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A New Method for Dissipative Dynamic Operator with Transmission Conditions

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Abstract In this paper, we investigate the spectral properties of a boundary value transmission problem generated by a dynamic equation on the union of two time scales. For such an analysis we assign a suitable dynamic operator which is in limit-circle case at infinity. We also show that this operator is a simple maximal dissipative operator. Constructing the inverse operator we obtain some information about the spectrum of the dissipative operator. Moreover, using the Cayley transform of the dissipative operator we pass to the contractive operator which is of the class C_0. With the aid of the minimal function we obtain more information on the dissipative operator. Finally, we investigate other properties of the contraction such that multiplicity of the contraction, unitary colligation with basic operator and CMV matrix representation associated with the contraction.

Keywords Time scale · Dissipative operator · Cayley transform · Completely non-unitary contraction · Unitary colligation · Characteristic function · CMV matrix

Mathematics Subject Classification Primary 34B20; Secondary 34N05 · 47B44

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1 Introduction

Operator theory is important to understand the nature of the spectral properties of an operator associated with a boundary value problem acting on a Hilbert space. To obtain such an information as is well known that the corresponding inner product is useful. Using the inner product an important class of operators consisting of all nonselfadjoint operators is defined as follows [1]

$$\operatorname{Im}(Af, f)_H \geq 0,\ f \in D(A), A : H \to H,$$

where $D(A)$ denotes the domain of the operator A. Such operators are called *dissipative*. A direct result on dissipative operators is that all eigenvalues lie in the closed upper half-plane. Therefore open lower half-plane does not belong to the point spectrum of A.

There is a connection between dissipative and contractive operators. Indeed, one may pass to a contraction using a dissipative operator and the fact the lower half-plane does not belong to the point spectrum of a dissipative operator. By a contraction it is meant a linear transformation acting from a Hilbert space H into another Hilbert space H' such that

$$\|Th\|_{H'} \leq \|h\|_H, \quad h \in H.$$

Note that H and H' may be identical. This property allows one to define selfadjoint operators on H and H', respectively, as follows [2,3]

$$D_T = \left(I_H - T^*T\right)^{1/2}, \quad D_{T^*} = \left(I_{H'} - TT^*\right)^{1/2},$$

where I_H and $I_{H'}$ denote the identity operators on H and H', respectively. D_T and D_{T^*} are bounded by 0 and 1 and are called *defect operators* of T. The spaces

$$\mathfrak{D}_T = \overline{D_T H}, \quad \mathfrak{D}_{T^*} = \overline{D_{T^*} H}$$

are called *defect spaces* and the numbers

$$\mathfrak{d}_T = \dim \mathfrak{D}_T, \quad \mathfrak{d}_{T^*} = \dim \mathfrak{D}_{T^*}$$

are called *defect numbers* of T.

Characteristic function theory on contractions was developed by Sz.-Nagy and Foiaş [2,3]. By the characteristic function of a contraction it is meant the transformation from $\mathfrak{D}_T$ into $\mathfrak{D}_{T^*}$ as follows

$$\Theta_T(\mu) = \left[-T + \mu D_{T^*}\left(I - \mu T^*\right)^{-1} D_T\right] \mid \mathfrak{D}_T,$$

where μ is a complex number such that $I - \mu T^*$ is boundedly invertible. With the aid of the characteristic function, Sz.-Nagy and Foiaş investigated several properties of a contraction.

One of the important types of contractions on a Hilbert space is the *completely non-unitary* (c.n.u.) contractions. Recall that a contraction acting on a Hilbert space H is called c.n.u. if for no nonzero reducing subspace $\mathcal{L}$ for T is $T \mid \mathcal{L}$ a unitary operator. A special class consisting of c.n.u. contractions is the class C_0. The class C_0 consists of those c.n.u. contractions T for which there exists a nonzero function $u \in H^\infty$ such that $u(T) = 0$, where H^p $(0 < p \leq \infty)$ denotes the Hardy class of functions u, holomorphic on the unit disc $\boldsymbol{D}$ and the corresponding norm

$$\|u\|_p = \begin{cases} \sup\limits_{0<r<1} \left[\frac{1}{2\pi} \int\limits_0^{2\pi} \left| u(re^{it}) \right|^p dt \right]^{1/p}, & 0 < p < \infty, \\ \sup\limits_{\lambda \in \boldsymbol{D}} |u(\lambda)|, \; p = \infty, \end{cases}$$

is finite. It should be noted that for $T \in C_0$ the function u can always be taken to be inner. Then a natural question arises as whether for every contraction $T \in C_0$ there exits an inner function u with $u(T) = 0$ such that every other function $v \in H^\infty$ with $v(T) = 0$ is a multiple of u. Such a function is called a *minimal function* of T and denoted by m_T. Minimal function m_T is determined up to a constant factor of modulus one. Note that for every contraction $T \in C_0$ there exits a minimal function m_T. Using the minimal function m_T some spectral properties of the contraction T can be investigated. Indeed, the zeros of the minimal function m_T in the open disc $\boldsymbol{D}$ and of the complement, in the unit circle $\boldsymbol{C}$, of the union of the arcs of $\boldsymbol{C}$ on which m_T is analytic and the spectrum of the contraction $T \in C_0$ coincide with each other. Moreover, the points of the spectrum in the interior of the unit circle $\boldsymbol{C}$ are eigenvalues of T. As a characteristic value of T, λ has finite index, equal to its multiplicity as a zero of m_T. Completeness of the root vectors of T associated with the points of the spectrum of T in $\boldsymbol{D}$ can be proved as showing that the minimal function m_T is a Blaschke product.

In the literature, there are some theorems or methods to investigate the complete spectral properties of a dissipative operators acting on a Hilbert space. For instance, Livšic's theorem, Lidskiĭ's theorem, Krein's theorem and Pavlov's method are used for such operators [1,4]. First three theorems are related with the compactness of the dissipative operators. To be more precise, we should note that these operators must have at least nuclear imaginary parts. Then the methods follow from the entire functions. However, in Pavlov's method, there is no need to obtain a compact dissipative operator. It is needed to construct the characteristic function of Sz.-Nagy–Foiaş that can be regarded as identical with scattering function of Lax–Phillips [5]. In the basic of this method, there is a decomposition of a direct sum Hilbert space $\mathbf{H} = L^2_- \oplus H \oplus L^2_+$ in which a selfadjoint dilation of the maximal dissipative differential operator acting in the Hilbert space H is constructed. The subspaces L^2_- and L^2_+ are the incoming and outgoing channels, respectively. With the aid of the selfadjoint operator, a unitary group $U(t)$ $(t \in \mathbb{R})$ is established. This unitary group and the incoming subspace $D_- = (L^2_-, 0, 0)$ and the outgoing subspace $D_+ = (0, 0, L^2_+)$ of $\mathbf{H}$ satisfy the following properties:

(i) $U(t)D_- \subset D_-, t \leq 0$; $U(t)D_+ \subset D_+, t \geq 0$,

(ii) $\bigcap_{t\le 0} U(t)D_- = \bigcap_{t\ge 0} U(t)D_+ = \{0\}$,

(iii) $\overline{\bigcup_{t\ge 0} U(t)D_-} = \overline{\bigcup_{t\le 0} U(t)D_+} = \mathbf{H}$,

(iv) $D_- \perp D_+$.

Then according to the scattering theory, a scattering function $S(\lambda)$, $\mathrm{Im}\lambda > 0$, acting from D_+ into D_- can be constructed. Using Fourier transformation $\mathcal{F}$ one obtains the functional model of c.n.u. continuous semigroup of contractions $Z(t)$ as follows

$$H = H_+^2 \ominus S(\varsigma)H_+^2,$$
$$PZ(t)f = P\left[e^{i\varsigma t}\mathcal{F}f\right], \quad f \in \mathbf{H},$$

where H_+^2 is the Hardy space, $\varsigma = (\lambda - i)/(\lambda + i)$ and P is the orthogonal projection onto H. Such contractions can be found in, for example, [4,6–12].

In the Pavlov's method it is used the connection between the continuous semigroup of contractions $\{Z(t)\}_{t\ge 0}$ and its cogenerator Z. Namely, every model of Z generates a model of $\{Z(t)\}_{t\ge 0}$. In this paper we do not use this connection but we use a different way. Namely, we use the characteristic function of a c.n.u. contraction of the class C_0 associated with a maximal dissipative differential operator. Then using the connection between the characteristic function and minimal function of the contraction, we obtain complete information about the spectral properties of the contraction and dissipative differential operator.

A contractive operator can be embedded into a unitary operator. This theory is the unitary colligation theory [13]. It should be noted that unitary colligation theory includes the Sz.-Nagy–Foiaş characteristic function theory. To be more precise, we should note that a contraction with its defect operators can be embedded into a unitary colligation [13,14]. Arlinskiĭ *et al.* studied the relation of c.n.u. contractions with rank-one defects and corresponding unitary colligations [14]. Therefore we introduce some results on the Cayley transform of the dissipative operator.

Jacobi matrices are useful to understand the characterization of selfadjoint, nonselfadjoint and unitary operators acting on separable Hilbert spaces. Indeed, multiplication operators on the Hilbert spaces $L^2(\mathbb{R})$ or $L^2(\boldsymbol{C})$ associated with the probability measure $\boldsymbol{m}$ on the real line or on the unit circle $\boldsymbol{C}$, respectively, is unitary equivalent to the selfadjoint or unitary operators with a simple spectrum acting on some Hilbert spaces [15]. Tri-diagonal Jacobi matrix representation of selfadjoint operators with simple spectrum was introduced by Stone [16]. The nonselfadjoint version of Stone's theorem has been introduced by Arlinskiĭ and Tsekanovskiĭ [17]. Moreover, the canonical matrix representation of unitary operators with simple spectrum has been introduced by Cantero et al. [18] with the help of five-diagonal unitary matrices called CMV matrices. Arlinskiĭ et al. [14] obtained a connection between truncated CMV matrix and Sz.-Nagy–Foiaş characteristic function. Therefore, we introduce truncated CMV matrix associated with the Cayley transform.

This paper is organized as follows. In Sect. 2, we introduce the second order dynamic equation defined on the union of two time scales. We consturct the associated boundary value transmission problem on time scale. This can be regarded as the generalization of the continuous boundary value transmission problems. In Sect. 3, we construct a dissipative differential operator N in an appropriate Hilbert space H and we introduce some theorems on N. In Sect. 4, the resolvent operator of N is established. Two entire functions are constructed whose zeros coincide with the eigenvalues of N. Moreover some spectral properties are investigated with the help of the resolvent operator. In Sect. 5, using the Cayley transform T of N a contractive operator is obtained whose domain is the whole Hilbert space H. It is shown that T belongs to the space C_0. Moreover, some spectral properties of both T and N are introduced. In Sect. 6, other properties of the contraction T are investigated. These properties are related with the weak contractions, multiplicity of weak contractions and associated results, unitary colligation theory and truncated CMV matrices.

Finally, we should note that the notations $\boldsymbol{C}$ and $\boldsymbol{D}$ will be used to denote the unit circle $\boldsymbol{C} = \{\mu : |\mu| = 1\}$ and unit disc $\boldsymbol{D} = \{\mu : |\mu| < 1\}$.

2 Dynamic Equation

Consider a time scale $\widetilde{\mathbb{T}}$ with $\min \widetilde{\mathbb{T}} = 0$ as a dense point and $\sup \widetilde{\mathbb{T}} = \infty$, where $\rho(x)$ is the backward jump operator on $\widetilde{\mathbb{T}}$ and suppose that $\sigma(x)$ is the forward jump operator on $\widetilde{\mathbb{T}}$. For details we refer to [19,20].

In this paper we consider the following dynamic expressions

$$\eta_k(y) = -\frac{1}{w_k(x)}(p_k(x)y^{\Delta_k})^{\nabla_k} + q_k(x)y,$$

on time scales $\mathbb{T}_k$, $k = 1, 2$, where $\mathbb{T}_1 = [0, b] \cap \widetilde{\mathbb{T}}$, $\mathbb{T}_2 = [b, \infty) \cap \widetilde{\mathbb{T}}$. We assume that

(i) $b \in \widetilde{\mathbb{T}}$ is a dense point such that $0 < b < \infty$,
(ii) y^{Δ_k} and y^{∇_k} are the delta- and nabla-derivative on $\mathbb{T}_{k,}$, respectively,
(iii) p_1, q_1 and w_1 and p_2, q_2 and w_2 are real-valued functions on $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively,
(iv) p_1^{-1}, q_1 and w_1 and p_2^{-1}, q_2 and w_2 are piecewise continuous functions on $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively,
(v) $w_1 > 0$ and $w_2 > 0$ are piecewise continuous functions on $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively.

We also assume that y, $p_1 y^{\Delta_1}$ and y, $p_2 y^{\Delta_2}$ are nabla-differentiable functions on $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively.

We consider the following dynamic equation

$$\eta(y) = \lambda y, \quad x \in \mathbb{T}, \tag{2.1}$$

where $\mathbb{T} = \mathbb{T}_1 \cup \mathbb{T}_2$ and

$$\eta = \eta_1 \oplus \eta_2.$$

Clearly $\mathbb{T}$ is again a time scale. As can be seen in the conditions (2.3) and (2.4) the solutions of (2.1) will be assumed to have a discontinuity at b. Therefore such a representation of $\mathbb{T}$ will make the resolvent operator of the corresponding dynamic operator clear in Sect. 4. The motivation such a representation follows from the papers [21–27].

Remark 2.1 As is well-known that time scale calculus makes it possible to unify very interesting closed subsets of the real-line. In particular in the case that $\mathbb{T} = \{0, 1, 2, \ldots, b-2, b+2, \ldots\}$ the expression (2.1) turns out to the second order difference equation

$$-p(n-1)y(n-1) + q_1(n)y(n) - p(n)y(n+1) = \lambda w(n)y(n), \quad n \in \mathbb{T},$$

where $q_1(n) = p(n-1) + p(n) + q(n)$. Shi and Sun [28] showed a very interesting detail appearing in the previous equation. In fact, they showed that the corresponding maximal operator is not well defined because of the term $y(-1)$. To overcome this, they followed the method of Coddington called linear subspace theory [29] and they constructed the minimal operator and selfadjoint extensions of the that minimal operator (subspace). This method is different from the known method and this is our task to overcome this problem for the dynamic equation in a forthcoming paper containing non-dense end points.

Let $L_w^2(\mathbb{T})$ be the Hilbert space consisting of all functions y such that

$$\int_0^\infty |y|^2 \, w\nabla x < \infty$$

with the inner product

$$(y, \chi) = \int_0^\infty y\overline{\chi} w\nabla x.$$

We denote by D being the subset of $L_w^2(\mathbb{T})$ consisting of all functions $y \in L_w^2(\mathbb{T})$ such that y, $p_1 y^{\Delta_1}$ and y, $p_2 y^{\Delta_2}$ are piecewise continuous functions on $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively, and $\eta(y) \in L_w^2(\mathbb{T})$. For arbitrary two functions $y, \chi \in D$ following Green's formula is valid

$$\int_0^\infty \{\eta(y)\chi - y\eta(\chi)\} \, w\nabla x = [y, \chi]_0^b + [y, \chi]_b^\infty$$

where $[y, \chi]_\xi^\zeta = [y, \chi](\zeta) - [y, \chi](\xi)$, $[y, \chi] = y\chi^{[\Delta_k]} - y^{[\Delta_k]}\chi$ $(x \in \mathbb{T}_k)$, $y^{[\Delta_k]} = p_k y^{\Delta_k}$ and

$$w(x) = \begin{cases} w_1(x), & x \in \mathbb{T}_1, \\ w_2(x), & x \in \mathbb{T}_2. \end{cases}$$

Green's formula implies the fact that the values $[y, \chi](\infty)$ and $[y, \overline{\chi}](\infty)$ exist and are finite. Latter one follows from the fact that p_k, q_k and w_k are real valued functions on each $\mathbb{T}_k$, $k = 1, 2$.

It should be noted that if $y(x, \lambda)$ and $\chi(x, \lambda)$ are the solutions of (2.1) corresponding to the parameter λ then $[y, \chi]$ is independent of x and depends only on λ on each $\mathbb{T}_k$,

$k = 1, 2$. Moreover the bracket $[y, \chi]$ coincides with the Wronskian of $y(x, \lambda)$ and $\chi(x, \lambda)$ on each $\mathbb{T}_k : W[y, \chi] = [y, \chi]$, $x \in \mathbb{T}_k$.

In this paper we assume that η is in the limit-circle case at singular point ∞. Limit point/circle theory for the linear dynamic equations have been studied in the literature [30–33].

Let

$$u(x) = \begin{cases} u_1(x), & x \in \mathbb{T}_1, \\ u_2(x), & x \in \mathbb{T}_2, \end{cases} \quad v(x) = \begin{cases} v_1(x), & x \in \mathbb{T}_1, \\ v_2(x), & x \in \mathbb{T}_2, \end{cases}$$

be the solutions of the equation $\eta(y) = 0$, $x \in \mathbb{T}$, satisfying

$$\begin{cases} u_1(d_1) = 1, & u_1^{[\Delta_1]}(d_1) = 0, \\ v_1(d_1) = 0, & v_1^{[\Delta_1]}(d_1) = 1, \end{cases} \begin{cases} u_2(d_2) = 1, & u_2^{[\Delta_2]}(d_2) = 0, \\ v_2(d_2) = 0, & v_2^{[\Delta_2]}(d_2) = 1, \end{cases}$$

where $d_1 \in \mathbb{T}_1$ and $d_2 \in \mathbb{T}_2$. Since limit-circle case holds for η, u and v belong to $L_w^2(\mathbb{T})$ and D. Therefore for arbitrary function $y \in D$, the values $[y, u](\infty)$ and $[y, v](\infty)$ exist and are finite. Moreover one finds

$$[u_1, v_1] \equiv 1, \quad x \in \mathbb{T}_1,$$

and

$$[u_2, v_2] \equiv 1, \quad x \in \mathbb{T}_2.$$

For $y \in D$ we consider the following conditions

$$y(0) + h_1(py')(0) = 0, \tag{2.2}$$

$$y(\rho^2(b)) = y(b-) = \gamma_1 y(\sigma^2(b)) = \gamma_1 y(b+), \tag{2.3}$$

$$y^{[\Delta_1]}(\rho^2(b)) = (py)'(b-) = \gamma_2 y^{[\Delta_2]}(\sigma^2(b)) = \gamma_2 (py)'(b+), \tag{2.4}$$

$$[y, u](\infty) - h_2[y, v](\infty) = 0, \tag{2.5}$$

where h_1 and h_2 are complex numbers such that $h_k = \text{Re}h_k + i\text{Im}h_k$ with $\text{Im}h_k > 0$, $k = 1, 2$, γ_1 and γ_2 are real numbers with $\gamma_1\gamma_2 > 0$. Boundary value transmission problems containing spectral parameter in the boundary conditions defined on the union of the continuous intervals have been investigated in some papers (for example, see, [11,12]). Moreover, for h_1 is real such problems have been studied in [34–37]. However, the problem (2.1)–(2.5) is new and contains several equations with appropriate boundary-transmission conditions.

3 Simple Dissipative Operator

In this section we shall construct a suitable operator corresponding to the problem (2.1)–(2.5).

Let $H = H_1 \oplus H_2$, $H_k = L^2_{w_k}(\mathbb{T}_k)$, $k = 1, 2$, be the Hilbert space with the inner product

$$\langle y, \chi \rangle_H = \int_0^b y_1 \chi_1 w_1 \nabla_1 x + \gamma_1 \gamma_2 \int_b^\infty y_2 \chi_2 w_2 \nabla_2 x,$$

where

$$y(x) = \begin{cases} y_1(x), & x \in \mathbb{T}_1, \\ y_2(x), & x \in \mathbb{T}_2, \end{cases} \quad \chi(x) = \begin{cases} \chi_1(x), & x \in \mathbb{T}_1, \\ \chi_2(x), & x \in \mathbb{T}_2. \end{cases}$$

It should be noted that the space H is, in fact, the direct sum space $L^2_w(\mathbb{T})$.

Let $D(N)$ be the set consisting of all functions $y \in D$ satisfying the conditions (2.2)–(2.5). We define the operator N on $D(N)$ as

$$Ny = \eta(y), \quad y \in D(N), \quad x \in \mathbb{T}.$$

Therefore the equation

$$Ny = \lambda y, \quad y \in D(N), \quad x \in \mathbb{T},$$

coincides with the problem (2.1)–(2.5).

Integration by parts implies that the adjoint operator N^* of N is the operator defined by

$$N^* y = \eta(y), \quad y \in D(N^*), \quad x \in \mathbb{T},$$

where $D(N^*)$ denotes the set of all functions $y \in D$ satisfying

$$\begin{aligned} & y(0) + \overline{h}_1 (py')(0) = 0, \\ & y(b-) = \gamma_1 y(b+), \\ & (py)'(b-) = \gamma_2 (py)'(b+), \\ & [y, u](\infty) - \overline{h}_2 [y, v](\infty) = 0. \end{aligned}$$

For

$$y(x) = \begin{cases} y_1(x), & x \in \mathbb{T}_1, \\ y_2(x), & x \in \mathbb{T}_2, \end{cases} \quad \chi(x) = \begin{cases} \chi_1(x), & x \in \mathbb{T}_1, \\ \chi_2(x), & x \in \mathbb{T}_2, \end{cases} \in D(N)$$

following equations are obtained:

$$\begin{aligned} & [y_1, \chi_1] = [y_1, u_1][\chi_1, v_1] - [y_1, v_1][\chi_1, u_1], \quad x \in \mathbb{T}_1, \\ & [y_2, \chi_2] = [y_2, u_2][\chi_2, v_2] - [y_2, v_2][\chi_2, u_2], \quad x \in \mathbb{T}_2. \end{aligned} \tag{3.1}$$

Then we obtain the following theorem.

Theorem 3.1 *N is dissipative in H.*

Proof Let $y \in D(N)$. Then we have

$$\langle Ny, y\rangle_H - \langle y, Ny\rangle_H = [y, \overline{y}]_0^b + \gamma_1\gamma_2[y, \overline{y}]_b^\infty. \tag{3.2}$$

Transmission conditions (2.3) and (2.4) give

$$[y, \overline{y}](b-) = \gamma_1\gamma_2[y, \overline{y}](b+). \tag{3.3}$$

On the other side we obtain from (2.2)

$$[y, \overline{y}](0) = -2i\mathrm{Im}h_1 \left|(py')(0)\right|^2. \tag{3.4}$$

Moreover (2.5) and (3.1) implies

$$[y, \overline{y}](\infty) = 2i\mathrm{Im}h_2 \left|[y, v](\infty)\right|^2. \tag{3.5}$$

Substituting (3.3)–(3.5) in (3.2) we obtain

$$\mathrm{Im}\,\langle Ny, y\rangle_H = \mathrm{Im}h_1 \left|(py)'(0)\right|^2 + \gamma_1\gamma_2\mathrm{Im}h_2 \left|[y, v](\infty)\right|^2 \tag{3.6}$$

and this completes the proof. □

Definition 3.2 A nonselfadjoint operator A acting on a Hilbert space H is called simple if there is no invariant subspace of H on which A has selfadjoint part there.

Theorem 3.3 *N is a simple dissipative operator in H.*

Proof Consider that N has a selfadjoint point in a nontrivial subspace H_s of H. Then for

$$y(x) = \begin{cases} y_1(x), & x \in \mathbb{T}_1, \\ y_2(x), & x \in \mathbb{T}_2, \end{cases} \in D(N) \cap H_s$$

one has from (3.6) that

$$0 = \mathrm{Im}h_1 \left|(py)'(0)\right|^2 + \gamma_1\gamma_2\mathrm{Im}h_2 \left|[y, v](\infty)\right|^2. \tag{3.7}$$

(3.7) is satisfied only if $(py)'(0) = 0$ and $[y, v](\infty) = 0$. Therefore (2.2) and (2.5) imply that $y(0) = 0$ and $[y, u](\infty) = 0$. Consequently, $y_1 \equiv 0$, $y_2 \equiv 0$ and $y \equiv 0$. This completes the proof. □

Corollary 3.4 *All eigenvalues of N lie in the open upper half-plane.*

4 Resolvent Operator

In this section we will construct the resolvent operator of N and we will investigate some spectral properties of N.

Now consider the solutions

$$\varphi(x,\lambda)=\begin{cases}\varphi_1(x,\lambda), & x\in\mathbb{T}_1,\\ \varphi_2(x,\lambda), & x\in\mathbb{T}_2,\end{cases}\quad \psi(x,\lambda)=\begin{cases}\psi_1(x,\lambda), & x\in\mathbb{T}_1,\\ \psi_2(x,\lambda), & x\in\mathbb{T}_2,\end{cases}$$

of the Eq. (2.1) satisfying

$$\begin{cases}\varphi_1(0,\lambda)=-h_1, & (p\varphi_1)'(0,\lambda)=1,\\ \varphi_2(b+,\lambda)=\frac{1}{\gamma_1}\varphi_1(b-,\lambda), & (p\varphi_2)'(b+,\lambda)=\frac{1}{\gamma_2}\varphi_1'(b-,\lambda),\end{cases} \tag{4.1}$$

and

$$\begin{cases}[\psi_2,u_2](\infty)=h_2, & [\psi_2,v_2](\infty)=1,\\ \psi_1(b-,\lambda)=\gamma_1\psi_2(b+,\lambda), & (p\psi_1)'(b-,\lambda)=\gamma_2\psi_2'(b+,\lambda).\end{cases} \tag{4.2}$$

If we denote by $\omega_k(\lambda)$ the Wronskian of $\varphi_k(x,\lambda)$ and $\psi_k(x,\lambda)$ such that $\omega_1(\lambda)=W[\varphi_1,\psi_1]$, $x\in\mathbb{T}_1$, and $\omega_2(\lambda)=W[\varphi_2,\psi_2]$, $x\in\mathbb{T}_2$, then constants of the Wronskian of $\varphi_k(x,\lambda)$ and $\psi_k(x,\lambda)$ on each interval and transmission conditions given in (4.1) and (4.2) imply

$$\omega(\lambda):=\omega_1(\lambda)=\gamma_1\gamma_2\omega_2(\lambda). \tag{4.3}$$

Note that the zeros of $\omega(\lambda)$ coincide with the eigenvalues of N. Moreover the zeros of the functions

$$\Delta_{h_1}(\lambda):=\psi(0,\lambda)+h_1(p\psi)'(0,\lambda)$$

and

$$\Delta_{h_2}(\lambda):=[\varphi(x,\lambda),u(x)](\infty)-h_2[\varphi(x,\lambda),v(x)](\infty)$$

coincide with the eigenvalues of N.

Let us consider the equation

$$(N-\lambda\mathbf{1})y=f,\quad x\in\mathbb{T}, \tag{4.4}$$

where λ is not an eigenvalue of N, $\mathbf{1}$ denotes the identity operator in H and

$$y=\begin{cases}y_1, & x\in\mathbb{T}_1\\ y_2, & x\in\mathbb{T}_2\end{cases}\in D_N,\quad f=\begin{cases}f_1, & x\in\mathbb{T}_1\\ f_2, & x\in\mathbb{T}_2\end{cases}\in H.$$

Equation (4.4) is equivalent to

$$\eta(y)-\lambda y=f,\quad x\in\mathbb{T},$$

subject to the conditions (2.2)–(2.5). One can infer that the solution y of (4.4) is in the form

$$y(x,\lambda)=\begin{cases} a_1\varphi_1(x,\lambda)+a_2\psi_1(x,\lambda)+\frac{1}{\omega_1(\lambda)}\int\limits_0^x[\varphi_1(x,\lambda)\psi_1(t,\lambda) \\ -\varphi_1(t,\lambda)\psi_1(x,\lambda)]\,w_1(t)f_1(t)\nabla_1(t), \quad x\in\mathbb{T}_1, \\ b_1\varphi_2(x,\lambda)+b_2\psi_2(x,\lambda)+\frac{1}{\omega_2(\lambda)}\int\limits_b^x[\varphi_2(x,\lambda)\psi_2(t,\lambda) \\ -\varphi_2(t,\lambda)\psi_2(x,\lambda)]\,w_2(t)f_2(t)\nabla_2(t), \quad x\in\mathbb{T}_2. \end{cases}$$

$\Delta-$differentiating yields

$$y^{[\Delta]}(x,\lambda)=\begin{cases} a_1\varphi_1^{[\Delta_1]}(x,\lambda)+a_2\psi_1^{[\Delta_1]}(x,\lambda)+\frac{1}{\omega_1(\lambda)}\int\limits_0^x\Big[\varphi_1^{[\Delta_1]}(x,\lambda)\psi_1(t,\lambda) \\ -\varphi_1(t,\lambda)\psi_1^{[\Delta_1]}(x,\lambda)\Big]\,w_1(t)f_1(t)\nabla_1(t), \quad x\in\mathbb{T}_1, \\ b_1\varphi_2^{[\Delta_2]}(x,\lambda)+b_2\psi_2^{[\Delta_2]}(x,\lambda)+\frac{1}{\omega_2(\lambda)}\int\limits_b^x\Big[\varphi_2^{[\Delta_2]}(x,\lambda)\psi_2(t,\lambda) \\ -\varphi_2(t,\lambda)\psi_2^{[\Delta_2]}(x,\lambda)\Big]\,w_2(t)f_2(t)\nabla_2(t), \quad x\in\mathbb{T}_2. \end{cases}$$

The condition (2.2) gives

$$a_2\Delta_{h_1}(\lambda)=0$$

and hence $a_2=0$. Moreover the condition (2.5) implies

$$b_1=-\frac{1}{\omega_2(\lambda)}\int\limits_b^\infty\psi_2(t,\lambda)w_2(t)f_2(t)\nabla_2(t).$$

Hence $y(x,\lambda)$ must be in the form

$$y(x,\lambda)=\begin{cases} \varphi_1(x,\lambda)\left[a_1+\frac{1}{\omega_1(\lambda)}\int\limits_0^x\psi_1(t,\lambda)w_1(t)f_1(t)\nabla_1(t)\right] \\ -\frac{\psi_1(x,\lambda)}{\omega_1(\lambda)}\int\limits_0^x\varphi_1(x,\lambda)w_1(t)f_1(t)\nabla_1(t), \quad x\in\mathbb{T}_1, \\ \varphi_2(x,\lambda)\left[-\frac{1}{\omega_2(\lambda)}\int\limits_x^\infty\psi_2(t,\lambda)w_2(t)f_2(t)\nabla_2(t)\right] \\ +\psi_2(x,\lambda)\left[b_2-\frac{1}{\omega_2(\lambda)}\int\limits_b^x\varphi_2(t,\lambda)w_2(t)f_2(t)\nabla_2(t)\right], \quad x\in\mathbb{T}_2. \end{cases}$$

Taking into account the transmission conditions (2.3) and (2.4), $y(x,\lambda)$ is found as

$$y(x,\lambda)=\begin{cases}-\frac{\varphi_1(x,\lambda)}{\omega_1(\lambda)}\int\limits_x^b \psi_1(t,\lambda)w_1(t)f_1(t)\nabla_1(t)-\frac{\psi_1(x,\lambda)}{\omega_1(\lambda)}\int\limits_0^x \varphi_1(t,\lambda)w_1(t)f_1(t)\nabla_1(t)\\ -\frac{\varphi_1(x,\lambda)}{\omega_2(\lambda)}\int\limits_b^\infty \psi_2(t,\lambda)w_2(t)f_2(t)\nabla_2(t), \quad x\in\mathbb{T}_1,\\ -\frac{\psi_2(x,\lambda)}{\omega_1(\lambda)}\int\limits_0^b \varphi_1(t,\lambda)w_1(t)f_1(t)\nabla_1(t)-\frac{\psi_2(x,\lambda)}{\omega_2(\lambda)}\int\limits_b^x \varphi_2(t,\lambda)w_2(t)f_2(t)\nabla_2(t)\\ -\frac{\varphi_2(x,\lambda)}{\omega_2(\lambda)}\int\limits_x^\infty \psi_2(t,\lambda)w_2(t)f_2(t)\nabla_2(t), \quad x\in\mathbb{T}_2.\end{cases} \tag{4.5}$$

Using (4.3) we shall construct the kernel

$$G(x,t,\lambda)=\begin{cases}-\frac{\psi(x,\lambda)\varphi(t,\lambda)}{\omega(\lambda)}, 0\le t\le x\le\infty, \quad t,x\ne b,\\ -\frac{\varphi(x,\lambda)\psi(t,\lambda)}{\omega(\lambda)}, 0\le x\le t\le\infty, \quad t,x\ne b.\end{cases}$$

Then (4.5) can be introduced as

$$y(x,\lambda)=\int\limits_0^b G(x,t,\lambda)w(t)f(t)\nabla_1(t)+\gamma_1\gamma_2\int\limits_b^\infty G(x,t,\lambda)w(t)f(t)\nabla_2(t)$$

or

$$y(x,\lambda)=\left\langle G(x,t,\lambda),\overline{f}(t)\right\rangle_H.$$

Let K_λ be the operator acting on H as follows

$$K_\lambda f=\left\langle G(x,t,\lambda),\overline{f}(t)\right\rangle_H, \tag{4.6}$$

where $f\in H$. Then K_λ is the inverse of the operator $N-\lambda\mathbf{1}$.

Definition 4.1 A dissipative operator A is called maximal dissipative if it does not have a proper dissipative extension.

Theorem 4.2 *N is maximal dissipative in H.*

Proof It is sufficient to show that the equality $(N-\lambda\mathbf{1})D(N)=H$, $\text{Im}\lambda<0$, holds, where $\mathbf{1}$ denotes the identity operator in H [2,3]. It is enough to use the method of variation of parameters given above. Let $y\in D(N)$ and $f\in H$. If one solve the equation

$$(N-\lambda\mathbf{1})y=f, \quad x\in\mathbb{T}, \quad \text{Im}\lambda<0,$$

then from (4.6) it is found that $y(x,\lambda)=K_\lambda f\in H$, $\text{Im}\lambda<0$. This completes the proof. □

Now let $\lambda = 0$. Note that zero is not an eigenvalue of N. Then the equation $Ny = f$, $x \in \mathbb{T}$, can be solved with a similar method given above. Hence the inverse of N is found as

$$Kf = \left\langle G(x,t), \overline{f}(t) \right\rangle_H ,$$

where $f \in H$. Since limit-circle case holds for η, K is a Hilbert–Schmidt operator. Therefore K and its imaginary part ImK are compact operators.

It is known that the nonreal spectrum of an operator with compact imaginary part consists of eigenvalues of finite algebraic multiplicities (dimensions of the corresponding root subspace) and the limit points of nonreal spectrum belong to the spectrum of the real part of the operator [17]. Therefore, together with the results given in [38] we obtain the following theorem.

Theorem 4.3 (i) *Eigenvalues of K are countable,*
(ii) *zero is the only possible limit point of the eigenvalues,*
(iii) *zero must belong to the spectrum of K, however, may not be an eigenvalue of K,*
(iv) *the nonreal spectrum of K consists of eigenvalues of finite algebraic multiplicities and limit points of nonreal spectrum belong to the spectrum of the real part ReK.*

Consequently, we obtain the following corollary.

Corollary 4.4 (i) *Eigenvalues of N are countable,*
(ii) *infinity is the only possible limit point of the eigenvalues of N,*
(iii) *infinity must belong to the spectrum of N, however, may not be an eigenvalue of N.*

The problem of completeness of eigen- and associated functions of N will be investigated in the next section.

5 Contractive Operator

First of all we shall remind the following nice connection between maximal dissipative operators and related contractions [2,3].

Lemma 5.1 (i) *Assume the operator L_0 is dissipative. Then the operator $T_0 = K(L_0) = (L_0 - iI)(L_0 + iI)^{-1}$ is a contraction from $(L_0 + iI)D(L_0)$ onto $(L_0 - iI)D(L_0)$ and $L_0 = i(I + T_0)(I - T_0)^{-1}$. For each contraction T_0 such that $1 \notin \sigma_p(T_0)$ (the point spectrum of the operator), operator $L_0 = K^{-1}(T_0)$, $D(L_0) = (I - T_0)D(T_0)$, is dissipative.*
(ii) *Each dissipative operator L_0 has a maximal dissipative extension L. A maximal dissipative operator is closed.*
(iii) *A maximal dissipative operator is maximal dissipative if and only if $T = K(L)$ is a contraction such that $D(T) = H$ and $1 \notin \sigma_p(T)$.*
(iv) *If L is a maximal dissipative operator, $L = K^{-1}(T)$, then $-L^*$ is also maximal dissipative, $L^* = -K^{-1}(T^*)$.*

(v) *If* L *is a maximal dissipative operator, then* $\sigma(T) \subset \overline{\mathbb{C}}_{+}$, $\left\|(L-\lambda I)^{-1}\right\| \leq |Im\lambda|^{-1}$, $\lambda \in \mathbb{C}_{-}$.

Let us consider the Cayley transform T of the dissipative operator N as follows

$$T = (N - i\mathbf{1})(N + i\mathbf{1})^{-1}.$$

Since N is maximal dissipative, domain of T is the whole Hilbert space H [2,3]. One immediately gets the following inequality

$$\|T\|_{H} < 1. \tag{5.1}$$

Indeed, for $(N + i\mathbf{1})^{-1} f = y$, a direct calculation gives

$$\|(N - i\mathbf{1})y\|_{H}^{2} < \|(N + i\mathbf{1})y\|_{H}^{2} \Leftrightarrow 2\mathrm{Im}\,\langle Ny, y\rangle_{H} > 0.$$

Definition 5.2 A contraction T on a Hilbert space H is called c.n.u. if for no non-zero reducing subspace $\mathcal{T}$ for T is $T \mid \mathcal{T}$ is a unitary operator.

(5.1) gives the following result.

Theorem 5.3 *T is c.n.u. contraction on H.*

Definition 5.4 [2,3,39,40] The classes $C_{0.}$ and $C_{.0}$ of contractions are defined as

$$\begin{aligned} T &\in C_{0.} \quad \text{if } T^{n} f \to 0 \text{ for all } f, \\ T &\in C_{.0} \quad \text{if } T^{*n} f \to 0 \text{ for all } f. \end{aligned}$$

Asymptotic classifications of $C_{0.}$ and $C_{.0}$ are given as

$$\begin{aligned} C_{0.} &= \left\{T : \|T\|_{H} \leq 1, \lim_{n} \|T^{n} f\|_{H} = 0 \text{ for every } f\right\}, \\ C_{.0} &= \left\{T : \|T\|_{H} \leq 1, \lim_{n} \|T^{*n} f\|_{H} = 0 \text{ for every } f\right\}. \end{aligned}$$

C_{00} is defined as $C_{00} = C_{0.} \cap C_{.0}$.

Theorem 5.5 $T \in C_{00}$.

Proof Following inequality holds:

$$\left\|T^{n} f\right\|_{H} \leq \|T\|_{H}^{n} \|f\|_{H}\,. \tag{5.2}$$

Then (5.1) and (5.2) give that $T \in C_{0.}$. Similarly the inequality

$$\left\|T^{*n} f\right\|_{H} \leq \left\|T^{*}\right\|_{H}^{n} \|f\|_{H}$$

and (5.1) imply $T \in C_{.0}$. This completes the proof. □

We denote by $D_T = (\mathbf{1} - T^*T)^{1/2}$ and $D_{T^*} = (\mathbf{1} - TT^*)^{1/2}$ the *defect operators* of T acting on H, $\mathfrak{D}_T = \overline{D_T H}$ and $\mathfrak{D}_{T^*} = \overline{D_{T^*} H}$ the *defect spaces* and $\mathfrak{d}_T = \dim \mathfrak{D}_T$ and $\mathfrak{d}_{T^*} = \dim \mathfrak{D}_{T^*}$ the *defect indices* of T.

Theorem 5.6 *The characteristic function* Θ_T *of* T *is*

$$\Theta_T(\mu) = \frac{\Delta_{h_1}(\lambda)}{\Delta_{\overline{h}_1}(\lambda)} = \frac{\Delta_{h_2}(\lambda)}{\Delta_{\overline{h}_2}(\lambda)}, \quad \mu = \frac{\lambda - i}{\lambda + i}, \quad Im\lambda > 0.$$

Proof Let $y \in D(N)$. Then for the equality $f = (N + i\mathbf{1})y$ we obtain the following [39]

$$D_T^2 f = (N + i\mathbf{1})y - (N^* + i\mathbf{1})\chi, \tag{5.3}$$

where

$$\chi = (N^* - i\mathbf{1})^{-1}(N - i\mathbf{1})y. \tag{5.4}$$

(5.4) implies that

$$(N - i\mathbf{1})y = (N^* - i\mathbf{1})\chi,$$

where $y \in D(N)$ and $\chi \in D(N^*)$. Hence $y - \chi$ is a solution of the equation

$$\eta(\tau) = i\tau, \quad \tau \in L_w^2(\mathbb{T}),$$

satisfying the conditions (2.3) and (2.4). Therefore we set

$$y - \chi = c\tau(x, i) = c\left(\varphi(x, i) + \psi(x, i)\right), \tag{5.5}$$

where c is a constant and

$$\tau(x, \lambda) = \begin{cases} \tau_1(x, \lambda), & x \in \mathbb{T}_1, \\ \tau_2(x, \lambda), & x \in \mathbb{T}_2. \end{cases}$$

Substituting (5.5) into (5.3) we obtain

$$D_T^2 f = (\eta + i\mathbf{1})(y - \chi) = 2ic\tau(x, i), \quad x \in \mathbb{T}. \tag{5.6}$$

Consequently $\mathfrak{D}_T$ is spanned by the function $\tau(x, i)$.

Now let $y \in D(N^*)$ and $f = (N^* - i\mathbf{1})y$. Then we get

$$D_{T^*}^2 f = (N^* - i\mathbf{1})y - (N - i\mathbf{1})\chi, \tag{5.7}$$

where

$$\chi = (N + i\mathbf{1})^{-1}(N^* + i\mathbf{1})y. \tag{5.8}$$

(5.8) shows that

$$(N^* + i\mathbf{1})y = (N + i\mathbf{1})\chi,$$

where $y \in D(N^*)$ and $\chi \in D(N)$. Now we set

$$y - \chi = d\tau(x, -i) = d\left(\varphi(x, -i) + \psi(x, -i)\right). \tag{5.9}$$

Substituting (5.9) into (5.7) one obtains

$$D_{T^*}^2 f = (\eta - i\mathbf{1})(y - \chi) = -2id\tau(x, -i), \quad x \in \mathbb{T}. \tag{5.10}$$

Hence $\mathfrak{D}_{T^*}$ is spanned by the function $\tau(x, -i)$.

(5.6) and (5.10) particularly imply that $D_T^2 \tau(x, i) = \tau(x, i)$ and $D_{T^*}^2 \tau(x, -i) = \tau(x, -i)$. Consequently, $T\tau(x, i) = 0$ and $T^*\tau(x, -i) = 0$. Moreover, $D_T \tau(x, i) = \tau(x, i)$ and $D_{T^*}\tau(x, -i) = \tau(x, -i)$.

Following equality is known [39]

$$D_{T^*}\Theta_T(\mu) = (\mu\mathbf{1} - T)(\mathbf{1} - \mu T^*)^{-1} D_T \mid \mathfrak{D}_T, \quad \mu \in \boldsymbol{D}. \tag{5.11}$$

(5.11) can be written as the following form

$$D_{T^*}\Theta_T(\mu) = -(N - \lambda\mathbf{1})(N + i\mathbf{1})^{-1}(N^* - i\mathbf{1})(N^* - \lambda\mathbf{1})^{-1} D_T \mid \mathfrak{D}_T, \quad \mu \in \boldsymbol{D}, \tag{5.12}$$

where

$$\lambda = i\frac{1 + \mu}{1 - \mu}.$$

Now let us consider the function

$$\tau(x, \zeta) + \theta\tau(x, \lambda).$$

It is clear that this sum does not belong to $D(N)$ unless θ is of the form

$$\theta = \theta_1(\lambda, \zeta, h) = -\frac{\Delta_{h_1}(\zeta)}{\Delta_{h_1}(\lambda)} \tag{5.13}$$

or

$$\theta = \theta_2(\lambda, \zeta, h) = -\frac{\Delta_{h_2}(\zeta)}{\Delta_{h_2}(\lambda)}. \tag{5.14}$$

Note that constants of the Wronskian of φ and ψ on each interval $\mathbb{T}_1$ and $\mathbb{T}_2$ and transmission conditions imply that (5.13) and (5.14) are identical.

Using the equation

$$(N - \lambda\mathbf{1})^{-1}\tau(x, \zeta) = \frac{\tau(x, \zeta) + \theta\tau(x, \lambda)}{\zeta - \lambda}$$

and (5.12) we obtain

$$D_{T^*}\Theta_T(\mu)\tau(x, i) = \theta_1 \frac{\Delta_{h_1}(\lambda)}{\Delta_{\overline{h}_1}(\lambda)}\tau(x, -i) = \theta_2 \frac{\Delta_{h_2}(\lambda)}{\Delta_{\overline{h}_2}(\lambda)}\tau(x, -i),$$

where

$$\theta_1 = -\frac{\Delta_{\overline{h}_1}(i)}{\Delta_{h_1}(-i)}, \quad \theta_2 = -\frac{\Delta_{\overline{h}_2}(i)}{\Delta_{h_2}(-i)}, \quad \mu = \frac{\lambda - i}{\lambda + i}.$$

Since $\tau(x,-i)=\overline{\tau(x,i)}$ it is obtained that $\|\tau(x,i)\|_H=\|\tau(x,-i)\|_H$. Moreover, $|\theta_1|=|\theta_2|=1$. Consequently the proof is completed. □

Corollary 5.7 $\mathfrak{d}_T=\mathfrak{d}_{T^*}=1.$

It is well known that the characteristic functions of T and T^* are connected by [2,3]

$$\Theta_{T^*}(\mu)=\Theta_T^*(\overline{\mu}),\quad \mu\in\boldsymbol{D}.$$

Therefore we obtain the following corollary.

Corollary 5.8 *The characteristic function of T^* is*

$$\Theta_{T^*}(\mu)=\frac{\Delta_{\overline{h}_1}\left(-i\frac{1+\mu}{1-\mu}\right)}{\Delta_{h_1}\left(-i\frac{1+\mu}{1-\mu}\right)}=\frac{\Delta_{\overline{h}_2}\left(-i\frac{1+\mu}{1-\mu}\right)}{\Delta_{h_2}\left(-i\frac{1+\mu}{1-\mu}\right)},\quad \mu=\frac{\lambda-i}{\lambda+i},\quad Im\lambda>0.$$

Theorem 5.5 and Corollary 5.7 give the following theorem.

Theorem 5.9 *T is a c.n.u. contraction of the class C_0. Moreover, the characteristic function $\Theta_T(\mu)$ of T coincides with the minimal function $m_T(\mu)$ of T.*

Remark 5.10 Since $T\in C_{.0}$, $\Theta_T(\mu)$ is an inner function.

Theorem 5.11 *Following inequalities hold in the closed upper half-plane:*

(i) $\mathrm{Im}\,\lambda\,\|\psi\|_H^2\geq\gamma_1\gamma_2\,\mathrm{Im}\,h_2,\ Im\lambda\geq 0;$
(ii) $Im\lambda\,\|\varphi\|_H^2\geq Im\,h_1,\ Im\lambda\geq 0.$

Proof Since $\Theta_T(\mu)$ is an inner function we have $|\Theta_T(\mu)|\leq 1$, $\mathrm{Im}\lambda\geq 0$. Then one has

$$\mathbf{1}-\Theta_T^*(\lambda)\Theta_T(\lambda)=\frac{2i\mathrm{Im}h_1[\psi,\overline{\psi}](0)}{\left|\Delta_{\overline{h}_1}(\lambda)\right|^2}.\tag{5.15}$$

On the other side using Green's formula we obtain

$$2i\mathrm{Im}\lambda\,\|\psi\|_H^2-2i\gamma_1\gamma_2\mathrm{Im}h_2=-[\psi,\overline{\psi}](0).\tag{5.16}$$

Substituting (5.16) into (5.15) we obtain (i).

(5.15) is also equivalent to

$$\mathbf{1}-\Theta_T^*(\lambda)\Theta_T(\lambda)=\frac{-2i\mathrm{Im}h_2[\varphi,\overline{\varphi}](\infty)}{\left|\Delta_{\overline{h}_2}(\lambda)\right|^2}.\tag{5.17}$$

A direct calculation shows that

$$2i\mathrm{Im}\lambda\,\|\varphi\|_H^2-2i\mathrm{Im}h_1=\gamma_1\gamma_2[\varphi,\overline{\varphi}](\infty).\tag{5.18}$$

Substituting (5.18) into (5.17) we obtain (ii). Therefore the proof is completed. □

Remark 5.12 Since T is a c.n.u. contraction,*1* can not be an eigenvalue of T. On the other side, the spectrum of T coincides with those μ belong to the disc $\boldsymbol{D}$ for which the operator $\Theta_T(\mu)$ is not boundedly invertible, together with those $\mu \in \boldsymbol{C}$ not lying on any of the open arcs of $\boldsymbol{C}$ on which $\Theta_T(\mu)$ is a unitary operator valued analytic function of μ and point spectrum of T coincides with those$\mu \in \boldsymbol{D}$ for which $\Theta_T(\mu)$ is not invertible at all. Since the zeros of $\Delta_{h_1}(\lambda)$ and $\Delta_{h_2}(\lambda)$, $\text{Im}\lambda > 0$, are eigenvalues of N, $\lambda = i(1+\mu)/(1-\mu)$ for $\lambda = is$, $\lim_{s\to\infty}(is) =: \lambda_\infty$ can not be a zero of $\Delta_{h_1}(\lambda)$ or $\Delta_{h_2}(\lambda)$ or equivalently an eigenvalue of N.

Theorem 5.13 *$\Theta_T(\mu)$ is a Blaschke product in the disc $\mu \in \boldsymbol{D}$.*

Proof For $\text{Im}\lambda > 0$, $\Theta_T(\lambda)$ has a factorization

$$\Theta_T(\lambda) = B(\lambda)\exp(i\lambda b), \quad b > 0,$$

where $B(\lambda)$ is a Blaschke product. Hence one has

$$|\Theta_T(\lambda)| \leq \exp(-b\text{Im}\lambda), \quad \text{Im}\lambda > 0. \tag{5.19}$$

For $\lambda_s := is$ we obtain from (5.19) that $\Delta_{h_1}(\lambda_s) \to 0$ or $\Delta_{h_2}(\lambda_s) \to 0$ as $s \to \infty$. In both cases, this means that λ_∞ is an eigenvalue of N. However, from Remark 5.12 this is not possible. Therefore this completes the proof. □

Since $\Theta_T(\mu)$ is the minimal function of T, from the well-known theorem of Sz.-Nagy–Foiaş (for example, see, [2], p. 135) we obtain the following theorem.

Theorem 5.14 *Root functions of T associated with the points of the spectrum of T in $\boldsymbol{D}$ span the Hilbert space H.*

Definition 5.15 Let all root functions of the operator L span the Hilbert space H. Such an operator is called complete operator. If every $L-$ invariant subspace is generated by root vectors of L belonging to the subspace then it is said L admits spectral synthesis.

Since every complete operator in C_0 admits spectral synthesis [39,40] we obtain the following.

Theorem 5.16 *T admits spectral synthesis.*

Since the root functions of T span H then those of N must span H [41] (p. 42). Therefore, we have the following.

Theorem 5.17 *Root functions of N associated with the point spectrum of N in the open upper half-plane $\text{Im}\lambda > 0$ span the Hilbert space H.*

Since all root functions of N coincide with the eigen- and associated functions of N and the eigenvalue problem of N coincides with the eigenvalue problem of the problem (2.1)–(2.5), we have the following corollary.

Corollary 5.18 (i) *Eigenvalues of (2.1)–(2.5) are countable in the open upper half-plane,*

(ii) *infinity is the only possible limit point of the eigenvalues of (2.1)–(2.5),*
(iii) *infinity must belong to the spectrum of (2.1)–(2.5), however, may not be an eigenvalue of (2.1)–(2.5),*
(iv) *the system of all eigen- and associated functions of the problem (2.1)–(2.5) span the Hilbert space H.*

6 Other Properties of the Contraction

6.1 Weak Contraction

An operator $A \geq 0$ on a Hilbert space H is said to be of *finite trace* if A is compact and its eigenvalues is finite. This sum is called the trace of A [2,3].

Definition 6.1 A contraction T on a Hilbert space H is called weak contraction if

(i) Its spectrum does not fill the unit disc $\boldsymbol{D}$,
(ii) $I_H - T^*T$ is of finite trace.

In particular, all contractions T with finite defect index $\mathfrak{d}_T$ such that its spectrum does not fill the closure of the unit disc $\boldsymbol{D}$ are weak contractions. Since the zeros of the minimal function m_T of a contraction $T \in C_0$ in the open unit disc $\boldsymbol{D}$ and of the complement, in the unit circle $\boldsymbol{C}$, of the union of the arcs of $\boldsymbol{C}$ on which m_T is analytic and the spectrum of T coincide and Δ_{h_1} (also Δ_{h_2}) is an entire function, we have following theorem.

Theorem 6.2 $T = (N - i\mathbf{1})(N + i\mathbf{1})^{-1}$ *is a weak contraction on* H.

Definition 6.3 [3] The cyclic multiplicity μ_T of an operator T defined on a complex Hilbert space H is the smallest cardinality of a subset $M \subset H$ with the property that the set

$$\left\{T^n h : h \in M, n \geq 0\right\}$$

generates H. The operator T is said to be multiplicity-free if $\mu_T = 1$.

Let T be a weak contraction of the class C_0 on a seperable Hilbert space H and let the characteristic function of T coincides with an inner function $\Theta(\mu)$. Then det $(\Theta(\mu))$ is an inner function that does not depend, up to a scalar factor of absolute value one, on the particular function Θ. This inner function is called the *characteristic determinant* of T and is denoted by d_T. Since the characteristic function $\Theta_T(\mu)$ of the contraction $T = (N - i\mathbf{1})(N + i\mathbf{1})^{-1}$ is a scalar function and coincides with the minimal function m_T, we have $d_T = m_T$. Thereofore according to [3], p. 437, we have the following theorem.

Theorem 6.4 $T = (N - i\mathbf{1})(N + i\mathbf{1})^{-1}$ *is multiplicity-free, i.e.,* $\mu_T = 1$.

In general, the adjoint of a multiplicity-free operator is not generally multiplicity-free. However, since $T \in C_0$ we can find the multiplicity of T^*. Before this, we shall give some definitions [2,3].

Definition 6.5 Let V be an isometry on the Hilbert space H. A subspace $\mathcal{L}$ of H is called a wandering space for V if $V^p\mathcal{L} \perp V^q\mathcal{L}$ for every pair of integers $p, q \geq 0$, $p \neq q$. An isometry V on H is called a unilateral shift if there exists in H a subspace $\mathcal{L}$ which is wandering for V and such that

$$H = \bigoplus_{0}^{\infty} V^n \mathcal{L}.$$

The dimension of $H \ominus V\mathcal{L}$ is called the multiplicity of the unilateral shift V.

Let S denote the unilateral shift of multiplicity one acting on H^2.

Definition 6.6 For each inner function $\varphi \in H^\infty$, the Jordan block $S(\varphi)$ is the operator defined on $H(\varphi) = H^2 \ominus \varphi H^2$ by $S(\varphi) = P_{H(\varphi)}S \mid H(\varphi)$ or equivalently, $S(\varphi)^* = S^* \mid H(\varphi)$.

Definition 6.7 By an affinity from H_1 to H_2 it is meant a linear, one-to-one and bicontinuous transformation X from H_1 onto H_2. Thus bounded operators, say S_1 on H_1 and S_2 on H_2, are said to be similar if there exists an affinity X from H_1 to H_2 such that $XS_1 = S_2X$ (and consequently $X^{-1}S_2 = S_1X$).

Definition 6.8 By a quasi-affinity from H_1 to H_2 it is meant a linear, one-to-one and continuous transformation X from H_1 onto a dense linear manifold in H_2 if S_1 and S_2 are bounded operators, S_1 on H_1 and S_2 on H_2, it is said that S_1 is a quasi-affine transform of S_2 if there exist a quasi-affinity X from H_1 to H_2 such that $XS_1 = S_2X$. The operators S_1 and S_2 are called quasi-similar if they are quasi-affine transforms of one another.

Definition 6.9 Let L be a bounded operator in H, and let $\mathcal{L}$ be a subspace of H. $\mathcal{L}$ is said to be hyperinvariant for L if it is invariant for every bounded operator which commutes with L.

Consequently, we can introduce the following theorem [3] (Chap. X, Sect. 4).

Theorem 6.10 (i) *T^* is multiplicity-free, i.e., $\mu_{T^*} = 1$,*
(ii) *T is quasi-similar to the Jordan block $S\left(\frac{\Delta_{h_1}}{\Delta_{\overline{h}_1}}\right) = S\left(\frac{\Delta_{h_2}}{\Delta_{\overline{h}_2}}\right)$,*
(iii) *$T \mid \mathcal{L}$ is multiplicity-free, i.e., $\mu_{T|\mathcal{L}} = 1$, where $\mathcal{L}$ is a invariant subspace of T,*
(iv) *$\mathcal{L}$ is hyperinvariant, where $\mathcal{L}$ is a invariant subspace of T.*

6.2 Unitary Colligation

It is known that Sz-Nagy–Foiaş characteristic function theory is a special case of the unitary colligation theory [13]. In fact every contraction T acting on a Hilbert space H can be included into the unitary colligation. By a unitary colligation it is meant a set $\mathbf{\Delta} = (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; T, F, G, S)$, where $\mathfrak{H}, \mathfrak{F}$ and $\mathfrak{G}$ are the seperable Hilbert spaces called, respectively, the inner, left-outer and right-outer spaces, such that the following block form

$$\boldsymbol{U}=\begin{pmatrix} T & F \\ G & S \end{pmatrix} : \mathfrak{H}\oplus\mathfrak{F}\rightarrow\mathfrak{H}\oplus\mathfrak{G}, \tag{6.1}$$

is a unitary mapping. $\boldsymbol{U}$ is called the connecting operator. Unitary colligation theory has been investigated in recent years by many authors. For example one may see the book [42] and references therein. Let P_1 and P_2 denote the orthogonal projections of $\mathfrak{H}\oplus\mathfrak{G}$ onto $\mathfrak{H}$ and $\mathfrak{G}$, respectively. Then following operators

$$T=P_1\left[\boldsymbol{U}\mid\mathfrak{H}\right],\quad F=P_1\left[\boldsymbol{U}\mid\mathfrak{F}\right],\quad G=P_2\left[\boldsymbol{U}\mid\mathfrak{H}\right],\quad S=P_2\left[\boldsymbol{U}\mid\mathfrak{F}\right]$$

are called the components of $\boldsymbol{\Delta}$ and T, F, G and S are called the basic, right-channelled, left-channelled, and duplicating operators, respectively. The unitary of $\boldsymbol{U}$ means that

$$\begin{aligned} T^{*}T+G^{*}G=I_{\mathfrak{H}},\ F^{*}F+S^{*}S=I_{\mathfrak{F}},\ T^{*}F+G^{*}S=0,\\ T^{*}T+F^{*}F=I_{\mathfrak{H}},\ GG^{*}+SS^{*}=I_{\mathfrak{G}},\ TG^{*}+FS^{*}=0. \end{aligned}$$

If one takes $F=D_{T^*}$, $G=D_T$, $S=-T^*$, $\mathfrak{F}=\mathfrak{D}_{T^*}$, $\mathfrak{G}=\mathfrak{D}_T$ then $\boldsymbol{U}$ also provides a unitary colligation.

The connecting operator $\boldsymbol{U}$ constructed in (6.1) can also be handled with a slightly different form:

$$\boldsymbol{U}=\begin{pmatrix} S & G \\ F & T \end{pmatrix} : \mathfrak{F}\oplus\mathfrak{H}\rightarrow\mathfrak{G}\oplus\mathfrak{H}, \tag{6.2}$$

or

$$\boldsymbol{U}=\begin{pmatrix} -T^{*} & D_T \\ D_{T^*} & T \end{pmatrix} : \mathfrak{D}_{T^*}\oplus H\rightarrow\mathfrak{D}_T\oplus H.$$

Consider the following subspaces of $\mathfrak{H}$ [14]

$$\begin{aligned} \mathfrak{H}^{(c)}&=\overline{span}\left\{T^{n}F\mathfrak{F}, n=0,1,\ldots\right\},\\ \mathfrak{H}^{(o)}&=\overline{span}\left\{T^{*n}G^{*}\mathfrak{G}, n=0,1,\ldots\right\}. \end{aligned}$$

$\mathfrak{H}^{(c)}$ and $\mathfrak{H}^{(o)}$ are called controllable and the observable subspace, respectively. Let

$$\left(\mathfrak{H}^{(c)}\right)^{\perp}:=\mathfrak{H}\ominus\mathfrak{H}^{(c)},\ \left(\mathfrak{H}^{(o)}\right)^{\perp}:=\mathfrak{H}\ominus\mathfrak{H}^{(o)}.$$

Then the unitary colligation is called prime if $\overline{\mathfrak{H}^{(c)}+\mathfrak{H}^{(o)}}=\mathfrak{H}$. This condition is equivalent to $\left(\mathfrak{H}^{(c)}\right)^{\perp}\cap\left(\mathfrak{H}^{(o)}\right)^{\perp}=\{0\}$. A unitary colligation $\boldsymbol{\Delta}=(\mathfrak{F},\mathfrak{G},\mathfrak{H};S,G,F,T)$ associated with (6.2) is prime if and only if T is c.n.u. contraction. Given a unitary colligation $\boldsymbol{\Delta}=(\mathfrak{F},\mathfrak{G},\mathfrak{H};S,G,F,T)$, its characteristic function $\Theta_{\boldsymbol{\Delta}}(\zeta)$ is defined by

$$\Theta_{\mathbf{\Delta}}(\zeta) = S + \zeta G(I_{\mathfrak{H}} - \zeta T)^{-1} F, \quad \zeta \in \boldsymbol{D}.$$

Following theorem describes all unitary colligations with basic operator T.

Theorem 6.11 ([14], p. 163) *Let T be a contraction with $\mathfrak{d}_T, \mathfrak{d}_{T^*} < \infty$ acting on Hilbert space H. Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are two Hilbert spaces such that $\dim \mathfrak{N} = \mathfrak{d}_T$ and $\dim \mathfrak{M} = \mathfrak{d}_{T^*}$. Then all unitary colligations with the basic operator T and left-outer and right-outer subspaces $\mathfrak{M}$ and $\mathfrak{N}$ take the form $\mathbf{\Delta} = (\mathfrak{M}, \mathfrak{N}, H; -KT^*M, KD_T, D_{T^*}M, T)$ such that*

$$\begin{pmatrix} -KT^*M & KD_T \\ D_{T^*}M & T \end{pmatrix} : \mathfrak{M} \oplus H \to \mathfrak{N} \oplus H,$$

where $K : \mathfrak{D}_T \to \mathfrak{N}$ and $M : \mathfrak{M} \to \mathfrak{D}_{T^*}$ are unitary operators. The characteristic function of $\mathbf{\Delta}$ is

$$\Theta_{\mathbf{\Delta}}(\zeta) = K\Theta_{T^*}(\zeta)M, \quad \zeta \in \boldsymbol{D}.$$

Now consider the unitary colligation $\mathbf{\Delta}_0 = (\mathfrak{D}_{T^*}, \mathfrak{D}_T, H; -T^*, D_T, D_{T^*}, T)$ with the characteristic function

$$\Theta_{\mathbf{\Delta}_0}(\zeta) = \left[-T^* + \zeta D_T(\mathbf{1} - \zeta T)^{-1} D_{T^*}\right] \mid \mathfrak{D}_{T^*}.$$

Note that $\Theta_{\mathbf{\Delta}_0}(\zeta)$ is also the characteristic function of T^*:

$$\Theta_{\mathbf{\Delta}_0}(\mu) = \frac{\Delta_{\overline{h}_1}\left(-i\frac{1+\mu}{1-\mu}\right)}{\Delta_{h_1}\left(-i\frac{1+\mu}{1-\mu}\right)} = \frac{\Delta_{\overline{h}_2}\left(-i\frac{1+\mu}{1-\mu}\right)}{\Delta_{h_2}\left(-i\frac{1+\mu}{1-\mu}\right)}, \quad \mu = \frac{\lambda - i}{\lambda + i}, \quad \mathrm{Im}\lambda > 0.$$

Recall that $\mathfrak{d}_T = \mathfrak{d}_{T^*} = 1$. Therefore one may consider the isometric mappings $K : \mathfrak{D}_T \to \mathbb{C}$ and $M : \mathbb{C} \to \mathfrak{D}_{T^*}$. Let $H^{(c)}$ and $H^{(o)}$ be the controllable and observable subspaces in H as follows

$$\begin{aligned} H^{(c)} &= \overline{span}\{T^n D_{T^*} M\mathbb{C}, n = 0, 1, \ldots\}, \\ H^{(o)} &= \overline{span}\left\{T^{*n}(KD_T)^*\mathbb{C}, n = 0, 1, \ldots\right\}, \end{aligned}$$

and $\left(H^{(c)}\right)^\perp = H \ominus H^{(c)}$ and $\left(H^{(o)}\right)^\perp = H \ominus H^{(o)}$. Then using the results of [14] we give the following.

Theorem 6.12 *$T = (N - i\mathbf{1})(N + i\mathbf{1})^{-1}$ can be included into the unitary colligation $\mathbf{\Delta}_0 = (\mathbb{C}, \mathbb{C}, H; -KT^*M, KD_T, D_{T^*}M, T)$ as*

$$\boldsymbol{U}_0 = \begin{pmatrix} -KT^*M & KD_T \\ D_{T^*}M & T \end{pmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H. \tag{6.3}$$

Let $\overrightarrow{1} = \binom{1}{\mathbf{0}} \in \mathbb{C} \oplus H$. Then $\left(H^{(c)}\right)^\perp = (\mathbb{C} \oplus H) \ominus \overline{span}\left\{\boldsymbol{U}_0^n \overrightarrow{1}, n = 0, 1, \ldots\right\}$, $\left(H^{(o)}\right)^\perp = (\mathbb{C} \oplus H) \ominus \overline{span}\left\{\boldsymbol{U}_0^{*n} \overrightarrow{1}, n = 0, 1, \ldots\right\}$ and (i) $\mathbf{\Delta}_0$ is prime, (ii) $\overrightarrow{1}$ is the *cyclic vector for* $\boldsymbol{U}_0 : \overline{span}\left\{\boldsymbol{U}_0^n \overrightarrow{1}, n \in \mathbb{Z}\right\} = \mathbb{C} \oplus H$*. All other unitary colligations with basic operator* T *and left- and right-outer spaces* $\mathbb{C}$ *are of the form* $\widetilde{\boldsymbol{\Delta}}_0 = (\mathbb{C}, \mathbb{C}, H; -d_1 d_2 T^*, d_1 D_T, d_2 D_{T^*}, T)$ *with*

$$\widetilde{\boldsymbol{U}}_0 = \begin{pmatrix} -\, d_1 d_2 K T^* M & d_1 K D_T \\ d_2 D_{T^*} M & T \end{pmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H,$$

where $|d_1| = |d_2| = 1$.

Since $\mathfrak{d}_{\mathcal{T}} = \mathfrak{d}_{\mathcal{T}^*} = 1$, we have the following theorem.

Theorem 6.13 *Let*

$$\boldsymbol{U}_0 = \begin{pmatrix} -\, K T^* M & K D_T \\ D_{T^*} M & T \end{pmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H,$$

is the prime unitary colligation with the characteristic function $\Theta_{\boldsymbol{\Delta}_0}$*. Let*

$$F(\mu) = \Big((\boldsymbol{U}_0 + \mu \boldsymbol{I}) \, (\boldsymbol{U}_0 - \mu \boldsymbol{I})^{-1} \overrightarrow{1}, \overrightarrow{1} \Big)_{\mathbb{C} \oplus H}, \quad \mu \in \boldsymbol{D},$$

where

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{1} \end{bmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H,$$

is the operator in $\mathbb{C} \oplus H$ *such that* 1 *is the scalar in* $\mathbb{C}$ *and* $\mathbf{1}$ *is the identity operator in* H*. Then*

$$F(\zeta) = \frac{1 + \mu \overline{\Theta_{\boldsymbol{\Delta}_0}(\mu)}}{1 - \mu \overline{\Theta_{\boldsymbol{\Delta}_0}(\mu)}}, \quad \mu \in \boldsymbol{D}.$$

6.3 Jacobi Matrix Representation

Jacobi matrix representation is important to understand the nature of the selfadjoint and unitary operators. Indeed, Stone proved that [15] every selfadjoint operator with simple spectrum is unitary equivalent to a tri-diagonal Jacobi matrix. Arlinskiĭ and Tsekanovskiĭ introduced a similar representation for dissipative operators with rank-one imaginary part [17]. Beside this a matrix representation for a unitary operator with a single spectrum introduced by Cantero, Moral and Velá zquez as a five-diagonal matrix representation called CMV matrices. Now we shall introduce this matrix representation and associated results. Note that one can find several papers including CMV matrix representation [18,43–48].

Let $\boldsymbol{m}$ be a probability measure on the unit circle $\boldsymbol{C}$. Then the moments of $\boldsymbol{m}$ is defined as

$$F(z) = F(z, \boldsymbol{m}) := \int_{\boldsymbol{C}} \frac{\zeta + z}{\zeta - z} d\boldsymbol{m}(\zeta) = 1 + 2 \sum_{n=1}^{\infty} \beta_n z^n, \quad \beta_n = \int_{\boldsymbol{C}} \zeta^{-n} d\boldsymbol{m}.$$

The function F is the Carathéodory function. Moreover it is an analytic function in the disc $\boldsymbol{D}$ and has the properties: $\mathrm{Re}F > 0,\ F(0) = 1$. Then one can define the following Schur function

$$f(z) = f(z, \boldsymbol{m}) := \frac{1}{z}\frac{F(z)-1}{F(z)+1}, \quad F(z) = \frac{1+zf(z)}{1-zf(z)}.$$

Schur function f becomes an analytic function in the disc $\boldsymbol{D}$ with $\sup_{\boldsymbol{D}} |f(z)| \leq 1$ [14]. Note that there is a connection between probability measures, Carathé odory function and Schur function. Under this correspondence $\boldsymbol{m}$ is trivial if and only if the associated Schur function is a finite Blaschke product. Let $f = f_0$ be a Schur function and not a finite Blaschke product. Then we let

$$f_{n+1}(z) = \frac{f_n(z) - \boldsymbol{\gamma}_n}{z\left(1 - \overline{\boldsymbol{\gamma}_n} f_n(z)\right)}, \quad \boldsymbol{\gamma}_n = f_n(0).$$

$\{f_n\}$ is an infinite sequence of Schur functions and neither of its terms is a finite Blaschke product. Here, the numbers $\{\boldsymbol{\gamma}_n\}$ are called the Schur parameters

$$\mathcal{S}f = \{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \ldots\}.$$

If a Schur function f is not a finite Blaschke product, the connection between the nontangental limit values $f(\zeta)$ and its Schur parameters $\{\boldsymbol{\gamma}_n\}$ is given by

$$\prod_{n=0}^{\infty}\left(1 - |\boldsymbol{\gamma}_n|^2\right) = \exp\left\{\int_{\boldsymbol{C}} \ln\left(1 - |f(\zeta)|^2\right) d\boldsymbol{m}\right\}.$$

Therefore the equation holds $\sum_{n=0}^{\infty} |\boldsymbol{\gamma}_n|^2 = \infty$ if anf only if $\ln\left(1 - |f(\zeta)|^2\right) \notin L^1(\boldsymbol{C})$.

Then we have the following.

Theorem 6.14 *There exists a probability measure* $\boldsymbol{m}$ *on* $\boldsymbol{C}$ *such that* $T = (N - i\mathbf{1})(N + i\mathbf{1})^{-1}$ *is unitary equivalent to the following operator*

$$\mathbb{L}h(\mu) = P_{\mathfrak{K}}(\mu h(\mu)), \quad h \in \mathfrak{K} := L^2(\boldsymbol{C}, d\boldsymbol{m}) \ominus \mathbb{C},$$

where $P_{\mathfrak{K}}$ *is the orthogonal projection in* $L^2(\boldsymbol{C}, d\boldsymbol{m})$ *onto* $\mathfrak{K}$. *The Schur function associated with* $\boldsymbol{m}$ *is the characteristic function* $\Theta_T(\mu)$ *of* N:

$$f(\mu) = \Theta_T(\mu) = \frac{\Delta_{h_1}(\lambda)}{\Delta_{\overline{h}_1}(\lambda)} = \frac{\Delta_{h_2}(\lambda)}{\Delta_{\overline{h}_2}(\lambda)}, \quad \mu = \frac{\lambda - i}{\lambda + i}, \quad Im\,\lambda > 0.$$

Let $\boldsymbol{m}$ be a nontrivial measure on the unit circle $\boldsymbol{C}$. Then the monic orthogonal polinomials $\Phi_n(z, \boldsymbol{m})$ are uniquely determined by

$$\Phi_n(z) = \prod_{j=1}^{n}(z - z_{n,j}), \quad \int\limits_{\boldsymbol{C}} \zeta^{-j}\Phi_n(\zeta)d\boldsymbol{m} = 0, \quad j = 0, 1, \ldots, n-1. \tag{6.4}$$

Therefore for $n \neq m$ it is obtained $(\Phi_n, \Phi_m) = 0$ on the Hilbert space $L^2(\boldsymbol{C}, d\boldsymbol{m})$. Following equation

$$\phi_n = \frac{\Phi_n}{\|\Phi_n\|}$$

constitute orthonormal polinomials.

It is known that the space of polinomials of degree at most n has dimension $n + 1$. Then this fact together with (6.4) imply the following:

$$\deg(P) \leq n, \quad P \perp \zeta^j, \quad j = 0, 1, \ldots, n-1 \Rightarrow P = c\Phi_n^*.$$

This shows that $\Phi_{n+1}(z) - z\Phi_n(z)$ is of degree n and orthogonal to z^j, $j = 1, 2, \ldots, n$. Furthermore,

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n(\boldsymbol{m})\Phi_n^*(z). \tag{6.5}$$

Here the complex numbers $\alpha_n(\boldsymbol{m})$ are called Verblunsky coefficient and the Eq. (6.5) is known as Szegö recurrences. Substituting the value $z = 0$ into (6.5), we get

$$\alpha_n(\boldsymbol{m}) = \alpha_n = -\overline{\Phi_{n+1}(0)}.$$

The equation

$$z\Phi_n(z) = \rho_n^{-2}\left(\Phi_{n+1}(z) + \overline{\alpha}_n\Phi_n^*(z)\right)$$

is known as the inverse Szegö recurrences. Here

$$\rho_j := \sqrt{1 - |\alpha_j|^2}, \quad 0 < \rho_j \leq 1, \quad |\alpha_j|^2 + \rho_j^2 = 1. \tag{6.6}$$

Consequently, the norm $\|\Phi_n\|$ in $L^2(\boldsymbol{C}, d\boldsymbol{m})$ may be determined as

$$\|\Phi_n\| = \prod_{j=0}^{n-1}\rho_j, \quad n = 1, 2, \ldots$$

The CMV basis $\{\chi_n\}$ is obtained by orthonormalizing the sequence $1, \zeta, \zeta^{-1}, \zeta^2, \zeta^{-2}, \ldots,$ and the matrix

$$\mathcal{C} = \mathcal{C}(\boldsymbol{m}) = \left\|c_{n,m}\right\|_{n,m=0}^{\infty} = \|(\zeta\chi_m, \chi_n)\|, \quad m, n \in \mathbb{Z}_+$$

is five-diagonal. An expression of $\{\chi_n\}$ is as follows:

$$\chi_{2n}(z) = z^{-n}\phi_{2n}^*(z), \quad \chi_{2n+1}(z) = z^{-n}\phi_{2n+1}^*(z), \quad n \in \mathbb{Z}_+.$$

Therefore one can find the matrix elements in terms of α's and ρ's as

$$\mathcal{C}\left(\{\alpha_n\}\right)=\begin{pmatrix} \overline{\alpha}_0 & \overline{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \cdots \\ \rho_0 & -\overline{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \cdots \\ 0 & \overline{\alpha}_2\rho_1 & -\overline{\alpha}_2\alpha_1 & \overline{\alpha}_3\rho_2 & \rho_3\rho_2 & \cdots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \cdots \\ 0 & 0 & 0 & \overline{\alpha}_4\rho_3 & -\overline{\alpha}_4\alpha_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Here α's are the Verblunsky coefficients and ρ's are as given in (6.6). $\mathcal{C}\left(\{\alpha_n\}\right)$ is the matrix representation of the unitary operator of multiplication by ζ in $L^2(\boldsymbol{C}, d\boldsymbol{m})$.

Finally we let the following matrix which is obtained from $\mathcal{C}\left(\{\alpha_n\}\right)$ by deleting the first row and the first column:

$$\mathcal{C}\left(\{\alpha_n\}\right)=\begin{pmatrix} -\overline{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \cdots \\ \overline{\alpha}_2\rho_1 & -\overline{\alpha}_2\alpha_1 & \overline{\alpha}_3\rho_2 & \rho_3\rho_2 & \cdots \\ \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \cdots \\ 0 & 0 & \overline{\alpha}_4\rho_3 & -\overline{\alpha}_4\alpha_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Recall that $\mathfrak{d}_T = \mathfrak{d}_{T^*} = 1$. Therefore we may introduce the following theorem [14].

Theorem 6.15 *$T = (N - i\mathbf{1})(N + i\mathbf{1})^{-1}$ is unitary equivalent to the operator acting on H determined by the truncated CMV matrix $\mathcal{C}\left(\{\alpha_n\}\right)$, where $\{\alpha_n\}$ are the Schur parameters of the characteristic function Θ_T of T.*

7 Conclusion and Remarks

One of the main aims of this paper is to investigate the spectral properties of the boundary value transmission problem (2.1)–(2.5) defined on the union of two time scales. It is well known that different time scales give rise to different boundary value (and transmission) problems. For this reason the problem (2.1)–(2.5) is new and contains interesting problems. To be more precise we shall give an example.

Let $\mathbb{T}_1 = [0, \epsilon_1] \cup \{1, \ldots, b-1\} \cup [\epsilon_2, b]$ and $\mathbb{T}_2 = [b, \infty)$, where $\epsilon_1, \epsilon_2 > 0$. In this case we have

$$\begin{aligned} \eta_1(y) &= -p(n-1)y(n-1) + q_1(n)y(n) - p(n)y(n+1) \\ &= \lambda w(n)y(n), \quad n \in \{1, \ldots, b-1\}, \end{aligned}$$

where $q_1(n) = p(n-1) + p(n) + q(n)$, and

$$\eta_2(y) = -(p(x)y')' + q(x)y = \lambda w(x)y, \quad x \in [0, \epsilon_1] \cup [\epsilon_2, b] \cup \mathbb{T}_2.$$

Moreover the conditions (2.1)–(2.5) must be understood as

$$\eta(y) = \lambda y, \quad x \in \mathbb{T}_1 \cup \mathbb{T}_2, \tag{7.1}$$

$$y(0) + h_1(py')(0) = 0, \tag{7.2}$$
$$y(b-) = \gamma_1 y(b+), \tag{7.3}$$
$$(py')(b-) = \gamma_2(py')(b+), \tag{7.4}$$
$$[y, u](\infty) - h_2[y, v](\infty) = 0, \tag{7.5}$$

where $b\pm$ should be understood as one approaches the point b with $b \pm \rho$, $\rho > 0$. The problem (7.1)–(7.5) is a new and interesting problem.

Even in the continuous case, such that $\mathbb{T}_1 = [0, b]$, $\mathbb{T}_2 = [b, \infty)$, the method constructed for the spectral analysis of the problem (2.1)–(2.5) is new in the literature.

Following theorem is important.

Theorem 7.1 ([49]) *Let A be an invertible operator. Then $-A$ is dissipative if and only if the inverse operator A^{-1} is dissipative.*

Therefore we obtain the following.

Theorem 7.2 *$-K$ is dissipative in H.*

Since the completeness of the root vectors of N and K (also $-K$) coincide we have the following.

Theorem 7.3 *The system of all root functions of K $(-K)$ span the Hilbert space H.*

Moreover we may introduce the following.

Theorem 7.4 *$-K$ is the simple dissipative operator in H.*

The proof follows from the fact the simplicity of the dissipative operator N. Indeed, if $-K$ has a nontrivial subspace of H in which $-K$ has a selfadjoint part there, then from the equation $y = Kf$ and the inner product in H, one obtains that N has a selfadjoint part in a nontrivial subspace of H. However, from Theorem 3.3, this is not possible

For the complete spectral analysis of the maximal dissipative operator N we use the Cayley transform T of the maximal dissipative operator N and we obtain that this Cayley transform T belongs to C_0. Theorem 6.2 implies that T is multiplicity-free and using the fact that the defect indices of T is finite and the spectrum of T does not fulfill the whole unit disc $\boldsymbol{D}$, we obtain that T is a weak contraction. However, according to [3], p. 437, this result can be obtained from the fact that every C_0-contraction with finite-multiplicity is a weak contraction.

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