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Weighted Harmonic Bloch Spaces on the Ball

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Abstract We study the family of weighted harmonic Bloch spaces b_α, $\alpha \in \mathbb{R}$, on the unit ball of $\mathbb{R}^n$. We provide characterizations in terms of partial and radial derivatives and certain radial differential operators that are more compatible with reproducing kernels of harmonic Bergman–Besov spaces. We consider a class of integral operators related to harmonic Bergman projection and determine precisely when they are bounded on L^∞_α. We define projections from L^∞_α to b_α and as a consequence obtain integral representations. We solve the Gleason problem and provide atomic decomposition for all b_α, $\alpha \in \mathbb{R}$. Finally we give an oscillatory characterization of b_α when $\alpha > -1$.

Keywords Harmonic Bloch space · Bergman space · Reproducing kernel · Radial fractional derivative · Bergman projection · Duality · Gleason problem · Atomic decomposition · Oscillatory characterization

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1 Introduction

For $n \geq 2$, let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$ and $\mathbb{S}$ be the unit sphere. We denote the space of complex-valued harmonic functions on $\mathbb{B}$ by $h(\mathbb{B})$. The well-known harmonic Bloch space b is the space of all $f \in h(\mathbb{B})$ such that

$$\sup_{x\in\mathbb{B}}(1-|x|^2)|\nabla f(x)| < \infty.$$

The space b is a member of the one-parameter family of weighted harmonic Bloch spaces b_α, $\alpha \in \mathbb{R}$. The aim of this work is to investigate the properties of this family in a detailed, systematic and unified way. The holomorphic counterpart of this family of spaces and the related little Bloch and Lipschitz spaces have been studied in [13,26].

To define b_α we need to introduce more definitions. We denote by L^∞ the Lebesgue class of essentially bounded functions on $\mathbb{B}$, and for $\alpha \in \mathbb{R}$ we define

$$L^\infty_\alpha = \{\varphi : (1-|x|^2)^\alpha \varphi(x) \in L^\infty\},$$

so that $L^\infty_0 = L^\infty$. The norm on L^∞_α is

$$\|\varphi\|_{L^\infty_\alpha} = \|(1-|x|^2)^\alpha \varphi(x)\|_{L^\infty}.$$

We will also use the following subspaces of L^∞_α:

$$\mathcal{C}_\alpha = \{\varphi \in L^\infty_\alpha : (1-|x|^2)^\alpha \varphi(x) \text{ is continuous on } \overline{\mathbb{B}}\},$$
$$\mathcal{C}_{\alpha 0} = \{\varphi \in \mathcal{C}_\alpha : (1-|x|^2)^\alpha \varphi(x) = 0 \text{ on } \partial\mathbb{B}\}.$$

Definition 1.1 For $\alpha > 0$, the weighted harmonic Bloch space b_α is $h(\mathbb{B}) \cap L^\infty_\alpha$ and the weighted harmonic little Bloch space $b_{\alpha 0}$ is $h(\mathbb{B}) \cap \mathcal{C}_{\alpha 0}$.

Obviously (for $\alpha > 0$), $b_{\alpha 0} = \{f \in b_\alpha : \lim_{|x|\to 1^-}(1-|x|^2)^\alpha f(x) = 0\}$. The norm on b_α (and $b_{\alpha 0}$) is the norm inherited from L^∞_α.

To extend the above definition to the range $\alpha \leq 0$, we need to consider growth rates of derivatives of $f \in h(\mathbb{B})$. For this we will employ three different types of differentiation. For the usual partial derivatives we will write

$$\partial^m f = \frac{\partial^{|m|} f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}},$$

where $m = (m_1, \ldots, m_n)$ is a multi-index, $m_1, \ldots, m_n$ are nonnegative integers and $|m| = m_1 + \cdots + m_n$.

It is well-known that $f \in h(\mathbb{B})$ has a homogeneous expansion $f = \sum_{k=0}^\infty f_k$, where f_k is a homogeneous harmonic polynomial of degree k and the series absolutely and uniformly converges on compact subsets of $\mathbb{B}$ (see [2]). The radial derivative $\mathcal{R}f$ of $f \in h(\mathbb{B})$ is defined as

$$\mathcal{R}f(x) = x \cdot \nabla f(x) = \sum_{k=0}^{\infty} k f_k(x), \tag{1}$$

and

$$\mathcal{R}^N f(x) = \mathcal{R}\mathcal{R}^{N-1} f(x) = \sum_{k=0}^{\infty} k^N f_k(x), \quad N = 2, 3, \ldots.$$

In addition to partial and radial derivatives we will extensively use certain radial fractional differential operators $D_s^t : h(\mathbb{B}) \to h(\mathbb{B})$, $(s, t \in \mathbb{R})$ introduced in [7] and [8]. These operators are defined in terms of reproducing kernels of harmonic Bergman–Besov spaces and are more convenient than partial or radial derivatives in studying harmonic function spaces. We will review properties of D_s^t in Sect. 2.3. For now, we only note that t determines the order of the differentiation and s plays a minor role.

The following theorem will enable us to define weighted harmonic Bloch space b_α for the whole range $\alpha \in \mathbb{R}$. We denote $\mathbb{N} = \{0, 1, 2, \ldots\}$ with 0 included.

Theorem 1.2 *Let $\alpha \in \mathbb{R}$ and $f \in h(\mathbb{B})$. The following are equivalent:*

(a) For every $N \in \mathbb{N}$ with $\alpha + N > 0$, we have $(1 - |x|^2)^N \partial^m f \in L_\alpha^\infty$ for every multi-index m with $|m| = N$.
(b) There exists an $N \in \mathbb{N}$ with $\alpha + N > 0$ such that $(1 - |x|^2)^N \partial^m f \in L_\alpha^\infty$ for every multi-index m with $|m| = N$.
(c) For every $N \in \mathbb{N}$ with $\alpha + N > 0$, we have $(1 - |x|^2)^N \mathcal{R}^N f \in L_\alpha^\infty$.
(d) There exists an $N \in \mathbb{N}$ with $\alpha + N > 0$ such that $(1 - |x|^2)^N \mathcal{R}^N f \in L_\alpha^\infty$.
(e) For every $s, t \in \mathbb{R}$ with $\alpha + t > 0$, we have $(1 - |x|^2)^t D_s^t f \in L_\alpha^\infty$.
(f) There exist $s, t \in \mathbb{R}$ with $\alpha + t > 0$ such that $(1 - |x|^2)^t D_s^t f \in L_\alpha^\infty$.

Moreover, if $\alpha + N > 0$ and $\alpha + t > 0$, then

$$\begin{aligned} \|(1 - |x|^2)^t D_s^t f\|_{L_\alpha^\infty} &\sim |f(0)| + \|(1 - |x|^2)^N \mathcal{R}^N f\|_{L_\alpha^\infty} \\ &\sim \sum_{|m| \le N-1} |(\partial^m f)(0)| + \sum_{|m|=N} \|(1 - |x|^2)^N \partial^m f\|_{L_\alpha^\infty}. \end{aligned} \tag{2}$$

A corresponding theorem holds when L_α^∞ is replaced by $\mathcal{C}_{\alpha 0}$.

Theorem 1.3 *Let $\alpha \in \mathbb{R}$ and $f \in h(\mathbb{B})$. The following are equivalent:*

(a) For every $N \in \mathbb{N}$ with $\alpha + N > 0$, we have $(1 - |x|^2)^N \partial^m f \in \mathcal{C}_{\alpha 0}$ for every multi-index m with $|m| = N$.
(b) There exists an $N \in \mathbb{N}$ with $\alpha + N > 0$ such that $(1 - |x|^2)^N \partial^m f \in \mathcal{C}_{\alpha 0}$ for every multi-index m with $|m| = N$.
(c) For every $N \in \mathbb{N}$ with $\alpha + N > 0$, we have $(1 - |x|^2)^N \mathcal{R}^N f \in \mathcal{C}_{\alpha 0}$.
(d) There exists an $N \in \mathbb{N}$ with $\alpha + N > 0$ such that $(1 - |x|^2)^N \mathcal{R}^N f \in \mathcal{C}_{\alpha 0}$.
(e) For every $s, t \in \mathbb{R}$ with $\alpha + t > 0$, we have $(1 - |x|^2)^t D_s^t f \in \mathcal{C}_{\alpha 0}$.
(f) There exist $s, t \in \mathbb{R}$ with $\alpha + t > 0$ such that $(1 - |x|^2)^t D_s^t f \in \mathcal{C}_{\alpha 0}$.

For $\alpha \geq 0$, equivalence of parts (a)–(d) of Theorems 1.2 and 1.3 are known and the main part of the above theorems is that they also hold for $\alpha < 0$. For $\alpha = 0$, see [4, Theorem 1.4] for the equivalence of parts (a)–(d) and an additional characterization with a different type of derivative. For $\alpha > 0$, see [18, Theorem 1.1] for the equivalence of parts (a) and (b) for the choices of $N = 0$ and $N = 1$.

Definition 1.4 Let $\alpha \in \mathbb{R}$. The weighted harmonic Bloch space b_α (respectively weighted harmonic little Bloch space $b_{\alpha 0}$) consists of those $f \in h(\mathbb{B})$ such that any one of the equivalent conditions of Theorem 1.2 (respectively Theorem 1.3) is satisfied.

If $\alpha > 0$ taking $N = 0$ in parts (b) of the above theorems shows that Definition 1.4 is consistent with Definition 1.1. Also, taking $N = 1$ in parts (b) of the above theorems shows that $b_0 = b$, the usual harmonic Bloch space and b_{00} is the usual harmonic little Bloch space: $b_{00} = \{f \in h(\mathbb{B}) : \lim_{|x| \to 1^-} (1 - |x|^2)|\nabla f(x)| = 0\}$.

We mention a few immediate consequences of Definition 1.4. First, for every $\alpha \in \mathbb{R}$, we have $b_{\alpha 0} \subset b_\alpha$. Also, if $f \in h(\overline{\mathbb{B}})$, then $f \in b_{\alpha 0}$; in particular every $b_{\alpha 0}$ (and b_α) contains harmonic polynomials and therefore is non-trivial. It is also clear that

$$b_\alpha \subset b_{\beta 0} \subset b_\beta \qquad (\text{for } \alpha < \beta). \tag{3}$$

The above inclusions are in fact strict (see Remark 4.9 below) and therefore all these spaces are different.

When $\alpha > 0$ we have a standard norm on b_α, but when $\alpha \leq 0$ we do not. For $\alpha \in \mathbb{R}$ if we pick any $N \in \mathbb{N}$ with $\alpha + N > 0$ or pick $s, t \in \mathbb{R}$ with $\alpha + t > 0$, each term in (2) is a norm on b_α. Since all these norms are equivalent, there is no essential difference in choosing any one of them; and we will denote any one of these norms by $\| \cdot \|_{b_\alpha}$ without indicating the dependence on N or s, t.

For $s, t \in \mathbb{R}$ and $f \in h(\mathbb{B})$ we will write

$$I_s^t f(x) := (1 - |x|^2)^t D_s^t f(x).$$

It is clear from Theorem 1.2 that given $\alpha \in \mathbb{R}$, if t is chosen to satisfy $\alpha + t > 0$, then $f \in b_\alpha$ if and only if $I_s^t f \in L_\alpha^\infty$ and $\|I_s^t f\|_{L_\alpha^\infty}$ is a norm on b_α.

Our next aim is to write b_α (respectively $b_{\alpha 0}$) as quotient spaces of L_α^∞ (respectively $\mathcal{C}_\alpha$ or $\mathcal{C}_{\alpha 0}$) by using Bergman–Besov projections and obtain integral representations for elements of b_α. For this we need more definitions.

Let ν be the volume measure on $\mathbb{B}$ normalized so that $\nu(\mathbb{B}) = 1$. For $q \in \mathbb{R}$ we define the weighted volume measures

$$d\nu_q(x) = \frac{1}{V_q}(1 - |x|^2)^q \, d\nu(x).$$

These measures are finite only when $q > -1$ and in this case we choose V_q so that $\nu_q(\mathbb{B}) = 1$. For $q \leq -1$, we set $V_q = 1$. For $1 \leq p < \infty$, we denote the Lebesgue classes with respect to ν_q by L_q^p.

For $1 \leq p < \infty$ and $q > -1$ the weighted harmonic Bergman space b^p_q is $h(\mathbb{B}) \cap L^p_q$. It is well-known that the space b^2_q is a reproducing kernel Hilbert space with kernel $R_q(x, y)$. In [7,8] the spaces b^p_q and the reproducing kernels $R_q(x, y)$ are extended to the whole range $q \in \mathbb{R}$. We will give a review of these in Sect. 2.2.

Definition 1.5 For $s \in \mathbb{R}$, the harmonic Bergman–Besov projection is

$$Q_s\varphi(x) = \int_{\mathbb{B}} R_s(x, y)\varphi(y)d\nu_s(y),$$

for suitable φ.

Theorem 1.6 *Let $\alpha \in \mathbb{R}$. The operator $Q_s : L^\infty_\alpha \to b_\alpha$ is bounded if and only if*

$$s > \alpha - 1. \tag{4}$$

For an s satisfying (4), if t satisfies

$$\alpha + t > 0, \tag{5}$$

then for $f \in b_\alpha$,

$$Q_s I^t_s f = \frac{V_{s+t}}{V_s} f, \tag{6}$$

and therefore Q_s is onto. Also, $Q_s : \mathcal{C}_\alpha \to b_{\alpha 0}$ or $Q_s : \mathcal{C}_{\alpha 0} \to b_{\alpha 0}$ is bounded (and onto) if and only if (4) holds.

By (6) we have the following integral representation: For $f \in b_\alpha$, if (4) and (5) holds, then

$$f(x) = \frac{V_s}{V_{s+t}} \int_{\mathbb{B}} R_s(x, y)I^t_s f(y)\, d\nu_s(y) = \int_{\mathbb{B}} R_s(x, y)D^t_s f(y)\, d\nu_{s+t}(y). \tag{7}$$

This representation is very fruitful and we will use it many times in Sects. 5 and 6.

The case $\alpha = 0$ of Theorem 1.6 is proved earlier in [4,11] and [14] where the authors use different differential operators than our D^t_s. In [18] a different integral representation valid for $\alpha > -1$ is given. We note that Theorem 1.6 covers all $\alpha \in \mathbb{R}$, gives a necessary and sufficient condition for the boundedness of the projection operator Q_s and provides a simple reproducing formula.

For the holomorphic analogue of Theorem 1.6 for the full range $-\infty < \alpha < \infty$, see [13,26].

This paper is organized as follows. In Sect. 2 we collect some known facts which we will need in the sequel. In Sect. 3 we will define a class of integral operators related to harmonic Bergman projection and determine when they are bounded on L^∞_α and $\mathcal{C}_{\alpha 0}$. In Sect. 4 we will prove Theorems 1.2 and 1.3 and derive basic properties of the spaces b_α and $b_{\alpha 0}$. We will also determine when $R_q(x, \zeta)$, $\zeta \in \mathbb{S}$ belongs to b_α (or

$b_{\alpha 0}$) and show that all b_α and $b_{\alpha 0}$ are distinct. In Sect. 5 we will prove Theorem 1.6 and as a consequence we will show that the dual of Bergman–Besov space b^1_q (for every $q \in \mathbb{R}$) is b_α and its pre-dual is $b_{\alpha 0}$ under suitable pairings.

Finally in Sect. 6 we will solve the Gleason problem and obtain atomic decomposition for all $\alpha \in \mathbb{R}$. We will also give an oscillatory characterization of b_α for $\alpha > -1$.

2 Preliminaries

For two positive expressions X and Y we will write $X \sim Y$ if X/Y is bounded above and below by some positive constants. We will denote these constants whose exact values are inessential by a generic upper case C. We will also write $X \lesssim Y$ to mean $X \le CY$.

The Pochhammer symbol $(a)_b$ is defined by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function. By Stirling formula

$$\frac{(a)_c}{(b)_c} \sim c^{a-b} \quad (c \to \infty). \tag{8}$$

For $x \in \overline{\mathbb{B}}$, $y \in \overline{\mathbb{B}}$, we will use the notation

$$[x, y] = \sqrt{1 - 2x \cdot y + |x|^2|y|^2}.$$

It is easy to see that when x, y are nonzero

$$[x, y] = \left| |y|x - \frac{y}{|y|} \right| = \left| |x|y - \frac{x}{|x|} \right|,$$

and when $y = \zeta \in \mathbb{S}$, we have $[x, \zeta] = |x - \zeta|$.

2.1 Zonal Harmonics

Let $\mathcal{H}_k(\mathbb{R}^n)$ denote the space of all homogeneous harmonic polynomials on $\mathbb{R}^n$ of degree k. The restriction of $f_k \in \mathcal{H}_k(\mathbb{R}^n)$ to the unit sphere $\mathbb{S}$ is called a spherical harmonic and the space of spherical harmonics of degree k is denoted by $\mathcal{H}_k(\mathbb{S})$. The finite-dimensional space $\mathcal{H}_k(\mathbb{S}) \subset L^2(\mathbb{S})$ is a reproducing kernel Hilbert space: For $\zeta \in \mathbb{S}$, there exists (real-valued) $Z_k(\cdot, \zeta)$ such that

$$f_k(\zeta) = \int_{\mathbb{S}} f_k(\eta) Z_k(\eta, \zeta) d\sigma(\eta) \quad (\forall f_k \in \mathcal{H}_k(\mathbb{S})),$$

where $d\sigma$ is normalized surface area measure on $\mathbb{S}$. The spherical harmonic $Z_k(\cdot, \zeta)$ is called zonal harmonic of degree k with pole ζ. It can be extended to $\mathbb{R}^n \times \mathbb{R}^n$ by making it homogeneous in each variable: If $x = |x|\eta$, $y = |y|\zeta$ with $\eta, \zeta \in \mathbb{S}$,

$$Z_k(x, y) = |x|^k |y|^k Z_k(\eta, \zeta), \quad k = 1, 2, \ldots$$

For $k = 0$, we set $Z_0(x, y) \equiv 1$. For future reference we state the following properties of Z_k (for details see Chapter 5 of [2]).

Lemma 2.1 *The following properties hold:*

(a) $Z_k(x, y)$ *is real-valued and symmetric in its variables.*
(b) $Z_k(x, 0) = Z_k(0, y) = 0$*, for every* $x, y \in \mathbb{R}^n$, $k = 1, 2, \ldots$
(c) For $k \geq 1$ *and* $\zeta \in \mathbb{S}$, $\max_{\eta \in \mathbb{S}} |Z_k(\eta, \zeta)| = Z_k(\zeta, \zeta)$ *and* $Z_k(\zeta, \zeta) \sim k^{n-2}$. *Therefore* $|Z_k(x, y)| \lesssim |x|^k |y|^k k^{n-2}$.
(d) If $f_k \in \mathcal{H}_k(\mathbb{R}^n)$*, then* $f_k(x) = \int_{\mathbb{S}} f_k(\eta) Z_k(x, \eta) d\sigma(\eta)$.
(e) If $f_k \in \mathcal{H}_k(\mathbb{R}^n)$ *and* $l \neq k$*, then* $\int_{\mathbb{S}} f_k(\eta) Z_l(x, \eta) d\sigma(\eta) = 0$.

2.2 Harmonic Bergman–Besov Spaces and Reproducing Kernels

Let $1 \leq p < \infty$ and $q > -1$. The weighted harmonic Bergman space b^p_q consists of all $f \in h(\mathbb{B})$ such that

$$\|f\|_{b^p_q} = \left(\int_{\mathbb{B}} |f|^p d\nu_q \right)^{1/p} = \left(\frac{1}{V_q} \int_{\mathbb{B}} |f(x)|^p (1 - |x|^2)^q d\nu(x) \right)^{1/p} < \infty.$$

It is well-known that the space b^2_q is a reproducing kernel Hilbert space with reproducing kernel $R_q(x, y)$:

$$f(x) = \int_{\mathbb{B}} R_q(x, y) f(y) d\nu_q(y), \quad \forall f \in b^2_q \quad (q > -1). \tag{9}$$

It is also well-known that $R_q(x, y)$ has the series expansion (see [16])

$$R_q(x, y) = \sum_{k=0}^{\infty} \frac{(1 + n/2 + q)_k}{(n/2)_k} Z_k(x, y) \quad (q > -1),$$

where the series absolutely and uniformly converges on $K \times \mathbb{B}$, for any compact subset K of $\mathbb{B}$. $R_q(x, y)$ is real-valued, symmetric in the variables x and y and harmonic with respect to each variable as these properties hold for $Z_k(x, y)$.

The family of weighted Bergman spaces can be extended to all $q \in \mathbb{R}$ in the following way: Pick a nonnegative integer N such that

$$q + pN > -1. \tag{10}$$

The harmonic Bergman–Besov space b^p_q consists of all $f \in h(\mathbb{B})$ such that

$$(1-|x|^2)^N \partial^m f \in L^p_q,$$

for every multi-index m with $|m| = N$. When $q > -1$, choosing $N = 0$ shows that $b^p_q = h(\mathbb{B}) \cap L^p_q$ is the usual weighted Bergman space. The harmonic Bergman–Besov spaces are studied in detail in [7,8] where it is shown that the choice of N is irrelevant as long as (10) is satisfied. In [7,8] these spaces are called Besov spaces, whereas in the literature the spaces b^p_{-n} are usually called Besov spaces.

For every $q \in \mathbb{R}$, the space b^2_q is a reproducing kernel Hilbert space with kernel

$$R_q(x, y) = \sum_{k=0}^{\infty} \gamma_k(q) Z_k(x, y), \tag{11}$$

where (see [7, Theorem 3.7], [8, Theorem 1.3])

$$\gamma_k(q) := \begin{cases} \dfrac{(1+n/2+q)_k}{(n/2)_k}, & \text{if } q > -(1+n/2); \\ \dfrac{(k!)^2}{(1-(n/2+q))_k (n/2)_k}, & \text{if } q \le -(1+n/2). \end{cases} \tag{12}$$

For $q > -1$, we endow b^2_q with the canonical inner product $\langle f, g\rangle = \int_{\mathbb{B}} f \overline{g} d\nu_q$ and R_q is the reproducing kernel with respect to this inner product. For $q \le -1$, there is no standard inner product, there are many possible choices each leading to a different reproducing kernel (see [8, Theorem 5.2] for the inner product leading to above R_q). The above choice of R_q follows [3] and [12], where *holomorphic* Bergman–Besov spaces are studied.

We list a few simple properties that we will use later: For every $q \in \mathbb{R}$ we have $\gamma_0(q) = 1$ and therefore by Lemma 2.1(b),

$$R_q(x, 0) = R_q(0, y) = 1, \quad \forall x, y \in \mathbb{B} \quad (\forall q \in \mathbb{R}). \tag{13}$$

Checking the two cases in (12), we have by (8)

$$\gamma_k(q) \sim k^{1+q} \quad (k \to \infty). \tag{14}$$

For each $x \in \mathbb{B}$, $R_q(x, \cdot)$ is harmonic on $\overline{\mathbb{B}}$ and if $K \subset \mathbb{B}$ is compact and m is a multi-index

$$|\partial^m R_q(x, y)| \lesssim 1, \quad \forall x \in K, \quad y \in \overline{\mathbb{B}}, \tag{15}$$

where differentiation is performed with respect to x.

2.3 The Operators D_s^t

Let $s, t \in \mathbb{R}$. The radial differential operator $D_s^t : h(\mathbb{B}) \to h(\mathbb{B})$ is defined in the following way (see [7,8]): If $f = \sum_{k=0}^{\infty} f_k$ is the homogeneous expansion, then

$$D_s^t f := \sum_{k=0}^{\infty} \frac{\gamma_k(s+t)}{\gamma_k(s)} f_k := \sum_{k=0}^{\infty} d_k(s,t) f_k. \tag{16}$$

By (14),

$$d_k(s,t) = \frac{\gamma_k(s+t)}{\gamma_k(s)} \sim k^t \quad (k \to \infty), \tag{17}$$

and therefore roughly speaking D_s^t multiplies the k^{th} homogeneous term by k^t. The exact form of D_s^t is chosen in order to have the relation

$$D_s^t R_s(x,y) = R_{s+t}(x,y), \tag{18}$$

where differentiation is performed on either of the variables x or y. For every $s \in \mathbb{R}$, $D_s^0 = I$, the identity. The additive property

$$D_{s+t}^u D_s^t = D_s^{u+t} \tag{19}$$

shows that every D_s^t is invertible with the two-sided inverse D_{s+t}^{-t}:

$$D_{s+t}^{-t} D_s^t = D_s^t D_{s+t}^{-t} = I. \tag{20}$$

The following lemma is Theorem 3.2 of [8].

Lemma 2.2 *Equip $h(\mathbb{B})$ with the topology of uniform convergence on compact subsets. Then $D_s^t : h(\mathbb{B}) \to h(\mathbb{B})$ is continuous for every $s, t \in \mathbb{R}$.*

In some cases we can write D_s^t as an integral operator. To see this we first show that we can push D_s^t into some certain integrals.

Lemma 2.3 *Let $c \in \mathbb{R}$ and $\varphi \in L_c^1$. For every $s, t \in \mathbb{R}$,*

$$D_s^t \int_{\mathbb{B}} R_c(x,y)\varphi(y)d\nu_c(y) = \int_{\mathbb{B}} D_s^t R_c(x,y)\varphi(y)d\nu_c(y).$$

Proof Since, for fixed x the series expansion (11) uniformly converges for $y \in \mathbb{B}$,

$$\int_{\mathbb{B}} R_c(x,y)\varphi(y)d\nu_c(y) = \sum_{k=0}^{\infty} \gamma_k(c) \int_{\mathbb{B}} Z_k(x,y)\varphi(y)d\nu_c(y) =: \sum_{k=0}^{\infty} \gamma_k(c) p_k(x).$$

As $Z_k(\cdot, y)$ is a homogeneous harmonic polynomial of degree k, so is p_k and the series on the right is homogeneous expansion. Therefore by (16),

$$D_s^t \int_{\mathbb{B}} R_c(x,y)\varphi(y)d\nu_c(y) = \sum_{k=0}^{\infty} d_k(s,t)\gamma_k(c) \int_{\mathbb{B}} Z_k(x,y)\varphi(y)d\nu_c(y)$$
$$= \int_{\mathbb{B}} D_s^t R_c(x,y)\varphi(y)d\nu_c(y),$$

where in the last equality we use uniform convergence of $\sum_{k=0}^{\infty} d_k(s,t)\gamma_k(c)Z_k(x,\cdot)$ (which follows from Lemma 2.1(c), (14) and (17)). □

If $c = s$, the following Corollary follows from (18).

Corollary 2.4 *Let $s \in \mathbb{R}$ and $\varphi \in L_s^1$. For every $t \in \mathbb{R}$,*

$$D_s^t \int_{\mathbb{B}} R_s(x,y)\varphi(y)d\nu_s(y) = \int_{\mathbb{B}} R_{s+t}(x,y)\varphi(y)d\nu_s(y).$$

Corollary 2.5 *Let $s > -1$ and $f \in L_s^1 \cap h(\mathbb{B})$. For every $t \in \mathbb{R}$,*

$$D_s^t f(x) = \int_{\mathbb{B}} R_{s+t}(x,y)f(y)d\nu_s(y). \tag{21}$$

Proof It is standard that the reproducing formula (9) remains true for all $f \in b_q^1$ $(q > -1)$. Therefore

$$f(x) = \int_{\mathbb{B}} R_s(x,y)f(y)d\nu_s(y).$$

We apply D_s^t to both sides and use the previous corollary. □

The operator D_s^t as an integral operator as in (21) appears in [11].

2.4 Estimates of Reproducing Kernels

For $a_j, b_j > 0$ $(j = 1, \dots J)$ and $x \in \mathbb{B}$, $y \in \overline{\mathbb{B}}$, let

$$W(x,y) = \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)\cdots\Gamma(a_J+k)}{\Gamma(b_1+k)\cdots\Gamma(b_J+k)} Z_k(x,y). \tag{22}$$

Note that by (12), $R_q(x,y)$ is of the form (22) for every $q \in \mathbb{R}$. The following estimates for $W(x,y)$ and its partial derivatives are taken from Section 7 of [8].

Lemma 2.6 *Let $a_j, b_j > 0$ ($j = 1, \dots J$) and m be a multi-index. Set $c = n - 1 + (a_1 + \cdots + a_J) - (b_1 + \cdots + b_J) + |m|$. Then for every $x \in \mathbb{B}$, $y \in \overline{\mathbb{B}}$,*

$$\left|(\partial^m W)(x, y)\right| \lesssim \begin{cases} 1, & \text{if } c < 0; \\ 1 + \log \dfrac{1}{[x, y]}, & \text{if } c = 0; \\ \dfrac{1}{[x, y]^c}, & \text{if } c > 0, \end{cases}$$

where differentiation is performed with respect to the first variable.

Checking the two cases of (12), one immediately obtains the following estimate for reproducing kernels.

Lemma 2.7 *Let $q \in \mathbb{R}$ and m be a multi-index. Then for every $x \in \mathbb{B}$, $y \in \overline{\mathbb{B}}$,*

$$\left|(\partial^m R_q)(x, y)\right| \lesssim \begin{cases} 1, & \text{if } q + |m| < -n; \\ 1 + \log \dfrac{1}{[x, y]}, & \text{if } q + |m| = -n; \\ \dfrac{1}{[x, y]^{n+q+|m|}}, & \text{if } q + |m| > -n. \end{cases}$$

The $q \geq -1$ part of the above lemma is proved in many places including [4, 11].

Since by (16), $D_s^t R_q(x, y)$ is also of the form (22), we have the following estimate.

Lemma 2.8 *Let $q, s, t \in \mathbb{R}$ and m be a multi-index. Then for every $x \in \mathbb{B}$, $y \in \overline{\mathbb{B}}$,*

$$\left|\partial^m (D_s^t R_q)(x, y)\right| \lesssim \begin{cases} 1, & \text{if } q + t + |m| < -n; \\ 1 + \log \dfrac{1}{[x, y]}, & \text{if } q + t + |m| = -n; \\ \dfrac{1}{[x, y]^{n+q+t+|m|}}, & \text{if } q + t + |m| > -n. \end{cases}$$

When $y = \zeta \in \mathbb{S}$ and $x = r\zeta$, $0 \leq r < 1$, the following two-sided estimate follows from part (c) of Lemma 2.1 and (14).

Lemma 2.9 *Let $\zeta \in \mathbb{S}$ and $0 \leq r < 1$. Then*

$$|R_q(r\zeta, \zeta)| \sim \begin{cases} 1, & \text{if } q < -n; \\ 1 + \log \dfrac{1}{1 - r^2}, & \text{if } q = -n; \\ \dfrac{1}{(1 - r^2)^{q+n}}, & \text{if } q > -n. \end{cases}$$

For $q > -1$ the following estimate on weighted integrals of R_q is proved in various places. For the whole range $q \in \mathbb{R}$, it is a special case of [8, Theorem 1.5].

Lemma 2.10 *Let $q \in \mathbb{R}$ and $c > -1$. Then for $x \in \mathbb{B}$,*

$$\int_{\mathbb{B}} |R_q(x, y)|(1 - |y|^2)^c d\nu(y) \sim \begin{cases} 1, & \textit{if } q < c; \\ 1 + \log \dfrac{1}{1 - |x|^2}, & \textit{if } q = c; \\ \dfrac{1}{(1 - |x|^2)^{q-c}}, & \textit{if } q > c. \end{cases}$$

We will also need the following integral estimate. For a proof see [15, Proposition 2.2] or [17, Lemma 4.4].

Lemma 2.11 *Let $a > -1$ and $c \in \mathbb{R}$. Then for $x \in \mathbb{B}$,*

$$\int_{\mathbb{B}} \frac{(1 - |y|^2)^a}{[x, y]^{n+a+c}} d\nu(y) \sim \begin{cases} 1, & \textit{if } c < 0; \\ 1 + \log \dfrac{1}{1 - |x|^2}, & \textit{if } c = 0; \\ \dfrac{1}{(1 - |x|^2)^{c}}, & \textit{if } c > 0. \end{cases}$$

We mention one more integral estimate.

Lemma 2.12 *Let $a > -1$, $c > 0$ and $0 \le r < 1$. Then*

$$\int_0^1 \frac{(1 - t^2)^a}{(1 - r^2 t^2)^{1+a+c}} dt \lesssim \frac{1}{(1 - r^2)^c}.$$

For a proof see, for example, [11, Lemma 2.1].

3 A Class of Integral Operators

In this section we will consider a class of integral operators and determine when they are bounded on L^∞_α or $\mathcal{C}_{\alpha 0}$.

For $a, c \in \mathbb{R}$ we define

$$T_{a,c}\, \varphi(x) = (1 - |x|^2)^a \int_{\mathbb{B}} R_{a+c}(x, y)\, \varphi(y)(1 - |y|^2)^c d\nu(y)$$
$$S_{a,c}\, \varphi(x) = (1 - |x|^2)^a \int_{\mathbb{B}} \big|R_{a+c}(x, y)\big|\, \varphi(y)(1 - |y|^2)^c d\nu(y)$$
$$E_{a,c}\, \varphi(x) = (1 - |x|^2)^a \int_{\mathbb{B}} \frac{1}{[x, y]^{n+a+c}}\, \varphi(y)(1 - |y|^2)^c d\nu(y)$$

The following theorem determines exactly when the above operators are bounded from L^∞_α to L^∞_α. Later, we will invoke this theorem many times.

Theorem 3.1 *Let $\alpha, a, c \in \mathbb{R}$. The following are equivalent:*

(a) $T_{a,c}$ is bounded on L^∞_α.

(b) $S_{a,c}$ *is bounded on* L^∞_α.
(c) $E_{a,c}$ *is bounded on* L^∞_α.
(d) $a+\alpha>0$ *and* $c>\alpha-1$.

Before proving this theorem we first show the following lemma. Recall that by (13), $R_q(0,y)=1$ for every $q\in\mathbb{R}$ and $y\in\mathbb{B}$. The lemma below shows that if x stays close to 0, then $R_q(x,y)$ is uniformly away from 0 for every $y\in\mathbb{B}$.

Lemma 3.2 *Let* $q\in\mathbb{R}$. *There exists* $\epsilon>0$ *such that for all* $|x|<\epsilon$ *and for all* $y\in\mathbb{B}$, *we have* $R_q(x,y)\geq 1/2$.

Proof Since $\gamma_0(q)=1$ and $Z_0(x,y)\equiv 1$, we have

$$R_q(x,y)=\sum_{k=0}^{\infty}\gamma_k(q)Z_k(x,y)=1+\sum_{k=1}^{\infty}\gamma_k(q)Z_k(x,y).$$

By (14) and Lemma 2.1(c), for $|x|\leq 1/2$,

$$\left|\sum_{k=1}^{\infty}\gamma_k(q)Z_k(x,y)\right|\lesssim\sum_{k=1}^{\infty}k^{1+q}k^{n-2}|x|^k|y|^k\lesssim|x|\sum_{k=1}^{\infty}k^{n+q-1}\left(\frac{1}{2}\right)^{k-1}\lesssim|x|.$$

Hence, for small enough ϵ, $\left|\sum_{k=1}^{\infty}\gamma_k(q)Z_k(x,y)\right|<1/2$ for $|x|<\epsilon$ and the lemma follows. □

Proof of Theorem 3.1 We first show the equivalence (a) ⇔ (b) ⇔ (d).

(b) ⇒ (a): This is immediate by the inequality $|T_{a,c}\varphi(x)|\leq S_{a,c}(|\varphi|)(x)$.

(a) ⇒ (d): We first show that $c>\alpha-1$. Let $\varphi(x)=(1-|x|^2)^{-\alpha}$. Then $\varphi\in L^\infty_\alpha$ and with ϵ as in Lemma 3.2, for $|x|<\epsilon$ we have

$$T_{a,c}\varphi(x)\geq(1-|x|^2)^a\int_{\mathbb{B}}\frac{1}{2}(1-|y|^2)^{c-\alpha}d\nu(y).$$

If $c\leq\alpha-1$, the last integral will be divergent and $T_{a,c}\varphi$ couldn't be in L^∞_α.

To see that $a+\alpha>0$, we again let $\varphi(x)=(1-|x|^2)^{-\alpha}$ and integrate in polar coordinates to obtain

$$\begin{aligned}T_{a,c}\varphi(x)&=(1-|x|^2)^a\int_{\mathbb{B}}R_{a+c}(x,y)(1-|y|^2)^{c-\alpha}d\nu(y)\\&=(1-|x|^2)^a\int_0^1 n\rho^{n-1}(1-\rho^2)^{c-\alpha}\int_{\mathbb{S}}R_{a+c}(x,\rho\eta)d\sigma(\eta)d\rho.\end{aligned}$$

By mean-value property the integral over $\mathbb{S}$ is $R_{a+b}(x,0)$ which is 1 by (13). So,

$$T_{a,c}\varphi(x)=\frac{\Gamma(n/2+1)\Gamma(c-\alpha+1)}{\Gamma(n/2+c-\alpha+1)}(1-|x|^2)^a=C(1-|x|^2)^a.$$

Since $T_{a,c}\,\varphi \in L^\infty_\alpha$ we must have $a+\alpha \geq 0$. What remains is to show that $a+\alpha = 0$ is not possible. So, suppose $a+\alpha = 0$. For $x_0 \in \mathbb{B}$, let

$$\varphi_{x_0}(y) = \begin{cases} (1-|y|^2)^{-\alpha} \dfrac{|R_{a+c}(x_0,y)|}{R_{a+c}(x_0,y)} & \text{if } R_{a+c}(x_0,y) \neq 0; \\ (1-|y|^2)^{-\alpha} & \text{if } R_{a+c}(x_0,y) = 0. \end{cases}$$

Clearly, $\|\varphi_{x_0}\|_{L^\infty_\alpha} = 1$. On the other hand by Lemma 2.10 we have

$$\begin{aligned} T_{a,c}\,\varphi_{x_0}(x_0) &= (1-|x_0|^2)^a \int_{\mathbb{B}} |R_{a+c}(x_0,y)|(1-|y|^2)^{c-\alpha} d\nu(y) \\ &\sim (1-|x_0|^2)^a \left(1 + \log \frac{1}{1-|x_0|^2}\right). \end{aligned}$$

This implies, by continuity of $T_{a,c}\,\varphi_{x_0}$, that

$$\begin{aligned} \|T_{a,c}\,\varphi_{x_0}\|_{L^\infty_\alpha} = \|(1-|x|^2)^\alpha T_{a,c}\,\varphi_{x_0}(x)\|_{L^\infty} &\geq (1-|x_0|^2)^\alpha T_{a,c}\,\varphi_{x_0}(x_0) \\ &\gtrsim 1 + \log \frac{1}{1-|x_0|^2}. \end{aligned}$$

Since $\|\varphi_{x_0}\|_{L^\infty_\alpha} = 1$, we get a contradiction with boundedness of $T_{a,c}$.

(d) $\Rightarrow$ (b): Suppose $a+\alpha > 0$ and $c > \alpha - 1$. Let $\varphi \in L^\infty_\alpha$. Then almost everywhere $|\varphi(y)| \leq \|\varphi\|_{L^\infty_\alpha}(1-|y|^2)^{-\alpha}$ and it follows from Lemma 2.10 that

$$\begin{aligned} |S_{a,c}\,\varphi(x)| &\leq (1-|x|^2)^a \int_{\mathbb{B}} |R_{a+c}(x,y)|\,|\varphi(y)|(1-|y|^2)^c d\nu(y) \\ &\leq \|\varphi\|_{L^\infty_\alpha}(1-|x|^2)^a \int_{\mathbb{B}} |R_{a+c}(x,y)|(1-|y|^2)^{c-\alpha} d\nu(y) \\ &\lesssim \|\varphi\|_{L^\infty_\alpha}(1-|x|^2)^a \frac{1}{(1-|x|^2)^{a+\alpha}}. \end{aligned}$$

Hence $\|S_{a,c}\,\varphi\|_{L^\infty_\alpha} \lesssim \|\varphi\|_{L^\infty_\alpha}$.

We next show (c) $\Leftrightarrow$ (d).

(c) $\Rightarrow$ (d): To see that $c > \alpha - 1$, we let $\varphi(x) = (1-|x|^2)^{-\alpha}$. Note that for $|x| < 1/2$ we have $1/2 \leq [x,y] = \big||x|y - y/|y|\big| \leq 3/2$. Therefore, for $|x| < 1/2$,

$$E_{a,c}\,\varphi(x) \gtrsim (1-|x|^2)^a \int_{\mathbb{B}} (1-|y|^2)^{c-\alpha} d\nu(y).$$

Since $E_{a,c}\,\varphi \in L^\infty_\alpha$, we must have $c-\alpha > -1$. That $a+\alpha \leq 0$ is not possible follows from Lemma 2.11: Letting again $\varphi(y) = (1-|y|^2)^{-\alpha}$, we have

$$E_{a,c}\,\varphi(x) = (1-|x|^2)^a \int_{\mathbb{B}} \frac{(1-|y|^2)^{c-\alpha}}{[x,y]^{n+a+c}} d\nu(y).$$

If $a+\alpha<0$, then by Lemma 2.11, the above integral is ~ 1 and if $a+\alpha=0$, it is $\sim 1+\log(1-|x|^2)^{-1}$. In each case $E_{a,c}\,\varphi$ cannot belong to L^∞_α.

(d) $\Rightarrow$ (c): This part easily follows from Lemma 2.11. □

Remark 3.3 Theorem 3.1 remains true when L^∞_α is replaced with $\mathcal{C}_{\alpha 0}$. This can be verified by repeating the above proof with making appropriate modifications (for example, we change $\varphi(x)=(1-|x|^2)^{-\alpha}$ with $\varphi(x)=(1-|x|^2)^{-\alpha}/\big(1+\log(1-|x|^2)^{-1}\big)$, etc.). We omit the details.

4 Proofs of Theorems 1.2 and 1.3

Before dealing with the general case $\alpha\in\mathbb{R}$, we will first consider the case $\alpha>0$. As is mentioned before when $\alpha\geq 0$ the equivalence of parts (a)–(d) of Theorems 1.2 and 1.3 are known. Nevertheless for the convenience of the reader and to make this work self-contained we will not refer to other sources and give a complete proof.

For future reference we record the following simple lemma which is a special case of the reproducing formula (7).

Lemma 4.1 *Let $\alpha>0$ and $s>\alpha-1$. If $f\in b_\alpha$, then*

$$f(x)=\int_{\mathbb{B}}R_s(x,y)f(y)d\nu_s(y)=\frac{1}{V_s}\int_{\mathbb{B}}R_s(x,y)f(y)(1-|y|^2)^s d\nu(y).$$

Proof The conditions imply $f\in b^1_s$ and the lemma follows from the reproducing formula (9) which is well-known to be true when $f\in b^1_q$. □

We begin the proof of Theorem 1.2 with the following lemma. This lemma is standard and can be proved by more elementary techniques. We include a proof for completeness and to illustrate how it follows from the reproducing formula, the kernel estimates and Theorem 3.1. Later, we will employ this technique many times.

Lemma 4.2 *Let $\alpha>0$ and $f\in h(\mathbb{B})$. The following are equivalent:*

(a) $f\in b_\alpha$.
(b) $(1-|x|^2)|\nabla f(x)|\in L^\infty_\alpha$.
(c) $(1-|x|^2)\mathcal{R}f(x)\in L^\infty_\alpha$.

Moreover,

$$\|f-f(0)\|_{b_\alpha}\sim\|(1-|x|^2)\,|\nabla f(x)|\,\|_{L^\infty_\alpha}\sim\|(1-|x|^2)\mathcal{R}f(x)\|_{L^\infty_\alpha}. \tag{23}$$

Proof (a) $\Rightarrow$ (b): Let $f\in b_\alpha$. Pick $s>\alpha-1$. By Lemma 4.1,

$$f(x)-f(0)=\frac{1}{V_s}\int_{\mathbb{B}}R_s(x,y)\big(f(y)-f(0)\big)(1-|y|^2)^s d\nu(y).$$

Taking partial derivative we obtain

$$\frac{\partial f}{\partial x_i}(x)=\frac{1}{V_s}\int_{\mathbb{B}}\frac{\partial}{\partial x_i}R_s(x,y)\big(f(y)-f(0)\big)(1-|y|^2)^s d\nu(y),$$

where changing the order of the derivative and integral is easily justified using (15). Applying Lemma 2.7, we get

$$(1-|x|^2)\Big|\frac{\partial f}{\partial x_i}(x)\Big| \lesssim (1-|x|^2)\int_{\mathbb{B}} \frac{1}{[x,y]^{n+s+1}}\big|f(y)-f(0)\big|(1-|y|^2)^s d\nu(y),$$

and part (b) now follows from Theorem 3.1.

(b) $\Rightarrow$ (c): This immediately follows from (1).

(c) $\Rightarrow$ (a): Let $M := \|(1-|x|^2)\mathcal{R}f(x)\|_{L^\infty_\alpha}$. Then

$$|\mathcal{R}f(x)| \leq \frac{M}{(1-|x|^2)^{\alpha+1}}, \quad \text{for } x\in\mathbb{B}. \tag{24}$$

By calculus and (1),

$$f(x)-f(0) = \int_0^1 x\cdot\nabla f(tx)\,dt = \int_0^{1/2}\frac{\mathcal{R}f(tx)}{t}dt + \int_{1/2}^1 \frac{\mathcal{R}f(tx)}{t}dt =: I_1 + I_2.$$

To estimate I_1 note that Cauchy's estimate and (24) implies, for $|x|\leq 1/2$,

$$|\nabla\mathcal{R}f(x)| \leq C\sup_{|y|=3/4}|\mathcal{R}f(y)| \lesssim M. \tag{25}$$

Since $\mathcal{R}f(0)=0$, we have $\mathcal{R}f(x)=\int_0^1 x\cdot\nabla\mathcal{R}f(tx)\,dt$ and using (25) we deduce $|\mathcal{R}f(x)|\lesssim M|x|$ for $|x|\leq 1/2$. Therefore

$$|I_1| \leq \int_0^{1/2}\frac{|\mathcal{R}f(tx)|}{t}\,dt \lesssim \int_0^{1/2}\frac{Mt|x|}{t}\,dt \lesssim M \leq \frac{M}{(1-|x|^2)^\alpha}.$$

For the second integral I_2 we use (24) and Lemma 2.12 to obtain

$$|I_2| \leq \int_{1/2}^1 \frac{|\mathcal{R}f(tx)|}{t}\,dt \lesssim \int_{1/2}^1 |\mathcal{R}f(tx)|\,dt \lesssim \int_0^1 \frac{M}{(1-t^2|x|^2)^{\alpha+1}}\,dt \lesssim \frac{M}{(1-|x|^2)^\alpha}.$$

Hence $\|f-f(0)\|_{b_\alpha} \lesssim \|(1-|x|^2)\mathcal{R}f(x)\|_{L^\infty_\alpha}$. □

We note that we can write (23) in the following form:

$$\|f\|_{b_\alpha} \sim |f(0)| + \|(1-|x|^2)\,|\nabla f(x)|\,\|_{L^\infty_\alpha} \sim |f(0)| + \|(1-|x|^2)\mathcal{R}f(x)\|_{L^\infty_\alpha}.$$

It is straightforward to extend the previous lemma to higher order derivatives.

Lemma 4.3 *Let $\alpha>0$ and $f\in h(\mathbb{B})$. The following are equivalent:*

(a) $f\in b_\alpha$.
(b) For every $N\in\mathbb{N}$, we have $(1-|x|^2)^N\partial^m f\in L^\infty_\alpha$ for every multi-index m with $|m|=N$.

(c) *There exists* $N \in \mathbb{N}$ *such that* $(1-|x|^2)^N \partial^m f \in L^\infty_\alpha$ *for every multi-index* m *with* $|m| = N$.
(d) *For every* $N \in \mathbb{N}$, *we have* $(1-|x|^2)^N \mathcal{R}^N f \in L^\infty_\alpha$.
(e) *There exists* $N \in \mathbb{N}$ *such that* $(1-|x|^2)^N \mathcal{R}^N f \in L^\infty_\alpha$.

Moreover,

$$\begin{aligned} \|f\|_{b_\alpha} &\sim \sum_{|m|\leq N-1} |(\partial^m f)(0)| + \sum_{|m|=N} \|(1-|x|^2)^N \partial^m f\|_{L^\infty_\alpha} \\ &\sim |f(0)| + \|(1-|x|^2)^N \mathcal{R}^N f\|_{L^\infty_\alpha}. \end{aligned} \tag{26}$$

Proof We show (a) ⇔ (b) ⇔ (c). The equivalence (a) ⇔ (d) ⇔ (e) can be justified similarly.

(a) ⇒ (b): Suppose $f \in b_\alpha$. By Lemma 4.2, $\dfrac{\partial f}{\partial x_i} \in b_{\alpha+1}$ for every $i = 1, 2, \ldots, n$. Applying Lemma 4.2 again we obtain $\dfrac{\partial^2 f}{\partial x_j \partial x_i} \in b_{\alpha+2}$ for every $i, j = 1, 2, \ldots, n$. We continue until we obtain $\partial^m f \in b_{\alpha+N}$ for every m with $|m| = N$.

(b) ⇒ (c): This part is clear.

(c) ⇒ (a): Suppose $(1-|x|^2)^N \partial^m f \in L^\infty_\alpha$, that is $\partial^m f \in b_{\alpha+N}$ for every multi-index m with $|m| = N$. Let m' be a multi-index with $|m'| = N-1$. Then $\dfrac{\partial}{\partial x_i}\partial^{m'} f \in b_{\alpha+N}$ for every $i = 1, 2, \ldots, n$ and Lemma 4.2 implies $\partial^{m'} f \in b_{\alpha+N-1}$. We repeat the same argument sufficiently many times until we obtain $f \in b_\alpha$.

It is not hard to verify (26) and we omit the details. □

We next show that instead of partial or radial derivatives we can use the operators D^t_s. We remain in the region $\alpha > 0$.

Lemma 4.4 *Let* $\alpha > 0$ *and* $f \in h(\mathbb{B})$. *The following are equivalent:*

(a) $f \in b_\alpha$.
(b) *For every* $s, t \in \mathbb{R}$ *with* $\alpha + t > 0$, *we have* $(1-|x|^2)^t D^t_s f \in L^\infty_\alpha$.
(c) *There exist* $s, t \in \mathbb{R}$ *with* $\alpha + t > 0$ *such that* $(1-|x|^2)^t D^t_s f \in L^\infty_\alpha$.

Moreover, $\|f\|_{b_\alpha} \sim \|(1-|x|^2)^t D^t_s f\|_{L^\infty_\alpha}$.

Proof (a) ⇒ (b): Suppose $f \in b_\alpha$. Pick $c > \alpha - 1$. By Lemma 4.1,

$$f(x) = \int_{\mathbb{B}} R_c(x, y) f(y) d\nu_c(y).$$

We apply D^t_s to both sides, push it into the integral by Lemma 2.3 and then use Lemma 2.8 (with $n + c + t > n + \alpha - 1 + t > n - 1 > 0$) to obtain

$$(1-|x|^2)^t |D^t_s f(x)| \lesssim (1-|x|^2)^t \int_{\mathbb{B}} \frac{1}{[x, y]^{n+c+t}} |f(y)|(1-|y|^2)^c d\nu(y).$$

Theorem 3.1 now implies $\|(1-|x|^2)^t D^t_s f(x)\|_{L^\infty_\alpha} \lesssim \|f\|_{L^\infty_\alpha}$ and part (b) follows.

With (b) ⇒ (c) being clear, we show (c) ⇒ (a): Suppose $(1-|x|^2)^t D_s^t f(x) \in L_\alpha^\infty$, that is $D_s^t f \in b_{\alpha+t}$. Pick c with $c > \alpha + t - 1$. By Lemma 4.1,

$$D_s^t f(x) = \int_{\mathbb{B}} R_c(x, y) D_s^t f(y) d\nu_c(y).$$

We apply D_{s+t}^{-t} to both sides, use (20) on the left, push D_{s+t}^{-t} into the integral by Lemma 2.3 and obtain

$$f(x) = \int_{\mathbb{B}} D_{s+t}^{-t} R_c(x, y) D_s^t f(y) d\nu_c(y).$$

Applying Lemma 2.8 shows (with $n + c - t > n + \alpha + t - 1 - t > n - 1 > 0$)

$$|f(x)| \lesssim \int_{\mathbb{B}} \frac{1}{[x, y]^{n+c-t}} (1-|x|^2)^t |D_s^t f(y)| (1-|y|^2)^{c-t} d\nu(y).$$

It now follows from Theorem 3.1 that $\|f\|_{L_\alpha^\infty} \lesssim \|(1-|x|^2)^t D_s^t f(x)\|_{L_\alpha^\infty}$. □

Before proving Theorem 1.2 for all $\alpha \in \mathbb{R}$ we mention one last elementary lemma. We include a proof for completeness.

Lemma 4.5 *Let $N \geq 1$ be an integer. Then*

$$\mathcal{R}^N = \sum_{1 \leq |m| \leq N} p_m \partial^m,$$

where p_m is a polynomial with degree equal to $|m|$.

Proof Let f be a smooth function. Then $\mathcal{R}f(x) = x \cdot \nabla f(x) = \sum_{i=1}^n x_i\, \partial f / \partial x_i$, so the lemma is true for $N = 1$. For $N = 2$ we compute

$$\mathcal{R}^2 f(x) = \sum_{j=1}^n x_j \frac{\partial}{\partial x_j} \left(\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} \right) = \sum_{i,j=1}^n x_i x_j \frac{\partial^2 f}{\partial x_j \partial x_i} + \sum_{j=1}^n x_j \frac{\partial f}{\partial x_j}$$

and the lemma is true for $N = 2$. The general case follows from induction. □

We are now ready to deal with the main part of Theorem 1.2, i.e. extending the previous lemmas to all $\alpha \in \mathbb{R}$.

Proof of Theorem 1.2 We will show (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (f) ⇒ (a); the implications (a) ⇒ (b), (c) ⇒ (d) and (e) ⇒ (f) being clear. We will refer many times to Lemmas 4.3 and 4.4 and in these cases we will make sure that the subscript of b is always greater than 0.

(b) ⇒ (c): Suppose there exists N_0 with $\alpha + N_0 > 0$ such that $(1-|x|^2)^{N_0} \partial^m f \in L_\alpha^\infty$, that is $\partial^m f \in b_{\alpha+N_0}$ for every multi-index m with $|m| = N_0$.

We first show that if m' is a multi-index with $|m'| < N_0$, then $\partial^{m'} f$ is also in $b_{\alpha+N_0}$: For $|m| = N_0$, by Lemma 4.3, $\partial^{m'}\partial^m f \in b_{\alpha+N_0+|m'|}$. Since by (3), $b_{\alpha+N_0+|m'|} \subset b_{\alpha+2N_0}$, we deduce that $\partial^m \partial^{m'} f \in b_{\alpha+2N_0}$ for every multi-index m with $|m| = N_0$ and it follows from Lemma 4.3 that $\partial^{m'} f \in b_{\alpha+N_0}$. Applying Lemma 4.5 now shows

$$\mathcal{R}^{N_0} f \in b_{\alpha+N_0}. \tag{27}$$

Suppose $N \in \mathbb{N}$ is such that $\alpha + N > 0$. If $N > N_0$, Lemma 4.3 and (27) implies $\mathcal{R}^N f = \mathcal{R}^{N-N_0}(\mathcal{R}^{N_0} f) \in b_{\alpha+N_0+(N-N_0)} = b_{\alpha+N}$. Similarly, if $N < N_0$, then $\mathcal{R}^{N_0} f = \mathcal{R}^{N_0-N}(\mathcal{R}^N f)$ and Lemma 4.3 and (27) implies $\mathcal{R}^N f \in b_{\alpha+N_0-(N_0-N)} = b_{\alpha+N}$.

(d) $\Rightarrow$ (e): Suppose there exists $N_0 \in \mathbb{N}$ with $\alpha + N_0 > 0$ such that $(1-|x|^2)^{N_0}\mathcal{R}^{N_0} f \in L^\infty_\alpha$, that is $\mathcal{R}^{N_0} f \in b_{\alpha+N_0}$. Take any $s, t \in \mathbb{R}$ such that $\alpha + t > 0$. Then by Lemma 4.4, we have $D^t_s(\mathcal{R}^{N_0} f) \in b_{\alpha+N_0+t}$. By considering their actions on homogeneous expansions it is clear that D^t_s and $\mathcal{R}^{N_0}$ commute. Therefore $\mathcal{R}^{N_0}(D^t_s f) \in b_{\alpha+N_0+t}$ and we conclude by Lemma 4.3 that $D^t_s f \in b_{\alpha+t}$.

(f) $\Rightarrow$ (a): Suppose there exists $s_0, t_0 \in \mathbb{R}$ with $\alpha + t_0 > 0$ such that $(1-|x|^2)^{t_0} D^{t_0}_{s_0} f \in L^\infty_\alpha$, that is $D^{t_0}_{s_0} \in b_{\alpha+t_0}$. Pick $c > \alpha + t_0 - 1$. Then by Lemma 4.1,

$$D^{t_0}_{s_0} f(x) = \int_{\mathbb{B}} R_c(x, y) D^{t_0}_{s_0} f(y) d\nu_c(y).$$

Applying $D^{-t_0}_{s_0+t_0}$ to both sides, using (20) on the left and pushing $D^{-t_0}_{s_0+t_0}$ into the integral by Lemma 2.3, we obtain

$$f(x) = \int_{\mathbb{B}} D^{-t_0}_{s_0+t_0} R_c(x, y)\, D^{t_0}_{s_0} f(y) d\nu_c(y).$$

Take $N \in \mathbb{N}$ with $\alpha + N > 0$ and let m be a multi-index with $|m| = N$. Then

$$\begin{aligned}\partial^m f(x) &= \partial^m \int_{\mathbb{B}} D^{-t_0}_{s_0+t_0} R_c(x, y) D^{t_0}_{s_0} f(y) d\nu_c(y) \\ &= \int_{\mathbb{B}} \partial^m \left(D^{-t_0}_{s_0+t_0} R_c(x, y)\right) D^{t_0}_{s_0} f(y) d\nu_c(y).\end{aligned}$$

Applying Lemma 2.8 (with $n + c - t_0 + N > n + \alpha + N - 1 > n - 1 > 0$), we get

$$(1-|x|^2)^N |\partial^m f(x)| \lesssim (1-|x|^2)^N \int_{\mathbb{B}} \frac{(1-|y|^2)^{t_0} |D^{t_0}_{s_0} f(y)|}{[x, y]^{n+c-t_0+N}} (1-|y|^2)^{c-t_0} d\nu(y).$$

Theorem 3.1 now implies that $(1-|x|^2)^N \partial^m f \in L^\infty_\alpha$.

By retracing the above proof it is not hard to see that (2) holds. □

Proof of Theorem 1.3 is similar to the proof of Theorem 1.2; the main difference is we refer to Remark 3.3 instead of Theorem 3.1. We omit the details.

We now show the basic properties of the spaces b_α and $b_{\alpha 0}$. As mentioned before, by (2) we can endow b_α (and its subspace $b_{\alpha 0}$) with many equivalent norms. In the sequel we will mainly use the norms induced by D^t_s: Given $\alpha \in \mathbb{R}$, pick any s, t with $\alpha + t > 0$, then $\|(1-|x|^2)^t D^t_s f\|_{L^\infty_\alpha} = \|I^t_s f\|_{L^\infty_\alpha}$ is a norm on b_α; all these norms are equivalent and we will denote any one of them by $\|\cdot\|_{b_\alpha}$ without indicating the dependence on s and t.

We first show that all b_α (resp. $b_{\alpha 0}$) are isomorphic. We emphasize that the proposition below is true for every $t \in \mathbb{R}$ without any restriction.

Proposition 4.6 *Let $\alpha \in \mathbb{R}$. For any $s, t \in \mathbb{R}$, the map $D^t_s : b_\alpha \to b_{\alpha+t}$ (resp. $D^t_s : b_{\alpha 0} \to b_{(\alpha+t)0}$) is an isomorphism and is an isometry when appropriate norms are used.*

Proof Pick u such that $\alpha + t + u > 0$. We endow b_α with the norm $\|f\|_{b_\alpha} = \|I^{t+u}_s f\|_{L^\infty_\alpha}$ and $b_{\alpha+t}$ with the norm $\|g\|_{b_{\alpha+t}} = \|I^u_{s+t} g\|_{L^\infty_{\alpha+t}}$. By (19),

$$\begin{aligned}\|D^t_s f\|_{b_{\alpha+t}} &= \|I^u_{s+t} D^t_s f\|_{L^\infty_{\alpha+t}} = \|(1-|x|^2)^u D^u_{s+t}(D^t_s f)\|_{L^\infty_{\alpha+t}} \\ &= \|(1-|x|^2)^u D^{u+t}_s f\|_{L^\infty_{\alpha+t}} = \|I^{u+t}_s f\|_{L^\infty_\alpha} = \|f\|_{b_\alpha}.\end{aligned}$$

□

For $0 < r < 1$, let $f_r : \mathbb{B} \to \mathbb{B}$, $f_r(x) = f(rx)$ be the dilate of f.

Corollary 4.7 *Let $\alpha \in \mathbb{R}$. The following properties hold:*

(a) b_α and $b_{\alpha 0}$ are complete spaces.
(b) Let $f \in b_\alpha$. Then $f_r \to f$ (as $r \to 1^-$) in b_α if and only if $f \in b_{\alpha 0}$.
(c) $b_{\alpha 0}$ is closure of polynomials in b_α.
(d) $b_{\alpha 0}$ is separable whereas b_α is inseparable.

Proof It is well known that these properties hold for b_0 and b_{00} (it is also elementary to verify them for $\alpha > 0$). The general case then follows from the isomorphism in Proposition 4.6, the fact that D^t_s maps polynomials to polynomials and the simple identity $D^t_s(f_r) = (D^t_s f)_r$. □

Fix $\zeta \in \mathbb{S}$. Then for any $q \in \mathbb{R}$, we have $R_q(\cdot, \zeta) \in h(\mathbb{B})$. In the next theorem we determine when $R_q(\cdot, \zeta)$ belongs to b_α (or $b_{\alpha 0}$) and therefore provide non-trivial (i.e. non-polynomial) examples of elements of b_α (or $b_{\alpha 0}$). This theorem will also allow us to distinguish between these spaces.

Theorem 4.8 *Let $q, \alpha \in \mathbb{R}$ and $\zeta \in \mathbb{S}$. Then*

(i) $R_q(\cdot, \zeta) \in b_\alpha$ if and only if $\alpha \geq n + q$.
(ii) $R_q(\cdot, \zeta) \in b_{\alpha 0}$ if and only if $\alpha > n + q$.

Proof Pick t large enough that $\alpha + t > 0$ and $n + q + t > 0$. By (18), we have

$$I^t_q R_q(x, \zeta) = (1-|x|^2)^t D^t_q R_q(x, \zeta) = (1-|x|^2)^t R_{q+t}(x, \zeta),$$

and Lemma 2.7 implies

$$|I_q^t R_q(x,\zeta)| \lesssim \frac{(1-|x|^2)^t}{[x,\zeta]^{n+q+t}} = \frac{(1-|x|^2)^t}{|x-\zeta|^{n+q+t}} \lesssim \frac{1}{(1-|x|^2)^{n+q}}.$$

Hence, if $\alpha \geq n+q$, then $I_q^t R_q(x,\zeta) \in L_\alpha^\infty$ and therefore $R_q(x,\zeta) \in b_\alpha$. Similarly, if $\alpha > n+q$, then $I_q^t R_q(x,\zeta) \in \mathcal{C}_{\alpha 0}$ and $R_q(x,\zeta) \in b_{\alpha 0}$.

For the reverse implications, note that by Lemma 2.9, we have

$$R_{q+t}(r\zeta,\zeta) \sim \frac{1}{(1-r^2)^{n+q+t}}, \quad 0 \leq r < 1,$$

and so $I_q^t R_q(r\zeta,\zeta) \sim (1-r^2)^{-(n+q)}$. Now, if $\alpha < n+q$, then $I_q^t R_q(x,\zeta) \notin L_\alpha^\infty$ and therefore $R_q(x,\zeta) \notin b_\alpha$. Similarly, if $\alpha \leq n+q$, then $I_q^t R_q(x,\zeta) \notin \mathcal{C}_{\alpha 0}$ and $R_q(x,\zeta) \notin b_{\alpha 0}$. □

Remark 4.9 Using the above theorem it is easy to see that the inclusions in (3) are strict. First, if $\alpha < \beta$, let $q = (\alpha+\beta)/2 - n$. Then we have $R_q(\cdot,\zeta) \in b_{\beta 0}$ but $R_q(\cdot,\zeta) \notin b_\alpha$. Next, for $\beta \in \mathbb{R}$, we have $R_{\beta-n}(\cdot,\zeta) \in b_\beta$ but $R_{\beta-n}(\cdot,\zeta) \notin b_{\beta 0}$.

5 Projections

In this section we will prove Theorem 1.6 and then apply it obtain duality results.

Proof of Theorem 1.6 We first show that $Q_s : L_\alpha^\infty \to b_\alpha$ is bounded if and only if $s > \alpha - 1$. Suppose $s > \alpha - 1$. For $\varphi \in L_\alpha^\infty$, the integral $\int_{\mathbb{B}} R_s(x,y)\varphi(y)d\nu_s(y) = Q_s\varphi(x)$ converges by (15) and is harmonic on $\mathbb{B}$. Pick t such that $\alpha + t > 0$. We need to show that $\|Q_s\varphi\|_{b_\alpha} = \|I_s^t Q_s\varphi\|_{L_\alpha^\infty} \lesssim \|\varphi\|_{L_\alpha^\infty}$. For this we note that by Corollary 2.4,

$$\begin{aligned} I_s^t Q_s\varphi(x) &= (1-|x|^2)^t D_s^t \int_{\mathbb{B}} R_s(x,y)\varphi(y)d\nu_s(y) \\ &= \frac{1}{V_s}(1-|x|^2)^t \int_{\mathbb{B}} R_{s+t}(x,y)\varphi(y)(1-|y|^2)^s d\nu(y) = \frac{1}{V_s} T_{s,t}\,\varphi(x), \end{aligned} \tag{28}$$

and Theorem 3.1 shows $\|I_s^t Q_s\varphi\|_{L_\alpha^\infty} \lesssim \|\varphi\|_{L_\alpha^\infty}$.

If $s \leq \alpha - 1$, let $\varphi(x) = (1-|x|^2)^{-\alpha}$ and ϵ be as in Lemma 3.2. Then for $|x| < \epsilon$,

$$Q_s\varphi(x) = \int_{\mathbb{B}} R_s(x,y)\varphi(y)d\nu_s(y) \geq \frac{1}{2V_s}\int_{\mathbb{B}} (1-|y|^2)^{s-\alpha} d\nu(y).$$

Since the integral on the right diverges, $Q_s\varphi$ can not be in L_α^∞.

We now show (6). Suppose (4) and (5) hold. Then $s+t > -1$. If $f \in b_\alpha$, we have $(1-|x|^2)^t D_s^t f \in L_\alpha^\infty$ and so $|D_s^t f(x)| \lesssim (1-|x|^2)^{-(\alpha+t)}$. This shows $D_s^t f \in L_{s+t}^1$ and applying Corollary 2.5 and (20), we obtain

$$\begin{aligned} Q_s I_s^t f(x) &= \frac{1}{V_s} \int_{\mathbb{B}} R_s(x,y) I_s^t f(y)(1-|y|^2)^s d\nu(y) \\ &= \frac{V_{s+t}}{V_s} \int_{\mathbb{B}} R_s(x,y) D_s^t f(y) d\nu_{s+t}(y) \\ &= \frac{V_{s+t}}{V_s} D_{s+t}^{-t} D_s^t f(x) = \frac{V_{s+t}}{V_s} f(x). \end{aligned}$$

$Q_s : L^\infty_\alpha \to b_\alpha$ is onto since $f \in b_\alpha$ implies $I_s^t f \in L^\infty_\alpha$ and $Q_s\big((V_s/V_{s+t}) I_s^t f\big) = f$ by above. Hence all the claims related to b_α hold.

We next show that under (4), Q_s maps $\mathcal{C}_\alpha$ (and therefore $\mathcal{C}_{\alpha 0}$) into $b_{\alpha 0}$. For this we first show that if p is a polynomial, then $Q_s((1-|x|^2)^{-\alpha} p)$ is a harmonic polynomial of the same degree and therefore belongs to $b_{\alpha 0}$. By linearity of Q_s, we can assume p is a homogeneous polynomial. Then, by [2, Theorem 5.7], there exists $p_j \in \mathcal{H}_j(\mathbb{R}^n)$ such that

$$p = p_k + |x|^2 p_{k-2}(x) + \cdots + |x|^{2l} p_{k-2l}(x), \tag{29}$$

where k is the degree of p and $l = [k/2]$. We have

$$Q_s\big((1-|x|^2)^{-\alpha} p\big)(x) = \int_{\mathbb{B}} R_s(x,y)(1-|y|^2)^{-\alpha} p(y) d\nu_s(y)$$

and using uniform convergence of the series expansion (11), (29), integrating in polar coordinates and using Lemma 2.1 (d) and (e) we obtain that $Q_s((1-|x|^2)^{-\alpha} p)$ is a harmonic polynomial of degree k.

Now, if $\varphi \in \mathcal{C}_\alpha$, then $(1-|x|^2)^\alpha \varphi =: \psi \in C(\overline{\mathbb{B}})$. By Stone-Weierstrass theorem we can approximate ψ with polynomials and therefore we can find a sequence (p_i) of polynomials such that $\|\varphi - (1-|x|^2)^{-\alpha} p_i\|_{L^\infty_\alpha} \to 0$ (as $i \to \infty$). Boundedness of $Q_s : L^\infty_\alpha \to b_\alpha$ shows $Q_s\big((1-|x|^2)^{-\alpha} p_i\big) \to Q_s(\varphi)$ and by completeness of $b_{\alpha 0}$, we conclude $Q_s \varphi \in b_{\alpha 0}$.

To see that (4) is necessary for boundedness of Q_s on $\mathcal{C}_\alpha$ or $\mathcal{C}_{\alpha 0}$, suppose $s \le \alpha - 1$ and let $\varphi(x) = (1-|x|^2)^{-\alpha} / \big(1 + \log((1-|x|^2)^{-1}\big)$. Then $\varphi \in \mathcal{C}_{\alpha 0}$, but by Lemma 3.2, $Q_s \varphi(x)$ diverges for sufficiently small $|x|$.

That Q_s maps $\mathcal{C}_{\alpha 0}$ (and $\mathcal{C}_\alpha$) onto $b_{\alpha 0}$ follows from (6) and Theorem 1.3.

Remark 5.1 In case $\alpha > 0$, we have $b_\alpha \subset L^\infty_\alpha$ and for an s satisfying (4), Q_s is a true projection on L^∞_α with range b_α (that $Q_s^2 = Q_s$ follows from (6) by choosing $t = 0$). When $\alpha \le 0$, b_α is no longer a subspace of L^∞_α but for t satisfying (5), $I_s^t(b_\alpha)$ is an isometric copy of b_α in L^∞_α. Under (4) and (5), the operator

$$P = \frac{V_s}{V_{s+t}} I_s^t Q_s$$

satisfies $P^2 = P$ by (6) and is a projection on L^∞_α with range $I_s^t(b_\alpha)$, the isometric copy .

We record the following relations between the operators Q_s, I_s^t and $T_{s,t}$.

Corollary 5.2 *Let $\alpha \in \mathbb{R}$. Suppose s, t satisfy (4) and (5). The following operator identities hold:*

(a) $Q_s I_s^t = \dfrac{V_{s+t}}{V_s} I$ *on* b_α.
(b) $T_{s,t} I_s^t = V_{s+t}\, I_s^t$ *on* b_α.
(c) $I_s^t Q_s = \dfrac{1}{V_s} T_{s,t}$ *on* L^∞_α.
(d) $Q_s T_{s,t} = V_{s+t}\, Q_s$ *on* L^∞_α.

Proof Part (a) is just (6) and part (c) is (28). The other parts follow from these. □

An immediate consequence of (7) is the following growth estimate. The $\alpha > 0$ part of this estimate is clear by definition of b_α.

Corollary 5.3 *Let $\alpha \in \mathbb{R}$ and $f \in b_\alpha$. Then*

$$|f(x)| \lesssim \|f\|_{b_\alpha} \begin{cases} (1-|x|^2)^{-\alpha}, & \text{if } \alpha > 0; \\ 1 + \log \dfrac{1}{1-|x|^2}, & \text{if } \alpha = 0; \\ 1, & \text{if } \alpha < 0, \end{cases}$$

for every $x \in \mathbb{B}$.

Proof Pick s, t such that (4) and (5) holds. Then by (7),

$$f(x) = \frac{V_s}{V_{s+t}} \int_{\mathbb{B}} R_s(x, y) I_s^t f(y)\, d\nu_s(y).$$

Using $|I_s^t f(y)| \lesssim (1-|x|^2)^{-\alpha} \|I_s^t f\|_{L^\infty_\alpha}$ and $\|f\|_{b_\alpha} = \|I_s^t f\|_{L^\infty_\alpha}$ and applying Lemma 2.10, we get the above estimates. □

5.1 Duality

When $1 < p < \infty$ and $q > -1$, it is well-known that $(b^p_q)'$, the dual of the harmonic Bergman space b^p_q, can be identified with $b^{p'}_q$, where $1/p + 1/p' = 1$. It is shown in [8, Theorem 13.4] that this statement is true for all $q \in \mathbb{R}$. Our aim in this subsection is to show that $(b^1_q)'$ can be identified with b_α and $(b_{\alpha 0})'$ can be identified with b^1_q. Here, $q, \alpha \in \mathbb{R}$ without any restriction and the aforementioned identification can be obtained using many different pairings. More precisely, we have the following.

Theorem 5.4 *Let $q \in \mathbb{R}$. Pick s, t such that*

$$s > q, \tag{30}$$
$$q + t > -1. \tag{31}$$

The dual of b^1_q can be identified with b_α (for any $\alpha \in \mathbb{R}$) under the pairing

$$\langle f, g\rangle = \int_{\mathbb{B}} I^t_s f \, \overline{I^{s-q-\alpha}_{t+q+\alpha} g} \, d\nu_{q+\alpha}, \qquad (f \in b^1_q, \ g \in b_\alpha). \tag{32}$$

Before proving Theorem 5.4 we need to review a few facts from [8]. Recall that in Sect. 2.2 we defined harmonic Bergman–Besov spaces b^p_q in terms of partial derivatives. Analogous to Theorems 1.2 and 1.3 we have the following: Given $1 \le p < \infty$ and $q \in \mathbb{R}$, pick $s, t \in \mathbb{R}$ with $q + pt > -1$. Then $f \in h(\mathbb{B})$ belongs to b^p_q if and only if $I^t_s f \in L^p_q$ and $\|I^t_s f\|_{L^p_q}$ is a norm on b^p_q (see Theorem 1.2 of [8]).

The following projection theorem for b^p_q spaces is Theorem 1.4 of [8].

Theorem A (See [8]). *Let $1 \le p < \infty$ and $q \in \mathbb{R}$. Then $Q_s : L^p_q \to b^p_q$ is bounded (and onto) if and only if*

$$q + 1 < p(s + 1). \tag{33}$$

Given s satisfying (33) if t satisfies

$$q + pt > -1, \tag{34}$$

then for $f \in b^p_q$, we have $Q_s I^t_s f = \dfrac{V_{s+t}}{V_s} f$.

The following theorem is Corollary 11.1 of [8]. It is similar to Corollary 5.2 but it is for the spaces L^p_q and b^p_q.

Theorem B (See [8]) *Let $1 \le p < \infty$ and $q \in \mathbb{R}$. If (33) and (34) holds, then*

(a) $Q_s I^t_s = \dfrac{V_{s+t}}{V_s} I$ *on* b^p_q.
(b) $T_{s,t} I^t_s = V_{s+t} \, I^t_s$ *on* b^p_q.
(c) $I^t_s Q_s = \dfrac{1}{V_s} T_{s,t}$ *on* L^p_q.
(d) $Q_s T_{s,t} = V_{s+t} \, Q_s$ *on* L^p_q.

We are now ready to prove Theorem 5.4.

Proof of Theorem 5.4 First, note that (30) and (31) are just (33) and (34) with $p = 1$. Let $t' = s - q - \alpha$ and $s' = t + q + \alpha$. Then by (30) and (31), we have

$$s' > \alpha - 1, \tag{35}$$
$$\alpha + t' > 0. \tag{36}$$

If $g \in b_\alpha$, then $I^{t'}_{s'} g \in L^\infty_\alpha$ by (36) and if $f \in b^1_q$, then $I^t_s f \in L^1_q$ by (31). Therefore the pairing (32) defines a linear functional on b^1_q.

Conversely, let $\mathcal{L} \in (b^1_q)'$. We will show that there exists $g \in b_\alpha$ such that $\mathcal{L}(f) = \langle f, g\rangle$. We begin by observing that $\mathcal{L} \circ Q_s \in (L^1_q)'$ by Theorem A and (30). Therefore by Riesz representation theorem there exists $\chi \in L^\infty$ such that for $\varphi \in L^1_q$,

$$\mathcal{L}\, Q_s(\varphi) = \int_{\mathbb{B}} \varphi\, \overline{\chi}\, d\nu_q = \int_{\mathbb{B}} \varphi\, \overline{\psi}\, d\nu_{q+\alpha},$$

where we set $\psi(x) := (1-|x|^2)^{-\alpha}\chi(x)$. It is clear that $\psi \in L^\infty_\alpha$. For $f \in b^1_q$ we have $I^t_s f \in L^1_q$ and by first part (a) and then part (b) of Corollary B,

$$\mathcal{L}(f) = \frac{V_s}{V_{s+t}}\, \mathcal{L} Q_s I^t_s f = \frac{V_s}{V_{s+t}} \int_{\mathbb{B}} I^t_s f\, \overline{\psi}\, d\nu_{q+\alpha} = \frac{V_s}{V^2_{s+t}} \int_{\mathbb{B}} T_{s,t} I^t_s f\, \overline{\psi}\, d\nu_{q+\alpha}.$$

Explicitly writing the action of $T_{s,t}$, and then applying Fubini's theorem and Corollary 5.2 (c), we obtain

$$\mathcal{L}(f) = \frac{V_s}{V^2_{s+t}} \int_{\mathbb{B}} I^t_s f\, \overline{T_{s',t'}\psi}\, d\nu_{q+\alpha} = \frac{V_s V_{s'}}{V^2_{s+t}} \int_{\mathbb{B}} I^t_s f\, \overline{I^{t'}_{s'} Q_{s'}\psi}\, d\nu_{q+\alpha}.$$

Let $g = \dfrac{V_s V_{s'}}{V^2_{s+t}} Q_{s'}\psi$. Then $g \in b_\alpha$ by Theorem 1.6 and (35), and $\mathcal{L}(f) = \langle f, g\rangle$. Retracing the above proof it is easy to see that $\|g\|_{b_\alpha} \sim \|\mathcal{L}\|$. Finally uniqueness of g follows from the uniqueness part of Riesz representation theorem. □

We now consider the dual of $b_{\alpha 0}$.

Theorem 5.5 *Let $\alpha \in \mathbb{R}$. Pick s, t such that*

$$s > \alpha - 1,$$
$$\alpha + t > -1.$$

The dual of $b_{\alpha 0}$ can be identified with b^1_q (for any $q \in \mathbb{R}$) under the pairing

$$\langle f, g\rangle = \int_{\mathbb{B}} I^t_s f\, \overline{I^{s-q-\alpha}_{t+q+\alpha}\, g}\, d\nu_{q+\alpha}, \qquad (f \in b_{\alpha 0},\ g \in b^1_q).$$

Proof The proof is very similar to the previous proof. We interchange the roles of Theorem 1.6 and Theorem A, and also interchange the roles of Corollary 5.2 and Corollary B, and make minor modifications. We omit the details. □

For $\alpha = 0$ and $q > -1$ Theorems 5.4 and 5.5 have been proved in [11,14,22]. For the *holomorphic* analogues of these theorems with the full range $\alpha \in \mathbb{R}$ and $q \in \mathbb{R}$, see [13,26]. The pairings in [13] are exact holomorphic counterparts of our pairings, whereas in [26] slightly different pairings (involving a limit) are used.

6 Gleason Problem, Atomic Decomposition and Oscillatory Characterization

6.1 Gleason Problem

Let $a \in \mathbb{B}$. The Gleason problem for weighted harmonic Bloch (or little Bloch) spaces is to determine whether there exist bounded operators $A_1, A_2, \dots, A_n : b_\alpha \to b_\alpha$ (or $b_{\alpha 0} \to b_{\alpha 0}$) such that

$$f(x) - f(a) = \sum_{j=1}^{n} (x_j - a_j)\, A_j f(x).$$

This problem is solved in the affirmative for $a = 0, \alpha = 0$ in [4] and for general $a \in \mathbb{B}$ and $\alpha > -1$ in [18]. In [10] this problem is solved not just for $\mathbb{B}$ but for bounded convex domains with C^2 boundary, but still with the restriction $\alpha > -1$.

Our aim here is to solve Gleason problem for all $\alpha \in \mathbb{R}$. The main ingredients of our proof are the reproducing formula (7) and the kernel estimates of Sect. 2.4.

Theorem 6.1 *Let $\alpha \in \mathbb{R}$ and $a \in \mathbb{B}$. There exist bounded linear operators $A_1, A_2, \dots, A_n$ on b_α (respectively $b_{\alpha 0}$) such that for all $f \in b_\alpha$ (respectively $b_{\alpha 0}$)*

$$f(x) - f(a) = \sum_{j=1}^{n} (x_j - a_j)\, A_j f(x), \quad \forall x \in \mathbb{B}. \tag{37}$$

Proof Let $f \in b_\alpha$. For $x \in \mathbb{B}$, by calculus,

$$\begin{aligned} f(x) - f(a) &= \int_0^1 \nabla f\big(\tau x + (1-\tau)a\big) \cdot (x - a)\, d\tau \\ &= \sum_{j=1}^{n} (x_j - a_j) \int_0^1 \partial_j f\big(\tau x + (1-\tau)a\big)\, d\tau, \end{aligned}$$

where we write $\partial_j f$ for $\partial f / \partial x_j$. Defining A_j by $A_j f(x) = \int_0^1 \partial_j f\big(\tau x + (1-\tau)a\big) d\tau$, it is obvious that (37) holds. It is also clear by differentiating under the integral that $A_j f \in h(\mathbb{B})$. We proceed to show that A_j is bounded on b_α. For this we pick $N \in \mathbb{N}$ with $\alpha + N > 0$. By Theorem 1.2 it suffices to show that

$$\sum_{|m| \le N-1} |(\partial^m A_j f)(0)| + \sum_{|m| = N} \|(1 - |x|^2)^N \partial^m A_j f\|_{L^\infty_\alpha} \lesssim \|f\|_{b_\alpha}.$$

We choose $s, t \in \mathbb{R}$ so that (4) and (5) holds and in addition we choose $s > -n$. Then $I^t_s f \in L^\infty_\alpha$ and $\|f\|_{b_\alpha} \sim \|I^t_s f\|_{L^\infty_\alpha}$. By (7),

$$f(x) = \frac{V_s}{V_{s+t}} \int_{\mathbb{B}} R_s(x, y)\, I^t_s f(y)\, d\nu_s(y),$$

and so

$$A_j f(x) = \frac{V_s}{V_{s+t}} \int_0^1 \int_{\mathbb{B}} (\partial_j R_s)(\tau x + (1-\tau)a, y)\, I_s^t f(y)\, d\nu_s(y)\, d\tau,$$

where pushing the derivative into the integral is possible by (15). Let m be a multi-index with $|m| \leq N$. Differentiating and using the chain rule we obtain

$$\partial^m A_j f(x) = \frac{V_s}{V_{s+t}} \int_0^1 \tau^{|m|} \int_{\mathbb{B}} (\partial^m \partial_j R_s)(\tau x + (1-\tau)a, y)\, I_s^t f(y)\, d\nu_s(y)\, d\tau.$$

Application of Lemma 2.7 and Fubini's theorem gives

$$|\partial^m A_j f(x)| \lesssim \int_{\mathbb{B}} |I_s^t f(y)| \int_0^1 \frac{1}{[\tau x + (1-\tau)a, y]^{n+s+|m|+1}}\, d\tau\, d\nu_s(y).$$

The inner integral is estimated in [18, Lemma 2.1] where it is shown that

$$\int_0^1 \frac{1}{[\tau x + (1-\tau)a, y]^{n+s+|m|+1}}\, d\tau \lesssim \frac{1}{[x, y]^{n+s+|m|}}.$$

Therefore

$$|\partial^m A_j f(x)| \lesssim \int_{\mathbb{B}} \frac{1}{[x, y]^{n+s+|m|}}\, |I_s^t f(y)|\, (1-|y|^2)^s\, d\nu(y).$$

If $|m| = N$, then it follows from Theorem 3.1 that

$$\|(1-|x|^2)^N \partial^m A_j f\|_{L^\infty_\alpha} \lesssim \|I_s^t f\|_{L^\infty_\alpha}.$$

If $|m| \leq N-1$, then since $[0, y] = 1$, we have

$$\begin{aligned}|\partial^m A_j f(0)| &\lesssim \int_{\mathbb{B}} |I_s^t f(y)|(1-|y|^2)^s d\nu(y) \leq \|I_s^t f\|_{L^\infty_\alpha} \int_{\mathbb{B}} (1-|y|^2)^{s-\alpha} d\nu(y)\\ &\lesssim \|I_s^t f\|_{L^\infty_\alpha}.\end{aligned}$$

We conclude that A_j is bounded on b_α.

For $b_{\alpha 0}$ we repeat the same argument. In this case $f \in b_{\alpha 0}$ implies $I_s^t f \in \mathcal{C}_{\alpha 0}$ by Theorem 1.3 and at the end we refer to Remark 3.3 instead of Theorem 3.1. □

6.2 Atomic Decomposition

Atomic decomposition for the standard harmonic Bloch space b_0 (and the little Bloch space b_{00}) is obtained first in [6]. That result is slightly extended in [5] where a different proof based on Möbius transformations is given. Their result is generalized in [23] to standard harmonic Bloch space on smooth bounded domains.

Here, by using the isomorphism in Proposition 4.6, we will provide atomic decomposition for all b_α, $\alpha \in \mathbb{R}$.

We first introduce some definitions. For details see [1,5] or [6]. Let $a \in \mathbb{B}$. The canonical Möbius transformation φ_a on $\mathbb{B}$ that exchanges a and 0 is given by

$$\varphi_a(x) = \frac{(1-|a|^2(a-x)+|a-x|^2 a}{[x,a]^2}.$$

The pseudohyperbolic metric $\rho(x, y)$ on $\mathbb{B}$ is defined by $\rho(x, y) = |\varphi_x(y)|$. Clearly, $0 \le \rho(x, y) < 1$ and a straightforward computation shows $\rho(x, y) = \dfrac{|x-y|}{[x,y]}$. The pseudohyperbolic ball with center a and radius r, $0 < r < 1$ is $E_r(a) = \{x \in \mathbb{B} : \rho(x, a) < 1\}$.

Let $0 < r < 1$ and (x_m) be a sequence in $\mathbb{B}$. The sequence (x_m) is called r-separated if the pseudohyperbolic balls $E_r(x_m)$ are pairwise disjoint. The sequence (x_m) is called an r-lattice if $\bigcup_m E_r(x_m) = \mathbb{B}$ and (x_m) is $r/2$-separated.

The following theorem gives the atomic decomposition for b_0 and b_{00}. Here, $\ell^\infty = \{(x_m) : x_m \text{ is bounded}\}$ and $c_0 = \{(x_m) : \lim_{m\to\infty} x_m = 0\}$.

Theorem C (See [5,6]). *Let $s > -1$. There exists $\delta_0 = \delta_0(s) > 0$ with the following property: If (x_m) is a δ-lattice with $\delta < \delta_0$, then for $f \in b_0$, there exists $(\lambda_m) \in \ell^\infty$ such that $\|\lambda_m\|_{\ell^\infty} \sim \|f\|_{b_0}$ and*

$$f(x) = \sum_m \lambda_m (1-|x_m|^2)^{s+n} R_s(x, x_m).$$

If additionally $f \in b_{00}$, then $(\lambda_m) \in c_0$.

The following is generalization of the above theorem to all $\alpha \in \mathbb{R}$.

Theorem 6.2 *Let $\alpha \in \mathbb{R}$ and $s > \alpha - 1$. There exists $\delta_0 = \delta_0(\alpha, s) > 0$ with the following property: If (x_m) is a δ-lattice with $\delta < \delta_0$, then for $f \in b_\alpha$, there exists $(\lambda_m) \in \ell^\infty$ such that $\|\lambda_m\|_{\ell^\infty} \sim \|f\|_{b_\alpha}$ and*

$$f(x) = \sum_m \lambda_m (1-|x_m|^2)^{s-\alpha+n} R_s(x, x_m). \tag{38}$$

If additionally $f \in b_{\alpha 0}$, then $(\lambda_m) \in c_0$.

Remark 6.3 The index s of the reproducing kernel R_s in (38) need not be greater than -1, in these cases extended kernels are involved.

Proof of Theorem 6.2 Let $\delta_0 = \delta_0(s-\alpha)$ be as provided by Theorem C and (x_m) be a δ-lattice with $\delta < \delta_0$. Let $f \in b_\alpha$. Then by Proposition 4.6, $D_s^{-\alpha} f \in b_0$ and $\|D_s^{-\alpha} f\|_{b_0} \sim \|f\|_{b_\alpha}$. Applying Theorem C with s replaced by $s-\alpha$ we see that there exists $(\lambda_m) \in \ell^\infty$ such that

$$D_s^{-\alpha} f(x) = \sum_m \lambda_m (1-|x_m|^2)^{s-\alpha+n} R_{s-\alpha}(x, x_m), \tag{39}$$

with $\|\lambda_m\|_{\ell^\infty} \sim \|D_s^{-\alpha} f\|_{b_0} \sim \|f\|_{b_\alpha}$. We now apply $D_{s-\alpha}^{\alpha}$ to both sides of (39). Using the fact that (x_m) is a δ-lattice it is not hard to see that the series on the right of (39) uniformly converges on compact subsets. Therefore we can push $D_{s-\alpha}^{\alpha}$ into the sum by Lemma 2.2 and obtain

$$D_{s-\alpha}^{\alpha} D_s^{-\alpha} f(x) = \sum_m \lambda_m (1-|x_m|^2)^{s-\alpha+n} D_{s-\alpha}^{\alpha} R_{s-\alpha}(x, x_m).$$

(38) now follows from (20) and (18). The claim for $b_{\alpha 0}$ is justified similarly. □

6.3 Oscillatory Characterization

Throughout this subsection we will assume $\alpha > -1$. Then one can choose $N = 1$ in Theorem 1.2 (a) and therefore $f \in h(\mathbb{B})$ belongs to b_α if and only if $\|(1-|x|^2)\,|\nabla f(x)|\,\|_{L_\alpha^\infty} < \infty$. Our aim is to show that instead of ∇f one can characterize b_α in terms of growth rate of $|f(x)-f(y)|/|x-y|$. For the standard harmonic Bloch space $b_0 = b$, the following result is shown in [19].

Theorem D (See [19]) *Let $f \in h(\mathbb{B})$ Then $f \in b_0$ if and only if*

$$K(f) := \sup_{x,y\in\mathbb{B},\, x\neq y} (1-|x|^2)^{1/2}(1-|y|^2)^{1/2} \frac{|f(x)-f(y)|}{|x-y|} < \infty.$$

Moreover, $K(f) \sim \|(1-|x|^2)\,|\nabla f(x)|\,\|_{L^\infty} \sim \|f - f(0)\|_{b_0}$.

The analogue of Theorem D for standard *holomorphic* Bloch space $\mathcal{B}$ on the unit ball of $\mathbb{C}^n$ is proved in [9] for $n = 1$, in [21] still with $n = 1$ but with an elementary proof and in [20] for arbitrary n.

In [25], Zhao generalized the result of [20] in two directions. First he showed that the power $1/2$ of $1-|x|^2$ and $1-|y|^2$ can be replaced by λ and $1-\lambda$ for suitable λ, second a similar result holds for weighted holomorphic α-Bloch spaces $\mathcal{B}^\alpha$ for $-1 < \alpha \leq 1$. (We note that [25] uses a slightly different notation than ours and writes $\alpha + 1$ where we write α and therefore denotes the standard Bloch space by $\mathcal{B}^1$).

We will first show the harmonic counterpart of the main result of [25] and after that we will consider the case $\alpha > 1$.

Theorem 6.4 *Let $-1 < \alpha \leq 1$. Let λ satisfy the following properties:*

(1) $0 \leq \lambda \leq \alpha + 1$ if $-1 < \alpha < 0$,
(2) $0 < \lambda < 1$ if $\alpha = 0$,
(3) $\alpha \leq \lambda \leq 1$ if $0 < \alpha \leq 1$.

Then $f \in h(\mathbb{B})$ belongs to b_α if and only if

$$K_{\alpha,\lambda}(f) := \sup_{x,y\in\mathbb{B},\, x\neq y} (1-|x|^2)^{\lambda}(1-|y|^2)^{\alpha+1-\lambda} \frac{|f(x)-f(y)|}{|x-y|} < \infty. \tag{40}$$

Moreover, $K_{\alpha,\lambda}(f) \sim \|(1-|x|^2)\,|\nabla f(x)|\,\|_{L_\alpha^\infty} \sim \|f - f(0)\|_{b_\alpha}$.

Choosing in each case $\lambda = (\alpha + 1)/2$ we obtain the following symmetric form.

Corollary 6.5 *Let* $-1 < \alpha \leq 1$ *and* $f \in h(\mathbb{B})$. *Then*

$$f \in b_\alpha \iff \sup_{x,y\in\mathbb{B},\, x\neq y} (1-|x|^2)^{(\alpha+1)/2}(1-|y|^2)^{(\alpha+1)/2} \frac{|f(x)-f(y)|}{|x-y|} < \infty.$$

One can prove Theorem 6.4 following the arguments of [25]. Here we want to give a different proof based on the integral representation (7) and the following estimate of the oscillation of reproducing kernels.

Theorem E (See [24]) *Let* $s > -n$ *and* $0 \leq \tau \leq 1$. *Then*

$$\frac{|R_s(x,u) - R_s(y,u)|}{|x-y|} \lesssim \frac{1}{[x,y]^{1-\tau}} \left(\frac{1}{[x,u]^{n+s+\tau}} + \frac{1}{[y,u]^{n+s+\tau}} \right),$$

for every $x, y, u \in \mathbb{B}$ *with* $x \neq y$.

Proof of Theorem 6.4 Let $-1 < \alpha \leq 1$ and λ satisfy the given properties. Suppose additionally that in case $-1 < \alpha < 0$, we have $0 < \lambda < \alpha + 1$. We will separately consider the cases $-1 < \alpha < 0$ and $\lambda = 0$ or $\lambda = \alpha + 1$.

Pick any s, t such that (4) and (5) holds. Since $K_{\alpha,\lambda}(f - f(0)) = K_{\alpha,\lambda}(f)$, we can assume $f(0) = 0$. Then by Theorem 1.2, $\|f\|_{b_\alpha} = \|I_s^t f\|_{L^\infty_\alpha} \sim \|(1 - |x|^2)\, |\nabla f(x)|\,\|_{L^\infty_\alpha}$. By the integral representation (7), we have

$$f(x) = \frac{V_s}{V_{s+t}} \int_{\mathbb{B}} R_s(x,u) I_s^t f(u)\, d\nu_s(u)$$

and therefore

$$\frac{|f(x)-f(y)|}{|x-y|} \lesssim \int_{\mathbb{B}} \frac{|R_s(x,u) - R_s(y,u)|}{|x-y|} |I_s^t f(u)|\, (1-|u|^2)^s d\nu(u).$$

Using that $(1-|u|^2)^\alpha |I_s^t f(u)| \leq \|I_s^t f\|_{L^\infty_\alpha} = \|f\|_{b_\alpha}$ and applying Theorem E we obtain

$$K_{\alpha,\lambda}(f) \lesssim \|f\|_{b_\alpha} \frac{(1-|x|^2)^\lambda (1-|y|^2)^{\alpha+1-\lambda}}{[x,y]^{1-\tau}} \left(\int_{\mathbb{B}} \frac{(1-|u|^2)^{s-\alpha}}{[x,u]^{n+s+\tau}}\, d\nu(u) + \int_{\mathbb{B}} \frac{(1-|u|^2)^{s-\alpha}}{[y,u]^{n+s+\tau}}\, d\nu(u) \right). \tag{41}$$

We now choose $0 \leq \tau \leq 1$ in the following ranges:

(1) $-\alpha < \tau \leq \min\{\lambda - \alpha, 1 - \lambda\}$ if $-1 < \alpha < 0$ and $0 < \lambda < \alpha + 1$,
(2) $0 < \tau \leq \min\{\lambda, 1 - \lambda\}$ if $\alpha = 0$ and $0 < \lambda < 1$,
(3) $0 \leq \tau \leq \min\{\lambda - \alpha, 1 - \lambda\}$ if $0 < \alpha \leq 1$ and $\alpha \leq \lambda \leq 1$.

Then the following inequalities will hold:

$$\alpha + \tau > 0, \tag{42}$$
$$\tau \leq 1 - \lambda, \tag{43}$$
$$\tau \leq \lambda - \alpha. \tag{44}$$

Estimating the integrals in (41) by Lemma 2.11 (with $\alpha + \tau > 0$) gives

$$K_{\alpha,\lambda}(f) \lesssim \|f\|_{b_\alpha} \left(\frac{(1-|x|^2)^\lambda (1-|y|^2)^{\alpha+1-\lambda}}{[x,y]^{1-\tau}(1-|x|^2)^{\alpha+\tau}} + \frac{(1-|x|^2)^\lambda (1-|y|^2)^{\alpha+1-\lambda}}{[x,y]^{1-\tau}(1-|y|^2)^{\alpha+\tau}} \right).$$

Since $[x, y] \geq 1 - |x|$ and $[x, y] \geq 1 - |y|$ and $1 - \tau \geq \lambda$ and $1 - \tau \geq \alpha + 1 - \lambda$ by (43) and (44), we conclude $K_{\alpha,\lambda}(f) \lesssim \|f\|_{b_\alpha} \lesssim \|(1-|x|^2)\,|\nabla f(x)|\,\|_{L^\infty_\alpha}$.

We now deal with the remaining case $-1 < \alpha < 0$ and $\lambda = 0$ or $\lambda = \alpha + 1$. For this part we follow the proof of [19]. Let $-1 < \alpha < 0$ and $\lambda = 0$ (the case $\lambda = \alpha + 1$ will follow from symmetry) and let $f \in b_\alpha$. As before we can assume $f(0) = 0$ and therefore $\|(1-|x|^2)\,|\nabla f(x)|\,\|_{L^\infty_\alpha} \sim \|f\|_{b_\alpha}$. By calculus,

$$f(x) - f(y) = \int_0^1 \nabla f(tx + (1-t)y) \cdot (x - y)\, dt$$

and therefore

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \int_0^1 |\nabla f(tx + (1-t)y)|\, dt.$$

Using that $|\nabla f(tx + (1-t)y)| \lesssim (1 - |tx + (1-t)y|^2)^{-(\alpha+1)} \|f\|_{b_\alpha}$, we deduce

$$\frac{|f(x) - f(y)|}{|x - y|} \lesssim \|f\|_{b_\alpha} \int_0^1 \frac{dt}{(1 - |tx + (1-t)y|^2)^{\alpha+1}}.$$

Since $|tx + (1-t)y| \leq t|x| + (1-t)|y| \leq t + (1-t)|y|$, we have

$$1 - |tx + (1-t)y|^2 \geq 1 - |tx + (1-t)y| \geq (1-t)(1-|y|).$$

Hence

$$K_{\alpha,0} = (1-|y|^2)^{\alpha+1} \frac{|f(x) - f(y)|}{|x - y|} \lesssim \|f\|_{b_\alpha} \int_0^1 \frac{dt}{t^{\alpha+1}} \lesssim \|f\|_{b_\alpha}$$
$$\lesssim \|(1-|x|^2)\,|\nabla f(x)|\,\|_{L^\infty_\alpha}.$$

To see the "if" part, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the i^{th} slot be the standard i^{th} basis vector. Then

$$\frac{\partial f}{\partial x_i}(x) = \lim_{t \to 0} \frac{f(x + te_i) - f(x)}{t}.$$

By (40), we have

$$|f(x+te_i)-f(x)| \le \frac{t\,K_{\alpha,\lambda}(f)}{(1-|x|^2)^{\lambda}(1-|x+te_i|^2)^{\alpha+1-\lambda}}$$

and therefore

$$\left|\frac{\partial f}{\partial x_i}(x)\right| \le \frac{K_{\alpha,\lambda}(f)}{(1-|x|^2)^{\alpha+1}}.$$

Hence $\|(1-|x|^2)\,|\nabla f(x)|\,\|_{L^\infty_\alpha} \lesssim K_{\alpha,\lambda}(f)$. □

Remark 6.6 In case $-1<\alpha<0$, by Theorem 1.2 we have

$$f \in b_\alpha \iff \sup_{x\in\mathbb{B}} (1-|x|^2)^{\alpha+1}|\nabla f(x)| < \infty.$$

On the other hand choosing $\lambda=\alpha+1$ in Theorem 6.4 shows

$$f \in b_\alpha \iff \sup_{x,y\in\mathbb{B},\,x\neq y} (1-|x|^2)^{\alpha+1}\frac{|f(x)-f(y)|}{|x-y|} < \infty.$$

So in this case one can replace $|\nabla f|$ in the definition of b_α with $|f(x)-f(y)|/|x-y|$.

The conditions on α and λ in Theorem 6.4 are all unimprovable. As an example, let us consider the case $\alpha=0$ and show that Theorem 6.4 is not true for $\lambda=0$ (by symmetry this will show that Theorem 6.4 is not true also for $\lambda=1$).

Pick $\zeta\in\mathbb{S}$ and let $f(x)=R_{-n}(x,\zeta)$. Then $f\in b_0$ by Theorem 4.8. On the other hand by Lemma 2.9, for $x=r\zeta$,

$$f(r\zeta)=R_{-n}(r\zeta,\zeta)\sim 1+\log\frac{1}{1-r^2}.$$

Therefore

$$\sup_{x\in\mathbb{B},\,x\neq 0}\frac{|f(x)-f(0)|}{|x|} \ge \sup_{0<r<1}\frac{1}{r}\log\frac{1}{1-r^2}=\infty$$

and so,

$$K_{0,0}(f)=\sup_{x,y\in\mathbb{B},\,x\neq y}(1-|y|^2)\frac{|f(x)-f(y)|}{|x-y|}=\infty.$$

In the other cases we argue similarly: We let $f(x)=R_{\alpha-n}(x,\zeta)$ for $\zeta\in\mathbb{S}$. Then Theorem 4.8 implies $f\in b_\alpha$. On the other hand using the estimate for $f(r\zeta)$ in Lemma 2.9 one can easily show that if λ is outside the given ranges, then $K_{\alpha,\lambda}(f)=\infty$.

Note that in case $\alpha = 1$, the only choice for λ in Theorem 6.4 is $\lambda = 1$ and for $f \in h(\mathbb{B})$ we have

$$f \in b_1 \Longleftrightarrow \sup_{x,y\in\mathbb{B},\, x\neq y} (1-|x|^2)(1-|y|^2)\frac{|f(x)-f(y)|}{|x-y|} < \infty. \tag{45}$$

We now generalize (45) to $\alpha \geq 1$ in the following form

Theorem 6.7 *Let $\alpha \geq 1$ and $f \in h(\mathbb{B})$. Then $f \in b_\alpha$ if and only if*

$$M_\alpha(f) := \sup_{x,y\in\mathbb{B},\, x\neq y} \frac{(1-|x|^2)^\alpha(1-|y|^2)^\alpha}{[x,y]^{\alpha-1}} \frac{|f(x)-f(y)|}{|x-y|} < \infty.$$

Moreover, $M_\alpha(f) \sim \|f - f(0)\|_{b_\alpha}$.

Proof The proof is similar to the proof of Theorem 6.4. Let $f \in b_\alpha$. Pick $s > \alpha - 1$. By (7) we have (with $t = 0$)

$$f(x) = \int_{\mathbb{B}} R_s(x,u) f(u) d\nu_s(u)$$

and so

$$\frac{|f(x)-f(y)|}{|x-y|} \lesssim \int_{\mathbb{B}} \frac{|R_s(x,u)-R_s(y,u)|}{|x-y|} |f(u)| (1-|u|^2)^s d\nu(u).$$

Applying Theorem E with $\tau = 0$ and using that $|f(u)| \leq (1-|u|^2)^{-\alpha}\|f\|_{b_\alpha}$ we obtain

$$M_\alpha(f) \lesssim \|f\|_{b_\alpha} \frac{(1-|x|^2)^\alpha(1-|y|^2)^\alpha}{[x,y]^\alpha} \left(\int_{\mathbb{B}} \frac{(1-|u|^2)^{s-\alpha}}{[x,u]^{n+s}} d\nu(u) + \int_{\mathbb{B}} \frac{(1-|u|^2)^{s-\alpha}}{[y,u]^{n+s}} d\nu(u) \right).$$

Estimating the above integrals with Lemma 2.11 shows

$$M_\alpha(f) \lesssim \|f\|_{b_\alpha} \frac{(1-|x|^2)^\alpha(1-|y|^2)^\alpha}{[x,y]^\alpha} \left(\frac{1}{(1-|x|^2)^\alpha} + \frac{1}{(1-|y|^2)^\alpha} \right).$$

Since $[x,y] \geq 1-|x|$ and $[x,y] \geq 1-|y|$ we deduce $M_\alpha(f) \lesssim \|f\|_{b_\alpha}$. Replacing f with $f - f(0)$ and noting that $M_\alpha(f - f(0)) = M_\alpha(f)$ we conclude $M_\alpha(f) \lesssim \|f - f(0)\|_{b_\alpha}$.

The proof of "if" part is same as the proof of the "if" part of Theorem 6.4. We only note that $\lim_{t\to 0}[x, x+te_i] = [x,x] = 1-|x|^2$. □

With making suitable modifications in the above proof one can easily verify the following generalization of Theorem 6.7.

Theorem 6.8 *Let $\alpha \geq 1$, $\lambda_1, \lambda_2 \geq \alpha$ and $f \in h(\mathbb{B})$. Then $f \in b_\alpha$ if and only if*

$$M_{\alpha,\lambda_1,\lambda_2}(f) := \sup_{x,y\in\mathbb{B},\, x\neq y} \frac{(1-|x|^2)^{\lambda_1}(1-|y|^2)^{\lambda_2}}{[x,y]^{\lambda_1+\lambda_2-(\alpha+1)}} \frac{|f(x)-f(y)|}{|x-y|} < \infty.$$

Moreover, $M_{\alpha,\lambda_1,\lambda_2}(f) \sim \|f - f(0)\|_{b_\alpha}$.

Finally, we mention that all the results of this subsection have counterparts for little Bloch spaces. When $-1 < \alpha \leq 1$ the "little" counterpart of Theorem 6.4 is the following.

Theorem 6.9 *Let $-1 < \alpha \leq 1$. Let λ satisfy the following properties:*

(1) $0 < \lambda \leq \alpha + 1$ *if* $-1 < \alpha < 0$,
(2) $0 < \lambda < 1$ *if* $\alpha = 0$,
(3) $\alpha \leq \lambda \leq 1$ *if* $0 < \alpha \leq 1$.

Then $f \in h(\mathbb{B})$ belongs to $b_{\alpha 0}$ if and only if

$$\lim_{|x|\to 1^-} \left(\sup_{y\in\mathbb{B},\, y\neq x} (1-|x|^2)^{\lambda}(1-|y|^2)^{\alpha+1-\lambda} \frac{|f(x)-f(y)|}{|x-y|} \right) = 0.$$

We note that in case $-1 < \alpha < 0$ we need λ to be strictly greater then 0. For $\alpha \geq 1$ we have the following "little" counterpart of Theorem 6.8.

Theorem 6.10 *Let $\alpha \geq 1$, $\lambda_1, \lambda_2 \geq \alpha$ and $f \in h(\mathbb{B})$. Then*

$$f \in b_{\alpha 0} \iff \lim_{|x|\to 1^-} \left(\sup_{y\in\mathbb{B},\, y\neq x} \frac{(1-|x|^2)^{\lambda_1}(1-|y|^2)^{\lambda_2}}{[x,y]^{\lambda_1+\lambda_2-(\alpha+1)}} \frac{|f(x)-f(y)|}{|x-y|} \right) = 0.$$

Theorem 6.9 (respectively 6.10) can be proved by using Theorem 6.4 (respectively 6.8) and following the arguments of [19, proof of Theorem 3.2]. The details are straightforward and omitted.

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