Complex Anal. Oper. Theory (2018) 12:1991–2001
https://doi.org/10.1007/s11785-017-0716-y

Complex Analysis and Operator Theory

Lipschitz Continuity of Quasiconformal Mappings and of the Solutions to Second Order Elliptic PDE with Respect to the Distance Ratio Metric

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Received: 16 January 2017 / Accepted: 21 July 2017 / Published online: 20 August 2017

Abstract The main aim of this paper is to study the Lipschitz continuity of certain (K, K')-quasiconformal mappings with respect to the distance ratio metric, and the Lipschitz continuity of the solution of a quasilinear differential equation with respect to the distance ratio metric.

Keywords Quasiconformal mappings · Harmonic and polyharmonic mappings · Distance ratio metric · Lipschitz continuity

Mathematics Subject Classification Primary 30C62 · 30L10 · 31A30 · 31B30; Secondary 26A16 · 33C05 · 35J05 · 30C20

1 Introduction and Main Results

Martio [22] was the first who considered the study on harmonic quasiconformal mappings in $\mathbb{C}$. In the recent years the articles [8,14,16–18,26] brought much light on this topic. In [6,21], the Lipschitz characteristic of (K, K')-quasiconformal mappings

Communicated by Ronen Peretz.

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has been discussed. In [20], the authors proved that a K-quasiconformal harmonic mapping from the unit disk $\mathbb{D}$ onto itself is bi-Lipschitz with respect to hyperbolic metric, and also proved that a K-quasiconformal harmonic mapping from the upper half-plane $\mathbb{H}$ onto itself is bi-Lipschitz with respect to hyperbolic metric. In [23], the authors proved that a K-quasiconformal harmonic mapping from D to D' is bi-Lipschitz with respect to quasihyperbolic metrics on D and D', where D and D' are proper domains in $\mathbb{C}$. Important definitions will be included later in this section.

In [15], Kalaj considered the bi-Lipschitz continuity of K-quasiconformal solution of the inequality

$$|\Delta f| \leq B|Df|^2. \tag{1.1}$$

Here Δf represents the two-dimensional Laplacian of f defined by $\Delta f = f_{xx}+f_{yy} = 4f_{z\bar{z}}$ and the mapping f satisfying the Laplace equation $\Delta f = 0$ is called harmonic. For $z = x + iy$ and $f = u + iv$, Df denotes the Jacobian matrix

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

so that $J_f = |f_z|^2 - |f_{\bar{z}}|^2$ is the Jacobian of f.

The first aim of this paper is to consider the Lipschitz continuity of (K, K')-quasiconformal solution of the inequality (1.1) with respect to the distance ratio metric.

Theorem 1.1 *Let f be a (K, K')-quasiconformal C^2 mapping from the unit disk $\mathbb{D} = \{z : |z| < 1\}$ onto itself, satisfying the inequality (1.1) and $f(0) = 0$. Then f is Lipschitz continuous with respect to the distance ratio metric.*

The proof of Theorem 1.1 will be presented in Sect. 2. Before we proceed further, let us fix up further notation, preliminaries and remarks.

1.1 (K, K')-Quasiconformal Mappings

We say that a function $u : D \to \mathbb{R}$ is *absolutely continuous on lines*, *ACL* in brief, in the domain D if for every closed rectangle $R \subset D$ with sides parallel to the axes x and y, u is absolutely continuous on almost every horizontal line segment and almost every vertical line segment in R. Such a function has, of course, partial derivatives u_x and u_y a.e. in D (cf. [1]). Further, we say $u \in ACL^2$ if $u \in ACL$ and its partial derivatives are locally L^2 integrable in D.

A sense-preserving continuous mapping $f : D \to \Omega$ is said to be

1. (K, K')-*quasiregular* if f is ACL^2 in D, $J_f \neq 0$ a.e. in D and there are constants $K \geq 1$ and $K' \geq 0$ such that $|Df|^2 \leq KJ_f + K'$ a.e. in D, where $|Df| = |f_z| + |f_{\bar{z}}|$;
2. K-*quasiregular* if $K' = 0$.

In particular, f is called (K, K')-*quasiconformal* if f is a (K, K')-quasiregular homeomorphism; and f is K-*quasiconformal* if f is a K-quasiregular homeomorphism.

Here are some basic comments on these mappings. From [6, Example 2.1] and [21, Example 2.1] we know that there are (K, K')-quasiregular mappings which are not K_1-quasiregular for any $K_1 \geq 1$. Moreover, it is known that (see [6, Example 4.1]) there are (K, K')-quasiconformal mappings whose inverses are not (K_1, K_1')-quasiconformal for any $K_1 \geq 1$ and $K_1' \geq 0$.

Remark 1.2 If f is a (K, K')-quasiregular mapping, g is a analytic function and $|g'|$ is bounded by a constant L, then $f \circ g$ is $(K, K'L^2)$-quasiregular mapping.

A mapping $f : D \to \Omega$ is *proper* if the preimage of every compact set in Ω is compact in D (cf. [19, p. 4051] or [30, p. 17]).

1.2 The Distance Ratio Metric

For a subdomain $G \subset \mathbb{C}$ and for all $z, w \in G$, the distance ratio metric j_G is defined as

$$j_G(z, w) = \log\left(1 + \frac{|z-w|}{\min\{\delta_G(z), \delta_G(w)\}}\right),$$

where $\delta_G(z)$ denotes the Euclidean distance from z to ∂G. The distance ratio metric was introduced by Gehring and Palka [12] and in the above simplified form by Vuorinen [31]. However, the distance ratio metric j_G is not invariant under Möbius transformations. Therefore, it is natural to consider the Lipschitz continuity of conformal mappings or Möbius transformations with respect to the distance ratio metric. Gehring and Osgood [11] proved that the distance ratio metric is not altered by more than a factor of 2 under Möbius transformations.

Theorem A ([11, Proof of Theorem 4]) *If G and G' are proper subdomains of $\mathbb{R}^n$ and if f is a Möbius transformation of G onto G', then $j_{G'}(f(x), f(y)) \leq 2 j_G(x, y)$ for all $x, y \in G$.*

Recall that a mapping $f : D \to \Omega$ is said to be *Lipschitz continuous* (resp. Lipschitz continuous with respect to the distance ratio metric) if there exists a positive constant L_1 (resp. a positive constant L) such that for all $z, w \in D$,

$$|f(z) - f(w)| \leq L_1|z - w| \quad (\text{resp. } j_\Omega(f(z), f(w)) \leq L j_D(z, w)).$$

In 2011, Kalaj and Mateljević [17] proved that every quasiconformal C^2 diffeomorphism f from the domain Ω with $C^{1,\alpha}$ compact boundary onto the domain G with $C^{2,\alpha}$ compact boundary satisfying the *Poisson differential inequality*

$$|\Delta f| \leq B|Df|^2 + C \tag{1.2}$$

for some constants $B \geq 0$ and $C \geq 0$, is Lipschitz continuous respect to Euclidean metric. Clearly if $B = C = 0$ in (1.2), then f is harmonic.

Recently, the authors in [6, Theorem 1.1] proved the following theorem:

Theorem B *Suppose f is a proper (K, K')-quasiregular C^2 mapping of a Jordan domain D with $C^{1,\alpha}$ boundary onto a Jordan domain Ω with $C^{2,\alpha}$ boundary. If f satisfies the partial differential inequality (1.2) for constants $B > 0$ and $C \geq 0$, then f has bounded partial derivatives in D. In particular, f is Lipschitz continuous.*

Remark 1.3 From Theorem B we infer that if $f : \mathbb{D} \to \mathbb{D}$ satisfies the conditions of Theorem 1.1, then there exists a constant M such that $|Df| \leq M$. Hence for all $z, w \in \mathbb{D}$, we have $|f(z) - f(w)| \leq M|z - w|$ and $|f(z)| \leq M|z|$. If $M < 1$, we get

$$\begin{aligned} j_{\mathbb{D}}(f(z), f(w)) &= \log\left(1 + \frac{|f(z) - f(w)|}{\min\{\delta_{\mathbb{D}}(f(z)), \delta_{\mathbb{D}}(f(w))\}}\right) \\ &\leq \log\left(1 + \frac{M|z - w|}{M\ \min\{\delta_{\mathbb{D}}(z), \delta_{\mathbb{D}}(w)\}}\right) \\ &\leq j_{\mathbb{D}}(z, w), \end{aligned}$$

which clearly shows that $f : \mathbb{D} \to \mathbb{D}$ is a Lipschitz continuous function with respect to the distance ratio metric. □

In order to state our next result, we need to recall the definition of hypergeometric series. For $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \ldots$, the *hypergeometric* function is defined by the power series

$$F(a, b; c; z) := {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1,$$

where $(a)_0 = 1$ and $(a)_n = a(a + 1) \cdots (a + n - 1)$ for $n = 1, 2, \ldots$ are the *Pochhammer symbols*. Obviously, for $n = 0, 1, 2, \ldots$, $(a)_n = \Gamma(a + n)/\Gamma(a)$. In particular, for $a, b, c > 0$ and $a + b < c$, we have (cf. [3,4])

$$F(a, b; c; 1) = \lim_{z \to 1^-} F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} < \infty.$$

Consider the operator equation

$$T_\alpha(f) = 0 \ \text{ in } \mathbb{D}, \tag{1.3}$$

where $f : \mathbb{D} \to \mathbb{C}$, $\alpha \in \mathbb{R}$, and

$$T_\alpha = -\frac{\alpha^2}{4}(1 - |z|^2)^{-\alpha-1} + \frac{\alpha}{2}(1 - |z|^2)^{-\alpha-1}\left(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}\right) + (1 - |z|^2)^{-\alpha}\frac{\partial^2}{\partial z \partial \overline{z}}$$

is the *second order elliptic partial differential operator* defined on the unit disk $\mathbb{D}$. In the case of $\alpha = 0$, $T_\alpha(f) = 0$ is equivalent to saying that f is harmonic in $\mathbb{D}$. More generally, if f satisfies (1.3) with $\alpha = 2(n - 1)$, then f is *polyharmonic* (or *n-harmonic*) in $\mathbb{D}$, where $n \in \{1, 2, \ldots\}$ (cf. [2,5,9,27]). Recently, several new properties of polyharmonic mappings are discussed in [2]. The following result concerns the solutions to the equation (1.3).

Lemma C [25, Theorem 2.2] *Let* $\alpha \in \mathbb{R}$ *and* $f \in C^2(\mathbb{D})$*. Then* f *satisfies* (1.3) *if and only if it has a series expansion for* $z \in \mathbb{D}$ *of the form*

$$f(z) = \sum_{k=0}^{\infty} c_k F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; |z|^2\right) z^k + \sum_{k=1}^{\infty} c_{-k} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; |z|^2\right) \bar{z}^k, \tag{1.4}$$

where $\{c_k\}_{k=-\infty}^{\infty}$ *is a sequence of complex numbers satisfying*

$$\limsup_{|k|\to\infty} |c_k|^{\frac{1}{|k|}} \leq 1. \tag{1.5}$$

In particular, the expansion (1.4)*, subject to* (1.5)*, converges in* $C^\infty(\mathbb{D})$*, and every solution* f *of* (1.3) *is* C^∞*-smooth in* $\mathbb{D}$*.*

In [10,24], the authors gave some properties of solution to (1.3) whereas in [28,29], the authors considered the Lipschitz continuity of the distance-ratio metric under some Möbius automorphisms of the unit ball and conformal mappings from $\mathbb{D}$ to $\mathbb{D}$. In [7], the authors discussed the Lipschitz continuity of polyharmonic mappings with respect to the distance ratio metric. Thus, it is natural to investigate Lipschitz continuity of the solution of (1.3) in $\mathbb{D}$ with respect to the distance ratio metric. We now state our next result.

Theorem 1.4 *For* $\alpha > -1$*, let* $f : \mathbb{D} \to \mathbb{D}$ *be a* C^2*-solution to* (1.3) *with the series expansion of the form* (1.4) *and* $f(0) = 0$*. If*

$$\sum_{k=1}^{\infty} (|c_k| + |c_{-k}|) \sum_{n=0}^{\infty} \frac{(-\frac{\alpha}{2})_n (k - \frac{\alpha}{2})_n}{(k+1)_n n!} \leq 1, \tag{1.6}$$

then

$$j_{\mathbb{D}}(f(z), f(w)) \leq j_{\mathbb{D}}(z, w),$$

and this inequality is sharp. That is, f *is Lipschitz continuous with respect to the distance ratio metric.*

Remark 1.5 In Theorem 1.4, we restrict $\alpha > -1$, see [25, proposition 1.4] for the reason for this constraint.

The proof of Theorem 1.4 will be presented in Sect. 2.

2 Proof of Theorems 1.1 and 1.4

First we shall deal with the Lipschitz continuity of certain (K, K')-quasiconformal mappings and then consider the Lipschitz continuity of the solution to the differential operator T_α with respect to the distance ratio metric.

Lemma 2.1 *Assume the hypotheses of Theorem 1.1. Then there exists a constant $C(K, K', B)$ such that for $z \in \mathbb{D}$*

$$\frac{1-|z|^2}{1-|f(z)|^2} \leq C(B, K, K'). \tag{2.1}$$

Proof By assumption f is a (K, K')-quasiconformal C^2 mapping from $\mathbb{D}$ onto itself, satisfying the inequality (1.1) and $f(0) = 0$. For convenience, we denote the class of all such functions f by $\mathcal{QC}(\mathbb{D}, B, K, K')$. Then there is a positive constant A not depending on f such that the function φ_f, $f \in \mathcal{QC}(\mathbb{D}, B, K, K')$, defined by

$$\varphi_f(z) = -\frac{1}{A} + \frac{1}{A}e^{A(|f(z)|-1)}$$

is subharmonic in $\mathbb{D}$.

Now, let us prove the existence of such an A. Take

$$\psi(\rho) = -\frac{1}{A} + \frac{1}{A}e^{A(\rho-1)}.$$

Then $\psi'(\rho) = e^{A(\rho-1)}$ and $\psi''(\rho) = Ae^{A(\rho-1)}$. On the other hand, using $f_z = (1/2)(f_x - if_y)$ and $f_{\overline{z}} = (1/2)(f_x + if_y)$, we find that

$$|D|f||^2 = |f|_x^2 + |f|_y^2 = (|f|_z + |f|_{\overline{z}})^2 + i^2(|f|_z - |f|_{\overline{z}})^2 = 4|f|_z|f|_{\overline{z}},$$

and thus,

$$\Delta\varphi_f = \psi''(|f|)|D|f||^2 + \psi'(|f|)\Delta|f|. \tag{2.2}$$

Furthermore, put $s = f/|f|$. By elementary calculations we see that the following equalities hold:

$$\Delta|f| = \frac{|f_z|^2 + |f_{\overline{z}}|^2}{|f|} - 2\mathrm{Re}\left(\overline{f}^{\frac{1}{2}} f^{-\frac{3}{2}} f_z f_{\overline{z}}\right) + \mathrm{Re}\,(\overline{s}\Delta f)$$

and

$$|s_z|^2 = |s_{\overline{z}}|^2 = \frac{1}{4}\frac{|f_z|^2 + |f_{\overline{z}}|^2}{|f|^2} - \frac{1}{2|f|}\mathrm{Re}\left(\overline{f}^{\frac{1}{2}} f^{-\frac{3}{2}} f_z f_{\overline{z}}\right).$$

Then we know that

$$\Delta|f| = |f| \cdot |Ds|^2 + \mathrm{Re}\,(\overline{s}\Delta f). \tag{2.3}$$

We continue the discussion by setting $\rho = |f|$. According to [6, Lemma 3.1], we have

$$|D\rho| \geq \frac{|Df|}{K} - \frac{\sqrt{K'}}{K}, \tag{2.4}$$

Using (1.1), (2.2), (2.3) and (2.4), it follows finally that

$$\begin{aligned}\Delta\varphi_f &= e^{A(\rho-1)}\Big[A|D\rho|^2 + \rho|Ds|^2 + \operatorname{Re}(\overline{s}\Delta f)\Big]\\ &\geq e^{A(\rho-1)}\Big[\frac{A}{K^2}(|Df| - \sqrt{K'})^2 - B|Df|^2\Big]\\ &= e^{A(\rho-1)}\Big(\frac{A - BK^2}{K^2}|Df|^2 - \frac{2A\sqrt{K'}}{K^2}|Df| + \frac{AK'}{K^2}\Big).\end{aligned}$$

We obtain from Theorem B that f is Lipschitz continuous, and then there exists a constant M such that $|Df| \leq M$. Hence, if we choose an appropriate A, satisfying $A \neq BK^2$ and

$$\frac{A\sqrt{K'} - \sqrt{ABK'K^2}}{A - BK^2} \geq M,$$

i.e.

$$(\sqrt{K'} - M)^2A^2 + [2BMK^2(\sqrt{K'} - M) - BK^2K']A + B^2M^2K^4 \geq 0, \tag{2.5}$$

we obtain the inequality $\Delta\varphi_f(z) \geq 0$ for $|z| < 1$. We next show that this choice of A is possible.

If $M = \sqrt{K'}$, then there exists an appropriate value of A satisfying the inequality (2.5). If $M \neq \sqrt{K'}$ and $K' + 4M^2 - 4M\sqrt{K'} \leq 0$, then (2.5) holds for all A. If $M \neq \sqrt{K'}$ and $K' + 4M^2 - 4M\sqrt{K'} > 0$, then (2.5) holds for all

$$A \geq \frac{BK^2K' - 2BMK^2(\sqrt{K'} - M) + BK^2\sqrt{K'(K' + 4M^2 - 4M\sqrt{K'})}}{2(\sqrt{K'} - M)^2}.$$

In conclusion, there must exist an appropriate A such that $\Delta\varphi_f(z) \geq 0$ for $|z| < 1$.

Define

$$F(z) = \sup\{\varphi_f(z) :\ f \in \mathcal{QC}(\mathbb{D}, B, K, K')\}.$$

We prove that F is subharmonic in $\mathbb{D}$. By [13, Theorem 1.6.2], we only need to prove that F is continuous. Define $h(z) = e^{A(|z|-1)},\ |z| < 1$. Elementary calculations show that

$$h_z(z) = \frac{A}{2}\frac{\overline{z}}{|z|}e^{A(|z|-1)} \quad \text{and} \quad h_{\overline{z}}(z) = \frac{A}{2}\frac{z}{|z|}e^{A(|z|-1)}.$$

Then $|Dh| = |h_z| + |h_{\overline{z}}| = Ae^{A(|z|-1)} < A$ which implies that

$$|h(z) - h(z')| \leq A|z - z'| \quad \text{for } z, z' \in \mathbb{D}.$$

According to Theorem B, we know that f is Lipschitz continuous. Therefore

$$|\varphi_f(z) - \varphi_f(z')| = \frac{1}{A}\left|e^{A(|f(z)|-1)} - e^{A(|f(z')|-1)}\right| \leq |f(z) - f(z')| \leq M|z - z'|,$$

where M is a constant. Hence, $|F(z) - F(z')| \leq M|z - z'|$ so that F is continuous. Finally, from the similar proof of [15, Lemma 2.3], we complete the proof. □

2.1 Proof of Theorem 1.1

From the hypotheses of Theorem 1.1 and Lemma 2.1, we obtain that

$$\frac{1 - |z|^2}{1 - |f(z)|^2} \leq C(K, K', B)$$

and thus, we obtain that

$$\frac{1 - |z|}{1 - |f(z)|} \leq C(K, K', B)\frac{1 + |f(z)|}{1 + |z|} \leq 2C(K, K', B).$$

Moreover, from Theorem B, we see that f is Lipschitz continuous and therefore, there exists a constant M_1 such that $|Df| \leq M_1$. Now, we choose an appropriate constant M satisfying $M > \max\{M_1, 1/(2C(K, K', B))\}$ so that $|Df| \leq M$. Consequently, using the Bernoulli inequality, for any two points z and w in $\mathbb{D}$, we have

$$\begin{aligned} j_{\mathbb{D}}(f(z), f(w)) &= \log\left(1 + \frac{|f(z) - f(w)|}{\min\{\delta_{\mathbb{D}}(f(z)), \delta_{\mathbb{D}}(f(w))\}}\right) \\ &\leq \log\left(1 + 2C(K, K', B)M\frac{|z - w|}{\min\{\delta_{\mathbb{D}}(z), \delta_{\mathbb{D}}(w)\}}\right) \\ &\leq 2C(K, K', B)Mj_{\mathbb{D}}(z, w) \end{aligned}$$

and thus, the proof of the theorem is complete. □

2.2 Proof of Theorem 1.4

For convenience, let $g(t) = F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; t)$. For $z, w \in \mathbb{D}$, let us assume that $|f(z)| \geq |f(w)|$. Then

$$
\begin{aligned}
&|f(z)-f(w)| \\
&\quad = \left|\sum_{k=1}^{\infty} c_k(|g(|z|^2)z^k - g(|w|^2)w^k) + \sum_{k=1}^{\infty} c_{-k}(|g(|z|^2)\overline{z}^k - g(|w|^2)\overline{w}^k)\right| \\
&\quad \le |z-w| \sum_{k=1}^{\infty} \frac{|g(|z|^2)z^k - g(|z|^2)w^k + g(|z|^2)w^k - g(|w|^2)w^k|}{|z-w|}|c_k| \\
&\qquad + |z-w| \sum_{k=1}^{\infty} \frac{|g(|z|^2)\overline{z}^k - g(|z|^2)\overline{w}^k + g(|z|^2)\overline{w}^k - g(|w|^2)\overline{w}^k|}{|z-w|}|c_{-k}| \\
&\quad \le |z-w| \sum_{k=1}^{\infty}(|c_k|+|c_{-k}|)\Bigg(\sum_{n=0}^{\infty} \frac{(-\frac{\alpha}{2})_n(k-\frac{\alpha}{2})_n}{(k+1)_n n!} \\
&\qquad \times |z|^{2n}(|z|^{k-1}+|z|^{k-2}|w|+\cdots+|w|^{k-1}) \\
&\qquad + \sum_{n=0}^{\infty} \frac{(-\frac{\alpha}{2})_n(k-\frac{\alpha}{2})_n}{(k+1)_n n!}|w|^k(|z|^{2n-1}+|z|^{2n-2}|w|+\cdots+|w|^{2n-1})\Bigg) \\
&\quad \le |z-w| \sum_{k=1}^{\infty}(|c_k|+|c_{-k}|) \sum_{n=0}^{\infty} \frac{(-\frac{\alpha}{2})_n(k-\frac{\alpha}{2})_n}{(k+1)_n n!} \sum_{s=0}^{2n+k-1} |z|^s,
\end{aligned}
$$

and

$$
\begin{aligned}
1-|f(z)| &\ge \sum_{k=1}^{\infty}(|c_k|+|c_{-k}|) \sum_{n=0}^{\infty} \frac{(-\frac{\alpha}{2})_n(k-\frac{\alpha}{2})_n}{(k+1)_n n!} - |f(z)| \\
&\ge \sum_{k=1}^{\infty}(|c_k|+|c_{-k}|)\Big(\sum_{n=0}^{\infty} \frac{(-\frac{\alpha}{2})_n(k-\frac{\alpha}{2})_n}{(k+1)_n n!} - g(|z|^2)|z|^k\Big) \\
&= \sum_{k=1}^{\infty}(|c_k|+|c_{-k}|) \sum_{n=0}^{\infty} \frac{(-\frac{\alpha}{2})_n(k-\frac{\alpha}{2})_n}{(k+1)_n n!}(1-|z|^{2n+k}) \\
&= (1-|z|) \sum_{k=1}^{\infty}(|c_k|+|c_{-k}|) \sum_{n=0}^{\infty} \frac{(-\frac{\alpha}{2})_n(k-\frac{\alpha}{2})_n}{(k+1)_n n!} \sum_{s=0}^{2n+k-1} |z|^s,
\end{aligned}
$$

so that, using the Bernoulli inequality, we have

$$
\begin{aligned}
j_{\mathbb{D}}(f(z), f(w)) &= \log\left(1+\frac{|f(z)-f(w)|}{1-|f(z)|}\right) \\
&\le \log\left(1+\frac{|z-w|}{1-|z|}\right) \\
&\le j_{\mathbb{D}}(z,w).
\end{aligned}
$$

As in [7, Theorem 7], the mapping $f(z) = |z|^{2(p-1)}z^m$ or $f(z) = |z|^{2(p-1)}\overline{z}^m$ for $p, m \geq 1$, shows the sharpness in the last inequality. The proof of the theorem is complete. $\square$

Acknowledgements The authors thank the referee for his/her careful reading and many useful comments. The first author was supported by Centre for International Co-operation in Science (CICS) through the award of "INSA JRD-TATA Fellowship" and was completed during her visit to the Indian Statistical Institute (ISI), Chennai Centre. The research was partly supported by NSF of China (No. 11571216 and No. 11671127). The second author is on leave from the IIT Madras.

Compliance of Ethical Standards

Conflict of interest The authors declare that there is no conflict of interest regarding the publication of this paper.

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