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Bloch Space of a Bounded Symmetric Domain and Composition Operators

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Abstract
We generalize a number of finite dimensional results on Bloch functions to infinite dimensional bounded symmetric domains. In particular, we characterize the Bloch space as well as the little Bloch space of a Hilbert ball, and give one sufficient and several necessary conditions for a composition operator on a Bloch space to be an isometry. We also answer some open questions of Allen and Colonna concerning Bloch functions and composition operators.

Keywords Bloch space · Bounded symmetric domain · Composition operator · JB*-triple

Mathematics Subject Classification 47B38 · 32A18 · 32M15

1 Introduction

Recently, we introduced the concept of a Bloch function on a bounded symmetric domain which can be infinite dimensional and derived some basic properties of Bloch spaces and composition operators in this setting, generalising several finite dimensional results [5,9]. In this paper, we refine and develop some results in [5] thereby extending a number of finite dimensional results on Bloch spaces and composition operators, as well as answering two open questions in [1].

For finite dimensional bounded homogeneous domains, the Bloch functions are usually defined in terms of the Bergman metric (cf. [21]), which is not available in infinite dimension. Instead, albeit equivalent, we define in [5] the Bloch functions on a

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bounded symmetric domain by the Kobayashi metric. We begin by showing in Proposition 2.6 the relationship between the Bloch seminorm and the Kobayashi distance, which gives a definitive answer to the open question (1) in [1, p. 687]. Characterizations of Bloch functions and the little Bloch space on a bounded symmetric domain have been given in [5]. In the course of studying Hankel operators, Holland and Walsh [15, Theorem 3] found an alternative characterization of Bloch functions on the open unit disc in $\mathbb{C}$. Stroethoff [20] gave a simpler proof of this result by exploiting the Möbius invariance, which also enabled him to give an analogous characterization of the little Bloch space of the unit disc. We extend both results to Hilbert balls in Proposition 2.8 and Theorem 2.11. A similar result for Bloch spaces of Euclidean balls has also been obtained in [19]. However, these characterizations of Bloch functions are special for Hilbert balls and we give an example to show that they are not valid even for the bidisc. It should be noted that, after this paper has been written, we learned that Deng and Ouyang [6] had defined Bloch spaces on bounded symmetric domains using the Carathéodory differential metric, and Krantz and Ma [17] had also introduced the concept of Bloch functions on finite dimensional strongly pseudoconvex domains using the Kobayashi metric. Nevertheless, our results do not overlap with those in [6,17].

Given a holomorphic mapping $\varphi : \mathbb{B}_X \to \mathbb{B}_Y$ between bounded symmetric domains, we show in [5] that the composition operator C_φ between their Bloch spaces is bounded. For a self-map φ on $\mathbb{B}_X$, we give upper and lower bounds for the norm $\|C_\varphi\|$, which generalizes the results obtained by Xiong [24] for the unit disc and by Allen and Colonna [1] for finite dimensional homogeneous domains. Finally, we prove one sufficient and several necessary conditions for C_φ to be an isometry, extending the results in [1]. We also give a positive answer to the question in [1, Remark 6.1].

In the following, all Banach spaces are over the complex field and we will adopt the following notation. Given a Banach space X with open unit ball $\mathbb{B}_X$ and a domain Ω in a Banach space Y, we will denote by $H(\mathbb{B}_X, \Omega)$ the space of holomorphic mappings from $\mathbb{B}_X$ into Ω. The space $H(\mathbb{B}_X, \mathbb{B}_X)$ of self-mappings will be abbreviated to $H(\mathbb{B}_X)$. For a mapping $f \in H(\mathbb{B}_X, Y)$, let $Df(z)$ denote the Fréchet derivative of f at $z \in \mathbb{B}_X$. A mapping $f \in H(\mathbb{B}_X, Y)$ is said to be *biholomorphic* if $f(\mathbb{B}_X)$ is a domain in Y and the holomorphic inverse f^{-1} exists on $f(\mathbb{B}_X)$. Let $L(X, Y)$ be the Banach space of continuous linear operators from X into Y, and let I_X be the identity in $L(X) = L(X, X)$.

Our approach to infinite dimensional bounded symmetric domains relies on Kaup's seminal result asserting that a bounded domain in a complex Banach space is a symmetric domain if and only if it is biholomorphic to the open unit ball of a JB*-triple (see [16]). A JB*-triple is a complex Banach space equipped with a Hermitian Jordan structure. By identifying a bounded symmetric domain with the open unit ball of a JB*-triple, one can make use of the Jordan algebraic structures to study function theory of these domains.

Unless stated otherwise, we will denote throughout by $\mathbb{B}_X$ a bounded symmetric domain realized as the open unit ball of a JB*-triple X. The latter means that X is a complex Banach space equipped with a continuous Jordan triple product

$$(x, y, z) \in X \times X \times X \mapsto \{x, y, z\} \in X$$

which is symmetric bilinear in the outer variables, conjugate linear in the middle variable, and satisfies

(i) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$,
(ii) the linear operator $a \,\square\, a : x \in X \mapsto \{a, a, x\} \in X$ is Hermitian with non-negative spectrum,
(iii) $\|\{x, x, x\}\| = \|x\|^3$

for all $a, b, x, y, z \in X$, where the identity (i) is called the *Jordan triple identity* and the operator $a \,\square\, b : x \in X \mapsto \{a, b, x\}$ is called a *box operator*.

JB*-triples include Hilbert spaces and C*-algebras. The Jordan triple product in a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ is given by

$$\{x, y, z\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x) \quad (x, y, z \in H).$$

The open unit ball $\mathbb{B}_H$ of H is a symmetric domain, which will be called a *Hilbert ball*. For a C*-algebra $\mathcal{A}$, the Jordan triple product is given by $\{a, b, c\} = (ab^*c + cb^*a)/2$ for $a, b, c \in \mathcal{A}$.

To study Bloch functions on a bounded symmetric domain $\mathbb{B}_X$ realized as the open unit ball of a JB*-triple X, we often make use of the Bergman operator $B(x, y) : X \to X$ and the Möbius transformation $g_a : \mathbb{B}_X \to \mathbb{B}_X$. They are defined as follows. Given $x, y \in X$, we define

$$B(x, y)z = z - 2(x \,\square\, y)(z) + \{x, \{y, z, y\}, x\} \quad (z \in X).$$

For $\|x\| < 1$, the operator $B(x, x)$ has non-negative spectrum [3, Lemma 2.5.21] and hence the square roots $B(x, x)^{\pm 1/2}$ exist. For each $a \in \mathbb{B}_X$, the Möbius transformation g_a, induced by a, is a biholomorphic mapping given by

$$g_a(x) = a + B(a, a)^{1/2}(I_X + x \,\square\, a)^{-1}(x).$$

The inverse of g_a is g_{-a}. If $\mathbb{B}_X$ is the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, then we have the familiar form $g_a(z) = \dfrac{a + z}{1 + \overline{a}z}$. We refer to [3,18] for further details of JB*-triples and references.

2 Bloch Functions

In this section, we generalize some finite dimensional characterizations of Bloch functions to infinite dimension. Bloch functions on bounded homogeneous domains in $\mathbb{C}^n$ have been defined and studied in [8,21]. For a Hilbert ball, they have also been defined in [2]. Unifying these definitions, we introduced in [5] the concept of a Bloch function on a bounded symmetric domain, defined in terms of the Kobayashi metric instead of the Bergman metric for the finite dimensional case since the latter is not available in infinite dimension. Nevertheless, our definition is equivalent to the other ones just mentioned. Because of this difference, it would be desirable to clarify the relationship between the Bloch seminorm and the Kobayashi metric. This will be shown in Proposition 2.6 below. Let us first recall our definition of a Bloch function.

Definition 2.1 Let $\mathbb{B}_X$ be a bounded symmetric domain and let $\kappa(z, x)$ denote the infinitesimal Kobayashi metric for $\mathbb{B}_X$. A complex function $f \in H(\mathbb{B}_X, \mathbb{C})$ is called a *Bloch function* if it has finite *Bloch seminorm*

$$\|f\|_{\mathcal{B}(\mathbb{B}_X),s} := \sup\{Q_f(z) : z \in \mathbb{B}_X\} < \infty,$$

where

$$Q_f(z) = \sup\left\{\frac{|Df(z)x|}{\kappa(z,x)} : x \in X \setminus \{0\}\right\}$$

and

$$\kappa(z, x) = \|B(z, z)^{-1/2}x\|.$$

For finite dimensional domains $\mathbb{B}_X$, Bloch functions are also defined in terms of the Bergman metric $H_z(x, \overline{x})$ (cf. [21, Definition 3.3]), namely, a holomorphic function f on $\mathbb{B}_X$ is a Bloch function if

$$\beta_f := \sup\{Q^h_f(z) : z \in \mathbb{B}_X\} < \infty,$$

where

$$Q^h_f(z) = \sup\left\{\frac{|Df(z)x|}{H_z(x,\overline{x})^{1/2}} : x \in X \setminus \{0\}\right\}.$$

These two definitions are equivalent in view of the inequality

$$\kappa(z, x) \le H_z(x, \overline{x})^{1/2} \le \sqrt{2}c(\mathbb{B}_X)^{1/2}\kappa(z, x) \quad (z \in \mathbb{B}_X, x \in X)$$

where

$$c(\mathbb{B}_X) = \frac{1}{2}\sup_{x\in\mathbb{B}_X} H_0(x, \overline{x})$$

and we have

$$\|f\|_{\mathcal{B}(\mathbb{B}_X),s} \ge \beta_f \ge \frac{1}{\sqrt{2c(\mathbb{B}_X)}}\|f\|_{\mathcal{B}(\mathbb{B}_X),s}. \tag{2.1}$$

The constant $c(\mathbb{B}_X)$ was defined in [11] and plays an important role in the formulation of various distortion theorems and the Bloch constant (cf. [4,10–14]).

Let $\mathrm{Aut}(\mathbb{B}_X)$ be the automorphism group of $\mathbb{B}_X$, consisting of biholomorphic mappings of $\mathbb{B}_X$. For $f \in H(\mathbb{B}_X, \mathbb{C})$, it has been shown in [5] that the Bloch seminorm satisfies

$$\|f\|_{\mathcal{B}(\mathbb{B}_X),s} = \sup\{\|D(f \circ g)(0)\| : g \in \mathrm{Aut}(\mathbb{B}_X)\}. \tag{2.2}$$

The class of all Bloch functions on $\mathbb{B}_X$ forms a complex Banach space in the Bloch norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_{\mathcal{B}(\mathbb{B}_X),s}$$

which is denoted by $\mathcal{B}(\mathbb{B}_X)$ and called the *Bloch space* on $\mathbb{B}_X$ (cf. [5]). If $\dim X < \infty$, we will denote by $(\mathcal{B}(\mathbb{B}_X), \beta)$ the Banach space of Bloch functions f on $\mathbb{B}_X$ with respect to the norm $|f(0)| + \beta_f$. In this setting, the two spaces $\mathcal{B}(\mathbb{B}_X)$ and $(\mathcal{B}(\mathbb{B}_X), \beta)$ are linearly isomorphic.

We will denote by ρ the Kobayashi distance on $\mathbb{B}_X$, which is the integral form of the Kobayashi metric κ. For $a, b \in \mathbb{B}_X$, we have $\rho(a, b) = \tanh^{-1} \|g_{-a}(b)\|$, where g_{-a} is the Möbius transformation induced by $-a$. The Bergman distance on $\mathbb{B}_X$ will be denoted by h. For the Euclidean unit ball B_n in $\mathbb{C}^n$, we have

$$h(a, b) = \sqrt{n+1} \tanh^{-1} \|g_{-a}(b)\|.$$

We have given several characterizations of a Bloch function in [5, Theorem 3.8]. An examination of the proof of this theorem reveals that a holomorphic function f on $\mathbb{B}_X$ is a Bloch function if and only if it is a Lipschitz mapping as a function from $\mathbb{B}_X$, equipped with the Kobayashi distance, to $\mathbb{C}$ in the Euclidean distance.

Lemma 2.2 *Let $f \in H(\mathbb{B}_X, \mathbb{C})$. Then f is a Bloch function if and only if the following inequality holds:*

$$|f(z) - f(w)| \leq \|f\|_{\mathcal{B}(\mathbb{B}_X),s}\, \rho(z, w), \quad \forall z, w \in \mathbb{B}_X.$$

Actually, the Lipschitz constant in this case is given by the Bloch seminorm of f, as shown in the following generalization of [1, Theorem 3.1], where it has been shown that

$$\beta_f = \sup_{z \neq w} \frac{|f(z) - f(w)|}{h(z, w)}$$

for $f \in (\mathcal{B}(\mathbb{B}_X), \beta)$ when X is finite dimensional.

Proposition 2.3 *Let f be a Bloch function on a bounded symmetric domain $\mathbb{B}_X$. Then we have*

$$\|f\|_{\mathcal{B}(\mathbb{B}_X),s} = \sup_{z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)}.$$

Proof Write

$$\mathrm{lip}(f) = \sup_{z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)}.$$

By Lemma 2.2, we have $\mathrm{lip}(f) \le \|f\|_{\mathcal{B}(\mathbb{B}_X),s}$. On the other hand, for $g \in \mathrm{Aut}(\mathbb{B}_X)$, any $t \in (0, 1)$ and $w \in \mathbb{B}_X$, we have

$$\begin{aligned}|f(g(tw)) - f(g(0))| &\le \mathrm{lip}(f)\rho(g(tw), g(0)) \\ &= \mathrm{lip}(f)\rho(tw, 0) \\ &= \mathrm{lip}(f) \tanh^{-1}(t\|w\|).\end{aligned}$$

It follows that

$$|D(f \circ g)(0)w| \le \mathrm{lip}(f)\|w\|$$

and hence $\|f\|_{\mathcal{B}(\mathbb{B}_X),s} \le \mathrm{lip}(f)$ by (2.2). This completes the proof. □

Using Proposition 2.3, we obtain the following generalization of [1, Theorem 3.3] to infinite dimensional bounded symmetric domains, by similar arguments. We omit the proof, but recall that a sequence $\{f_n\}$ of functions on a domain $D \subset X$ converges locally uniformly to a function f if and only if it converges uniformly on every closed ball strictly contained in D (cf. [7]).

Proposition 2.4 *Let $\{f_n\}$ be a sequence of Bloch functions on a bounded symmetric domain $\mathbb{B}_X$ converging locally uniformly to some $f \in H(\mathbb{B}_X, \mathbb{C})$. If the sequence $\{\|f_n\|_{\mathcal{B}(\mathbb{B}_X),s}\}$ is bounded, then f is a Bloch function and*

$$\|f\|_{\mathcal{B}(\mathbb{B}_X),s} \le \liminf_{n\to\infty} \|f_n\|_{\mathcal{B}(\mathbb{B}_X),s}.$$

In other words, the Bloch seminorm $\|\cdot\|_{\mathcal{B}(\mathbb{B}_X),s}$ on $\mathcal{B}(\mathbb{B}_X)$ is lower semi-continuous in the locally uniform topology.

For $x \in X \setminus \{0\}$, the set

$$T(x) = \{\ell_x \in X^* : \ \ell_x(x) = \|x\|, \ \|\ell_x\| = 1\}$$

of support functionals of x is nonempty by the Hahn-Banach theorem. We give an example of Bloch functions which will be useful later.

Example 2.5 Let $\mathbb{B}_X$ be the open unit ball of a JB*-triple X. For any $a \in \mathbb{B}_X \setminus \{0\}$ with support functional ℓ_a, let $f_a(z) = \psi(l_a(z))$ for $z \in \mathbb{B}_X$, where

$$\psi(\zeta) := \tanh^{-1} \zeta = \frac{1}{2} \log \frac{1+\zeta}{1-\zeta} \quad (\zeta \in \mathbb{D}).$$

Then $f_a \in \mathcal{B}(\mathbb{B}_X)$ and $\|f_a\|_{\mathcal{B}(\mathbb{B}_X),s} \le \|\psi\|_{\mathcal{B}(\mathbb{D}),s} = 1$ by [5, Proposition 3.15]. Since $Q_{f_a}(0) = 1$, we have $\|f_a\|_{\mathcal{B}(\mathbb{B}_X)} = \|\psi\|_{\mathcal{B}(\mathbb{D}),s} = 1$.

For a finite dimensional domain $\mathbb{B}_X$, it has been observed in [1] that

$$h(z, w) \ge \sup\{|f(z) - f(w)| : f \in (\mathcal{B}(\mathbb{B}_X), \beta), \beta_f \le 1\}$$

for $z, w \in \mathbb{B}_X$. In view of the fact that this inequality becomes an equality for the one-dimensional open unit disc $\mathbb{B}_X = \mathbb{D}$ (cf. [25, Theorem 5.1.7] and [26, Theorem 3.9]), a natural question has been posed in [1, Open question (1), p. 687] asking if the equality also holds in higher dimensions. We show below that the equality actually holds for all dimensions as soon as the Bergman distance is replaced by the Kobayashi distance.

Proposition 2.6 *Let $\mathbb{B}_X$ be a bounded symmetric domain. Then for any $z, w \in \mathbb{B}_X$, we have*

$$\rho(z, w) = \sup\{|f(z) - f(w)| : f \in \mathcal{B}(\mathbb{B}_X), \|f\|_{\mathcal{B}(\mathbb{B}_X),s} \leq 1\}. \tag{2.3}$$

Proof By Proposition 2.3, we have

$$\sup\{|f(z) - f(w)| : f \in \mathcal{B}(\mathbb{B}_X), \|f\|_{\mathcal{B}(\mathbb{B}_X),s} \leq 1\} \leq \rho(z, w).$$

To prove the reverse inequality, fix any $a \in \mathbb{B}_X \setminus \{0\}$ and let $f_a \in \mathcal{B}(\mathbb{B}_X)$ be the function defined in Example 2.5. Then

$$\begin{aligned}\rho(a, 0) = \tanh^{-1} \|a\| &= |f_a(a) - f_a(0)| \\ &\leq \sup\{|f(a) - f(0)| : f \in \mathcal{B}(\mathbb{B}_X), \|f\|_{\mathcal{B}(\mathbb{B}_X),s} \leq 1\}.\end{aligned}$$

By composing with a Möbius transformation of $\mathbb{B}_X$, we have

$$\begin{aligned}\rho(z, w) &= \rho(g_{-w}(z), g_{-w}(w)) \\ &\leq \sup\{|f(g_{-w}(z)) - f(g_{-w}(w))| : f \in \mathcal{B}(\mathbb{B}_X), \|f\|_{\mathcal{B}(\mathbb{B}_X),s} \leq 1\} \\ &= \sup\{|f(z) - f(w)| : f \in \mathcal{B}(\mathbb{B}_X), \|f\|_{\mathcal{B}(\mathbb{B}_X),s} \leq 1\}\end{aligned}$$

for any $z, w \in \mathbb{B}_X$, since $g_{-w}(w) = 0$ and also $\|f \circ g_{-w}\|_{\mathcal{B}(\mathbb{B}_X),s} \leq 1$ for any $f \in \mathcal{B}(\mathbb{B}_X)$ with $\|f\|_{\mathcal{B}(\mathbb{B}_X),s} \leq 1$ and for any $w \in \mathbb{B}_X$. This completes the proof. □

We now turn to the Bloch space and the little Bloch space of a Hilbert ball. We prove a lemma first. Let $\mathbb{B}$ be the open unit ball of a complex Banach space. For $f \in H(\mathbb{B}, \mathbb{C})$, we define

$$S(f) = \sup_{z,w\in\mathbb{B}, z\neq w} (1 - \|z\|^2)^{1/2}(1 - \|w\|^2)^{1/2} \frac{|f(z) - f(w)|}{\|z - w\|}.$$

Lemma 2.7 *Let $\mathbb{B}$ be the unit ball of a complex Banach space X and let $f \in H(\mathbb{B}, \mathbb{C})$. Then $S(f) < \infty$ if and only if $\sup_{z\in\mathbb{B}}(1 - \|z\|^2)\|Df(z)\| < \infty$.*

Proof Let $M = \sup_{z\in\mathbb{B}}(1 - \|z\|^2)\|Df(z)\|$. If $M < \infty$, then similar arguments to those in the proof of [19, Theorem 3.2] yields $S(f) \leq 2\pi M < \infty$.

Assume, on the other hand, $S(f) < \infty$. Let $w \in X$ with $\|w\| = 1$ and $z + tw \in \mathbb{B}$ for small $t > 0$. Then for each $z \in \mathbb{B}$, we have

$$(1 - \|z\|^2)^{1/2}(1 - \|z + tw\|^2)^{1/2} \frac{|f(z) - f(z + tw)|}{t} \leq S(f).$$

Letting $t \to 0+$, we have $(1 - \|z\|^2)|Df(z)w| \leq S(f)$. Since $z \in \mathbb{B}$ and $w \in X$ with $\|w\| = 1$ are arbitrary, we have $\sup_{z \in \mathbb{B}}(1 - \|z\|^2)\|Df(z)\| \leq S(f) < \infty$. This completes the proof. □

By [2], a holomorphic function f on a Hilbert ball $\mathbb{B}_H$ is a Bloch function if and only if $\sup_{z \in \mathbb{B}_H}(1 - \|z\|^2)\|Df(z)\| < \infty$. The above lemma gives immediately the following generalization of Holland-Walsh's characterization [15, Theorem 3] of Bloch functions on the open unit disc in $\mathbb{C}$, which has also been extended to the Euclidean balls in [19].

Proposition 2.8 *Let $\mathbb{B}_H$ be a Hilbert ball and let $f \in H(\mathbb{B}_H, \mathbb{C})$. Then f is a Bloch function if and only if $S(f) < \infty$.*

Given a bounded symmetric domain $\mathbb{B}_X$, we have shown in [5] that a Bloch function f on $\mathbb{B}_X$ satisfies $\sup_{z \in \mathbb{B}_X}(1 - \|z\|^2)\|Df(z)\| < \infty$. This gives the following result.

Proposition 2.9 *Let f be a Bloch function on a bounded symmetric domain $\mathbb{B}_X$. Then we have $S(f) < \infty$.*

The question of the converse of the previous result is a delicate issue. The following counter example for the bidisc suggests that the criterion of Bloch functions in Proposition 2.8 for Hilbert balls is atypical for bounded symmetric domains. We recall that the negative square root $B(z, z)^{-1/2}$ of the Bergman operator for the bidisc $\mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$ is given by

$$B(z, z)^{-1/2}(x) = \left(\frac{x_1}{1 - |z_1|^2}, \frac{x_2}{1 - |z_2|^2} \right)$$

for $z = (z_1, z_2) \in \mathbb{D} \times \mathbb{D}$ and $x = (x_1, x_2) \in \mathbb{C}^2$.

Example 2.10 Let $D = \mathbb{D} \times \mathbb{D}$ be the bidisc and let $f : D \to \mathbb{C}$ be defined by

$$f(z_1, z_2) = (1 - z_2) \log \frac{1}{1 - z_1}$$

for $(z_1, z_2) \in D$. Then we have $S(f) < \infty$, but f is not a Bloch function. We first show that $\sup_{z \in D}(1 - \|z\|^2)\|Df(z)\| < \infty$ which is equivalent to $S(f) < \infty$ by Lemma 2.7. Indeed, we have, for $(x_1, x_2) \in \mathbb{C}^2$,

$$Df(z_1, z_2)(x_1, x_2) = \frac{(1 - z_2)x_1}{1 - z_1} - x_2 \log \frac{1}{1 - z_1}.$$

Observe that

$$\begin{aligned} \|Df(z_1, z_2)\| &\leq \left| \frac{1 - z_2}{1 - z_1} \right| + \left| \log \frac{1}{1 - z_1} \right| \\ &\leq \frac{|1 - z_2|}{|1 - z_1|} + \frac{2}{|1 - z_1|} + \log 2 + \pi \end{aligned}$$

where $1/|1-z_1| \geq 1/2$ implies $2/|1-z_1| \geq \log(2/|1-z_1|) \geq |\log(1/|1-z_1|)| - \log 2$.

Pick $z = (z_1, z_2) \in D$. In the case of $|z_1| > |z_2|$, we have

$$\begin{aligned}(1-\|z\|^2)\|Df(z)\| &\leq (1-|z_1|^2)\left(\frac{|1-z_2|}{|1-z_1|} + \frac{2}{|1-z_1|} + \log 2 + \pi\right)\\ &\leq (1+|z_1|)(|1-z_2| + 2 + (\log 2 + \pi)(1-|z_1|)).\end{aligned}$$

In the case of $|z_1| \leq |z_2|$, we have $|1-z_1| \geq 1-|z_1| \geq 1-|z_2|$ and hence

$$\begin{aligned}(1-\|z\|^2)\|Df(z)\| &= (1-|z_2|^2)\|Df(z)\|\\ &\leq (1+|z_2|)(|1-z_2| + 2 + (\log 2 + \pi)(1-|z_2|)).\end{aligned}$$

We deduce from these inequalities that

$$\sup_{z\in D}(1-\|z\|^2)\|Df(z)\| < \infty.$$

On the other hand, for $z = (z_1, 0) \in D$, we have

$$\left|Df(z_1, 0)\left(\frac{1-|z_1|^2}{2},\ 1\right)\right| = \left|\frac{1-|z_1|^2}{2(1-z_1)} - \log\frac{1}{1-z_1}\right|$$

and

$$\kappa(z, ((1-|z_1|^2)/2,\ 1)) = \|B(z,z)^{-1/2}((1-|z_1|^2)/2,\ 1)\| = \|(1/2, 1)\| = 1.$$

Hence

$$\begin{aligned}Q_f(z_1, 0) &\geq \left|\frac{1-|z_1|^2}{2(1-z_1)} - \log\frac{1}{1-z_1}\right|\\ &\geq \left|\log\frac{1}{1-z_1}\right| - \frac{1-|z_1|^2}{2|1-z_1|}\\ &\geq \left|\log\frac{1}{1-z_1}\right| - \frac{1+|z_1|}{2} \geq \left|\log\frac{1}{1-z_1}\right| - 1\end{aligned}$$

from which we conclude that f is not a Bloch function.

We now prove the following generalization of Stroethoff's characterization of the little Bloch space on the unit disc [20]. The case of the Euclidean unit balls has been shown in [19]. We denote by $\mathcal{B}(\mathbb{B}_H)_0$ the *little Bloch space*, which is defined to be the closure in $\mathcal{B}(\mathbb{B}_H)$ of the polynomial functions on the Hilbert ball $\mathbb{B}_H$ (cf. [23]).

Theorem 2.11 *Let $\mathbb{B}_H$ be a Hilbert ball and let $f \in H(\mathbb{B}_H, \mathbb{C})$. Then $f \in \mathcal{B}(\mathbb{B}_H)_0$ if and only if*

$$\lim_{\|z\|\to 1}\ \sup_{w\in\mathbb{B}, w\neq z}(1-\|z\|^2)^{1/2}(1-\|w\|^2)^{1/2}\frac{|f(z)-f(w)|}{\|z-w\|} = 0. \tag{2.4}$$

Proof Let $f \in \mathcal{B}(\mathbb{B}_H)_0$ and let $f_t(z) = f(tz)$ for $t \in (0, 1)$. Then as in the proof of [19, Theorem 3.2], we have

$$\sup_{w \in \mathbb{B}, w \neq z} (1 - \|z\|^2)^{1/2} (1 - \|w\|^2)^{1/2} \frac{|f(z) - f(w)|}{\|z - w\|}$$
$$\leq C \frac{t}{1 - t^2} (1 - \|z\|^2)^{1/2} \|f\|_{\mathcal{B}} + 2\pi \|f - f_t\|_{\mathcal{B}}.$$

For any given $\varepsilon > 0$, there exists $t_0 \in (0, 1)$ such that $\|f - f_{t_0}\|_{\mathcal{B}} < \varepsilon/4\pi$ as in the proof of [22, Proposition 3.2]. Then there exists $\delta \in (0, 1)$ such that

$$C \frac{t_0}{1 - t_0^2} (1 - \|z\|^2)^{1/2} \|f\|_{\mathcal{B}} < \varepsilon/2$$

for z with $\delta < \|z\| < 1$. Therefore, (2.4) holds.

Conversely, assume that (2.4) holds. For any given $\varepsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$\sup_{w \in \mathbb{B}, w \neq z} (1 - \|z\|^2)^{1/2} (1 - \|w\|^2)^{1/2} \frac{|f(z) - f(w)|}{\|z - w\|} < \varepsilon$$

for z with $\delta < \|z\| < 1$. Analogous to the proof of Lemma 2.7, we have $(1 - \|z\|^2)\|Df(z)\| \leq \varepsilon$ for z with $\delta < \|z\| < 1$. Therefore, $f \in \mathcal{B}(\mathbb{B}_H)_0$. This completes the proof. □

3 Composition Operators

In this section, we consider composition operators on the Bloch space of a bounded symmetric domain. Our main task is to estimate the norm of a composition operator and determine when it is an isometry. Let $\mathbb{B}_X$ be a bounded symmetric domain. Given a holomorphic self-mapping φ on $\mathbb{B}_X$, we denote the composition operator with symbol φ by

$$C_\varphi : f \in \mathcal{B}(\mathbb{B}_X) \mapsto f \circ \varphi \in \mathcal{B}(\mathbb{B}_X)$$

and define the *Kobayashi constant* of φ by

$$K_\varphi := \sup_{z \in \mathbb{B}_X} \sup_{x \in X \setminus \{0\}} \frac{\kappa(\varphi(z), D\varphi(z)x)}{\kappa(z, x)}.$$

In contrast to the *Bergman constant* B_φ defined in [1] (cf. [1, Open question (3), p.687]), we always have $K_\varphi \leq 1$ by the contractive property of the Kobayashi metric.

Since the operator norm of C_φ depends on the norm of the underlying Banach space, it should be pointed out that in the literature for finite dimensional domains $\mathbb{B}_X$, the operator C_φ is considered on the space $(\mathcal{B}(\mathbb{B}_X), \beta)$ defined by the Bergman metric, whereas we consider C_φ on the Bloch space $(\mathcal{B}(\mathbb{B}_X), \|\cdot\|_{\mathcal{B}})$ defined by the Kobayashi metric.

Observing that the Kobayashi metric is invariant under the automorphisms of $\mathbb{B}_X$, the following result can be verified readily. It is analogous to the result in [1, Corollary 5.1] for the Bergman constant B_φ.

Proposition 3.1 *Let $\mathbb{B}_X$ be a bounded symmetric domain and let $\varphi \in H(\mathbb{B}_X)$. Then for all $S \in \mathrm{Aut}(\mathbb{B}_X)$, $K_{\varphi\circ S} = K_\varphi = K_{S\circ\varphi}$.*

The following theorem is a generalization of [1, Theorem 3.2 and Corollary 3.1] and [24, Theorem 2 and Corollary 1] to bounded symmetric domains. The lower bound is an improvement if $\mathbb{B}_X$ is not the Euclidean unit ball of $\mathbb{C}^n$.

Theorem 3.2 *Let $\mathbb{B}_X$ be a bounded symmetric domain and let $\varphi \in H(\mathbb{B}_X)$. Then C_φ is a bounded operator on $\mathcal{B}(\mathbb{B}_X)$ and we have*

$$\max\{1,\ \rho(\varphi(0),0)\} \le \|C_\varphi\| \le \max\left\{1,\ \rho(\varphi(0),0) + K_\varphi\right\}.$$

In particular, $\|C_\varphi\| = 1$ if $\varphi(0) = 0$.

Proof The boundedness of C_φ has been proved in [5, Theorem 5.1]. As in the proof of [1, Theorem 3.2], we have

$$\|C_\varphi\| \le \max\left\{1, \rho(\varphi(0),0) + K_\varphi\right\}.$$

Next, we prove the lower estimate of $\|C_\varphi\|$. Let f_1 be the constant function 1 on $\mathbb{B}_X$. Then $1 = \|C_\varphi(f_1)\|_{\mathcal{B}(\mathbb{B}_X)} = \|f_1\|_{\mathcal{B}(\mathbb{B}_X)}$. This implies that $\|C_\varphi\| \ge 1$. Therefore, if $\varphi(0) = 0$, then we have $1 \le \|C_\varphi\| \le \max\{1, K_\varphi\} = 1$ and thus, $\|C_\varphi\| = 1$. We will consider the case $\varphi(0) \neq 0$. Let $f_2(z) = \psi(l_{\varphi(0)}(z))$ for $z \in \mathbb{B}_X$, where

$$\psi(\zeta) = \tanh^{-1}\zeta \quad (\zeta \in \mathbb{D}).$$

Then $\|f_2\|_{\mathcal{B}(\mathbb{B}_X)} = 1$ by Example 2.5. Therefore, we have

$$\|C_\varphi\| \ge \|f_2 \circ \varphi\|_{\mathcal{B}(\mathbb{B}_X)} \ge |f_2(\varphi(0))| = \tanh^{-1}\|\varphi(0)\| = \rho(\varphi(0),0).$$

This completes the proof. □

A similar argument as in the proof of [1, Theorem 5.1] yields the following more general result for bounded symmetric domains.

Proposition 3.3 *Let $\mathbb{B}_X$ be a bounded symmetric domain and let $\varphi \in H(\mathbb{B}_X)$ with $\varphi(0) = 0$. If there is a sequence $\{S_j\}$ in $\mathrm{Aut}(\mathbb{B}_X)$ such that $\{\varphi \circ S_j\}$ converges locally uniformly to the identity mapping on $\mathbb{B}_X$, then C_φ is an isometry on $\mathcal{B}(\mathbb{B}_X)$.*

Finally, we deduce in the next theorem some necessary conditions for C_φ to be an isometry, which generalizes [1, Theorem 6.1 (a), (b), Proposition 7.1] to bounded symmetric domains. Condition (a) below gives a positive answer to [1, Remark 6.1]. Let $\sigma(C_\varphi)$ be the spectrum of C_φ.

Theorem 3.4 *Let $\mathbb{B}_X$ be a bounded symmetric domain and let C_φ be an isometry on $\mathcal{B}(\mathbb{B}_X)$ for some $\varphi \in H(\mathbb{B}_X)$. Then we have*

(a) $\varphi(0) = 0$.
(b) $\|l_a \circ \varphi\|_{\mathcal{B}(\mathbb{B}_X),s} = 1$ *for each* $a \in X \setminus \{0\}$ *and support functional* $l_a \in T(a)$.
(c) $K_\varphi = 1$.
(d) *If φ is not injective, then $\sigma(C_\varphi) = \overline{\mathbb{D}}$.*
(e) *If* $\dim X = n$ *and $\varphi = (\varphi_1, \ldots, \varphi_n)$, then the components $\varphi_1, \ldots, \varphi_n$ are linearly independent.*

Proof (a) For any $l \in X^* \setminus \{0\}$, let $f = l \circ g_{-a} \in H(\mathbb{B}_X, \mathbb{C})$, where $a = \varphi(0)$ and g_{-a} is the Möbius transformation induced by $-a$. Then we have

$$\begin{aligned} |l(-a)| + \|l \circ g_{-a}\|_{\mathcal{B}(\mathbb{B}_X),s} &= \|f\|_{\mathcal{B}(\mathbb{B}_X)} \\ &= \|C_\varphi(f)\|_{\mathcal{B}(\mathbb{B}_X)} \\ &= \|l \circ g_{-a} \circ \varphi\|_{\mathcal{B}(\mathbb{B}_X),s} \\ &\le \|l \circ g_{-a}\|_{\mathcal{B}(\mathbb{B}_X),s}. \end{aligned}$$

Therefore, we have $l(-a) = 0$ for any $l \in X^* \setminus \{0\}$. This implies that $\varphi(0) = a = 0$.

(b) Since $\varphi(0) = 0$ and C_φ is an isometry, we have

$$\|l_a \circ \varphi\|_{\mathcal{B}(\mathbb{B}_X),s} = \|l_a \circ \varphi\|_{\mathcal{B}(\mathbb{B}_X)} = \|l_a\|_{\mathcal{B}(\mathbb{B}_X)} = 1.$$

(c) Since

$$Q_{f \circ \varphi}(z) \le K_\varphi Q_f(\varphi(z)), \quad f \in \mathcal{B}(\mathbb{B}_X), z \in \mathbb{B}_X$$

holds, we have

$$\|C_\varphi(f)\|_{\mathcal{B}(\mathbb{B}_X),s} \le K_\varphi \|f\|_{\mathcal{B}(\mathbb{B}_X),s}, \quad f \in \mathcal{B}(\mathbb{B}_X).$$

Thus, if C_φ is an isometry, then $\varphi(0) = 0$ and $K_\varphi = 1$.

(d) We use an argument similar to that in the proof of [1, Proposition 7.1]. By [1, Lemma 7.1], it suffices to show that C_φ is not invertible. Since C_φ is an isometry, it is injective. We show that C_φ is not surjective. Suppose, for contradiction, that C_φ is surjective. As φ is not injective, there are distinct $\zeta, \eta \in \mathbb{B}_X$ such that $\varphi(\zeta) = \varphi(\eta)$. Let $a = \zeta - \eta \in X \setminus \{0\}$ and let $\ell_a \in T(a)$. Let

$$g(z) = \ell_a(z) - \ell_a(\eta), \quad z \in \mathbb{B}_X.$$

Then $g \in \mathcal{B}(\mathbb{B}_X)$ and $g(\zeta) = \ell_a(a) = \|a\| \neq 0$. By the surjectivity assumption of C_φ, there exists $f \in \mathcal{B}(\mathbb{B}_X)$ such that $C_\varphi(f) = g$. This gives $f(\varphi(\zeta)) = C_\varphi(f)(\zeta) = g(\zeta) \neq 0$ and $f(\varphi(\eta)) = C_\varphi(f)(\eta) = g(\eta) = 0$. This contradicts $\varphi(\zeta) = \varphi(\eta)$ and therefore C_φ is surjective.

(e) The proof is same as in the proof of [1, Theorem 6.1 (a)].

□

Example 3.5 We now give an example for $\varphi \in H(\mathbb{B}_X)$ such that $K_\varphi = 1$ and C_φ is not an isometry on $\mathcal{B}(\mathbb{B}_X)$. Let $X = X_1 \oplus X_2$ and $\varphi(z_1, z_2) = (z_1, 0)$ for $(z_1, z_2) \in X$. Then $K_\varphi = 1$. Let $f(\cdot) = l_{(0,a_2)}(\cdot)$, where $a_2 \in \mathbb{B}_{X_2}$. Then $f \in \mathcal{B}(\mathbb{B}_X)$, $\|f \circ \varphi\|_{\mathcal{B}(\mathbb{B}_X)} = 0$ and

$$\|f\|_{\mathcal{B}(\mathbb{B}_X)} \geq Q_f(0) = 1.$$

Thus C_φ is not an isometry on $\mathcal{B}(\mathbb{B}_X)$.

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