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Normal Truncated Toeplitz Operators

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Abstract The characterization of normal truncated Toepltiz operators is first given by Chalendar and Timotin. We give an elementary proof of their result without using the algebraic properties of truncated Toeplitz operators.

Keywords Truncated Toeplitz operator · Normal operator

Mathematics Subject Classification Primary 47

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. Let L^2 denote the Lebesgue space of square integrable functions on the unit circle $\partial\mathbb{D}$. The Hardy space H^2 is the subspace of analytic functions on $\mathbb{D}$ whose Taylor coefficients are square summable. Then it can also be identified with the subspace of L^2 of functions whose negative Fourier coefficients vanish. Let P and $P^\perp$ be the orthogonal projections from L^2 to H^2 and $[H^2]^\perp$, respectively. Here $[H^2]^\perp$ is the orthogonal complement of H^2 in L^2. For $f \in L^\infty$, the space of essentially bounded Lebesgue measurable functions on $\partial\mathbb{D}$, the Toeplitz operator T_f with symbol $f \in L^\infty$ is defined by

$$T_f h = P(fh),$$

for $h \in H^2$.

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An analytic function θ is called an inner function if $|\theta| = 1$ a.e. on $\mathbb{T}$. For each non-constant inner function θ, the so-called model space is

$$K_\theta = H^2 \ominus \theta H^2.$$

It is a reproducing kernel Hilbert space with reproducing kernels

$$k_w^\theta(z) = \frac{1 - \overline{\theta(w)}\theta(z)}{1 - \bar{w}z}.$$

Let P_θ denote the orthogonal projection from L^2 onto K_θ,

$$P_\theta f = Pf - \theta P(\bar{\theta} f). \tag{1.1}$$

For $\varphi \in L^2$, the truncated Toeplitz operator A_ϕ is defined by

$$A_\varphi^\theta f = P_\theta(\varphi f),$$

on the dense subset $K_\theta \cap H^\infty$ of K_θ. In particular, $K_\theta \cap H^\infty$ contains all reproducing kernels k_w^θ. The operator A_φ^θ may be extended to a bounded operator on K_θ even for unbounded symbols φ. The symbol φ is never unique and it is proved in [2] that

$$A_\varphi^\theta = 0$$

if and only if

$$\varphi \in \theta H^2 + \overline{\theta H^2}.$$

If $\theta(0) = 0$, then A_φ^θ has a unique symbol

$$\varphi \in K_\theta + \overline{K_\theta}.$$

The set of all bounded truncated Toeplitz operators is denoted by $\mathcal{T}_\theta$.

Recall that a bounded operator T on a Hilbert space $\mathcal{H}$ is normal if $T^*T = TT^*$. The characterization of normal truncated Toepltiz operators is first given by Chalendar and Timotin using the algebraic properties of truncated Toeplitz operators obtained by Sarason [2] and Sedlock [3].

Theorem 1.1 [1, Theorem 6.2] *Let θ be a non-constant inner function vanishing at* 0. *Then A_φ^θ is normal if and only if one of the following holds*

(1) *A_φ^θ belongs to $\mathscr{B}_\theta^\alpha$, for some unimodular constant α.*
(2) *A_φ^θ is a linear combination of a self-adjoint truncated Toeplitz operator and the identity.*

Here $\mathscr{B}_\theta^\alpha$ is a class of truncated Toeplitz operators introduced in [3]. In this note, we give an elementary proof of their result.

2 Proof of the Main Result

In this section we offer a proof of our characterization of normal truncated Toepltiz operators A^θ_φ. We begin with some reduction. Notice that for any constant C, $A^\theta_{\varphi+C} = A^\theta_\varphi + CI$, which implies A^θ_φ is normal if and only if $A^\theta_{\varphi+C}$ is normal. Thus we may assume, without losing of generality, that $\varphi(0) = 0$.

For $a \in \mathbb{D}$, let u_a be the Möbius transform

$$u_a(z) = \frac{z-a}{1-\bar{a}z}.$$

The Crofoot transform is the unitary operator $J : K_\theta \to K_{u_a\circ\theta}$ defined by

$$J(f) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}\theta} f.$$

It is proved in [2] that

$$J\mathcal{T}_\theta J^* = \mathcal{T}_{u_a\circ\theta}.$$

Taking $a = \theta(0)$, we see that it is sufficient to consider the normal truncated Toeplitz operators for $\theta(0) = 0$. In this case, constant functions are in K_θ. Write $\varphi = \varphi_1 + \overline{\varphi_2}$, where φ_1, φ_2 are in K_θ. We may also assume $\varphi_1(0) = \varphi_2(0) = 0$.

It is easy to see that

$$(A^\theta_\varphi)^* = A^\theta_{\bar{\varphi}}.$$

Our approach to characterizing normal truncated Toeplitz operators starts with a computation of

$$||A^\theta_\varphi u||^2 - ||(A^\theta_\varphi)^* u||^2.$$

Lemma 2.1 *Let θ be a non-constant inner function. Suppose*

$$\varphi = \varphi_1 + \overline{\varphi_2},$$

where φ_1, φ_2 are in K_θ. Then for every $u \in K_\theta \cap H^\infty$,

$$\begin{aligned} &||A^\theta_\varphi u||^2 - ||(A^\theta_\varphi)^* u||^2 \\ &\quad = ||P^\perp(\bar{\theta}\varphi_1 u)||^2 - ||P(\bar{\varphi}_1 u)||^2 - (||P^\perp(\bar{\theta}\varphi_2 u)||^2 - ||P(\bar{\varphi}_2 u)||^2). \end{aligned}$$

Proof By (1.1), we have for every $u \in K_\theta \cap H^\infty$

$$\begin{aligned} A^\theta_\varphi u &= P_\theta(\varphi u) \\ &= P(\varphi u) - \theta P(\bar{\theta}\varphi u) \end{aligned}$$

$$= \varphi_1 u + P(\bar{\varphi}_2 u) - \theta P(\bar{\theta}\varphi_1 u + \bar{\theta}\bar{\varphi}_2 u)$$
$$= \varphi_1 u - \theta P(\bar{\theta}\varphi_1 u) + P(\bar{\varphi}_2 u).$$

Then

$$\begin{aligned} ||A^\theta_\varphi u||^2 &= ||(\varphi_1 u - \theta P(\bar{\theta}\varphi_1 u)) + P(\bar{\varphi}_2 u)||^2 \\ &= ||\varphi_1 u - \theta P(\bar{\theta}\varphi_1 u)||^2 + ||P(\bar{\varphi}_2 u)||^2 + 2\,\mathrm{Re}\langle \varphi_1 u - \theta P(\bar{\theta}\varphi_1 u), P(\bar{\varphi}_2 u)\rangle \\ &= ||\bar{\theta}\varphi_1 u||^2 - ||P(\bar{\theta}\varphi_1 u)||^2 + ||P(\bar{\varphi}_2 u)||^2 \\ &\quad + 2\,\mathrm{Re}\langle \varphi_1 u - \theta P(\bar{\theta}\varphi_1 u), P(\bar{\varphi}_2 u)\rangle \\ &= ||P^\perp(\bar{\theta}\varphi_1 u)||^2 + ||P(\bar{\varphi}_2 u)||^2 + 2\,\mathrm{Re}\langle \varphi_1 u - \theta P(\bar{\theta}\varphi_1 u), P(\bar{\varphi}_2 u)\rangle. \end{aligned}$$

And

$$\begin{aligned} \langle \varphi_1 u - \theta P(\bar{\theta}\varphi_1 u), P(\bar{\varphi}_2 u)\rangle &= \langle \varphi_1 u, P(\bar{\varphi}_2 u)\rangle - \langle \theta P(\bar{\theta}\varphi_1 u), P(\bar{\varphi}_2 u)\rangle \\ &= \langle \varphi_1 u, \bar{\varphi}_2 u\rangle - \langle P(\bar{\theta}\varphi_1 u), \bar{\theta} P(\bar{\varphi}_2 u)\rangle \\ &= \langle \varphi_1 u, \bar{\varphi}_2 u\rangle - \langle P(\bar{\theta}\varphi_1 u), \bar{\theta} u\bar{\varphi}_2 - \bar{\theta} P^\perp(\bar{\varphi}_2 u)\rangle \\ &= \langle \varphi_1 u, \bar{\varphi}_2 u\rangle. \end{aligned}$$

Thus

$$||A^\theta_\varphi u||^2 = ||P^\perp(\bar{\theta}\varphi_1 u)||^2 + ||P(\bar{\varphi}_2 u)||^2 + 2\,\mathrm{Re}\langle \varphi_1 u, \bar{\varphi}_2 u\rangle. \tag{2.1}$$

Similarly

$$||(A^\theta_\varphi)^* u||^2 = ||A^\theta_{\varphi_2 + \bar{\varphi}_1} u||^2 = ||P^\perp(\bar{\theta}\varphi_2 u)||^2 + ||P(\bar{\varphi}_1 u)||^2 + 2\,\mathrm{Re}\langle \varphi_2 u, \bar{\varphi}_1 u\rangle. \tag{2.2}$$

Subtracting (2.2) from (2.1), we get the desired identity. □

For $w \in \mathbb{D}$, let

$$k_w(z) = \frac{1}{1 - \bar{w}z}$$

be the reproducing kernel of H^2.

First we show that if A^θ_φ is normal then φ_1/φ_2 is a unimodular function.

Lemma 2.2 *Let θ be a non-constant inner function vanishing at 0. Suppose $\varphi = \varphi_1 + \overline{\varphi_2}$, where φ_1, φ_2 are in K_θ, and $\varphi_1(0) = \varphi_2(0) = 0$. If A^θ_φ is normal then*

$$|\varphi_1| = |\varphi_2|,$$

a.e. on $\mathbb{T}$.

Proof By Lemma 2.1, A^θ_φ is normal implies

$$||P^\perp(\bar{\theta}\varphi_1 u)||^2 - ||P(\bar{\varphi}_1 u)||^2 = ||P^\perp(\bar{\theta}\varphi_2 u)||^2 - ||P(\bar{\varphi}_2 u)||^2, \tag{2.3}$$

for every $u \in K_\theta \cap H^\infty$. Take $u = 1$, we get

$$||P^\perp(\bar{\theta}\varphi_1)||^2 - ||P(\bar{\varphi_1})||^2 = ||P^\perp(\bar{\theta}\varphi_1)||^2 - ||P(\bar{\varphi_2})||^2.$$

Since

$$P^\perp(\bar{\theta}\varphi_j) = \bar{\theta}\varphi_j, \tag{2.4}$$

and

$$P(\bar{\varphi_j}) = 0, \tag{2.5}$$

we have

$$||\varphi_1|| = ||\varphi_2||. \tag{2.6}$$

Next we consider the reproducing kernels of K_θ:

$$k_w^\theta(z) = \frac{1 - \overline{\theta(w)}\theta(z)}{1 - \bar{w}z},$$

and take $u = u_w = k_w^\theta + 1$ in (2.3). Using (2.4) and (2.5), we have

$$\begin{aligned} ||P^\perp(\bar{\theta}\varphi_j u_w)||^2 &= ||P^\perp(\bar{\theta}\varphi_j k_w^\theta)||^2 + ||P^\perp(\bar{\theta}\varphi_j)||^2 + 2\mathrm{Re}\,\langle P^\perp(\bar{\theta}\varphi_j k_w^\theta), P^\perp(\bar{\theta}\varphi_j)\rangle \\ &= ||P^\perp(\bar{\theta}\varphi_j k_w^\theta)||^2 + ||\bar{\theta}\varphi_j||^2 + 2\mathrm{Re}\,\langle P^\perp(\bar{\theta}\varphi_j k_w^\theta), \bar{\theta}\varphi_j\rangle, \end{aligned}$$

and

$$\begin{aligned} ||P(\bar{\varphi_j}u_w)||^2 &= ||P(\bar{\varphi_j}k_w^\theta)||^2 + ||P(\bar{\varphi_j})||^2 + 2\mathrm{Re}\,\langle P(\bar{\varphi_j}k_w^\theta), P(\bar{\varphi_j})\rangle \\ &= ||P(\bar{\varphi_j}k_w^\theta)||^2. \end{aligned}$$

This together with Lemma 2.1 and (2.6) implies

$$\mathrm{Re}\,\langle P^\perp(\bar{\theta}\varphi_1 k_w^\theta), \bar{\theta}\varphi_1\rangle = \mathrm{Re}\,\langle P^\perp(\bar{\theta}\varphi_2 k_w^\theta), \bar{\theta}\varphi_2\rangle. \tag{2.7}$$

Since

$$k_w^\theta = (1 - \overline{\theta(w)}\theta)k_w,$$

we get

$$\begin{aligned} P^\perp(\bar{\theta}\varphi_j k_w^\theta) &= P^\perp(\bar{\theta}\varphi_j(1 - \overline{\theta(w)}\theta)k_w) = P^\perp(\bar{\theta}\varphi_j k_w) - \overline{\theta(w)}P^\perp(\varphi_j k_w) \\ &= P^\perp(\bar{\theta}\varphi_j k_w). \end{aligned}$$

Hence

$$\begin{aligned} &\mathrm{Re}\,\langle P^\perp(\bar{\theta}\varphi_j k_w^\theta), \bar{\theta}\varphi_j\rangle \\ &= \mathrm{Re}\,\langle P^\perp(\bar{\theta}\varphi_j k_w), \bar{\theta}\varphi_j\rangle \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{Re}\langle \bar{\theta}\varphi_j k_w, \bar{\theta}\varphi_j\rangle \\
&= \operatorname{Re}\int_0^{2\pi} \frac{|\varphi_j(e^{it})|^2}{1-\bar{w}e^{it}}\frac{dt}{2\pi} \\
&= \int_0^{2\pi} |\varphi_j(e^{it})|^2\Big(\operatorname{Re}\frac{1}{1-\bar{w}e^{it}}\Big)\frac{dt}{2\pi} \\
&= \frac{1}{2}\int_0^{2\pi} |\varphi_j(e^{it})|^2\Big(1+\operatorname{Re}\frac{1+\bar{w}e^{it}}{1-\bar{w}e^{it}}\Big)\frac{dt}{2\pi} \\
&= \frac{1}{2}||\varphi_j||^2 + \frac{1}{2}\int_0^{2\pi} |\varphi_j(e^{it})|^2\Big(\operatorname{Re}\frac{1+\bar{w}e^{it}}{1-\bar{w}e^{it}}\Big)\frac{dt}{2\pi} \\
&= \frac{1}{2}(||\varphi_j||^2 + \widehat{|\varphi_j|^2}(w)).
\end{aligned}
$$

The last equality holds because

$$\operatorname{Re}\frac{1+\bar{w}e^{it}}{1-\bar{w}e^{it}}$$

is the Poisson kernel at w. Here $\widehat{|\varphi_j|^2}$ is the harmonic extension of the function $|\varphi_j|^2$. It follows from (2.7) and (2.6) that

$$\widehat{|\varphi_1|^2}(w) = \widehat{|\varphi_2|^2}(w).$$

Let $w \to \zeta \in \mathbb{T}$ nontangentially, we see that

$$|\varphi_1| = |\varphi_2|,$$

a.e. on $\mathbb{T}$. □

Let U is the unitary operator on L^2 defined by

$$Uh(z) = \bar{z}\tilde{h}(z),$$

where $\tilde{h}(z) = h(\bar{z})$. Let V_θ be the operator

$$V_\theta h = P(\theta h),$$

for $h \in L^2$. Consider the decomposition

$$[H^2]^\perp = \bar{\theta}K_\theta \oplus \bar{\theta}[H^2]^\perp.$$

It is easy to check that V_θ maps $\bar{\theta}K_\theta$ onto K_θ, and maps $\bar{\theta}[H^2]^\perp$ to 0. Thus V_θ maps $[H^2]^\perp$ onto K_θ. Since U maps H^2 onto $[H^2]^\perp$, we see that

$$V_\theta U : H^2 \to K_\theta$$

is also onto.

We shall use the following identity.

Lemma 2.3 *Let θ be an inner function and let g be in H^2. Then for every function $f \in H^\infty$*

$$||P(\bar{g}V_\theta Uf)|| = ||P^\perp(\bar{\theta}gf^*)||,$$

where $f^*(z) = \overline{f(\bar{z})}$.

Proof Notice that for all $h \in L^2$, we have

$$(Uh)^* = U(h^*)$$

and

$$(Ph)^* = P(h^*).$$

Thus

$$\begin{aligned} P(\bar{g}V_\theta Uf) &= P(\bar{g}P(\theta Uf)) = P(\bar{g}\theta Uf) = P(\bar{z}\theta\bar{g}\tilde{f}) \\ &= PU((\bar{\theta}g)^* f) = P(U(\bar{\theta}gf^*))^* \\ &= (PU(\bar{\theta}gf^*))^* = (UP^\perp(\bar{\theta}gf^*))^*. \end{aligned}$$

Here we used $PU = UP^\perp$ in the last equality. Since $||h|| = ||h^*||$, for all $h \in L^2$ and U is an isometry, we get the desired identity. □

The following result is well-known (see e.g. [4, Lemma 8]).

Theorem 2.1 *If $f \in H^2$, then for every $w \in \mathbb{D}$,*

$$P(\bar{f}k_w) = \overline{f(w)}k_w.$$

Now we can prove the main result.

Theorem 2.2 *Let θ be a non-constant inner function vanishing at 0. Suppose $\varphi = \varphi_1 + \overline{\varphi_2}$, where φ_1, φ_2 are in K_θ. Then A_φ^θ is normal if and only if either*

$$\varphi_2 - \varphi_2(0) = \alpha(\varphi_1 - \varphi_1(0))$$

or

$$\varphi_2 - \varphi_2(0) = \alpha\theta(\overline{\varphi_1} - \overline{\varphi_1(0)}),$$

for some unimodular constant α.

Proof We may assume $\varphi_1(0) = \varphi_2(0) = 0$. Sufficiency follows easily from Lemma 2.1.

Suppose A_φ^θ is normal. By (2.3) and Lemma 2.2, we have

$$||P(\bar{\varphi_1}u)||^2 + ||P(\bar{\theta}\varphi_1 u)||^2 = ||P(\bar{\varphi_2}u)||^2 + ||P(\bar{\theta}\varphi_2 u)||^2, \tag{2.8}$$

for every $u \in K_\theta \cap H^\infty$. According to the discussion before Lemma 2.3, if we write $u = V_\theta U f$, where $f \in H^\infty$, (2.8) is equivalent to

$$||P(\bar{\varphi_1}V_\theta U f)||^2 + ||P(\bar{\theta}\varphi_1 V_\theta U f)||^2 = ||P(\bar{\varphi_2}V_\theta U f)||^2 + ||P(\bar{\theta}\varphi_2 V_\theta U f)||^2,$$

for every $f \in H^\infty$. Using Lemma 2.3 and that $f \mapsto f^*$ is a bijection on H^∞, we have

$$||P^\perp(\bar{\theta}\varphi_1 f)||^2 + ||P^\perp(\bar{\varphi_1} f)||^2 = ||P^\perp(\bar{\theta}\varphi_2 f)||^2 + ||P^\perp(\bar{\varphi_2} f)||^2, \tag{2.9}$$

for every $f \in H^\infty$. By Lemma 2.2,

$$||\bar{\theta}\varphi_1 f|| = ||\bar{\theta}\varphi_2 f||,$$

and

$$||\bar{\varphi_1} f|| = ||\bar{\varphi_2} f||.$$

We see that (2.9) implies

$$||P(\bar{\theta}\varphi_1 f)||^2 + ||P(\bar{\varphi_1} f)||^2 = ||P(\bar{\theta}\varphi_2 f)||^2 + ||P(\bar{\varphi_2} f)||^2, \tag{2.10}$$

for every $f \in H^\infty$.

Take $f = k_w$ in (2.10). By Theorem 2.1, we get

$$|\varphi_1(w)|^2 + |(\theta\bar{\varphi_1})(w)|^2 = |\varphi_2(w)|^2 + |(\theta\bar{\varphi_2})(w)|^2, \tag{2.11}$$

for every $w \in \mathbb{D}$. Here $(\theta\bar{\varphi_1})(w)$ means $\langle \theta\bar{\varphi_1}, k_w\rangle$.

On the other hand, using Lemma 2.2, we have

$$\begin{aligned} \varphi_1(w)(\theta\bar{\varphi_1})(w) &= \langle \varphi_1(\theta\bar{\varphi_1}), k_w\rangle = \langle \theta|\varphi_1|^2, k_w\rangle = \langle \theta|\varphi_2|^2, k_w\rangle \\ &= \langle \varphi_2(\theta\bar{\varphi_2}), k_w\rangle = \varphi_2(w)(\theta\bar{\varphi_2})(w). \end{aligned} \tag{2.12}$$

for every $w \in \mathbb{D}$.

Multiplying both sides of (2.11) by $|\varphi_2(w)|^2$ and using (2.12), we have

$$\begin{aligned} |\varphi_1(w)\varphi_2(w)|^2 + |\varphi_2(w)(\theta\bar{\varphi_1})(w)|^2 &= |\varphi_2(w)|^4 + |\varphi_2(w)(\theta\bar{\varphi_2})(w)|^2 \\ &= |\varphi_2(w)|^4 + |\varphi_1(w)(\theta\bar{\varphi_1})(w)|^2, \end{aligned}$$

which is equivalent to

$$(|\varphi_1(w)|^2 - |\varphi_2(w)|^2)(|(\theta\bar{\varphi_1})(w)|^2 - |\varphi_2(w)|^2) = 0.$$

Thus for every $w \in \mathbb{D}$, either

$$|\varphi_1(w)| = |\varphi_2(w)|,$$

or

$$|\varphi_2(w)| = |(\theta\bar{\varphi_1})(w)|.$$

Then it follows from the properties of analytic functions that either

$$\varphi_1 = \alpha\varphi_2,$$

or

$$\varphi_2 = \alpha\theta\bar{\varphi_1},$$

for some unimodular constant α. □

Remark 2.1 The characterization given in Theorem 2.2 is equivalent to that in Theorem 1.1. In fact, if we write $\varphi = \varphi_1 + \bar{\varphi_2} + \varphi(0)$, where φ_1, φ_2 are in $K_\theta \cap zH^2$, it is shown in [1, Section 5] that $A_\varphi^\theta \in \mathscr{B}_\theta^\alpha$ if and only if $\theta\bar{\varphi_2} = \alpha\varphi_1$.

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