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Compactness of Hankel Operators with Continuous Symbols

Timothy G. Clos[1] · **Sönmez Şahutoğlu**[1]

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Abstract Let Ω be a bounded convex Reinhardt domain in $\mathbb{C}^2$ and $\phi \in C(\overline{\Omega})$. We show that the Hankel operator H_ϕ is compact if and only if ϕ is holomorphic along every non-trivial analytic disc in the boundary of Ω.

Keywords Hankel operator · Reinhardt · Compact · Convex

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Let Ω be a domain in $\mathbb{C}^n$ and let $L^2(\Omega)$ and $A^2(\Omega)$ denote square integrable functions on Ω and the Bergman space on Ω (the set of square integrable holomorphic functions on Ω), respectively. Since $A^2(\Omega)$ is a closed subspace in $L^2(\Omega)$ the Bergman projection $P : L^2(\Omega) \to A^2(\Omega)$, the orthogonal projection, exists. Furthermore, let $H_\phi f = (I - P)(\phi f)$ for all $f \in A^2(\Omega)$ and $\phi \in L^\infty(\Omega)$. We note that H_ϕ is called the Hankel operator with symbol ϕ. We refer the reader to [9,11] and references there in for more information on these operators.

Hankel operators form an active research area in operator theory. Our interest lies in their compactness properties in relation to the behavior of the symbols on the boundary of the domain. On the unit disc $\mathbb{D}$ in $\mathbb{C}$ Axler [1] showed that, for f holomorphic on the unit disc $\mathbb{D}$, the Hankel operator $H_{\overline{f}}$ is compact on $A^2(\mathbb{D})$ if and only if f is in

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✉ Sönmez Şahutoğlu
Sonmez.Sahutoglu@utoledo.edu

Timothy G. Clos
Timothy.Clos@rockets.utoledo.edu

1 Department of Mathematics and Statistics, University of Toledo, Toledo, OH 43606, USA

the little Bloch space (that is, $(1-|z|^2)|f'(z)| \to 0$ as $|z| \to 1$). This result has been extended into higher dimensions by Peloso [8] in case the domain is smooth bounded and strongly pseudoconvex. The same year, Li [7] characterized bounded and compact Hankel operators on strongly pseudoconvex domains for symbols that are square integrable only. Recently, Čučković and Şahutoğlu [2, Theorem 3] gave a characterization for compactness of Hankel operators on smooth bounded convex domains in $\mathbb{C}^2$ with symbols smooth up to the boundary. We note that even though they stated their result for smooth domains and smooth symbols on the closure, examination of the proof shows that C^1-smoothness of the domain and the symbol is sufficient. They proved the following theorem.

Theorem (Čučković–Şahutoğlu) *Let Ω be a C^1-smooth bounded convex domain in $\mathbb{C}^2$ and $\phi \in C^1(\overline{\Omega})$. Then the Hankel operator H_ϕ is compact on $A^2(\Omega)$ if and only if $\phi \circ f$ is holomorphic for any holomorphic function $f : \mathbb{D} \to b\Omega$.*

In this paper we prove a similar result with symbols that are only continuous up to the boundary. The first result in this direction was proven by Le in [6]. He showed that for $\Omega = \mathbb{D}^n$, the polydisc in $\mathbb{C}^n$, and $\phi \in C(\overline{\Omega})$, the Hankel operator H_ϕ is compact on $A^2(\Omega)$ if and only if $\phi = f + g$ where f and g are continuous on $\overline{\Omega}$, $f = 0$ on $b\Omega$, and g is holomorphic on Ω. We prove the following theorem, generalizing Le's result in $\mathbb{C}^2$.

Theorem 1 *Let Ω be a bounded convex Reinhardt domain in $\mathbb{C}^2$ and $\phi \in C(\overline{\Omega})$. Then the Hankel operator H_ϕ is compact on $A^2(\Omega)$ if and only if $\phi \circ f$ is holomorphic for any holomorphic function $f : \mathbb{D} \to b\Omega$.*

We note that in the theorem above there is no regularity restriction on the domain, but the class of domains is smaller than the one considered in [2]. It would be interesting to know if the same result is still true on convex domains in $\mathbb{C}^n$.

Proof of Theorem 1

Let us start by some notation. We denote

$$\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}, S_r = \{z \in \mathbb{C} : |z| = r\},$$
$$A(0, \delta_1, \delta_2) = \{z \in \mathbb{C} : \delta_1 < |z| < \delta_2\}$$

for $r, \delta_1, \delta_2 > 0$.

In the next lemma we prove that any analytic disc $\Delta_0 \subset b\Omega$ is contained in a disc that intersects the coordinate axis. This allows us to simplify the problem for convex Reinhardt domains, since any disc in $b\Omega$ must be horizontal or vertical.

Lemma 1 *Let Ω be a bounded convex Reinhardt domain in $\mathbb{C}^2$ and $\Delta \subset b\Omega$ be a non-trivial analytic disc. Then there exists $r > 0$ and $p \in \mathbb{C}$ such that either $\Delta \subset \mathbb{D}_r \times \{p\} \subset b\Omega$ or $\Delta \subset \{p\} \times \mathbb{D}_r \subset b\Omega$.*

Proof Suppose that $F(\mathbb{D}) = \Delta$ is a non-trivial disc in $b\Omega$ where $F(\xi) = (f(\xi), g(\xi))$. Then either $f'g' \equiv 0$ or there exists $\xi_0 \in \mathbb{D}$ such that $f'(\xi_0)g'(\xi_0) \neq 0$. In case

$f'g' \equiv 0$, by identity principle, we conclude that either $f' \equiv 0$ or $g' \equiv 0$. That is, either f or g is constant.

On the other hand, if $f'(\xi_0)g'(\xi_0) \neq 0$ then the disc Δ is a smooth complex curve in a neighborhood $F(\xi_0)$. Furthermore, the fact that Ω is Reinhardt domain in $\mathbb{C}^2$ implies that $b\Omega$ is smooth locally in a neighborhood of $F(\xi_0)$. This can be seen as follows: Without loss of generality we assume that $f(\xi_0) \neq 0$. Let $\xi_0 = x_0 + iy_0$ and

$$G(x, y, \theta) = (e^{i\theta} f(x+iy), g(x+iy)).$$

Then one can show that the image of G is a smooth surface in $\mathbb{C}^2$ near $G(\xi_0, 0) = F(\xi_0)$ as the Jacobian of G is of rank 3 at $(\xi_0, 0)$. Since $b\Omega$ is a 3 dimensional surface we conclude that the boundary of Ω is smooth near near $F(\xi_0)$ as it can be seen as the image of $G(x, y, \theta)$. Then we can apply [3, Lemma 2] (since $b\Omega$ is smooth near $F(\xi_0)$) and use the identity principle to conclude that either f or g is constant. We reach a contradiction with the assumption that $f(\xi_0) \neq 0$. Therefore, either Δ is flat and horizontal (g is constant) or flat and vertical (f is constant).

For the rest of the proof, without loss of generality, we assume that Δ is horizontal. There exists $p \in \mathbb{C}$, $\delta_1 > 0$, and $\delta_2 > 0$ such that

$$A(0, \delta_1, \delta_2) \times \{p\} \subset b\Omega.$$

The assumption that Ω is convex and Reinhardt implies that Ω is complete. So,

$$\{(z, w) \in \mathbb{C}^2 : |z| \leq \delta_2, |w| \leq |p|\} \subset \overline{\Omega}. \tag{1}$$

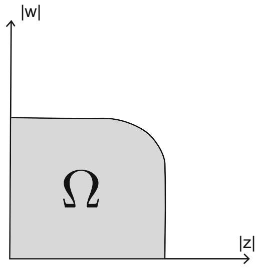
Next, we will show that $\{(z, w) \in \mathbb{C}^2 : |z| \leq \delta_1, |w| > |p|\} \cap \Omega = \emptyset$. Suppose that there exists $(z_0, w_0) \in \{(z, w) \in \mathbb{C} : |z| \leq \delta_1, |w| > |p|\} \cap \Omega$ and let $z \in \mathbb{C}$ such that $|z| = \delta_2$. We choose $\lambda > 0$ small enough such that $(|z| - \lambda, |p| - \lambda) \in \Omega$ and the line segment joining $(|z| - \lambda, |p| - \lambda)$ with (z_0, w_0), called L_1, is such that

$$L_1 \cap (A(0, \delta_1, \delta_2) \times \{|p|e^{i\theta} : 0 \leq \theta \leq 2\pi\}) \neq \emptyset.$$

However, since $\overline{A(0, \delta_1, \delta_2)} \times \{|p|e^{i\theta} : 0 \leq \theta \leq 2\pi\} \subset b\Omega$, we conclude $L_1 \cap b\Omega \neq \emptyset$. Since the initial and terminal points of L_1 lie in Ω and Ω is convex, we arrive at a contradiction. This shows that $\{(z, w) \in \mathbb{C}^2 : |z| \leq \delta_1, |w| > |p|\} \cap \Omega = \emptyset$. Combining this with (1) we conclude that $\{(z, w) \in \mathbb{C}^2 : |z| \leq \delta_2, |w| = |p|\} \subset b\Omega$. □

We take this opportunity to correct a typo in [3, Lemma 2]. In the statement of the lemma, the word "complete" should be "convex". The lemma is proven for the correct domains: piecewise smooth bounded convex Reinhardt domains in $\mathbb{C}^2$.

Remark 1 Lemma 1 implies that if $\Omega \subset \mathbb{C}^2$ is a bounded convex Reinhardt domain, then any horizontal analytic disc in $b\Omega$ is contained in $\mathbb{D}_r \times S_q$ for some $r > 0$ and $q > 0$. Likewise, any vertical analytic disc in $b\Omega$ is contained in $S_{q'} \times \mathbb{D}_{r'}$ for some $r' > 0$ and $q' > 0$.



As in [3] we represent a complete Reinhardt domain $\Omega \subset \mathbb{C}^2$ as union of horizontal slices. In other words, let H_Ω be an open disc in $\mathbb{C}$ such that

$$\Omega = \bigcup_{w \in H_\Omega} \Delta_w \times \{w\} \tag{2}$$

where $\Delta_w = \{z \in \mathbb{C} : |z| < r_w\}$ is the slice of Ω at w level. That is, $(z, w) \in \Omega$ if and only if $|z| < r_w$.

Lemma 2 ([3]) *Let $\phi \in C(\mathbb{C})$ and $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Then*

$$\|H_\phi^{\mathbb{D}_r} f\|_{L^2(\mathbb{D}_r)} \to \|H_\phi^{\mathbb{D}_{r_0}} f\|_{L^2(\mathbb{D}_{r_0})}$$

as $r \to r_0$.

Lemma 1, Lemma 2, and [3, Lemma 3] imply the following corollary.

Corollary 1 *Let Ω be a bounded convex Reinhardt domain in $\mathbb{C}^2$, $\phi \in C(\mathbb{C})$, and $\Delta_{w_0} \times \{w_0\}$ be a non-trivial analytic disc in $b\Omega$ where $w_0 \in bH_\Omega$. Then*

$$\lim_{H_\Omega \ni w \to w_0} \|H_\phi^{\Delta_w}(1)\|_{L^2(\Delta_w)} = \|H_\phi^{\Delta_{w_0}}(1)\|_{L^2(\Delta_{w_0})}.$$

Lemma 3 *Let Ω be a bounded convex Reinhardt domain in $\mathbb{C}^2$ and $\phi \in C(\overline{\Omega})$. Furthermore, let $w_0 \in bH_\Omega$ and $\phi_0(z, w) = \phi(z, w_0)$. Assume that H_ϕ is compact on $A^2(\Omega)$ and $\{g_j\}$ is a bounded sequence in $A^2(H_\Omega)$ such that $g_j \to 0$ uniformly on $H_\Omega \setminus V$ as $j \to \infty$ for any open set V containing w_0. Then $H_{\phi_0} g_j \to 0$ as $j \to \infty$.*

Proof We note that $g_j \to 0$ weakly in $A^2(\Omega)$ as $j \to 0$. Hence, by compactness of H_ϕ we have $\|H_\phi g_j\|_{L^2(\Omega)} \to 0$ as $j \to \infty$. Now, we write

$$\|H_{\phi_0} g_j\|_{L^2(\Omega)} \leq \|H_{\phi - \phi_0} g_j\|_{L^2(\Omega)} + \|H_\phi g_j\|_{L^2(\Omega)}.$$

So, we just consider the first term on the right hand side of the above inequality. Since $\{g_j\}$ is a bounded sequence, there exists $M > 0$ such that $\|g_j\|^2_{L^2(\Omega)} \leq M$ for all $j \in \mathbb{N}$. Furthermore, since $\phi - \phi_0$ is continuous on $\overline{\Omega}$ and $\phi - \phi_0 = 0$ on $\overline{\Delta_{w_0}}$ for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup\{|\phi(z, w) - \phi_0(z, w)|^2 : (z, w) \in \overline{\Omega}, |w - w_0| \leq \delta\} < \frac{\varepsilon}{2M}.$$

We note that, below, $V(\Omega)$ denotes the volume of Ω with respect to Lebesgue measure.

$$\begin{aligned}\|H_{\phi-\phi_0}g_j\|^2_{L^2(\Omega)} &\leq \|(\phi-\phi_0)g_j\|^2_{L^2(\{(z,w)\in\Omega:|w-w_0|\leq\delta\})}\\ &\quad + V(\Omega)\|(\phi-\phi_0)g_j\|^2_{L^\infty(\{(z,w)\in\Omega:|w-w_0|>\delta\})}\\ &< \frac{\varepsilon}{2} + V(\Omega)\|(\phi-\phi_0)g_j\|^2_{L^\infty(\{(z,w)\in\Omega:|w-w_0|>\delta\})}.\end{aligned}$$

Since $(\phi-\phi_0)\in C(\overline{\Omega})$ and $g_j \to 0$ uniformly on $\{(z,w)\in\Omega : |w-w_0|>\delta\}$ as $j\to\infty$, we conclude that for any $\delta,\varepsilon>0$ there exists $j_0\in\mathbb{N}$ such that

$$V(\Omega)\|(\phi-\phi_0)g_j\|^2_{L^\infty(\{(z,w)\in\Omega:|w-w_0|>\delta\})} < \frac{\varepsilon}{2}$$

for $j\geq j_0$. Therefore,

$$\|H_{\phi-\phi_0}g_j\|^2_{L^2(\Omega)} < \varepsilon$$

for $j\geq j_0$ and the proof of the lemma is complete. □

Before we state the next lemma some explanation about the notation is in order. We think of the operators as defined on spaces on Ω unless the domain is indicated as a superscript. For instance, for an open subset V of Ω the operators H^V_ϕ and P^V are defined on $A^2(V)$ and $L^2(V)$, respectively; whereas, H_ϕ and P are defined on $A^2(\Omega)$ and $L^2(\Omega)$, respectively. Furthermore, in the next two lemmas, we think of ϕ as a function of z (or as a function of (z,w) but independent of w). For instance, ϕ is a function of z in $H^{\Delta_w}_\phi$ and a function (z,w) (but independent of w) in H_ϕ.

The following lemma is a special case of equation (3) in [3, pg. 637] for $\phi=\psi_0=\phi_0$ and $f_1=f_2\equiv 1$.

Lemma 4 ([3]) *Let Ω be a bounded convex Reinhardt domain in $\mathbb{C}^2$ and $\phi\in C(\overline{\Omega})$ such that $\phi(z,w)=\phi(z,0)$ for $(z,w)\in\Omega$. Then*

$$\begin{aligned}\|H_\phi g\|^2_{L^2(\Omega)} &= \int_{H_\Omega}|g(w)|^2\int_{\Delta_w}|H^{\Delta_w}_\phi(1)(z)|^2 dV(z)dV(w)\\ &\quad + \int_\Omega (H_\phi g)(z,w)\overline{P^{\Delta_w}(\phi)(z)g(w)}dV(z,w)\end{aligned}$$

for $g\in A^2(H_\Omega)$

Similarly the following lemma is included in [3, pg 640] again for $\phi=\psi_0=\phi_0$ and $f_1=f_2\equiv 1$.

Lemma 5 ([3]) *Let Ω be a bounded convex Reinhardt domain in $\mathbb{C}^2$ and $\phi\in C(\overline{\Omega})$ such that $\phi(z,w)=\phi(z,0)$ for $(z,w)\in\Omega$. Assume that $\{g_j\}$ is a bounded sequence in $A^2(H_\Omega)$ such that $g_j\to 0$ uniformly on $\overline{H}_\Omega\backslash V$ for any open set V containing $w_0\in H_\Omega$. Then*

$$\int_\Omega (H_\phi g_j)(z,w)\overline{P^{\Delta_w}(\phi)(z)g_j(w)}dV(w,z)\to 0 \quad \textit{as } j\to\infty.$$

The next lemma allows us to approximate the symbol with smooth appropriate symbols. We define $\Gamma_\Omega \subset b\Omega$ to be the closure of the union of all non-trivial analytic discs in $b\Omega$. That is,

$$\Gamma_\Omega = \overline{\bigcup\{f(\mathbb{D}) : f : \mathbb{D} \to b\Omega \text{ is non-constant holomorphic mapping}\}}. \quad (3)$$

Lemma 6 *Let Ω be a bounded convex Reinhardt domain in $\mathbb{C}^2$ that is not the product of two discs. Assume that $\Gamma_\Omega \neq \emptyset$ and $\phi \in C(\overline{\Omega})$ such that $\phi \circ f$ is holomorphic for any holomorphic function $f : \mathbb{D} \to b\Omega$. Then there exists $\{\psi_n\} \subset C^\infty(\overline{\Omega})$ such that*

i. *$\psi_n \circ f$ is holomorphic for all n and for any holomorphic function $f : \mathbb{D} \to b\Omega$,*
ii. *$\|\psi_n - \phi\|_{L^\infty(\Gamma_\Omega)} \to 0$ as $n \to \infty$.*

Proof Let $\Delta_1 = \mathbb{D}_{r_1} \times S_{s_1}$ be the family of horizontal analytic discs in $b\Omega$ as outlined in Lemma 1. Then for $0 < r < 1$ we define

$$\phi_r(z, w) = \phi(rz, w).$$

Since $\phi \in C(\overline{\Omega})$, one can show that

$$\phi_r \to \phi \text{ uniformly on } \overline{\Omega} \text{ as } r \to 1^-.$$

We consider ϕ, restricted to $\overline{\Delta_1}$, to be a function of (z, θ) for $z \in \overline{\mathbb{D}}_{r_1}$ and periodic in $\theta \in \mathbb{R}$ with period 2π. By assumption, the function $\phi_r(., \theta)$ is holomorphic on a neighborhood of $\overline{\mathbb{D}}_{r_1}$ for every $\theta \in \mathbb{R}$. Let $\gamma \in C_0^\infty((-1, 1))$ be such that $\gamma \geq 0$ and $\int_{-1}^1 \gamma(\theta)d\theta = 1$. Similarly, let $\chi \in C_0^\infty(\mathbb{D}_{r_1})$ be such that $\chi \geq 0$ and $\int_{\mathbb{D}_{r_1}} \chi(z)dV(z) = 1$. Now, we define $\gamma_\delta(\theta) = \delta^{-1}\gamma(\theta/\delta)$ and $\chi_\varepsilon(z) = \varepsilon^{-2}\chi(z/\varepsilon)$. Notice that $\{\gamma_\delta\}_{\delta>0}$ and $\{\chi_\varepsilon\}_{\varepsilon>0}$ are approximate identities. We define the convolution

$$C^\phi_{r,\varepsilon}(z, \theta) = \int_{-\pi}^{\pi} \int_{\mathbb{D}_{r_1}} \phi(r(z - \alpha), (\theta - \theta'))\chi_\varepsilon(\alpha)\gamma_\varepsilon(\theta')dV(\alpha)d\theta'.$$

One can show that for $\varepsilon > 0$ sufficiently small (depending on r) the function $C^\phi_{r,\varepsilon}(., \theta)$ is holomorphic on a neighborhood of $\overline{\mathbb{D}}_{r_1}$ for every $\theta \in \mathbb{R}$. Also the assumption that $\phi \in C(\overline{\Omega})$ implies that

$$C^\phi_{r,\varepsilon} \to \phi_r \text{ uniformly on } \overline{\Delta_1} \text{ as } \varepsilon \to 0^+$$

for all $0 < r < 1$. Therefore, the functions $C^\phi_{r,\varepsilon}$ are holomorphic “along” horizontal analytic discs in $b\Omega$ for small $\varepsilon > 0$. Now, we extend $C^\phi_{r,\varepsilon}$ as a C^∞-smooth function onto $\overline{\Omega}$ and call this extension $\widetilde{C}^\phi_{r,\varepsilon}$.

If $b\Omega$ contains non-trivial vertical analytic discs Δ_2 then we can use a similar construction on Δ_2. That is, using the regularization procedure outlined above in this proof, we can construct a collection of functions $\widetilde{B}^{\phi}_{r,\varepsilon} \in C^{\infty}(\overline{\Omega})$ such that $\widetilde{B}^{\phi}_{r,\,\varepsilon}$ are holomorphic "along" any vertical analytic disc in Δ_2 for small $\varepsilon > 0$ and

$$\widetilde{B}^{\phi}_{r,\varepsilon} \to \phi_r \text{ uniformly on } \overline{\Delta_2} \text{ as } \varepsilon \to 0^+$$

for all $0 < r < 1$. Since Ω is not the product of discs, (hence $\overline{\Delta_1} \cap \overline{\Delta_2} = \emptyset$), there exists open sets F and G such that $\overline{\Delta_1} \subset F$, $\overline{\Delta_2} \subset G$, and $\overline{F} \cap \overline{G} = \emptyset$. Then we choose $\chi_F, \chi_G \in C^{\infty}_0(\mathbb{C}^2)$ such that $0 \leq \chi_G, \chi_F \leq 1$, $\chi_G \equiv 1$ on G, $\chi_F \equiv 1$ on F, and $\chi_F + \chi_G \equiv 1$ on $\overline{\Omega}$.

We define

$$\phi^{r,\,\varepsilon} = \chi_F \widetilde{C}^{\phi}_{r,\,\varepsilon} + \chi_G \widetilde{B}^{\phi}_{r,\,\varepsilon}. \tag{4}$$

By construction, $\chi_F \equiv 0$ on G and $\chi_G \equiv 0$ on F. Furthermore, $\widetilde{C}^{\phi}_{r,\,\varepsilon}$ is holomorphic along Δ_1, and $\widetilde{B}^{\phi}_{r}$ is holomorphic along Δ_2 for small $\varepsilon > 0$. For $n = 1, 2, \ldots$ we choose $r_n = (n-1)/n$ and $\varepsilon_n \to 0^+$ so that

i. $\phi^{r_n,\,\varepsilon_n} \circ h$ is holomorphic for all n and every holomorphic $h : \mathbb{D} \to b\Omega$,
ii. $\phi^{r_n,\,\varepsilon_n} \to \phi$ uniformly on Γ_Ω as $n \to \infty$.

Finally, we finish the proof by defining $\psi_n = \phi^{r_n,\,\varepsilon_n}$. □

Let X and Y be two normed linear spaces and $T : X \to Y$ be a bounded linear operator. We define the essential norm of T, denoted by $\|T\|_e$, as

$$\|T\|_e = \inf\{\|T - K\| : K : X \to Y \text{ is a compact operator}\}$$

where $\|\cdot\|$ denotes the operator norm.

Lemma 7 *Let Ω be a bounded convex domain in $\mathbb{C}^n$ and $\Gamma_\Omega \neq \emptyset$ be defined as in* (3). *Assume that $\{\phi_n\} \subset C(\overline{\Omega})$ is a sequence such that $\phi_n \to 0$ uniformly on Γ_Ω as $n \to \infty$. Then $\lim_{n\to\infty} \|H_{\phi_n}\|_e = 0$.*

Proof Let $\varepsilon > 0$. Then there exists N such that $\sup\{|\phi_n(z, w)| : (z, w) \in \Gamma_\Omega\} < \varepsilon$ for $n \geq N$. For $n \geq N$ we choose an open neighborhood $U_{n,\,\varepsilon}$ of Γ_Ω such that $|\phi_n(z, w)| < \varepsilon$ for $(z, w) \in U_{n,\,\varepsilon}$. Furthermore, we choose a smooth cut-off function $\chi_{n,\,\varepsilon} \in C^{\infty}_0(U_{n,\,\varepsilon})$ such that $0 \leq \chi_{n,\,\varepsilon} \leq 1$ and $\chi_{n,\,\varepsilon} = 1$ on a neighborhood of Γ_Ω.

Let us define

$$\phi_{1,\,n,\,\varepsilon} = \chi_{n,\,\varepsilon}\phi_n \quad \text{and} \quad \phi_{2,\,n,\,\varepsilon} = (1 - \chi_{n,\,\varepsilon})\phi_n.$$

Then $\phi_n = \phi_{1,\,n,\,\varepsilon} + \phi_{2,\,n,\,\varepsilon}$ and $|\phi_{1,\,n,\,\varepsilon}| < \varepsilon$ on $\overline{\Omega}$ while $\phi_{2,\,n,\,\varepsilon} = 0$ on a neighborhood of Γ_Ω in $\overline{\Omega}$. Furthermore,

$$\|H_{\phi_{1,\,n,\,\varepsilon}}\|_e \leq \|H_{\phi_{1,\,n,\,\varepsilon}}\| \leq \sup\{|\phi_{1,\,n,\,\varepsilon}(z, w)| : (z, w) \in \overline{\Omega}\} < \varepsilon.$$

Next we will show that $H_{\phi_{2,\,n,\,\varepsilon}}$ is compact. Since $\phi_{2,\,n,\,\varepsilon} = 0$ on a neighborhood of Γ_Ω in $\overline{\Omega}$, using convolution with approximate identity, one can choose $\{\psi_{k,\,n,\,\varepsilon}\} \subset C^{\infty}(\overline{\Omega})$ such that $\psi_{k,\,n,\,\varepsilon} = 0$ on a neighborhood of Γ_Ω in $\overline{\Omega}$ for all k and $\psi_{k,\,n,\,\varepsilon} \to$

$\phi_{2,n,\varepsilon}$ uniformly on $\overline{\Omega}$ as $k \to \infty$. We choose finitely many open balls $U_j = B(p_j, r_j)$ for $j = 1, \ldots, N$ such that $\Gamma_\Omega \subset \cup_{j=1}^N U_j$, $p_j \in \Gamma_\Omega$, and $\psi_{k,n,\varepsilon} = 0$ on U_j for all j. Then we cover $b\Omega \setminus \cup_{j=1}^N U_j$ by finitely many open balls $U_j = B(p_j, r_j)$ for $j = N+1, \ldots, M$ such that $p_j \in b\Omega$ and $U_j \cap \Gamma_\Omega = \emptyset$ for $j = N+1, \ldots, M$.

Below R_V denotes the restriction operator onto $V \subset \Omega$. That is, $R_V f = f|_V$ for $f \in A^2(\Omega)$. We note that $U_j \cap \Omega$ is a bounded convex domain with no analytic disc in the boundary for all $j = N+1, \ldots, M$. Then [4, Theorem 1.1] (see also [10, Theorem 4.26]) implies that the $\overline{\partial}$-Neumann operator on $U_j \cap \Omega$ is compact (for $j = N+1, \ldots, M$) and [10, Proposition 4.1], in turn, implies that the Hankel operator $H^{U_j \cap \Omega}_{R_{U_j \cap \Omega}(\psi_{k,n,\varepsilon})} R_{U_j \cap \Omega}$ is compact for $j = N+1, \ldots, M$.

Therefore, we have chosen finitely many balls $U_j = B(p_j, r_j)$ for $j = 1, \ldots, M$ such that

i. $p_j \in b\Omega$ and $b\Omega \subset \cup_{j=1}^M U_j$,
ii. the operator $H^{U_j \cap \Omega}_{R_{U_j \cap \Omega}(\psi_{k,n,\varepsilon})} R_{U_j \cap \Omega} = 0$ for $p_j \in \Gamma_\Omega$,
iii. the operator $H^{U_j \cap \Omega}_{R_{U_j \cap \Omega}(\psi_{k,n,\varepsilon})} R_{U_j \cap \Omega}$ is compact for $p_j \notin \Gamma_\Omega$.

So, the local Hankel operators $H^{U_j \cap \Omega}_{R_{U_j \cap \Omega}(\psi_{k,n,\varepsilon})} R_{U_j \cap \Omega}$ are compact for all $j = 1, \ldots, M$. Now we use [2, Proposition 1, (ii)] to conclude that $H_{\psi_{k,n,\varepsilon}}$ is compact. Hence $H_{\phi_{2,n,\varepsilon}}$ is compact and $\|H_{\phi_n}\|_e \le \varepsilon$ for $n \ge N$. Therefore, $\lim_{n\to\infty} \|H_{\phi_n}\|_e = 0$. □

We will now show one implication of the main theorem on non-product domains if the symbol is smooth up to the boundary.

Lemma 8 *Let $\Omega \subset \mathbb{C}^2$ be a bounded convex Reinhardt domain that is not the product of two discs and $\phi \in C^\infty(\overline{\Omega})$. Assume that $\phi \circ f$ is holomorphic for any holomorphic function $f : \mathbb{D} \to b\Omega$. Then H_ϕ is compact on $A^2(\Omega)$.*

Proof If $b\Omega$ does not contain any non-trivial analytic disc the $\overline{\partial}$-Neumann operator is compact [10, Theorem 4.26] (see also [4, Theorem 1.1]). Furthermore, if the $\overline{\partial}$-Neumann operator is compact then H_ϕ is compact for all $\phi \in C(\overline{\Omega})$ [10, Proposition 4.1]. So if $b\Omega$ does not contain any non-trivial analytic disc, there is nothing to prove as the operator H_ϕ is compact. Lemma 1 implies that the analytic discs in $b\Omega$ are flat and horizontal or flat and vertical. We assume that there are non-trivial vertical and horizontal analytic discs in $b\Omega$ as the proof is even simpler if there are no vertical or horizontal discs. Let Δ_1 and Δ_2 be the horizontal and the vertical discs in $b\Omega$. So there exists $0 < r_1 < s_2, 0 < r_2 < s_1$ (since Ω is not product of two discs) such that

$$\Delta_1 = \mathbb{D}_{r_1} \times S_{s_1} \quad \text{and} \quad \Delta_2 = S_{s_2} \times \mathbb{D}_{r_2}.$$

We note that $\Gamma_\Omega = \overline{\Delta_1} \cup \overline{\Delta_2}$ and $\overline{\Delta_1} \cap \overline{\Delta_2} = \emptyset$. Let us define

$$\phi_1(z,w) = \phi(z,w) - (|w|^2 - s_1^2)\frac{1}{w}\frac{\partial \phi(z,w)}{\partial \overline{w}}$$

for $w \neq 0$. We note that ϕ_1 is a C^∞-smooth function on $\overline{\Omega}$ for $w \neq 0$ and $\phi_1 = \phi$ on Δ_1. Furthermore, using the fact that $\phi(., w)$ is holomorphic on $\mathbb{D}_{r_1}$ for $|w| = s_1$, one can verify that $\overline{\partial}\phi_1 = 0$ on Δ_1. Similarly we define

$$\phi_2(z, w) = \phi(z, w) - (|z|^2 - s_2^2)\frac{1}{z}\frac{\partial\phi(z, w)}{\partial\overline{z}}$$

and one can verify that $\phi_2 = \phi$ and $\overline{\partial}\phi_2 = 0$ on Δ_2.

We choose $\chi_1, \chi_2 \in C^\infty(\overline{\Omega})$ such that

i. $\chi_1 \equiv 1$ on a neighborhood of $\overline{\Delta_1}$ and $\chi_1 \equiv 0$ on a neighborhood of $\overline{\Delta_2} \cup \{(z, w) \in \overline{\Omega} : |w| = 0\}$,
ii. $\chi_2 \equiv 1$ on a neighborhood of $\overline{\Delta_2}$ and $\chi_2 \equiv 0$ on a neighborhood of $\overline{\Delta_1} \cup \{(z, w) \in \overline{\Omega} : |z| = 0\}$.

Then we define

$$\psi = \chi_1\phi_1 + \chi_2\phi_2 \in C^\infty(\overline{\Omega}).$$

We note that $\psi = \phi$ and $\overline{\partial}\psi = 0$ on Γ_Ω. Lemma 7 implies that $H_{\phi-\psi}$ is compact on $A^2(\Omega)$. To finish the proof we only need to show that H_ψ is compact. This can be done exactly in the same manner as the proof of H_β^Ω is compact in [2, pp 3740]. □

Proposition 1 *Let $f \in C(\overline{\mathbb{D}^2})$ such that $f(e^{i\theta}, .)$ and $f(., e^{i\theta})$ are holomorphic on $\mathbb{D}$ for each fixed θ. Then H_f is compact on $A^2(\mathbb{D}^2)$.*

Proof Let $\mathbb{T}^2 = \{(z, w) \in \mathbb{C}^2 : |z| = |w| = 1\}$ be the distinguished boundary and

$$F_N(z, w) = \sum_{|m|,|j|\leq N}\left(1 - \frac{|m|}{N+1}\right)\left(1 - \frac{|j|}{N+1}\right)a_{mj}(f)z^m w^j$$

where

$$a_{mj}(f) = \int_{\mathbb{T}^2} f(\zeta_1, \zeta_2)\zeta_1^{-m}\zeta_2^{-j}\,d\sigma(\zeta)$$

and σ is the normalized Lebesgue measure on $\mathbb{T}^2$. We let $S_{N,2}$ be the N-th Fejér kernel on $\mathbb{T}^2$. As in [5, Chapter I, Section 9], it is just the product of the N-th Fejér kernels on the circle. Since $f \in C(\mathbb{T}^2)$, and the convolution $S_{N,2} * f = F_N$, Fejér's Theorem on Cesàro summability (see, for example, [5, Section 9.2, pg 64] for homogeneous Banach spaces) implies that

$$\|F_N - f\|_{L^\infty(\mathbb{T}^2)} \to 0$$

as $N \to \infty$.

Now we claim that $a_{mj}(P) = 0$ for any holomorphic polynomial P and $m \leq -1$ or $j \leq -1$. Let

$$P(z, w) = \sum_{l,k=0}^{n} b_{lk} z^l w^k$$

and $m \leq -1$ or $j \leq -1$. Then

$$\begin{aligned} a_{mj}(P) &= \sum_{l,k=0}^{n} b_{lk} \langle \zeta_1^l \zeta_2^k, \zeta_1{}^m \zeta_2{}^j \rangle_{L^2(\mathbb{T}^2)} \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \sum_{l,k=0}^{n} b_{lk} e^{i\theta_1 l} e^{i\theta_2 k} e^{-i\theta_1 m} e^{-i\theta_2 j} d\theta_1 d\theta_2 \\ &= 0. \end{aligned} \tag{5}$$

Next we will show that $a_{mj}(f) = 0$ for $m \leq -1$ or $j \leq -1$. Without loss of generality, we suppose that $j \leq -1$. Since $f(e^{i\theta_1}, .)$ is holomorphic on $\mathbb{D}$, using Mergelyan's Theorem, there exists a sequence of holomorphic polynomials $\{P_{n,\theta_1}\}_{n \in \mathbb{N}}$ converging to f uniformly on $\overline{\mathbb{D}}$ as $n \to \infty$. Let us define $P_{n,\theta_1,r}(\xi) = P_{n,\theta_1}(r\xi)$ and $f_r(z, w) = f(z, rw)$ for $0 < r < 1$. Then $P_{n,\theta_1,r} \to f_r(e^{i\theta_1}, .)$ uniformly on $\overline{\mathbb{D}}$ as $n \to \infty$. As we have computed above in (5), one can show that $a_{mj}(P_{n,\theta_1,r}) = 0$ for all $m \in \mathbb{Z}$, $n \in \mathbb{N}$, and $0 < r < 1$. So by taking limits as $n \to \infty$ we have $a_{mj}(f_r) = 0$ for all $0 < r < 1$. Finally taking the limit as $r \to 1^-$ we conclude that $a_{mj}(f) = 0$ for $j \leq -1$. The proof for $m \leq -1$ is similar. Hence we have shown that $a_{mj}(f) = 0$ for $j \leq -1$ or $m \leq -1$.

We define

$$G_N(z, w) = \sum_{0 \leq m, j \leq N} \left(1 - \frac{m}{N+1}\right)\left(1 - \frac{j}{N+1}\right) a_{mj}(f) z^m w^j.$$

Since we have shown $G_N \equiv F_N$ on $\mathbb{T}^2$, we have $\|G_N - f\|_{L^\infty(\mathbb{T}^2)} \to 0$ as $N \to \infty$. Since $(G_N - f)(e^{i\theta}, w)$ is holomorphic in w and $(G_N - f)(z, e^{i\theta})$ is holomorphic in z, using the Maximum Modulus Principle for holomorphic functions, we have

$$\|G_N - f\|_{L^\infty(b\mathbb{D}^2)} \leq \|G_N - f\|_{L^\infty(\mathbb{T}^2)}.$$

So $\|G_N - f\|_{L^\infty(b\Omega)} \to 0$ as $N \to \infty$. Then Lemma 7 implies that $\|H_{G_N - f}\|_e \to 0$ as $N \to \infty$. Furthermore, $\|H_f\|_e = \|H_{G_n - f}\|_e$ as $H_{G_N} = 0$. Therefore, we conclude that $\|H_f\|_e = 0$. That is, H_f is compact on $A^2(\mathbb{D}^2)$. □

Remark 2 Even though we stated the previous proposition on $\mathbb{D}^2$ the same proof, with trivial modifications, works on products of two discs.

Now we are ready for the proof of the main result.

Proof of Theorem 1 First we will prove the sufficiency. Assume that H_ϕ is compact on $A^2(\Omega)$. If there is no non-trivial analytic disc in the boundary of Ω then there is nothing to prove. So assume that $\Delta = f(\mathbb{D})$ is a non-trivial disc in $b\Omega$ such that $\phi \circ f$ is not holomorphic. Without loss of generality we may assume that Δ is horizontal as the proof for vertical discs is similar. Let us fix $(z_0, w_0) \in \Delta$ and define $\alpha_j = (j-1)/j$. Then one can check that $\|(w - w_0)^{-\alpha_j}\|_{L^2(H)} \to \infty$ as $j \to \infty$. Let us define

$$g_j(w) = \frac{a_j}{(w - w_0)^{\alpha_j}}$$

where $a_j = 1/\|(w - w_0)^{-\alpha_j}\|_{L^2(H_\Omega)}$. Then $\|g_j\|_{L^2(H_\Omega)} = 1$ for all j. Furthermore, $g_j \to 0$ uniformly on any compact subset in Ω as $j \to \infty$. Without loss of generality, we assume that Δ is the largest horizontal disc in $b\Omega$ passing through (z_0, w_0) and ϕ_0 be a continuous function on $\mathbb{C}^2$ such that $\phi_0(z, w) = \phi(z, w_0)$ for all $(z, w) \in \Omega$. That is, ϕ_0 is the extension of $\phi|_\Delta$ to $\mathbb{C}$ in z. Since ϕ_0 is not holomorphic (as a function of z) on Δ we have $H^\Delta_{\phi_0}(1) \neq 0$. That is, $\|H^\Delta_{\phi_0}(1)\|_{L^2(\Delta)} > 0$. Then by Corollary 1, there exists $\beta > 0$ and $\delta > 0$ such that if $w \in H_\Omega$ and $|w - w_0| < \delta$, then

$$\|H^{\Delta_w}_{\phi_0}(1)\|_{L^2(\Delta_w)} > \beta.$$

Let us define $K = \{w \in H_\Omega : |w - w_0| \leq \delta\}$. Then

$$\begin{aligned}
&\int_{H_\Omega} |g_j(w)|^2 \int_{\Delta_w} |H^{\Delta_w}_{\phi_0}(1)(z)|^2 dV(z) dV(w) \\
&\geq \int_K |g_j(w)|^2 \int_{\Delta_w} |H^{\Delta_w}_{\phi_0}(1)(z)|^2 dV(z) dV(w) \\
&\geq \beta^2 \|g_j\|^2_{L^2(K)}.
\end{aligned}$$

However, since $\|g_j\|^2_{L^2(H_\Omega)} = 1$ for all j and $g_j \to 0$ uniformly on any compact set away from w_0 we conclude that $\|g_j\|^2_{L^2(K)} \geq 1/2$ for large j. Therefore, for large j we have

$$\int_{H_\Omega} |g_j(w)|^2 \int_{\Delta_w} |H^{\Delta_w}_{\phi_0}(1)(z)|^2 dV(z) dV(w) \geq \frac{\beta^2}{2} > 0.$$

Then Lemma 4 and Lemma 5 imply that $\|H_{\phi_0} g_j\|^2_{L^2(\Omega)}$ does not converge to 0 as $j \to \infty$. This contradicts Lemma 3 as we have assumed that H_ϕ is compact.

Finally we will prove the necessity. We assume $\phi \in C(\overline{\Omega})$ is such that $\phi \circ f$ is holomorphic for any holomorphic function $f : \mathbb{D} \to b\Omega$. Furthermore, we assume that Ω is not the product of two discs as that case is covered in Proposition 1. Lemma 6 implies that there exists a family of functions $\{\psi_n\} \subset C^\infty(\overline{\Omega})$ such that

i. $\psi_n \circ f$ is holomorphic for any n and any holomorphic $f : \mathbb{D} \to b\Omega$,
ii. $\psi_n \to \phi$ uniformly on Γ_Ω as $n \to \infty$.

Lemma 8 implies that H_{ψ_n} is compact and Lemma 7 implies that $\|H_{\phi-\psi_n}\|_e \to 0$ as $n \to \infty$. Therefore,

$$\|H_\phi\|_e = \|H_\phi\|_e - \|H_{\psi_n}\|_e \leq \|H_{\phi-\psi_n}\|_e.$$

This implies $\|H_\phi\|_e = 0$, proving that H_ϕ is compact on $A^2(\Omega)$. □

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