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Fractional Sturm–Liouville Equations: Self-Adjoint Extensions

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Abstract

In this report we study a fractional analogue of Sturm–Liouville equation. A class of self-adjoint fractional Sturm–Liouville operators is described. We give a biological interpretation of the fractional order equation and nonlocal boundary conditions that arise in describing the systems separated by a membrane. In particular, the connection with so called "fractional kinetic" equations is observed. Also, some spectral properties of the fractional kinetic equations are derived. An application to the anomalous diffusion of particles in a heterogeneous system of the fractional Sturm–Liouville equations is discussed.

Keywords Fractional kinetic equation · Caputo derivative · Riemann–Liouville derivative · Green's formula · Self-adjoint problem · Conservation law · The extension theory

Mathematics Subject Classification 34K08 · 47G20 · 45J05

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1 Introduction

Many processes can be described by systems separated by a membrane, especially, in biology and engineering [1,4]. In [7] authors studied anomalous diffusion of particles in a heterogeneous system separated by a semipermeable membrane, where the particle dynamics was governed by fractional diffusion equations in the bulk and by kinetic equations on the membrane, which characterizes an interface between two different media. For $0 < \beta < 1$ the mathematical model that was offered in [7] is

$$\mathcal{D}^{\beta}_{0+,t}u(t,x) + k\frac{\partial^2}{\partial x^2}u(t,x) = 0, \tag{1.1}$$

where k is a diffusion coefficient and, u represents the density function of particles. The fractional time derivative given in the Eq. (1.1) is the Riemann–Liouville type integro-differential operator.

One observes a new mathematical model to describe behaviours of the particle distribution at the interface and in the bulk can be stated, that is, the anomalous diffusion governed by a fractional kinetic equation

$$\mathcal{D}^{\beta}_{0+,t}u(t,x) + F^{2\alpha}_{x}u(t,x) = f(t,x), \tag{1.2}$$

for some $\alpha > 0$, with the initial condition

$$u(0,x) = \varphi(x), \quad x \in [0,1]. \tag{1.3}$$

Here f and φ are to defined later. In this note we introduce a fractional type kinetic equation $F^{2\alpha}_{x}$. By other words, to have "good" properties, we need $F^{2\alpha}_{x}$ to be a well-posed operator with, maybe, discrete spectrum and corresponding eigenfunctions form a basis in a Hilbert space. In this case, it is easy to show that the anomalous diffusion (1.2) with the initial condition (1.3) is well-posed with natural requirements on f and φ.

To have things that are mentioned, we are interested in studying symmetric fractional order kinetic type operators. In general, fractional order operators are not symmetric, and in all researches related to investigations of spectral properties only non self-adjoint problems are considered. In the recent manuscript [6] one symmetric fractional order differential operator is described by Klimek and Agrawal in the weighted class of continuous functions. Here, we show self-adjointness of a fractional kinetic operator of Caputo–Riemann–Liouville type, and continue researches started in [12], where the authors began to describe self-adjoint fractional order differential operators.

One of our main goals in this report is to establish, by using techniques offered in the recent papers of Ruzhansky et al. [3,9,10], an analogue of the Green's formula for fractional kinetic equations with further applications in describing a class of self-adjoint operators, and by the duality, it let us define fractional kinetic operators on the space of distributions.

Here, fractional kinetic equations of the Caputo and Riemann–Liouville type are objects of our investigations. At the end, we describe a class of self-adjoint problems related to this fractional kinetic equation in $L^2(0, 1)$, and formulate several statements on the spectral properties. In some point, we can say that it is found a symmetric Caputo–Riemann–Liouville operator of the order 2α (with $\frac{1}{2} < \alpha < 1$). In appreciate sense, it can be also interpreted as an analogue of the classical Sturm–Liouville operator.

2 Preliminaries

Here, we recall definitions and properties of fractional integration and differentiation operators [5,8,11].

Definition 2.1 Let f be a function defined on the interval $[0, 1]$. Assume that the following integrals exist

$$I_0^\alpha [f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t \in (0, 1],$$

and

$$I_1^\alpha [f](t) = \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} f(s)ds, \quad t \in [0, 1).$$

Then we call them the left-side, and the right-side, Riemann–Liouville integral operator of the fractional order $\alpha > 0$, respectively.

Definition 2.2 Define left-side and right-side Riemann–Liouville differential operators of the fractional order α $(0 < \alpha < 1)$ by

$$D_0^\alpha [f](t) = \frac{d}{dt} I_0^{1-\alpha} [f](t) \text{ and } D_1^\alpha [f](t) = -\frac{d}{dt} I_1^{1-\alpha} [f](t),$$

respectively.

Definition 2.3 For $0 < \alpha < 1$ we say that the actions

$$\mathcal{D}_0^\alpha [f](t) = D_0^\alpha [f(t) - f(0)] \text{ and } \mathcal{D}_1^\alpha [f](t) = D_1^\alpha [f(t) - f(1)],$$

are left-side and right-side differential operators of the fractional order α $(0 < \alpha < 1)$ in the Caputo sense, respectively.

Note that in monographs [5,8,11], authors studied different types of fractional differentiations and their properties. In what follows we formulate statements of necessary properties of integral and integro-differential operators of the Riemann–Liouville type and fractional Caputo operators.

Property 2.4 [8, p. 34] *Let* $u, v \in L^2(0, 1)$ *and* $0 < \alpha < 1$. *Then we have the formula of integration by parts*

$$\left(I_1^{\beta} u, v\right) = \left(u, I_0^{\beta} v\right).$$

Here, by $(\cdot, \cdot)$ *we denote the inner product of the Hilbert space* $L^2(0, 1)$.

Let us formulate Theorem 3.2 of the book [11]:

Theorem 2.5 *Assume that* $\varphi \in H^{\gamma}([0, 1])$, $\gamma \geq 0$. *Then the fractional integral* $I_0^{\alpha}\varphi$, $\alpha > 0$ *has the form*

$$I_0^{\alpha}\varphi = \sum_{k=0}^{m} \frac{\varphi^{(k)}(0)}{\Gamma(\alpha + k + 1)} x^{\alpha+k} + \psi(x),$$

where m *is the greatest integer such that* $m < \gamma$; *and* $\psi \in H^{\gamma+\alpha}([0, 1])$, *if* $\gamma + \alpha$ *is not integer, or if* $\gamma, \alpha \in \mathbb{Z}$.

3 Main Results

In what follows, we assume that $\frac{1}{2} < \alpha < 1$. Now, let us consider

$$\mathcal{L}u(x) := \mathcal{D}_1^{\alpha}\left[D_0^{\alpha}[u]\right](x), \ 0 < x < 1. \tag{3.1}$$

Here, our aim is to investigate spectral properties of operators generated by the fractional kinetic equation (3.1) in $L^2(0, 1)$. To start, we define an operator in the Hölder classes. Consider the spectral problem

$$\mathcal{L}u(x) = \lambda u(x), \ 0 < x < 1, \tag{3.2}$$

in the space $H_0^{2\alpha+o}([0, 1]) := \{\varphi \in H^{2\alpha+o}([0, 1]) : \varphi(0) = 0, \ldots, \varphi^{(m)}(0) = 0\}$, where $m = [2\alpha+o]$, and $H^{2\alpha+o}([0, 1])$ is the Hölder space with the parameter $2\alpha+o$. Here o is a sufficiently small positive number such that $o < 1 - \alpha$. By other words, we deal with the following spaces

$$\begin{aligned} H_0^{2\alpha+o}([0, 1]) &:= \{\varphi \in H^{2\alpha+o}([0, 1]) : \varphi(0) = 0, \varphi'(0) = 0\}, \\ H_0^{\alpha+o}([0, 1]) &:= \{\varphi \in H^{\alpha+o}([0, 1]) : \varphi(0) = 0\}, \\ H_0^{o}([0, 1]) &:= \{\varphi \in H^{o}([0, 1]) : \varphi(0) = 0\}. \end{aligned}$$

From the book of Samko et al. [11, Chapter 1, Theorem 3.2] it follows that the integro-differential operator $\mathcal{L}$ is well-defined on $H_0^{2\alpha+o}([0, 1])$.

Hence, the functionals

$$\begin{aligned} \xi_1^-(u) &:= I_0^{1-\alpha}[u](0), \quad \xi_2^-(u) := I_0^{1-\alpha}[u](1), \\ \xi_1^+(u) &:= D_0^{\alpha}[u](0) \quad \text{and} \quad \xi_2^+(u) := D_0^{\alpha}[u](1), \end{aligned}$$

are well-defined for all $u \in H_0^{2\alpha+o}([0,1])$ and, it is easy to see that

$$\xi_1^-(u) := I_0^{1-\alpha}[u](0) = \lim_{\varepsilon\to 0+}\frac{1}{\Gamma(1-\alpha)}\int_0^{\varepsilon}(\varepsilon-s)^{-\alpha}u(s)ds,$$

$$\xi_2^-(u) := I_0^{1-\alpha}[u](1) = \frac{1}{\Gamma(1-\alpha)}\int_0^{1}(1-s)^{-\alpha}u(s)ds,$$

$$\xi_1^+(u) := D_0^{\alpha}[u](0) = \lim_{\varepsilon\to 0+}\frac{1}{\Gamma(1-\alpha)}\int_0^{\varepsilon}(\varepsilon-s)^{-\alpha}u'(s)ds,$$

and

$$\xi_2^+(u) := D_0^{\alpha}[u](1) = \frac{1}{\Gamma(1-\alpha)}\int_0^{1}(1-s)^{-\alpha}u'(s)ds.$$

Indeed, ξ_1^- and ξ_1^+ are functionals of taking trace of the density function u and its velocity u' at zero. Functional ξ_2^- and ξ_2^+ play role of conservation laws. Anticipating results, "boundary" conditions arise later will express relationships between the conservation laws and information at the origin of the the density function and its changing velocity function.

Denote by $\mathcal{L}_0$ an operator generated by the fractional differential expression (3.1) with "boundary" conditions

$$\xi_2^-(u) = 0 \;\text{ and }\; \xi_1^+(u) = 0. \tag{3.3}$$

Then due to the definitions and properties given by Sect. 2 (see, [11, Chapter 1]) for

$$f \in \tilde{H}_0^o([0,1]) := \{v \in H_0^o([0,1]) : \int_0^1 v(s)s^{2\alpha}ds = 0 \text{ and } \int_0^1 v(s)s^{2\alpha-1}ds = 0\}$$

an inverse operator to $\mathcal{L}_0$ has the form

$$\mathcal{L}_0^{-1}f(x) = I_0^{\alpha}I_1^{\alpha}f(x) := \int_0^1 K(x,s)f(s)ds, \;\; 0 < x < 1,$$

as $\mathcal{L}_0^{-1} : \tilde{H}_0^o \to H_0^{2\alpha+o}$, with the symmetric kernel $K(\cdot,\cdot)$ from $L^2(0,1)\otimes L^2(0,1)$. Since, $S := span\{x^k,\ k \in \mathbb{N}\} \subset H_0^o([0,1])$, and powers of the sets S and $\tilde{S} := \{v \in S : \int_0^1 v(s)s^{2\alpha}ds = 0 \text{ and } \int_0^1 v(s)s^{2\alpha-1}ds = 0\}$ are equal then we conclude that a closure of the space $\tilde{H}_0^o([0,1])$ by the L^2-norm is $L^2(0,1)$. Hence, $\mathcal{L}_0^{-1}$ has a continuous continuation to a compact operator in $L^2(0,1)$. Compactness implies the fact that there exists non empty discrete spectrum with the eigenfunctions form an orthogonal basis in the space $L^2(0,1)$.

Denote by λ_k, $k \in \mathbb{N}$ eigenvalues of the spectral problem (3.2)–(3.3) in the ascending order and by u_k, $k \in \mathbb{N}$ corresponding eigenfunctions, i.e.

$$\mathcal{D}_1^\alpha \left[D_0^\alpha \left[u_k \right] \right](x) = \lambda_k u_k(x), \; 0 < x < 1,$$
$$\xi_2^-(u_k) = 0, \;\; \xi_1^+(u_k) = 0$$

for all $k \in \mathbb{N}$. Thus, the domain of the operator $\mathcal{L}_0$

$$\mathrm{Dom}(\mathcal{L}_0) := \{u \in H_0^{2\alpha+o}([0,1]) : \xi_2^-(u) = 0, \quad \xi_1^+(u) = 0\}$$

is not empty.

Now, we introduce the space of test functions $C_{\mathcal{L}_0}^\infty([0,1])$ (for more details, see [3,9,10]) as follows:

$$C_{\mathcal{L}_0}^\infty([0,1]) := \bigcap_{k=1}^{\infty} \mathrm{Dom}(\mathcal{L}_0^k),$$

where $\mathrm{Dom}(\mathcal{L}_0^k)$ is a domain of $\mathcal{L}_0^k$. Here $\mathcal{L}_0^k$ stands for the k times iterated $\mathcal{L}_0$ with the domain

$$\mathrm{Dom}(\mathcal{L}_0^k) := \{\mathcal{L}_0^{k-j-1} u \in \mathrm{Dom}(\mathcal{L}_0), \, j = 0, 1, \ldots, k-1\}$$

for $k \geq 2$. Since the linear combination of all eigenfunctions is in $C_{\mathcal{L}_0}^\infty([0,1])$ the space of test functions is not empty as a set. For further properties of the space $C_{\mathcal{L}_0}^\infty([0,1])$ we refer to the papers [9,10], where the properties of the test functions based on a basis are studied. The dual space to $C_{\mathcal{L}_0}^\infty([0,1])$ we denote by $\mathcal{D}'_{\mathcal{L}_0}(0,1)$ (the space of continuous functionals on $C_{\mathcal{L}_0}^\infty([0,1])$).

Now, we are in a way to define a fractional derivation of generalized functions. To begin, note that for all $u, v \in C_{\mathcal{L}_0}^\infty([0,1])$ we get

$$\left(\mathcal{D}_1^\alpha \left[D_0^\alpha u \right], v \right) = \left(u, \mathcal{D}_1^\alpha \left[D_0^\alpha v \right] \right). \tag{3.4}$$

Here, both sides exist in the classical sense.

As the result, one takes the Green's formula:

Lemma 3.1 *Let $u, v \in H_0^{2\alpha+o}([0,1])$. Then the following Green's formula makes a sense*

$$(\mathcal{D}_1^\alpha \left[D_0^\alpha u \right], v) - \left(u, \mathcal{D}_1^\alpha \left[D_0^\alpha \right] v \right) = \sum_{i=1}^{2} [\xi_i^-(u)\xi_i^+(v) - \xi_i^-(v)\xi_i^+(u)]. \tag{3.5}$$

Since $u, v \in C^{\infty}_{\mathcal{L}_0}([0,1])$ the identity (3.5) implies (3.4).

Define an action of the operator $\mathcal{L}$ on a generalized function $u \in \mathcal{D}'_{\mathcal{L}_0}(0,1)$. Put

$$(\mathcal{L}u, v) := (u, \mathcal{D}^{\alpha}_{1}\left[D^{\alpha}_{0} v\right]) \tag{3.6}$$

for all $v \in C^{\infty}_{\mathcal{L}_0}([0,1])$. The term $(u, \mathcal{D}^{\alpha}_{1}\left[D^{\alpha}_{0} v\right])$ exists due to the fact that $v \in C^{\infty}_{\mathcal{L}_0}([0,1])$ also involves $\mathcal{D}^{\alpha}_{1}\left[D^{\alpha}_{0} v\right] \in C^{\infty}_{\mathcal{L}_0}([0,1])$. Thus, the action of $\mathcal{L}$ introduced by the Formula (3.6) is well defined on the space of generalized functions $\mathcal{D}'_{\mathcal{L}_0}(0,1)$.

Now, we consider the following expression

$$\mathcal{L}u(x) := \mathcal{D}^{\alpha}_{1}\left[D^{\alpha}_{0}\left[u\right]\right](x),\ 0 < x < 1, \tag{3.7}$$

in the space $L^2(0,1)$. To define correctly $\mathcal{L}$ in $L^2(0,1)$, we introduce the space $W^{2\alpha}_2(0,1)$ as a closure of $H^{2\alpha+o}_0([0,1])$ by the norm

$$\|u\|_{W^{2\alpha}_2(0,1)} := \|u\|_{L^2(0,1)} + \|\mathcal{D}^{\alpha}_{1} D^{\alpha}_{0} u\|_{L^2(0,1)}.$$

Indeed, the space $W^{2\alpha}_2(0,1)$ with the introduced norm is a Banach one. Moreover, it is the Hilbert space with the scalar product

$$(u, v)_{W^{2\alpha}_2(0,1)} := (u, v) + (\mathcal{D}^{\alpha}_{1} D^{\alpha}_{0} u, \mathcal{D}^{\alpha}_{1} D^{\alpha}_{0} v).$$

We define $\mathcal{L}_m$ as an operator acting from $L^2(0,1)$ to $L^2(0,1)$ by the Formula (3.7) with the domain

$$\mathrm{Dom}(\mathcal{L}_m) = \left\{u \in W^{2\alpha}_2(0,1) :\ \ \xi^{-}_1(u) = \xi^{-}_2(u) = \xi^{+}_1(u) = \xi^{+}_2(u) = 0\right\}.$$

Also, introduce an operator $\mathcal{L}_M : L^2(0,1) \to L^2(0,1)$ generated by the Expression (3.7) with the domain $\mathrm{Dom}(\mathcal{L}_M) := \left\{u \in W^{2\alpha}_2(0,1)\right\}$.

Let us introduce a class of matrices 2×4. This, to define boundary forms for $\mathcal{D}^{\alpha}_{1}\left[D^{\alpha}_{0}\left[u\right]\right]$.

Definition 3.2 We call

$$\omega := \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} \end{pmatrix}$$

S-matrix, if it can be represented in one of the following forms:

$$\begin{pmatrix} 1 & 0 & r & c \\ 0 & 1 & -c & d \end{pmatrix}, \begin{pmatrix} d & 1 & 0 & r \\ c & 0 & 1 & d \end{pmatrix},$$
$$\begin{pmatrix} 1 & d & r & 0 \\ 0 & c & -d & 1 \end{pmatrix}, \begin{pmatrix} r & c & 1 & 0 \\ -c & d & 0 & 1 \end{pmatrix},$$

where $r, c, d \in \mathbb{R}$. Here, the matrices

$$\begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} \end{pmatrix}, \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} \\ \gamma\omega_{21} & \gamma\omega_{22} & \gamma\omega_{23} & \gamma\omega_{24} \end{pmatrix} \quad (\gamma \neq 0),$$
$$\begin{pmatrix} \omega_{11} \pm \omega_{21} & \omega_{12} \pm \omega_{22} & \omega_{13} \pm \omega_{23} & \omega_{14} \pm \omega_{24} \\ \omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} \end{pmatrix} \quad \begin{pmatrix} \omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} \\ \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} \end{pmatrix}$$

are the same.

Then, by using the extension theory, we prove:

Theorem 3.3 *Let ω be an S-matrix. Then an operator $\mathcal{L}_\omega$ generated by*

$$\mathcal{D}_1^\alpha D_0^\alpha u(x) = f(x), \quad 0 < x < 1,$$

for $u \in W_2^{2\alpha}(0,1)$ with "boundary" conditions

$$\omega_{11}\xi_1^-(u) + \omega_{12}\xi_2^-(u) + \omega_{13}\xi_1^+(u) + \omega_{14}\xi_2^+(u) = 0,$$
$$\omega_{21}\xi_1^-(u) + \omega_{22}\xi_2^-(u) + \omega_{23}\xi_1^+(u) + \omega_{24}\xi_2^+(u) = 0,$$

is a self-adjoint extension of $\mathcal{L}_m$ in $W_2^{2\alpha}(0,1)$.

Remark 3.4 In general, for $\alpha < 1/2$ Theorem 3.3 does not hold.

The following result gives a class of positive operators:

Lemma 3.5 *Let ω be one of the following:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \rho & 1 & 0 & 0 \\ 0 & 0 & 1 & \rho \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.8}$$

Then for all $\rho \in \mathbb{R}$ the operator $\mathcal{L}_\omega$ is positive in $L^2(0,1)$.

Finally, from Delgado and Ruzhansky's paper [2] follows:

Corollary 3.6 *Assume that ω is from (3.8). Then the inverse operator $\mathcal{L}_\omega^{-1}$ acting on $L^2(0,1)$ is from the Schatten classes $S_p\left(L^2(0,1)\right)$ for all $p > \frac{2}{1+4\alpha}$.*

As an conclusion, we mention that it is introduced the conservation laws or so called mathematical "boundary conditions". This is a subject to consider them in combination with the Cauchy problem for the Eq. (1.2). One says the conservation laws naturally come from applications and they will stand for non-local type boundary operators. It is natural due to the observations made in [7].

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