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Existence of Positive Solution for Kirchhoff Problems

K. Ben Ali[1,3] · M. Bezzarga[4] · A. Ghanmi[2,3] · K. Kefi[3,5]

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Abstract In this work, we study the following Kirchhoff type problem

$$-\Big(a + b\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = g(x)u^{-\gamma} + \lambda f(x,u), \quad \text{in } \Omega,$$
$$u = 0, \quad \text{on } \partial\Omega,$$

where $p \geq 2$, Ω is a regular bounded domain in $\mathbb{R}^N$, $(N \geq 3)$. Firstly, for $p > 2$, we prove under some appropriate conditions on the singularity and the nonlinearity the existence of nontrivial weak solution to this problem. For $p = 2$, we show, under

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✉ A. Ghanmi
Abdeljabbar.ghanmi@lamsin.rnu.tn

K. Ben Ali
benali.khaled@yahoo.fr

M. Bezzarga
mounir.bezzarga@yahoo.fr

K. Kefi
khaled_kefi@yahoo.fr

1 Jazan Technical College, P.O. Box 241, Jazan 45952, Saudi Arabia

2 Mathematics Department, Faculty of Sciences and Arts Khulais, University of Jeddah, KSA, Jeddah, Saudi Arabia

3 Mathematics Department, Faculty of Sciences Tunis El Manar, 1060 Tunis, Tunisia

4 Mathematics Department, IPEIT, 2 Rue Jaweher Lel Nehru, 1008 Tunis, Tunisia

5 Community College of Rafha, Northern border university, KSA, Arar, Saudi Arabia

supplementary condition, the positivity of this solution. Moreover, in the case $\lambda = 0$ we prove an uniqueness result. We use the variational method to prove our main results.

Keywords Kirchhoff type equation · Singularity problem · Variational methods · Resonance · Positive solution · Mountain pass lemma

Mathematics Subject Classification 35B09 · 35B33 · 35J20

1 Introduction and Main Results

In this article, we consider the Kirchhoff type problem

$$(\mathbf{P}_{\lambda,p})\begin{cases} -\Big(a+b\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = g(x)u^{-\gamma} + \lambda f(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $p \geq 2$, $\Omega \subset \mathbb{R}^N$, $(N \geq 3)$ is a bounded regular domain, $a, b \geq 0$, $a + b > 0$, $0 < \gamma < 1$ and $\lambda \geq 0$ are parameters.

Note that, the existence and multiplicity of solutions for the following problem

$$\begin{cases} -\Big(a+b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, has been extensively studied (see [1–21]). This type of problem is related to the stationary analogue of the following problem

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ, ρ_0, h, E, and L are constants, which extends the classical d'Alembert wave equation, by considering the effects of the changes in the length of the strings during the vibrations. For more detail we refer the reader to [10] and the references therein. Some important results concerning problem of the form (1.1) are given in [11,12,14,15]. Problems like (1.1) are also introduced as models for other physical phenomena as, biological systems where u describes a process which depends on the average of itself.

An interesting generalization of problem (1.1) is

$$\begin{cases} -\Big(a+b\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = f(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

The problems of this type is important and arises in an interesting physical context. Much interest has grown on singular problems (see for example [12,13,15]).

However, the singular Kirchhoff type problems have few been considered, exept for [9,10]. Before giving our main results let us recall literature concerning related nonlinear equations. Liu and Sun in [10] have investigated problem (1.1) with $f(x,u) = g(x)u^{-\gamma} + \lambda h(x)\frac{u^p}{|x|^s}$, and $g, h \in C(\overline{\Omega})$, $0 \le s < 1$, $3 < p < 5-2s$. They proved that the non-degenerate case of problem (1.1) has at least two positive solutions for $\lambda > 0$ small enough using the Nehari manifold. By the variational methods, they obtained that problem (1.1) has at least two positive solutions for $\mu > 0$ small enough.

Liao et al. [13] considered the following problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = g(x)u^{-\gamma} - \lambda u^p, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \ge 3)$ is a bounded domain, $0 < \gamma < 1$, $\lambda \ge 0$, $0 < p \le p^*-1$ and $a, b \ge 0$, $a + b > 0$ are parameters. The coefficient $g \in L^{\frac{2^*}{2^*+\gamma-1}}(\Omega)$ with $g(x) > 0$ for almost every $x \in \Omega$ and $2^* = \dfrac{2N}{N-2}$ denotes the critical Sobolev exponent. Using the minimax method and some analysis techniques, they obtained the uniqueness of positive solutions for problem (1.2).

Inspired by the above articles, in this paper, we would like to generalize problem (1.2). More precisely, we investigate the existence of solutions for problem $(\mathbf{P}_{\lambda,p})$ by using variational methods. Under some appropriate conditions we prove the positivity and uniqueness of solution in the case $p = 2$.

In the sequel, for Hölder argument reason, we suppose that the function $g \in L^{\frac{p^*}{p^*+\gamma-1}}(\Omega)$ with $g(x) > 0$ for almost every $x \in \Omega$ and $p^* = \dfrac{Np}{N-p}$ denotes the critical Sobolev exponent for the embedding $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$ for $q \in [1, \frac{Np}{N-p}]$. $f \in C\left(\overline{\Omega} \times \mathbb{R}, \mathbb{R}\right)$ is positively homogeneous of degree $r-1$ where $1 < r < p$. more precisely we assume the following:

(H1) $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function such that

$$f(x, tu) = t^{r-1} f(x, u), \ (t > 0) \ \text{ for all } \ x \in \overline{\Omega}, \ u \in \mathbb{R}.$$

(H2) $f(x,t) \ge 0$ in $\Omega_1 \subset\subset \Omega$ such that $|\Omega_1| > 0$.

Note that in the case $1 < r < p$ we have the validity of the coercivity properties of the functional energy associated with the problem $(\mathbf{P}_{\lambda,p})$.

Remark 1.1 (i) A more general condition on f is the Ambrosetti–Rabinowitz condition, but in this case, we can't prove that Lemma 2.2 hold true.

(ii) Put $F(x,s) := \int_0^s f(x,t)dt$, then, assumption (**H1**), f leads to the so-called Euler identity

$$\begin{aligned} uf(x,u) &= rF(x,u), \\ F(x,u) &\le K|u|^r \quad \text{for some constant K,} \end{aligned} \tag{1.3}$$

and $f(x,0) = 0 = (\partial f/\partial t)(x,0)$ for every $t \in \mathbb{R}$.

Our main results are the following:

Theorem 1.2 *Assuming the hypotheses* (**H1**) *and* (**H2**) *holds, then for all* $\lambda \geq 0$, *problem* $(\boldsymbol{P}_{\lambda,p})$ *has at least one non trivial weak solution with negative energy.*

Next, we give tow Theorems concerning the uniqueness and positivity of solution in the special case when $p = 2$.

Theorem 1.3 *For* $\lambda = 0$ *and under the same assumptions of Theorem* 1.2*, the solution given in Theorem* 1.2 *for* $(\boldsymbol{P}_{0,2})$ *is unique.*

Theorem 1.4 *Under the same assumptions of Theorem* 1.2*. If* $\Omega = \Omega_1$*, then, the solution given in Theorem* 1.2 *for* $(\boldsymbol{P}_{\lambda,2})$ *is positive.*

2 Proof of Theorem 1.2

In this case, we consider the Kirchhoff type problem

$$(\mathbf{P}_{\lambda,p}) \begin{cases} -\left(a + b\int_\Omega |\nabla u|^p dx\right)\Delta_p u = g(x)u^{-\gamma} + \lambda f(x,u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

For $u \in W_0^{1,p}(\Omega)$, we define the energy functional associated to the above problem:

$$I_\lambda(u) = \frac{a}{p}\int_\Omega |\nabla u|^p dx + \frac{b}{2p}\left(\int_\Omega |\nabla u|^p dx\right)^2 - \frac{1}{1-\gamma}\int_\Omega g(x)|u|^{1-\gamma}dx - \lambda\int_\Omega F(x,u(x))dx,$$

where $W_0^{1,p}(\Omega)$ is a Sobolev space equipped with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}.$$

Note that a function u is called a weak solution of $(\mathbf{P}_{\lambda,p})$ if $u \in W_0^{1,p}(\Omega)$ satisfies the following:

$$\left(a+b\int_\Omega |\nabla u|^p dx\right)\int_\Omega |\ \nabla u\ |^{p-1}\ .\nabla\varphi dx - \int_\Omega g(x)|u|^{-\gamma}\varphi dx - \lambda\int_\Omega f(x,u)\varphi dx = 0, \tag{2.1}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

In order to prove Theorem 1.2, we show firstly that I_λ attains his global minimizer in $W_0^{1,p}(\Omega)$. For this purpose, we need the following lemmas:

Lemma 2.1 I_λ *is coercive and bounded from below on* $W_0^{1,p}(\Omega)$.

Proof Combining Hölder and Sobolev inequalities, it follows that

$$\begin{aligned}
I_\lambda(u) &= \frac{a}{p}\int_\Omega |\nabla u|^p dx + \frac{b}{2p}\Big(\int_\Omega |\nabla u|^p dx\Big)^2 \\
&\quad - \frac{1}{1-\gamma}\int_\Omega g(x)|u|^{1-\gamma}dx - \lambda\int_\Omega F(x,u(x))dx, \\
&\geq \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{1}{1-\gamma}|g|_{\frac{p^*}{p^*+\gamma-1}}|u|_{p^*}^{1-\gamma} - \lambda K|u|_p^r|\Omega|^{\frac{p-r}{p}}, \\
&\geq \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{C}{1-\gamma}|g|_{\frac{p^*}{p^*+\gamma-1}}\|u\|^{1-\gamma} - \lambda K\|u\|^r|\Omega|^{\frac{p-r}{p}},
\end{aligned}$$

where $C > 0$ is a constant. Since $1 < r < p$, this ends the proof. □

Thus

$$m_\lambda = \inf_{u\in W_0^{1,p}(\Omega)} I_\lambda(u)$$

is well defined. Let us show that, $m_\lambda < 0$.

Lemma 2.2 *There exist $\varphi \in W_0^{1,p}(\Omega)$ such that $\varphi \geq 0, \varphi \not\equiv 0$ and $I_\lambda(t\varphi) < 0$ for $t > 0$ and small enough.*

Proof Let $\varphi \in C_0^\infty(\Omega)$ such that $\text{supp}(\varphi) \subset \Omega_1 \subset\subset \Omega, \varphi = 1$ in a subspace $\Omega' \subset \text{supp}(\varphi)$, $0 \leq \varphi \leq 1$ in Ω, then

$$\begin{aligned}
I_\lambda(t\varphi) &= a\frac{t^p}{p}\|\varphi\|^p + b\frac{t^{2p}}{2p}\|\varphi\|^{2p} - \frac{t^{1-\gamma}}{1-\gamma}\int_\Omega g(x)|\varphi|^{1-\gamma}dx \\
&\quad - \lambda\frac{t^r}{r}\int_{\Omega_1} F(x,\varphi)dx, \\
&\leq t^p\left[\frac{a}{p}\|\varphi\|^p + \frac{b}{2p}\|\varphi\|^{2p}\right] - \frac{t^{1-\gamma}}{1-\gamma}\int_\Omega g(x)|\varphi|^{1-\gamma}dx.
\end{aligned}$$

Consequently, $I_\lambda(t\varphi) < 0$ for $t < \delta^{\frac{1}{p-(1-\gamma)}}$ with

$$0 < \delta < \min\left\{1, \frac{\frac{1}{1-\gamma}\int_\Omega g(x)|\varphi|^{1-\gamma}dx}{\dfrac{a}{p}\|\varphi\|^p + \dfrac{b}{2p}\|\varphi\|^{2p}}\right\}$$

Finally, we point out that $\frac{a}{p}\|\varphi\|^p + \frac{b}{2p}\|\varphi\|^{2p} > 0$. In fact if $\frac{a}{p}\|\varphi\|^p + \frac{b}{2p}\|\varphi\|^{2p} = 0$, then $\varphi = 0$ in Ω which is a contradiction. □

Now, using Lemmas 2.1 and 2.2 one has:

Proposition 2.1 *Suppose that* $0 < \gamma < 1$, $\lambda \geq 0$, $a, b \geq 0$ *with* $a + b > 0$, $g \in L^{\frac{p^*}{p^*+\gamma-1}}(\Omega)$ *with* $g(x) > 0$ *for almost every* $x \in \Omega$ *and assuming the hypotheses* (**H1**) *and* (**H2**) *holds. Then* I_λ *attains his global minimizer in* $W_0^{1,p}(\Omega)$*, that is, there exists* $u_* \in W_0^{1,p}(\Omega)$ *such that* $I_\lambda(u_*) = m_\lambda < 0$.

Proof Let $\{u_n\}$ be a minimizing sequence, that is to say

$$I_\lambda(u_n) \to m_\lambda.$$

Suppose $\{u_n\}$ is not bounded, so

$$\|u_n\| \to +\infty \text{ as } n \to +\infty.$$

Since I_λ is coercive then

$$I_\lambda(u_n) \to +\infty \text{ as } \|u_n\| \to +\infty.$$

This contradicts the fact that $\{u_n\}$ is a minimizing sequence.

So, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore up to a subsequence, there exists $u_* \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_*, & \text{weakly in } W_0^{1,p}(\Omega), \\ u_n \to u_*, & \text{strongly in } L^s(\Omega), \quad 1 \leq s < p^* \\ u_n(x) \to u_*(x), & \text{a.e. in } \Omega, \end{cases} \tag{2.2}$$

Let, $M(t) = a + bt, \ t \geq 0$ and $\hat{M}(t) = \int_0^t M(s)ds$. The function M is positive, so $\hat{M}$ is increasing and the weak convergence of $\{u_n\}$ implies

$$\|u_*\| \leq \liminf_{n\to+\infty} \|u_n\|$$

from where

$$\hat{M}\left(\|u_*\|^p\right) \leq \hat{M}\left(\left(\liminf_{n\to+\infty} \|u_n\|\right)^p\right)$$

Since the function $\hat{M}$ is continuous and $\|u_n\| > 0$ we obtain

$$\hat{M}\left(\left(\liminf_{n\to+\infty} \|u_n\|\right)^p\right) = \hat{M}\left(\liminf_{n\to+\infty} \|u_n\|^p\right) = \liminf_{n\to+\infty} \hat{M}\left(\|u_n\|^p\right).$$

Hence,

$$\hat{M}(\|u_*\|)^p) \leq \liminf_{n\to+\infty} \hat{M}\left(\|u_n\|^p\right) = \liminf_{n\to+\infty} \left(\frac{a}{p}\|u_n\|^p + \frac{b}{2p}\|u_n\|^{2p}\right). \tag{2.3}$$

By Vital's theorem (see [17] pp. 113), we can claim that

$$\lim_{n\to\infty}\int_\Omega g(x)\,|u_n|^{1-\gamma}\,dx = \int_\Omega g(x)|u_*|^{1-\gamma}dx. \tag{2.4}$$

Indeed, we only need to prove that $\{\int_\Omega g(x)\,|u_n|^{1-\gamma}\,dx, n \in \mathbb{N}\}$ is equi-absolutely-continuous. Note that $\{u_n\}$ is bounded, by the Sobolev embedding theorem, there exits a constant $C > 0$, such that $|\, u_n \,|_{p^*} \leq C$. For every $\varepsilon > 0$, by the absolutely-continuity of $\int_\Omega |g(x)|^{\frac{p^*}{p^*+\gamma-1}}dx$, there exists $\delta > 0$ such that

$$\int_\Omega |g(x)|^{\frac{p^*}{p^*+\gamma-1}}dx \leq \varepsilon^{\frac{p^*}{p^*+\gamma-1}} \quad \text{for every} \quad E \subset \Omega \ \text{ with } \ meas E < \delta.$$

Consequently, by the Hölder inequality, we have:

$$\int_\Omega |g(x)|\,|\, u_n \,|^{1-\gamma}\, dx \leq \left|u_n^{1-\gamma}\right|_{p^*}\left(\int_\Omega |g(x)|^{\frac{p^*}{p^*+\gamma-1}}dx\right)^{\frac{p^*+\gamma-1}{p^*}} < C^{1-\gamma}\varepsilon.$$

Thus, claim (2.4) is valid.

Using inequality (1.3) and the Lebesgue dominated convergence theorem, we have:

$$u \mapsto \lambda\int_\Omega F(x, u(x))dx$$

is weakly continuous, then

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{1-\gamma}\int_\Omega g(x)|u_n|^{1-\gamma}dx - \lambda\int_\Omega F(x, u_n(x))dx \\ = \frac{1}{1-\gamma}\int_\Omega g(x)|u_*|^{1-\gamma}dx - \lambda\int_\Omega F(x, u_*(x))dx.\end{aligned} \tag{2.5}$$

Using (2.3), (2.4) and (2.5) we deduce that I_λ is weakly lower semi-continuous and consequently

$$m_\lambda \leq I_\lambda(u_*) \leq \liminf_{n\to+\infty} I_\lambda(u_n) = m_\lambda,$$

then

$$I_\lambda(u_*) = m_\lambda.$$

Similar to the arguments in [7,8,13], we can prove that u_* is a solution of problem $(\mathbf{P}_{\lambda,p})$. This shows that $(\mathbf{P}_{\lambda,p})$ has a negative energy solution. This completes the proof of Proposition 2.1 and Theorem 1.2 □

3 Proof of Theorems 1.3 and 1.4

In this case, we consider the Kirchhoff type problem for $p = 2$

$$(\mathbf{P}_{\lambda,2}) \begin{cases} -\Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = g(x)u^{-\gamma} + \lambda f(x,u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

For $u \in H_0^1(\Omega)$, we define

$$\begin{aligned} I_\lambda(u) = &\frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}\Big(\int_\Omega |\nabla u|^2 dx\Big)^2 \\ &- \frac{1}{1-\gamma}\int_\Omega g(x)|u|^{1-\gamma} dx - \lambda \int_\Omega F(x, u(x))dx, \end{aligned}$$

where $H_0^1(\Omega)$ is the Sobolev space equipped with the norm $\|u\| = \big(\int_\Omega |\nabla u|^2 dx\big)^{1/2}$. Note that a function u is called a weak solution of $(\mathbf{P}_{\lambda,2})$ if $u \in H_0^1(\Omega)$ and satisfies the following:

$$\Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\int_\Omega \nabla u.\nabla\varphi dx - \int_\Omega g(x)|u|^{-\gamma}\varphi dx - \lambda\int_\Omega f(x,u)\varphi dx = 0, \tag{3.1}$$

for all $\varphi \in H_0^1(\Omega)$.

In what follows, it is very important to mention that the Lemmas 2.1, 2.2 and Proposition 2.1 are also true for $p = 2$, so we have automatically the existence result of a solution denoted u_* and we focus the last part of this paper to prove the positivity of this solution when $\Omega = \Omega_1$ and a uniqueness result when $\lambda = 0$.

Remark 3.1 On one hand, to the best of our knowledge, the existence and uniqueness of solutions for problem $(\mathbf{P}_{\lambda,2})$ has not been studied up to now. The results that we obtain in Theorem 1.2 and Theorem 1.3 holds not only for the degenerate case, but also for the non-degenerate case. On the other hand, in references [7–9], problem (1.2) was considered only in dimension $N = 3$. However, we get the existence and uniqueness of solution for problem (1.1) in high dimensions, i.e. $N \geq 3$.

Remark 3.2 When $a = 1, b = 0$, problem $(\mathbf{P}_{\lambda,2})$ reduces to the classic semilinear singular equation. Theorem 1.2 is also true. Moreover, when $\lambda = 0$, our Theorem 1.2 is the corresponding result of [16]. We point out that the condition that $g \in L^{\frac{2^*}{2^*+\gamma-1}}(\Omega)$ is more general than the condition that $g \in L^\infty(\Omega)$ in [16].

3.1 Proof of Theorem 1.3

In this section, we prove that u_* is the unique solution of problem $(\mathbf{P}_{0,2})$. Assume that v_* is another positive solution of problem $(\mathbf{P}_{0,2})$. Since u_*, v_* are positive solutions of problem $(\mathbf{P}_{0,2})$, then it follows from (3.1) that

$$\left(a+b\|u_*\|^2\right)\int_\Omega (\nabla u_*, \nabla(u_*-v_*))\,dx - \int_\Omega g(x)u_*^{-\gamma}(u_*-v_*)dx = 0, \quad (3.2)$$

and

$$\left(a+b\|v_*\|^2\right)\int_\Omega (\nabla v_*, \nabla(u_*-v_*))dx - \int_\Omega g(x)v_*^{-\gamma}(u_*-v_*)dx = 0. \quad (3.3)$$

From (3.2) and (3.3), one obtains

$$\begin{aligned}0 &= \left(a+b\|u_*\|^2\right)\int_\Omega (\nabla u_*, \nabla(u_*-v_*))\,dx - \left(a+b\|v_*\|^2\right)\\ &\quad\times\int_\Omega (\nabla v_*, \nabla(u_*-v_*))\,dx - \int_\Omega g(x)\left(u_*^{-\gamma}-v_*^{-\gamma}\right)(u_*-v_*)\,dx,\\ &= a\|u_*-v_*\|^2 + b\Big[\|u_*\|^4 - \|u_*\|^2\int_\Omega (\nabla u_*, \nabla v_*)\,dx - \|v_*\|^2\\ &\quad\times\int_\Omega (\nabla v_*, \nabla v_*)\,dx + \|v_*\|^4\Big] - \int_\Omega g(x)\left(u_*^{-\gamma}-v_*^{-\gamma}\right)(u_*-v_*)\,dx. \quad (3.4)\end{aligned}$$

Denote

$$J(u_*, v_*) = \|u_*\|^4 - \|u_*\|^2\int_\Omega (\nabla u_*, \nabla v_*)dx - \|v_*\|^2\int_\Omega (\nabla v_*, \nabla v_*)dx + \|v_*\|^4.$$

By the Hölder inequality, one has

$$\begin{aligned}J(u_*, v_*) &\geq \|u_*\|^4 - \|u_*\|^3\|v_*\| - \|v_*\|^3\|u_*\| + \|v_*\|^4\\ &= (\|u_*\|-\|v_*\|)^2\left(\|u_*\|^2 + \|u_*\|\|v_*\| + \|v_*\|^2\right)\\ &\geq 0.\end{aligned}$$

Since $0 < \gamma < 1$, we have the following elementary inequality

$$(h^{-\gamma} - l^{-\gamma})(h-l)) \leq 0 \quad \forall\, h, l > 0.$$

Thus

$$\int_\Omega g(x)\left(u_*^{-\gamma}-v_*^{-\gamma}\right)(u_*-v_*)dx \leq 0.$$

Consequently, it follows from (3.4) that $a\|u_*-v_*\|^2 \leq 0$. If $a = 0$, one has $\|u_*\| = \|v_*\|$ and $J(u_*, v_*) = 0$. As a result,

$$J(u_*, v_*) = \|u_*\|^2\left(2\|u_*\|^2 - 2\int_\Omega (\nabla u_*, \nabla v_*)dx\right) = \|u_*\|^2\|u_*-v_*\|^2 = 0,$$

this implies $\|u_*-v_*\|^2 = 0$.

Thus, for every $a \geq 0$, one has $u_* = v_*$. Therefore u_* is the unique solution of problem $(\mathbf{P}_{\lambda,2})$. This completes the proof of Theorem 1.3.

3.2 Proof of Theorem 1.4

In what follows and without loss of generality, let us assume that $\Omega = \Omega_1$ and $u_* \geq 0$, we can prove the following:

Proposition 3.3 *If the set Ω_1 given by* **(H2)** *is such that $\Omega = \Omega_1$, then the solution u_* of problem $(\boldsymbol{P}_{\lambda,2})$ is nonnegative.*

Proof From Proposition 2.1, we have $I_\lambda(u_*) = m_\lambda < 0$, so $u_* \geq 0$ and $u_* \not\equiv 0$ in Ω. We prove that $u_*(x) > 0$ for almost every $x \in \Omega$. Since $u_*(x) \geq 0$ for all $x \in \Omega$, then $\forall\ \phi \in H_0^1(\Omega)$, $\phi \geq 0$, and $t > 0, t \in \mathbb{R}$, such that $u_* + t\phi \in H_0^1(\Omega)$, we have the following

$$\begin{aligned}
0 &\leq \liminf_{t\to 0} \frac{I_\lambda(u_* + t\phi) - I_\lambda(u_*)}{t} \\
&= a\int_\Omega (\nabla u_*, \nabla\phi)dx + b\|u_*\|^2 \int_\Omega (\nabla u_*, \nabla\phi)dx \\
&\quad - \limsup_{t\to 0^+} \frac{1}{1-\gamma}\int_\Omega g(x)\frac{\left((u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}\right)}{t}dx \\
&\quad - \lim_{t\to 0^+} \lambda \int_\Omega \frac{F(x, u_* + t\phi) - F(x, u_*)}{t}dx. \qquad (3.5)
\end{aligned}$$

Obviously, one gets

$$\int_\Omega \frac{F(x, u_* + t\phi) - F(x, u_*)}{t}dx = \int_\Omega f(x, u_* + t\eta\phi)\phi dx,$$
$$\int_\Omega \frac{\left((u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}\right)}{t}dx = (1-\gamma)\int_\Omega (u_* + t\phi)^{-\gamma}\phi dx,$$

where $0 < \eta < 1, \theta < 1$, and

$$f(x, u_* + \eta t\phi)\phi \to f(x, u_*)\phi, \qquad a.e.\ x \in \Omega,$$
$$(u_* + \theta t\phi)^{-\gamma}\phi \to u_*^{-\gamma}\phi, \qquad a.e.\ x \in \Omega,$$

as $t \to 0^+$.

For any $x \in \Omega$, put $h(t) = g(x)\dfrac{[u_*(x) + t\phi(x)]^{1-\gamma} - u_*^{1-\gamma}(x)}{(1-\gamma)t}$. then $h'(t) = g(x)\dfrac{u_*^{1-\gamma}(x) - [\gamma t\phi(x) + u_*(x)][u_*(x) + t\phi(x)]^{-\gamma}}{(1-\gamma)t^2} \leq 0$, which implies that h is non-increasing on $(0, \infty)$. Moreover, one has $\lim_{t\to 0^+} h(t) = g(x)u_*^{1-\gamma}(x)\phi(x)$ for

$x \in \Omega$, which may be $+\infty$ when $u_*(x) = 0$ and $\phi(x) > 0$. Consequently, by the Monotone Convergence Theorem (Beppo-Levi), we obtain

$$\lim_{t\to 0^+} \frac{1}{1-\gamma} \int_\Omega g(x) \frac{\Big((u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}\Big)}{t} dx = \int_\Omega g(x) u_*^{-\gamma} \phi dx, \tag{3.6}$$

which possibly equals to $+\infty$.

Moreover, using Lebesgue's dominated convergence theorem on the function f, one has:

$$\begin{aligned}\lim_{t\to 0^+} \int_\Omega \frac{F(x, u_* + t\phi) - F(x, u_*)}{t} dx &= \int_\Omega \lim_{t\to 0^+} f(x, u_* + t\phi)\phi dx \\ &= \int_\Omega f(x, u_*)\phi dx. \end{aligned} \tag{3.7}$$

Then, from (3.5),(3.6) and (3.7), we obtain

$$\int_\Omega g(x) u_*^{-\gamma} \phi dx \le \Big(a + b\|u_*\|^2\Big) \int_\Omega (\nabla u_*, \nabla \phi) dx - \int_\Omega f(x, u_*)\phi dx, \tag{3.8}$$

for all $\phi \in H_0^1(\Omega)$ with $\phi > 0$.

Thus, one has

$$\int_\Omega (\nabla u_*, \nabla \phi) dx \ge 0.$$

Since $u_* \ge 0$ and $u_* \not\equiv 0$, by strong maximum principal, it follows that

$$u_* > 0, \quad \forall x \in \Omega.$$

This completes the proof of Theorem 1.4. □

4 An Example

In this section, we give an example to illustrate our results. To this aim, we fix $p \ge 2$ and a bounded domain $\Omega \subset \mathbb{R}^3$. Let $g \in L^{\frac{p^*}{p^*+\gamma-1}}(\Omega)$ such that for almost every $x \in \Omega$ we have $g(x) > 0$. We consider the following elliptic problem

$$(\mathbf{P}_{\lambda,p}) \begin{cases} -\Big(a + b\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = g(x)u^{-\gamma} + \lambda h(x)|u(x)|^{r-2}u(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $0 < \gamma < 1 < r < p$, $a, b \ge 0$ such that $a + b > 0$ and h be a positive bounded function in Ω. It is easy to see that $f(x,t) = h(x)|t|^{r-2}t$ is positively homogeneous of degree $r - 1$, moreover, by a simple calculation we obtain $F(x,t) = h(x)|t|^r$ which is positively homogeneous of degree r. That is ($\mathbf{H}_1$) is satisfied. On the other hand, since $h > 0$, it is easy to see that for all $x \in \Omega$, we have $F(x,t) = h(x)|t|^r > 0$, that is ($\mathbf{H}_2$) is satisfied and $\Omega_1 = \Omega$. Hence, all conclusions of Theorems 1.2, 1.3 and 1.4 hold true.

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