} \left|\frac{\partial}{\partial z}\mathcal{G}_g(z)\right| &\leq \frac{1}{4\pi}\int_{\mathbb{D}} \frac{(1-|w|^2)}{|z-w||z\overline{w}-1|}|g(w)|dA(w) \\ &\leq \frac{\|g\|_{\infty}}{4\pi}\int_{\mathbb{D}} \frac{(1-|w|^2)}{|z-w||z\overline{w}-1|}dA(w) \\ &= \frac{\|g\|_{\infty}}{4\pi}\int_{\mathbb{D}} \frac{(1-|w|^2)}{|\zeta||1-\overline{z}w|^2}\frac{(1-|z|^2)^2}{|1-\overline{z}\zeta|^4}dA(\zeta) \\ &= \frac{\|g\|_{\infty}}{4\pi}\int_{\mathbb{D}} \frac{(1-|z|^2)(1-|\zeta|^2)}{|\zeta||1-\overline{z}\zeta|^4}dA(\zeta) \\ &= \frac{\|g\|_{\infty}(1-|z|^2)}{2}\int_0^1 \left[(1-r^2)\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{|1-\overline{z}re^{it}|^4}\right)\right]dr \end{aligned}$$

$$
\begin{aligned}
&= \frac{\|g\|_\infty (1-|z|^2)}{2} \int_0^1 \left[(1-r^2) \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} (n+1)(r\overline{z})^n e^{int} \right|^2 dt \right) \right] dr \\
&= \frac{\|g\|_\infty (1-|z|^2)}{2} \int_0^1 (1-r^2) \left[\sum_{n=0}^{\infty} (n+1)^2 |z|^{2n} r^{2n} \right] dr \\
&= \frac{\|g\|_\infty (1-|z|^2)}{2} \int_0^1 \frac{(1-r^2)(1+|z|^2 r^2)}{(1-|z|^2 r^2)^3} dr \\
&= \frac{\|g\|_\infty (1-|z|^2)}{2} \left[-\frac{1}{|z|^2} I_1 + \left(\frac{3}{|z|^2} - 1 \right) I_2 + 2 \left(1 - \frac{1}{|z|^2} \right) I_3 \right], \qquad (2.8)
\end{aligned}
$$

where

$$
I_1 = \int_0^1 \frac{dr}{1-r^2|z|^2} = \frac{1}{|z|} \log \frac{1+|z|r}{\sqrt{1-|z|^2 r^2}} \Bigg|_0^1 = \frac{1}{|z|} \log \frac{1+|z|}{\sqrt{1-|z|^2}}, \qquad (2.9)
$$

$$
\begin{aligned}
I_2 &= \int_0^1 \frac{dr}{(1-r^2|z|^2)^2} = \frac{1}{2|z|} \left(\log \frac{1+|z|r}{\sqrt{1-|z|^2 r^2}} + \frac{|z|r}{1-|z|^2 r^2} \right) \Bigg|_0^1 \\
&= \frac{1}{2|z|} \log \frac{1+|z|}{\sqrt{1-|z|^2}} + \frac{1}{2(1-|z|^2)} \qquad (2.10)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \int_0^1 \frac{dr}{(1-r^2|z|^2)^3} = \frac{1}{4|z|} \left(\frac{|z|r}{(1-r^2|z|^2)^2} + \frac{3}{2} \frac{|z|r}{1-r^2|z|^2} + \frac{3}{2} \log \frac{1+|z|r}{\sqrt{1-|z|^2 r^2}} \right) \Bigg|_0^1 \\
&= \frac{1}{4(1-|z|^2)^2} + \frac{3}{8(1-|z|^2)} + \frac{3}{8|z|} \log \frac{1+|z|}{\sqrt{1-|z|^2}}. \qquad (2.11)
\end{aligned}
$$

By (2.9), (2.10) and (2.11), we get

$$
\begin{aligned}
&-\frac{1}{|z|^2} I_1 + \left(\frac{3}{|z|^2} - 1 \right) I_2 + 2 \left(1 - \frac{1}{|z|^2} \right) I_3 \\
&= \frac{1}{4|z|^2} \left[\frac{1+|z|^2}{1-|z|^2} - \frac{(1-|z|^2)}{2|z|} \log \frac{1+|z|}{1-|z|} \right],
\end{aligned}
$$

which, together with (2.8), yields that

$$
\left| \frac{\partial}{\partial z} \mathcal{G}_g(z) \right| \leq \mu(|z|), \qquad (2.12)
$$

where

$$\mu(|z|)=\frac{\|g\|_{\infty}(1-|z|^2)}{8|z|^2}\left[\frac{1+|z|^2}{1-|z|^2}-\frac{(1-|z|^2)}{2|z|}\log\frac{1+|z|}{1-|z|}\right].$$

By a similar proof process of (2.12), we have

$$\left|\frac{\partial}{\partial\overline{z}}\mathcal{G}_g(z)\right|\leq\mu(|z|). \tag{2.13}$$

By direct calculation (or by [19, Lemma 2.3]), we obtain

$$\lim_{|z|\to 0^+}\frac{\|g\|_{\infty}(1-|z|^2)}{8|z|^2}\left[\frac{1+|z|^2}{1-|z|^2}-\frac{(1-|z|^2)}{2|z|}\log\frac{1+|z|}{1-|z|}\right]=\frac{\|g\|_{\infty}}{3}, \tag{2.14}$$
$$\lim_{|z|\to 1^-}\frac{\|g\|_{\infty}(1-|z|^2)}{8|z|^2}\left[\frac{1+|z|^2}{1-|z|^2}-\frac{(1-|z|^2)}{2|z|}\log\frac{1+|z|}{1-|z|}\right]=\frac{\|g\|_{\infty}}{4}$$

and $\mu(|z|)$ is decreasing on $|z|\in(0,1)$.

On the other hand, since $\mathcal{P}_{\psi}$ is harmonic in $\mathbb{D}$, by [14, Theorem 3] (see also [11,12]), we see that, for $z\in\mathbb{D}$,

$$\|D_{\mathcal{P}_{\psi}}(z)\|\leq\frac{4\|\mathcal{P}_{\psi}\|_{\infty}}{\pi}\frac{1}{1-|z|^2}. \tag{2.15}$$

Hence (1.8) follows from (2.12), (2.13) and (2.15). Furthermore, applying (1.8) and (2.14), we get (1.9). The proof of this theorem is complete. □

Now we formulate the following well-known result.

Lemma 1 *The improper integral*

$$\int_0^{\frac{\pi}{2}}\log\sin x\,dx=\int_0^{\frac{\pi}{2}}\log\cos x\,dx=-\frac{\pi}{2}\log 2.$$

Lemma 2 *For $z\in\mathbb{D}\backslash\{0\}$, the improper integral*

$$\begin{aligned}\int_{\mathbb{D}}\frac{dA(w)}{|w||z-w|}&=\int_0^{2\pi}\log\left(1-r\cos t+\sqrt{1+r^2-2r\cos t}\right)dt\\&\quad-2\pi\log r+2\pi\log 2\\&\leq 2\pi\log 4(1+r)-2\pi\log r,\end{aligned}$$

where $r=|z|$.

Proof Let $z = re^{i\alpha}$ and $w = \rho e^{i\theta}$. Then

$$\begin{aligned}
\int_{\mathbb{D}} \frac{dA(w)}{|w||z-w|} &= \int_0^1 d\rho \int_0^{2\pi} \frac{d\theta}{\sqrt{r^2+\rho^2-2\rho r\cos(\theta-\alpha)}} \\
&= \int_0^1 d\rho \int_0^{2\pi} \frac{dt}{\sqrt{r^2+\rho^2-2\rho r\cos t}} \\
&= \int_0^{2\pi} dt \int_0^1 \frac{d\rho}{\sqrt{r^2+\rho^2-2\rho r\cos t}} \\
&= \int_0^{2\pi} \left\{ \frac{1}{2r\cos t}\left[\int_0^1 \frac{2\rho d\rho}{\sqrt{r^2+\rho^2-2\rho r\cos t}} \right.\right. \\
&\quad \left.\left. - \int_0^1 \frac{d(r^2+\rho^2-2\rho r\cos t)}{\sqrt{r^2+\rho^2-2\rho r\cos t}} \right] \right\} dt \\
&= \int_0^{2\pi} \left[\frac{1}{r\cos t} \int_0^1 \frac{\rho d\rho}{\sqrt{r^2+\rho^2-2\rho r\cos t}} \right. \\
&\quad \left. - \frac{1}{r\cos t}\left(\sqrt{1+r^2-2r\cos t}-r\right) \right] dt \\
&= \int_0^{2\pi} \left[\frac{1}{r\cos t} \int_0^1 \frac{\rho d\rho}{\sqrt{r^2+\rho^2-2\rho r\cos t}} \right. \\
&\quad \left. - \frac{\sqrt{1+r^2-2r\cos t}}{r\cos t} + \frac{1}{\cos t} \right] dt. \qquad (2.16)
\end{aligned}$$

By calculations, we get

$$\begin{aligned}
\int_0^1 \frac{\rho d\rho}{\sqrt{r^2+\rho^2-2\rho r\cos t}} &= H(\rho)|_0^1 \\
&= \sqrt{1+r^2-2r\cos t} \\
&\quad + r\cos t \log\left(1 - r\cos t + \sqrt{1+r^2-2r\cos t}\right) \\
&\quad - r - r\cos t \log r(1-\cos t), \qquad (2.17)
\end{aligned}$$

where

$$H(\rho) = \sqrt{\rho^2+r^2-2r\rho\cos t} + r\cos t \log\left(\rho - r\cos t + \sqrt{r^2+\rho^2-2\rho r\cos t}\right).$$

By (2.16), (2.17) and Lemma 1, we see that

$$\begin{aligned}
\int_{\mathbb{D}} \frac{dA(w)}{|w||z-w|} &= \int_0^{2\pi} \log\left(1 - r\cos t + \sqrt{1+r^2-2r\cos t}\right) dt \\
&\quad - \int_0^{2\pi} \log r(1-\cos t) dt
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{2\pi} \log\left(1 - r\cos t + \sqrt{1 + r^2 - 2r\cos t}\right)dt \\
&\quad - 2\pi \log r - \int_0^{2\pi} \log\left(2\sin^2\frac{t}{2}\right)dt \\
&= \int_0^{2\pi} \log\left(1 - r\cos t + \sqrt{1 + r^2 - 2r\cos t}\right)dt \\
&\quad - 2\pi \log 2r - 8\int_0^{\frac{\pi}{2}} \log(\sin t)dt \\
&= \int_0^{2\pi} \log\left(1 - r\cos t + \sqrt{1 + r^2 - 2r\cos t}\right)dt \\
&\quad - 2\pi \log r + 2\pi \log 2 \\
&\le 2\pi \log 4(1 + r) - 2\pi \log r. \qquad (2.18)
\end{aligned}$$

The proof of this lemma is complete. □

Lemma E ([10, Lemma 1]) *Let f be a harmonic mapping of $\mathbb{D}$ into $\mathbb{C}$ such that $|f(z)| \le M$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$. Then $|a_0| \le M$ and for all $n \ge 1$,*

$$|a_n| + |b_n| \le \frac{4M}{\pi}.$$

Lemma 3 *For $x \in (0, 1)$, let*

$$\phi(x) = \frac{1}{\frac{4}{\pi}M_2 + \frac{2}{3}M_1} - \frac{4M_2}{\pi}\frac{x(2-x)}{(1-x)^2} - 2M_1\left[\log 4(1+x) - \log x\right](2+x)x,$$

where $M_2 > 0$ and $M_1 \ge 0$ are constant. Then ϕ is strictly decreasing and there is an unique $x_0 \in (0, 1)$ such that $\phi(x_0) = 0$.

Proof For $x \in (0, 1)$, let

$$f_1(x) = \frac{4M_2}{\pi}\frac{x(2-x)}{(1-x)^2}$$

and

$$f_2(x) = 2M_1\left[\log 2(1+x) - \log x + \log 2\right](2+x)x.$$

Since, for $x \in (0, 1)$,

$$f_1'(x) = \frac{8M_2}{\pi}\frac{1}{(1-x)^3} > 0$$

and

$$\begin{aligned} f_2'(x) &= 2M_1\left[2(x+1)\log\frac{4(1+x)}{x} - \frac{2+x}{1+x}\right] \\ &= 2M_1\left\{2(x+1)\left[\log 4 + \log\left(1+\frac{1}{x}\right)\right] - \frac{2+x}{1+x}\right\} \\ &\geq 2M_1\left\{2(x+1)\left[1+\frac{1}{1+x}\right] - \frac{2+x}{1+x}\right\} \\ &= 2M_1\frac{(2+x)(2x+1)}{1+x} \geq 0, \end{aligned}$$

we see that $f_1 + f_2$ is continuous and strictly increasing in $(0, 1)$. Then ϕ is continuous and strictly decreasing in $(0, 1)$, which, together with

$$\lim_{x\to 0^+}\phi(x) = \frac{1}{\frac{4}{\pi}M_2 + \frac{2}{3}M_1} \quad \text{and} \quad \lim_{x\to 1^-}\phi(x) = -\infty,$$

implies that there is an unique $x_0 \in (0, 1)$ such that $\phi(x_0) = 0$. □

Lemma 4 *For* $x \in (0, 1]$*, let*

$$\tau_1(x) = \frac{2 - r_0x}{(1-r_0x)^2} \quad \textit{and} \quad \tau_2(x) = x\big[\log 4(1+r_0x) - \log(r_0x)\big],$$

where $r_0 \in (0, 1)$ *is a constant. Then* τ_1 *and* τ_2 *are increasing functions in* $(0, 1]$.

Proof of Theorem 4 As before, by (2.1) with f in place of ψ, we have

$$f(z) = \mathcal{P}_f(z) - \mathcal{G}_g(z), \quad z \in \mathbb{D},$$

where $\mathcal{P}_f$ and $\mathcal{G}_g$ are defined in (2.1). By [19, Lemma 2.3], Theorem D and Lemma 2, we have

$$\begin{aligned} \left|\frac{\partial\mathcal{G}_g(z)}{\partial z} - \frac{\partial\mathcal{G}_g(0)}{\partial z}\right| &= \left|\frac{1}{4\pi}\int_{\mathbb{D}}\frac{(1-|w|^2)}{(z-w)(z\overline{w}-1)}g(w)dA(w)\right. \\ &\quad \left. - \frac{1}{4\pi}\int_{\mathbb{D}}\frac{(1-|w|^2)}{w}g(w)dA(w)\right| \\ &= \left|\frac{1}{4\pi}\int_{\mathbb{D}}\frac{z(1-|w|^2)(1+|w|^2-z\overline{w})}{w(z-w)(z\overline{w}-1)}g(w)dA(w)\right| \\ &\leq \frac{M_1|z|}{4\pi}\int_{\mathbb{D}}\frac{(1-|w|^2)\big|1+|w|^2-z\overline{w}\big|}{|w||z-w||1-z\overline{w}|}dA(w) \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{|z|(2+|z|)M_1}{4\pi}\int_{\mathbb{D}}\frac{(1+|w|)}{|w||z-w|}dA(w)\\
&\leq \frac{|z|(2+|z|)M_1}{2\pi}\int_{\mathbb{D}}\frac{1}{|w||z-w|}dA(w)\\
&\leq M_1\big[\log 4(1+|z|)-\log|z|\big]|z|(2+|z|). \qquad (2.19)
\end{aligned}
$$

By a similar argument, we get

$$
\begin{aligned}
\left|\frac{\partial\mathcal{G}_g(z)}{\partial\overline{z}}-\frac{\partial\mathcal{G}_g(0)}{\partial\overline{z}}\right| &= \left|\frac{1}{4\pi}\int_{\mathbb{D}}\frac{(1-|w|^2)}{(\overline{z}-\overline{w})(w\overline{z}-1)}g(w)dA(w)\right.\\
&\quad\left.-\frac{1}{4\pi}\int_{\mathbb{D}}\frac{(1-|w|^2)}{\overline{w}}g(w)dA(w)\right|\\
&\leq M_1\big[\log 4(1+|z|)-\log|z|\big]|z|(2+|z|). \qquad (2.20)
\end{aligned}
$$

On the other hand, $\mathcal{P}_f$ can be written by

$$
\mathcal{P}_f(z)=\sum_{n=0}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b}_n\overline{z}^n
$$

because $\mathcal{P}_f$ is harmonic in $\mathbb{D}$.

Since $|\mathcal{P}_f(z)|\leq M_2$ for $z\in\mathbb{D}$, by Lemma E, we have

$$
|a_n|+|b_n|\leq\frac{4M_2}{\pi} \qquad (2.21)
$$

for $n\geq 1$.

By (2.21), we see that

$$
\begin{aligned}
\left|\frac{\partial\mathcal{P}_f(z)}{\partial z}-\frac{\partial\mathcal{P}_f(0)}{\partial z}\right|+\left|\frac{\partial\mathcal{P}_f(z)}{\partial\overline{z}}-\frac{\partial\mathcal{P}_f(0)}{\partial\overline{z}}\right| &= \left|\sum_{n=2}^{\infty}na_nz^{n-1}\right|+\left|\sum_{n=2}^{\infty}nb_n\overline{z}^{n-1}\right|\\
&\leq\sum_{n=2}^{\infty}n\big(|a_n|+|b_n|\big)|z|^{n-1}\\
&\leq\frac{4M_2}{\pi}\sum_{n=2}^{\infty}n|z|^{n-1}\\
&=\frac{4M_2}{\pi}\frac{|z|(2-|z|)}{(1-|z|)^2}. \qquad (2.22)
\end{aligned}
$$

Applying Theorem 3, we obtain

$$1 = J_f(0) = \|D_f(0)\|\lambda(D_f(0)) \leq \lambda(D_f(0))\left(\frac{4}{\pi}M_2 + \frac{2}{3}M_1\right),$$

which gives that

$$\lambda(D_f(0)) \geq \frac{1}{\frac{4}{\pi}M_2 + \frac{2}{3}M_1}. \tag{2.23}$$

In order to prove the univalence of f in $\mathbb{D}_{r_0}$, we choose two distinct points $z_1, z_2 \in \mathbb{D}_{r_0}$ and let $[z_1, z_2]$ denote the segment from z_1 to z_2 with the endpoints z_1 and z_2, where r_0 satisfies the following equation

$$\frac{1}{\frac{4}{\pi}M_2 + \frac{2}{3}M_1} - \frac{4M_2}{\pi}\frac{r_0(2-r_0)}{(1-r_0)^2} - 2M_1\big[\log 4(1+r_0) - \log r_0\big](2+r_0)r_0 = 0.$$

By (2.19), (2.20), (2.22), (2.23), Lemmas 3 and 4, we have

$$\begin{aligned}
f(z_2) - f(z_1)| &= \left|\int_{[z_1,z_2]} f_z(z)dz + f_{\overline{z}}(z)d\overline{z}\right| \\
&\geq \left|\int_{[z_1,z_2]} f_z(0)dz + f_{\overline{z}}(0)d\overline{z}\right| \\
&\quad -\left|\int_{[z_1,z_2]} \big(f_z(z) - f_z(0)\big)dz + \big(f_{\overline{z}}(z) - f_{\overline{z}}(0)\big)d\overline{z}\right| \\
&\geq \lambda(D_f(0))|z_2 - z_1| \\
&\quad - \int_{[z_1,z_2]} \big(|f_z(z) - f_z(0)| + |f_{\overline{z}}(z) - f_{\overline{z}}(0)|\big)|dz| \\
&\geq \lambda(D_f(0))|z_2 - z_1| \\
&\quad - \int_{[z_1,z_2]} \left(\left|\frac{\partial \mathcal{G}_g(z)}{\partial z} - \frac{\partial \mathcal{G}_g(0)}{\partial z}\right| + \left|\frac{\partial \mathcal{G}_g(z)}{\partial \overline{z}} - \frac{\partial \mathcal{G}_g(0)}{\partial \overline{z}}\right|\right)|dz| \\
&\quad - \int_{[z_1,z_2]} \left(\left|\frac{\partial \mathcal{P}_f(z)}{\partial z} - \frac{\partial \mathcal{P}_f(0)}{\partial z}\right| + \left|\frac{\partial \mathcal{P}_f(z)}{\partial \overline{z}} - \frac{\partial \mathcal{P}_f(0)}{\partial \overline{z}}\right|\right)|dz| \\
&> |z_2 - z_1|\Big\{\lambda(D_f(0)) - \frac{4M_2}{\pi}\frac{r_0(2-r_0)}{(1-r_0)^2} \\
&\quad - 2M_1\big[\log 4(1+r_0) - \log r_0\big](2+r_0)r_0\Big\} \\
&\geq |z_2 - z_1|\left\{\frac{1}{\frac{4}{\pi}M_2 + \frac{2}{3}M_1} - \frac{4M_2}{\pi}\frac{r_0(2-r_0)}{(1-r_0)^2}\right. \\
&\quad \left. - 2M_1\big[\log 4(1+r_0) - \log r_0\big](2+r_0)r_0\right\} \\
&= 0,
\end{aligned} \tag{2.24}$$

which yields that $f(z_2) \neq f(z_1)$. The univalence of f follows from the arbitrariness of z_1 and z_2.

Now, for all $\zeta = r_0 e^{i\theta} \in \partial\mathbb{D}_{r_0}$, by (2.19), (2.20), (2.22), (2.23), Lemmas 3 and 4, we obtain

$$
\begin{aligned}
|f(\zeta) - f(0)| &= \left| \int_{[0,\zeta]} f_z(z)dz + f_{\overline{z}}(z)d\overline{z} \right| \\
&= \left| \int_{[0,\zeta]} f_z(0)dz + f_{\overline{z}}(0)d\overline{z} \right| \\
&\quad - \left| \int_{[0,\zeta]} \big(f_z(z) - f_z(0)\big)dz + \big(f_{\overline{z}}(z) - f_{\overline{z}}(0)\big)d\overline{z} \right| \\
&\geq \lambda(D_f(0))r_0 \\
&\quad - \int_{[0,\zeta]} \big(|f_z(z) - f_z(0)| + |f_{\overline{z}}(z) - f_{\overline{z}}(0)|\big)|dz| \\
&\geq \lambda(D_f(0))r_0 \\
&\quad - \int_{[0,\zeta]} \left(\left| \frac{\partial \mathcal{G}_g(z)}{\partial z} - \frac{\partial \mathcal{G}_g(0)}{\partial z} \right| + \left| \frac{\partial \mathcal{G}_g(z)}{\partial \overline{z}} - \frac{\partial \mathcal{G}_g(0)}{\partial \overline{z}} \right| \right) |dz| \\
&\quad - \int_{[0,\zeta]} \left(\left| \frac{\partial \mathcal{P}_f(z)}{\partial z} - \frac{\partial \mathcal{P}_f(0)}{\partial z} \right| + \left| \frac{\partial \mathcal{P}_f(z)}{\partial \overline{z}} - \frac{\partial \mathcal{P}_f(0)}{\partial \overline{z}} \right| \right) |dz| \\
&\geq \frac{r_0}{\frac{4}{\pi}M_2 + \frac{2}{3}M_1} - \frac{4M_2}{\pi} \int_{[0,\zeta]} \frac{|z|(2-|z|)}{(1-|z|)^2}|dz| \\
&\quad - 2M_1 \int_{[0,\zeta]} \big[\log 4(1+|z|) - \log|z|\big]|z|(2+|z|)|dz| \\
&= \frac{r_0}{\frac{4}{\pi}M_2 + \frac{2}{3}M_1} - \frac{4M_2 r_0^2}{\pi} \int_0^1 \frac{t(2-r_0 t)}{(1-r_0 t)^2}dt \\
&\quad - 2M_1 r_0^2 \int_0^1 \big[\log 4(1+r_0 t) - \log(r_0 t)\big]t(2+r_0 t)dt \\
&\geq \frac{r_0}{\frac{4}{\pi}M_2 + \frac{2}{3}M_1} - \frac{4M_2 r_0^2}{\pi} \frac{(2-r_0)}{(1-r_0)^2} \int_0^1 t dt \\
&\quad - 2M_1 r_0^2 (2+r_0) \int_0^1 \big[\log 4(1+r_0 t) - \log(r_0 t)\big]t dt \\
&\geq r_0 \left\{ \frac{1}{\frac{4}{\pi}M_2 + \frac{2}{3}M_1} - \frac{2M_2}{\pi} \frac{r_0(2-r_0)}{(1-r_0)^2} \right. \\
&\quad \left. - 2M_1 r_0(2+r_0)\big[\log 4(1+r_0) - \log r_0\big] \right\} \\
&= \frac{2M_2}{\pi} \frac{r_0^2(2-r_0)}{(1-r_0)^2}.
\end{aligned}
$$

Hence $f(\mathbb{D}_{r_0})$ contains an univalent disk $\mathbb{D}_{R_0}$ with

$$R_0 \geq \frac{2M_2}{\pi} \frac{r_0^2(2-r_0)}{(1-r_0)^2}.$$

The proof of this theorem is complete. □

Proof of Corollary 1 For $z_1, z_2 \in \mathbb{D}_{r_0}$, by (2.24), we see that there is a positive constant L_1 such that

$$L_1|z_1 - z_2| \leq |f(z_1) - f(z_2)|,$$

where r_0 satisfies the following equation

$$\frac{1}{\frac{4}{\pi}M_2 + \frac{2}{3}M_1} - \frac{4M_2}{\pi} \frac{r_0(2-r_0)}{(1-r_0)^2} - 2M_1\big[\log 4(1+r_0) - \log r_0\big](2+r_0)r_0 = 0.$$

On the other hand, for $z_1, z_2 \in \mathbb{D}_{r_0}$, we use Theorem 3 to get

$$\begin{aligned}
|f(z_2) - f(z_1)| &= \left|\int_{[z_1,z_2]} df(z)\right| \\
&\leq \int_{[z_1,z_2]} \|D_f(z)\||dz| \\
&\leq \int_{[z_1,z_2]} \left(\frac{4M_2}{\pi} \frac{1}{1-r_0^2} + \frac{2}{3}M_1\right)|dz| \\
&= \left(\frac{4M_2}{\pi} \frac{1}{1-r_0^2} + \frac{2}{3}M_1\right)|z_1 - z_2|,
\end{aligned}$$

where $[z_1, z_2]$ is the segment from z_1 to z_2 with the endpoints z_1 and z_2. Therefore, f is bi-Lipschitz in $\mathbb{D}_{r_0}$. □

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