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Spinor Spaces in Discrete Clifford Analysis

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Abstract In this paper we work in the 'split' discrete Clifford analysis setting, i.e. the m-dimensional function theory concerning null-functions, defined on the grid $\mathbb{Z}^m$, of the discrete Dirac operator ∂, involving both forward and backward differences, which factorizes the (discrete) Star-Laplacian. We show how the space $\mathcal{M}_k$ of discrete spherical monogenics homogeneous of degree k, is decomposable into irreducible $\mathfrak{so}(m)$-representations.

Keywords Discrete Clifford analysis · Irreducible representation · Orthogonal Lie algebra · Monogenic functions

Mathematics Subject Classification 17B15 · 47A67 · 20G05 · 15A66 · 39A12

1 Introduction

In particle physics, the (massless) Dirac operator is a well-studied operator in the setting of elementary particle physics [14]. This operator can be studied in a more mathematical setting, namely Euclidean Clifford analysis in general dimensions, where the Dirac operator factorizes the Laplace operator, making Clifford analysis a refinement

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of harmonic analysis. The Dirac operator is a Spin(m)-invariant operator, or equivalently, invariant under its Lie algebra $\mathfrak{so}(m)$.

The current paper deals with a special type of Clifford analysis, namely discrete Clifford analysis, where functions are defined on a grid $(h\mathbb{Z})^m$ rather than the continuous m-dimensional space $\mathbb{R}^m$. In recent years, a function theory studying discrete functions defined on the standard grid with mesh size 1 ($\mathbb{Z}^m$) has given rise to a discrete counterpart of Euclidean Clifford analysis [2,5]. Different choices have been made for the discrete Dirac operator [15,16,19], containing forward, backward and central differences.

The 'split' discrete Clifford analysis setting, which we are considering here (e.g. [4,7,13]) introduces a discrete Dirac operator ∂ defined using only forward and backward differences, which factorizes the discrete star Laplace operator Δ^* [20]. The function theory has already lead to a number of results regarding polynomial solutions of this differential operator, namely a Taylor series decomposition [11], a Cauchy–Kowalewskaya extension theorem [12], etc.

In this paper, we are however more interested in the representation theoretical aspect of this theory. It is well known that in classical harmonic analysis, the Laplace operator is a rotational invariant operator, or equivalently, invariant under the Lie algebra $\mathfrak{so}(m)$.

In [10], it has been shown that infinitesimal rotation operators can be defined in the split discrete Clifford analysis setting under which the star Laplacian is invariant. However, while the space $\mathcal{H}_k$ of discrete k-homogeneous harmonic polynomials is a representation of $\mathfrak{so}(m)$, contrary to the classical harmonic case, it is not irreducible. This has been shown in [9], where a full decomposition has been made into irreducible representations. The aim of this paper is to do the same for the space $\mathcal{M}_k$ of discrete monogenic polynomials of arbitrary degree of homogeneity k. In the process, we will be able to define spinor spaces in the discrete setting.

In classical harmonic analysis, the infinitesimal 'rotations', i.e. the elements of the orthogonal Lie algebra corresponding to the rotation group $SO(m)$, are given by the angular momentum operators $L_{a,b} = x_a \partial_{x_b} - x_b \partial_{x_a}$. These operators satisfy the commutation relations

$$[L_{a,b}, L_{c,d}] = \delta_{b,c}\, L_{a,d} - \delta_{b,d}\, L_{a,c} - \delta_{a,c}\, L_{b,d} + \delta_{a,d}\, L_{b,c},$$

which are exactly the defining relations of the special orthogonal Lie algebra $\mathfrak{so}(m)$ and they form endomorphisms of the space $\mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ of scalar-valued harmonic k-homogeneous polynomials, thus transforming the latter in an (irreducible) $\mathfrak{so}(m, \mathbb{C})$-representation. To establish $\mathcal{M}_k(\mathbb{R}^m, \mathbb{S})$, i.e. the spinor-valued homogeneous monogenics of degree k, classically as $\mathfrak{so}(m, \mathbb{C})$-representation, the following operators are considered

$$dR(e_{a,b}) : \mathcal{M}_k(\mathbb{R}^m, \mathbb{S}) \to \mathcal{M}_k(\mathbb{R}^m, \mathbb{S}), \qquad M_k \mapsto \left(L_{a,b} - \frac{1}{2}\, e_a\, e_b\right) M_k.$$

These operators are endomorphisms of the space of spinor-valued k-homogeneous polynomials in m vector variables which also satisfy the defining relations of $\mathfrak{so}(m, \mathbb{C})$:

$$\left[dR(e_{a,b}), dR(e_{c,d})\right] = \delta_{b,c}\, dR(e_{a,d}) - \delta_{b,d}\, dR(e_{a,c}) \\ -\delta_{a,c}\, dR(e_{b,d}) + \delta_{a,d}\, dR(e_{b,c}).$$

In [10], we developed similar operators in the discrete Clifford analysis setting: the angular momentum operators are discrete operators $L_{a,b} = \xi_a\, \partial_b + \xi_b\, \partial_a$, $a \neq b$. For $a = b$, we define $L_{aa} = 0$. Then the operators $\Omega_{a,b}$, acting on discrete functions f as $\Omega_{a,b}\, f = L_{a,b}\, f\, e_b\, e_a$, satisfy the defining relations of the orthogonal lie algebra $\mathfrak{so}(m)$:

$$\left[\Omega_{a,b}, \Omega_{c,d}\right] = \delta_{b,c}\, \Omega_{a,d} - \delta_{b,d}\, \Omega_{a,c} - \delta_{a,c}\, \Omega_{b,d} + \delta_{a,d}\, \Omega_{b,c}.$$

Furthermore, they are endomorphisms of the space $\mathcal{H}_k$ of Clifford-algebra valued homogeneous harmonics of degree k, since $\Omega_{a,b}$ commutes with $\mathfrak{sl}_2 = \left\{\Delta, \xi^2, \mathbb{E} + \frac{m}{2}\right\}$, for all (a, b). Here ξ^2 is the square of the discrete vector variable ξ and $\mathbb{E}$ is the discrete Euler operator (see Sect. 2). In [9], we showed that $\mathcal{H}_k$ is the sum of 2^{2m} isomorphic copies of the irreducible representation of $\mathfrak{so}(m, \mathbb{C})$ with highest weight $(k, 0, \ldots, 0)$.

The discrete Dirac operator ∂ is however not invariant under the operators $\Omega_{a,b}$, hence $\mathcal{M}_k$ cannot be expressed as $\mathfrak{so}(m, \mathbb{C})$-representation by means of these operators. Therefore, we considered in [10] the operators $L_{a,b} - \dfrac{1}{2}$ and the four-vector $V_{a,b} = e_a\, e_b\, e_a^{\perp}\, e_b^{\perp} = -e_a^{\perp}\, e_a\, e_b^{\perp}\, e_b$. Let the operator $dR(e_{a,b})$, $a \neq b$, act on discrete functions f as

$$dR(e_{a,b})\, f = V_{a,b} \left(L_{a,b} - \frac{1}{2}\right) f\, e_a^{\perp}\, e_b^{\perp}. \tag{1}$$

For $a = b$, we defined $dR(e_{a,a}) = 0$. Note that, for the sake of continuity, we use the same notation for the discrete and continuous rotation operators. From this point on, we always refer to the discrete versions unless stated otherwise. The operators $dR(e_{a,b})$ satisfy the defining relations of the special lie algebra $\mathfrak{so}(m)$:

$$\left[dR(e_{a,b}), dR(e_{c,d})\right] = \delta_{b,c}\, dR(e_{a,d}) - \delta_{b,d}\, dR(e_{a,c}) \\ -\delta_{a,c}\, dR(e_{b,d}) + \delta_{a,d}\, dR(e_{b,c}),$$

and commute with $\mathfrak{osp}(1|2) = \left\{\partial, \xi, \mathbb{E} + \frac{m}{2}\right\}$ which makes them endomorphisms of $\mathcal{M}_k$. As such, the space $\mathcal{M}_k$ is a reducible $\mathfrak{so}(m, \mathbb{C})$-representation. In [10], it was already suggested that $\mathcal{M}_k$ can be decomposed into irreducible parts of highest weight $(k)'_+ = \left(k + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ resp. $(k)'_- = \left(k + \frac{1}{2}, \frac{1}{2}, \ldots, -\frac{1}{2}\right)$, but this was left as an open conjecture. In the following sections, we will show how this decomposition is done exactly.

2 Preliminaries

Let $\mathbb{R}^m$ be the m-dimensional Euclidian space with orthonormal basis e_j, $j = 1, \ldots, m$ and consider the Clifford algebra $\mathbb{R}_{m,0}$ over $\mathbb{R}^m$. Passing to the so-called 'split' discrete setting [7,13], we embed the Clifford algebra $\mathbb{R}_{m,0}$ into the bigger complex one $\mathbb{C}_{2m,0}$, the underlying vector space of which has twice the dimension, and introduce forward and backward basis elements $\mathbf{e}_j^{\pm}$ satisfying the following anti-commutator rules:

$$\left\{\mathbf{e}_j^-, \mathbf{e}_\ell^-\right\} = \left\{\mathbf{e}_j^+, \mathbf{e}_\ell^+\right\} = 0, \qquad \left\{\mathbf{e}_j^+, \mathbf{e}_\ell^-\right\} = \delta_{j\ell}, \qquad j, \ell = 1, \ldots, m.$$

The connection to the original basis e_j is given by $\mathbf{e}_j^+ + \mathbf{e}_j^- = e_j$, $j = 1, \ldots, m$, which indeed implies that $e_j^2 = 1$. We will often write $e_j^{\perp} = \mathbf{e}_j^+ - \mathbf{e}_j^-$ and $\mathbf{e}_j^+ \wedge \mathbf{e}_j^- = \mathbf{e}_j^+\mathbf{e}_j^- - \mathbf{e}_j^-\mathbf{e}_j^+ = e_j^{\perp} e_j$.

Now consider the standard m-dimensional equidistant lattice $\mathbb{Z}^m$; the coordinates of a Clifford vector $\underline{x}$ will thus only take integer values. We construct a discrete Dirac operator factorizing the discrete Laplacian, using both forward and backward differences $\Delta_j^{\pm}$, $j = 1, \ldots, m$, acting on Clifford-valued functions f as follows:

$$\Delta_j^+[f](\cdot) = f(\cdot + e_j) - f(\cdot), \qquad \Delta_j^-[f](\cdot) = f(\cdot) - f(\cdot - e_j).$$

With respect to the $\mathbb{Z}^m$-grid, the usual definition of the discrete Laplacian in $\underline{x} \in \mathbb{Z}^m$ is

$$\Delta^*[f](\underline{x}) = \sum_{j=1}^{m} \Delta_j^+ \Delta_j^- [f] = \sum_{j=1}^{m} \big(f(\underline{x} + e_j) + f(\underline{x} - e_j)\big) - 2m\, f(\underline{x}).$$

This operator is also known as "Star Laplacian"; we will from now on simply write Δ. An appropriate definition of a discrete Dirac operator ∂ factorizing Δ, i.e. satisfying $\partial^2 = \Delta$, is obtained by combining the forward and backward basis elements with the corresponding forward and backward differences, more precisely

$$\partial = \sum_{j=1}^{m} \left(\mathbf{e}_j^+ \Delta_j^+ + \mathbf{e}_j^- \Delta_j^-\right).$$

Denote the co-ordinate difference operators $\partial_j = \mathbf{e}_j^+ \Delta_j^+ + \mathbf{e}_j^- \Delta_j^-$ and consider the discrete co-ordinate vector variables $\xi_j = \mathbf{e}_j^+ X_j^- + \mathbf{e}_j^- X_j^+$, $j = 1, \ldots, m$, with $X_j^{\pm}$ scalar operators. In order to receive an analogue of the classical Weyl relations $\partial_{x_j} x_k - x_k \partial_{x_j} = \delta_{jk}$, the co-ordinate vector variable operators ξ_j are defined by their interaction with the corresponding co-ordinate operators ∂_j, according to the skew Weyl relations, cf. [7]

$$\partial_j\, \xi_j - \xi_j\, \partial_j = 1, \;\; j = 1, \ldots, m,$$

which imply that $\partial_j\, \xi_j^k[1] = k\, \xi_j^{k-1}[1]$. The operators ξ_j and ∂_j furthermore satisfy the following anti-commutator relations:

$$\left\{\xi_j, \xi_k\right\} = \left\{\partial_j, \partial_k\right\} = \left\{\partial_j, \xi_k\right\} = 0, \qquad j \neq k,\ j, k = 1, \ldots, m$$

implying that $\partial_\ell\, \xi_j^k[1] = 0$, $j \neq \ell$.

The natural powers $\xi_j^k[1]$ of the operator ξ_j acting on the constant 1 are the basic discrete k-homogeneous polynomials of degree k in the variable x_j, i.e. $\mathbb{E}\, \xi_j^k[1] = k\, \xi_j^k[1]$, where $\mathbb{E} = \sum_{j=1}^m \xi_j\, \partial_j$ is the discrete Euler operator. They constitute a basis for all discrete polynomials. Explicit formulas for $\xi_j^k[1]$ are given for example in [7,12]; furthermore $\xi_j^k[1](x_j) = 0$ if $k \geqslant 2\, |x_j| + 1$.

A discrete function taking values in the Clifford algebra $\mathbb{C}_{2m}$ is discrete harmonic (resp. left discrete monogenic) in a domain $\Omega \subset \mathbb{Z}^m$ if $\Delta f(\underline{x}) = 0$ (resp. $\partial f(\underline{x}) = 0$), for all $\underline{x} \in \Omega$. The space of discrete harmonic (resp. monogenic) homogeneous polynomials of degree k (i.e. $\mathbb{E} f = kf$) is denoted $\mathcal{H}_k$ (resp. $\mathcal{M}_k$), while the space of all discrete harmonic (resp. monogenic) homogeneous polynomials is denoted $\mathcal{H}$ (resp. $\mathcal{M}$). It is clear that

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k, \qquad \mathcal{M} = \bigoplus_{k=0}^{\infty} \mathcal{M}_k.$$

The respective dimensions of these spaces over $\mathbb{C}$ are:

$$\dim(\mathcal{H}_k) = 2^{2m} \left(\binom{k+m-1}{k} - \binom{k+m-3}{k-2} \right),$$
$$\dim(\mathcal{M}_k) = 2^{2m} \binom{k+m-2}{k}.$$

The calculations are similar to the classical case (see e.g. [2]).

3 Orthogonal Lie Algebras

As the remainder of this paper will deal with the explicit construction of representations for the orthogonal algebra $\mathfrak{so}(m, \mathbb{C})$, we will start by briefly introducing this Lie algebra $\mathfrak{so}(m, \mathbb{C})$; a detailed description can be found for example in [17]. In even dimension $m = 2n$, the Lie algebra $\mathfrak{so}(m, \mathbb{C})$ is generated as a matrix algebra as follows. Define $E_{i,j}$ as the $(m \times m)$-matrix with entry 1 on position (i, j), while all other entries are 0. Then we can define the matrices

$$\begin{aligned} H_i &= E_{i,i} - E_{n+i,n+i} \\ X_{i,j} &= E_{i,j} - E_{n+j,n+i} \\ Y_{i,j} &= E_{i,n+j} - E_{j,n+i} \\ Z_{i,j} &= E_{n+i,j} - E_{n+j,i}, \end{aligned}$$

for all $1 \leqslant i, j \leqslant n$. Note that due to the symmetry of $Y_{i,j}$ and $Z_{i,j}$ in the indices, it suffices in fact that for those sets of operators, we pick $i < j$. The matrices H_i, $X_{i,j}$, $Y_{i,j}$ and $Z_{i,j}$ generate exactly the Lie algebra $\mathfrak{so}(2n, \mathbb{C})$ using the commutator as Lie bracket. When considering the same set of generators as $(2n + 1 \times 2n + 1)$-matrices, in addition with the matrices

$$\begin{aligned} U_i &= E_{i,2n+1} - E_{2n+1,n+i} \\ V_i &= E_{n+i,2n+1} - E_{2n+1,i}, \end{aligned}$$

for all $1 \leqslant i \leqslant n$, we find a set of generators for $\mathfrak{so}(2n + 1, \mathbb{C})$, again using the commutator as Lie bracket. From now on we consider these generators as abstract elements, satisfying the same relations as their matrix equivalents. Hence

$$\begin{aligned} \mathfrak{so}(2n, \mathbb{C}) &= \operatorname{span}_{\mathbb{C}} \left\{H_a, X_{a,b}, Y_{a,b}, Z_{a,b}, 1 \leqslant a, b \leqslant n,\ a \neq b\right\}, \\ \mathfrak{so}(2n+1, \mathbb{C}) &= \operatorname{span}_{\mathbb{C}} \left\{H_a, X_{a,b}, Y_{a,b}, Z_{a,b}, U_a, V_a, 1 \leqslant a, b \leqslant n,\ a \neq b\right\}. \end{aligned}$$

The Cartan subalgebra is chosen as

$$\mathfrak{h} = \{H_a, 1 \leqslant a \leqslant n\},$$

independently of the parity of the dimension, i.e. $\mathfrak{so}(2n, \mathbb{C})$ and $\mathfrak{so}(2n + 1, \mathbb{C})$ are both Lie algebras of rank n. The roots of $\mathfrak{so}(m, \mathbb{C})$ (see also [21]) are determined by considering the action of the Cartan algebra on the other generators of the adjoint representation of $\mathfrak{so}(m, \mathbb{C})$. Hence, for all $1 \leqslant a, b, c, d \leqslant n$:

$$\begin{aligned} \left[H_c, Y_{a,b}\right] &= (\delta_{ca} + \delta_{cb})\, Y_{a,b} = ((L_a + L_b)\,(H_c))\, Y_{a,b}, \\ \left[H_c, X_{a,b}\right] &= (\delta_{ca} - \delta_{cb})\, X_{a,b} = ((L_a - L_b)\,(H_c))\, X_{a,b}, \\ \left[H_c, Z_{a,b}\right] &= -\,(\delta_{ca} + \delta_{cb})\, Z_{a,b} = ((-L_a - L_b)\,(H_c))\, Z_{a,b}, \\ [H_c, U_a] &= \delta_{ca}\, U_a = (L_a(H_c))\, U_a, \\ [H_c, V_a] &= -\delta_{ca}\, U_a = (-L_a(H_c))\, U_a. \end{aligned}$$

Here $\{L_a, 1 \leqslant a \leqslant n\}$ is a basis of the dual vector space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$, i.e. $L_a\,(H_b) = \delta_{a,b}$. Note in particular that the Cartan subalgebra elements H_a appears in the commutator of a certain positive root with a negative root of the same index:

$$\left[Y_{a,b}, Z_{a,b}\right] = -H_a - H_b, \qquad \left[X_{a,b}, X_{b,a}\right] = H_a - H_b.$$

We thus deduce the following roots and root vectors.

Root	Root vector
$m = 2n$	
$L_a - L_b$	$X_{a,b}$
$L_a + L_b$	$Y_{a,b}$
$-L_a - L_b$	$Z_{a,b}$
$m = 2n + 1$	
$L_a - L_b$	$X_{a,b}$
$L_a + L_b$	$Y_{a,b}$
$-L_a - L_b$	$Z_{a,b}$
L_a	U_a
$-L_a$	V_a

By the usual convention (see e.g. [17]), we choose the positive roots in even dimension to be

$$\{L_a + L_b, L_a - L_b : 1 \leqslant a < b \leqslant n\}$$

and negative roots

$$\{-L_a - L_b, L_b - L_a : 1 \leqslant a < b \leqslant n\}\,.$$

In odd dimension, one chooses positive roots

$$\{L_a + L_b, L_a - L_b : 1 \leqslant a < b \leqslant n\} \cup \{L_a : 1 \leqslant a \leqslant n\}$$

and negative roots

$$\{-L_a - L_b, L_b - L_a : 1 \leqslant a < b \leqslant n\} \cup \{-L_a : 1 \leqslant a \leqslant n\}\,.$$

In [10], we introduced the algebra $\mathfrak{so}(m, \mathbb{C})$ (up to an isomorphism) in the discrete Clifford analysis context. The generators of $\mathfrak{so}(m, \mathbb{C})$ were not given in terms of the root vectors and Cartan subalgebra, but rather by the generators $\left\{dR(e_{a,b}) : 1 \leqslant a \neq b \leqslant m\right\}$, see (1) satisfying the defining relations of $\mathfrak{so}(m, \mathbb{C})$:

$$\begin{aligned}\left[dR(e_{a,b}), dR(e_{c,d})\right] = \delta_{a,d}\, dR(e_{b,c}) + \delta_{b,c}\, dR(e_{a,d})\\ -\delta_{a,c}\, dR(e_{b,d}) - \delta_{b,d}\, dR(e_{a,c}),\end{aligned} \tag{2}$$

see [10]. The next step is to identify both realisations of $\mathfrak{so}(m, \mathbb{C})$ in the discrete Clifford analysis setting, by determining the explicit expressions of the root vectors and Cartan subalgebra.

4 Decomposition of $\mathcal{M}_k$ in Irreducible Representations

Since the definition of the generators of $\mathfrak{so}(m, \mathbb{C})$ differs in even and odd dimensions, we have to make a distinction. We start with the even dimensional case.

4.1 Even Dimension $m = 2n$

Definition 1 We define the operators H_a, $X_{a,b}$, $Y_{a,b}$ and $Z_{a,b} \in \mathfrak{so}(m, \mathbb{C})$:

$$\begin{aligned}
H_a &= i\, dR(e_{2a-1,2a}), \qquad 1 \leqslant a \leqslant n, \\
X_{a,b} &= \frac{1}{2}\left(dR(e_{2a-1,2b-1}) + i\, dR(e_{2a-1,2b}) - i\, dR(e_{2a,2b-1}) + dR(e_{2a,2b})\right), \\
Y_{a,b} &= \frac{1}{2}\left(dR(e_{2a-1,2b-1}) - i\, dR(e_{2a-1,2b}) - i\, dR(e_{2a,2b-1}) - dR(e_{2a,2b})\right), \\
Z_{a,b} &= \frac{1}{2}\left(dR(e_{2a-1,2b-1}) + i\, dR(e_{2a-1,2b}) + i\, dR(e_{2a,2b-1}) - dR(e_{2a,2b})\right), \\
&\quad 1 \leqslant a, b \leqslant n.
\end{aligned}$$

Note that, because $dR(e_{a,b}) = -dR(e_{b,a})$, we find that $Y_{b,a} = -Y_{a,b}$ and $Z_{b,a} = -Z_{a,b}$. For $X_{a,b}$, we find that $X_{b,a} \neq X_{a,b}$ and that $X_{a,a} = H_a$, hence we will only consider couples (a, b) with $a \neq b$.

The original operators $dR(e_{a,b})$ can be reconstructed as linear combinations of the operators $X_{a,b}$, $Y_{a,b}$, $Z_{a,b}$ and H_a.

Straightforward calculations, which make use of (2) show that these operators satisfy the same commutator relations as their matrix equivalents. In particular, we have the following lemma.

Lemma 1 *The operators $H_c, X_{a,b}, Y_{a,b}$ and $Z_{a,b}$, $1 \leqslant a, b, c, d \leqslant n$, satisfy the commutation relations of $\mathfrak{so}(m, \mathbb{C})$:*

$$\begin{aligned}
\left[H_c, Y_{a,b}\right] &= (\delta_{ca} + \delta_{cb})\, Y_{a,b} = (L_a + L_b)\,(H_c)\, Y_{a,b}, \\
\left[H_c, X_{a,b}\right] &= (\delta_{ca} - \delta_{cb})\, X_{a,b} = (L_a - L_b)\,(H_c)\, X_{a,b}, \\
\left[H_c, Z_{a,b}\right] &= -\,(\delta_{ca} + \delta_{cb})\, Z_{a,b} = -\,(L_a + L_b)\,(H_c)\, Z_{a,b}, \\
\left[X_{a,b}, Y_{c,d}\right] &= \delta_{bc}\, Y_{a,d} - \delta_{bd}\, Y_{a,c}.
\end{aligned}$$

This means that the operators $X_{a,b}$ with $a < b$ and $Y_{a,b}$ are indeed positive root vectors corresponding to the roots $L_a - L_b$, resp. $L_a + L_b$. Furthermore, $X_{a,b}$ with $a > b$ and $Z_{a,b}$ are negative root vectors corresponding to the roots $L_a - L_b$ resp. $-L_a - L_b$.

Proof Since the commutator relations between the operators $dR(e_{a,b})$ are the same as those between the operators $\Omega_{a,b}$ of the harmonics, the proof is completely similar as the proof given in [9].

We already established in [10] that since the operators $dR(e_{a,b})$ are endomorphisms of $\mathcal{M}_k$, $\mathcal{M}_k$ is a representation of $\mathfrak{so}(2n, \mathbb{C})$ although this representation is not irreducible. The next step in our reasoning is to decompose $\mathcal{M}_k$ into irreducible representations of $\mathfrak{so}(m, \mathbb{C})$. This is done by splitting 1 into a sum of idempotents, see further. When constructing irreducible representations, the main target is to find so-called highest weight vectors. These are in our context functions belonging to $\mathcal{M}_k$ which:

- vanish under the action of all positive root vectors,
- are simultaneous eigenfunctions for the action of all H_a.

The property of such highest weight vectors is that they generate an entire irreducible representation by all consecutive actions of negative root vectors. The aim is to construct idempotents L such that there exists a discrete monogenic function P_k, such that $P_k L$ satisfies the conditions for a highest weight vector. Here $P_k\, L$ denotes the multiplication of P_k with the idempotent L. For a function $P_k\, L$ to be an eigenfunction of the maximal abelian subgroup $\mathfrak{h}$, it must hold that $L\, e_{2a-1}^{\perp}\, e_{2a}^{\perp}$ is again equal to L up to a (complex) constant. Consider, for $a = 1, \ldots, n$, the Clifford elements

$$L_{2a-1}^{\pm} = \left(\mathbf{e}_{2a-1}^{+}\mathbf{e}_{2a-1}^{-} \pm i\, \mathbf{e}_{2a-1}^{+}\right), \qquad L_{2a}^{\pm} = \left(\mathbf{e}_{2a}^{+}\mathbf{e}_{2a}^{-} \pm \mathbf{e}_{2a}^{+}\right),$$
$$M_{2a-1}^{\pm} = \left(\mathbf{e}_{2a-1}^{-}\mathbf{e}_{2a-1}^{+} \pm i\, \mathbf{e}_{2a-1}^{-}\right), \qquad M_{2a}^{\pm} = \left(\mathbf{e}_{2a}^{-}\mathbf{e}_{2a}^{+} \pm \mathbf{e}_{2a}^{-}\right).$$

For the rest of this article, we will need the following notations. For $F_a \in \left\{L_a^{\pm}, M_a^{\pm}\right\}$, $a = 1, \ldots, m$, denote

$$|F_a| = \begin{cases} 0, & \text{if } F_a = L_a^{+} \text{ or } M_a^{-} \\ 1, & \text{if } F_a = L_a^{-} \text{ or } M_a^{+} \end{cases} \quad \text{and} \quad \|F_a\| = \begin{cases} 0, & \text{if } F_a = L_a^{\pm}, \\ 1, & \text{if } F_a = M_a^{\pm}. \end{cases}$$

Furthermore, denote by $\widetilde{F}_a$ the idempotent

$$\widetilde{F}_a = \begin{cases} L_a^{\mp}, & \text{if } F_a = L_a^{\pm}, \\ M_a^{\mp}, & \text{if } F_a = M_a^{\pm}. \end{cases}$$

Then $|\widetilde{F}_a| = 1 - |F_a|$ and $\|\widetilde{F}_a\| = \|F_a\|$. Before we introduce the highest weight vectors, we will study the effect of multiplication by basis elements on these idempotents.

Lemma 2 *The multiplication by $e_a^{\perp}$ from the right on the idempotent $F_a \in \left\{L_a^{\pm}, M_a^{\pm}\right\}$ is:*

$$F_{2a-1}\, e_{2a-1}^{\perp} = (-1)^{|F_{2a}|+1}\, i\, F_{2a-1},$$
$$F_{2a}\, e_{2a}^{\perp} = (-1)^{|F_{2a}|+1}\, \widetilde{F}_{2a}.$$

As a result, for $1 \leqslant a \leqslant n$, we have that

$$F_{2a-1}\, F_{2a}\, e_{2a-1}^{\perp}\, e_{2a}^{\perp} = (-1)^{|F_{2a-1}|+|F_{2a}|+1}\, i\, F_{2a-1}\, F_{2a}.$$

Denote, for $1 \leqslant s_1 < s_2 \leqslant m$:

$$F^{s_1,s_2} = F_1\, F_2 \ldots\, F_{s_1-1}\, \widetilde{F}_{s_1}\, \widetilde{F}_{s_1+1} \ldots\, \widetilde{F}_{s_2-1}\, \widetilde{F}_{s_2}\, F_{s_2+1}\, F_{s_2+2} \ldots F_{m-1}\, F_m,$$

then we find that for $1 \leqslant a < b \leqslant n$ *and a general idempotent* $F = \prod_{s=1}^{m} F_s$*, with* $F_s \in \left\{L_s^{\pm}, M_s^{\pm}\right\}$*:*

$$\begin{aligned}
V_{2a-1,2b-1}\, F\, e_{2a-1}^{\perp}\, e_{2b-1}^{\perp} &= (-1)^{|F_{2a-1}|+|F_{2b-1}|+\|F_{2a-1}\|+\|F_{2b-1}\|+1}\, F^{2a,2b-1},\\
V_{2a-1,2b}\, F\, e_{2a-1}^{\perp}\, e_{2b}^{\perp} &= (-1)^{|F_{2a-1}|+|F_{2b}|+\|F_{2a-1}\|+\|F_{2b}\|}\, i\, F^{2a,2b-1},\\
V_{2a,2b-1}\, F\, e_{2a}^{\perp}\, e_{2b-1}^{\perp} &= (-1)^{|F_{2a}|+|F_{2b-1}|+\|F_{2a}\|+\|F_{2b-1}\|}\, i\, F^{2a,2b-1},\\
V_{2a,2b}\, F\, e_{2a}^{\perp}\, e_{2b}^{\perp} &= (-1)^{|F_{2a}|+|F_{2b}|+\|F_{2a}\|+\|F_{2b}\|}\, F^{2a,2b-1}.
\end{aligned}$$

Proof Note that

$$\begin{aligned}
L_{2a-1}^{\pm}\, e_{2a-1}^{\perp} &= \left(\mathbf{e}_{2a-1}^{+} \mp i\, \mathbf{e}_{2a-1}^{+}\mathbf{e}_{2a-1}^{-}\right) = \mp i\, L_{2a-1}^{\pm}, & L_{2a}^{\pm}\, e_{2a}^{\perp} &= \left(\mathbf{e}_{2a}^{+} \mp \mathbf{e}_{2a}^{+}\mathbf{e}_{2a}^{-}\right) = \mp L_{2a}^{\mp},\\
M_{2a-1}^{\pm}\, e_{2a-1}^{\perp} &= \left(-\mathbf{e}_{2a-1}^{-} \pm i\, \mathbf{e}_{2a-1}^{-}\mathbf{e}_{2a-1}^{+}\right) = \pm i\, M_{2a-1}^{\pm}, & M_{2a}^{\pm}\, e_{2a}^{\perp} &= \left(-\mathbf{e}_{2a}^{-} \pm \mathbf{e}_{2a}^{-}\mathbf{e}_{2a}^{+}\right) = \pm M_{2a}^{\mp}.
\end{aligned}$$

We may indeed summarize this as

$$F_{2a-1}\, e_{2a-1}^{\perp} = (-1)^{|F_{2a-1}|+1}\, i\, F_{2a-1}, \qquad F_{2a}\, e_{2a}^{\perp} = (-1)^{|F_{2a}|+1}\, \widetilde{F}_{2a}.$$

From this it follows that

$$\widetilde{F}_{2a-1}\, e_{2a-1}^{\perp} = (-1)^{|F_{2a-1}|}\, i\, \widetilde{F}_{2a-1}, \qquad \widetilde{F}_{2a}\, e_{2a}^{\perp} = (-1)^{|F_{2a}|}\, F_{2a}.$$

Hence we find that

$$F_{2a-1}\, F_{2a}\, e_{2a-1}^{\perp}\, e_{2a}^{\perp} = F_{2a-1}\, e_{2a-1}^{\perp}\, \widetilde{F}_{2a}\, e_{2a}^{\perp} = (-1)^{|F_{2a}|+|F_{2a}|+1}\, i\, F_{2a-1}\, F_{2a}.$$

Also important to note is that $e_a^{\perp}\, e_a\, L_a^{\pm} = L_a^{\pm}$ and $e_a^{\perp}\, e_a\, M_a^{\pm} = -M_a^{\pm}$ so for the idempotent $F = \prod_{s=1}^{m} F_s$, we find that

$$V_{a,b}\, F = -e_a^{\perp}\, e_a\, e_b^{\perp}\, e_b\, F = (-1)^{1+\|F_a\|+\|F_b\|}\, F.$$

We thus get that

$$\begin{aligned}
V_{2a-1,2b-1}\, F\, e_{2a-1}^{\perp}\, e_{2b-1}^{\perp} &= (-1)^{1+\|F_{2a-1}\|+\|F_{2b-1}\|}\, F_1\, F_2 \ldots F_m\, e_{2a-1}^{\perp}\, e_{2b-1}^{\perp}\\
&= (-1)^{1+\|F_{2a-1}\|+\|F_{2b-1}\|}\, F_1 \ldots F_{2a-2}\, F_{2a-1}\, e_{2a-1}^{\perp}\, \widetilde{F}_{2a}\\
&\quad \ldots \widetilde{F}_{2b-1}\, e_{2b-1}^{\perp}\, F_{2b}\, F_{2b+1} \ldots F_m\\
&= (-1)^{|F_{2a-1}|+|F_{2b-1}|+\|F_{2a-1}\|+\|F_{2b-1}\|}\, i^2\, F_1\, F_2\\
&\quad \ldots F_{2a-2}\, F_{2a-1}\, \widetilde{F}_{2a}\\
&\quad \ldots \widetilde{F}_{2b-1}\, F_{2b}\, F_{2b+1} \ldots F_m\\
&= (-1)^{|F_{2a-1}|+|F_{2b-1}|+\|F_{2a-1}\|+\|F_{2b-1}\|+1}\, F^{2a,2b-1}.
\end{aligned}$$

Analogously, we find that

$$\begin{aligned}V_{2a-1,2b}\, F\, e_{2a-1}^{\perp}\, e_{2b}^{\perp} &= (-1)^{1+\|F_{2a-1}\|+\|F_{2b}\|}\, F_1\, F_2 \ldots F_m\, e_{2a-1}^{\perp}\, e_{2b}^{\perp} \\ &= (-1)^{1+\|F_{2a-1}\|+\|F_{2b}\|}\, F_1 \ldots F_{2a-2}\, F_{2a-1}\, e_{2a-1}^{\perp}\, \widetilde{F}_{2a} \\ &\quad \ldots \widetilde{F}_{2b}\, e_{2b}^{\perp}\, F_{2b+1}\, F_{2b+2} \ldots F_m \\ &= (-1)^{|F_{2a-1}|+|F_{2b}|+\|F_{2a-1}\|+\|F_{2b}\|}\, i\, F_1\, F_2 \ldots F_{2a-2}\, F_{2a-1}\, \widetilde{F}_{2a} \\ &\quad \ldots \widetilde{F}_{2b-1}\, F_{2b}\, F_{2b+1} \ldots F_m \\ &= (-1)^{|F_{2a-1}|+|F_{2b}|+\|F_{2a-1}\|+\|F_{2b}\|}\, i\, F^{2a,2b-1}.\end{aligned}$$

Also

$$\begin{aligned}V_{2a,2b-1}\, F\, e_{2a}^{\perp}\, e_{2b-1}^{\perp} &= (-1)^{1+\|F_{2a}\|+\|F_{2b-1}\|}\, F_1\, F_2 \ldots F_m\, e_{2a}^{\perp}\, e_{2b-1}^{\perp} \\ &= (-1)^{1+\|F_{2a}\|+\|F_{2b-1}\|}\, F_1 \ldots F_{2a-1}\, F_{2a}\, e_{2a}^{\perp}\, \widetilde{F}_{2a+1} \\ &\quad \ldots \widetilde{F}_{2b-1}\, e_{2b-1}^{\perp}\, F_{2b}\, F_{2b+1} \ldots F_m \\ &= (-1)^{|F_{2a}|+|F_{2b-1}|+\|F_{2a}\|+\|F_{2b-1}\|}\, i\, F_1\, F_2 \ldots F_{2a-1}\, \widetilde{F}_{2a}\, \widetilde{F}_{2a+1} \\ &\quad \ldots \widetilde{F}_{2b-1}\, F_{2b}\, F_{2b+1} \ldots F_m \\ &= (-1)^{|F_{2a}|+|F_{2b-1}|+\|F_{2a}\|+\|F_{2b-1}\|}\, i\, F^{2a,2b-1}.\end{aligned}$$

Finally

$$\begin{aligned}V_{2a,2b}\, F\, e_{2a}^{\perp}\, e_{2b}^{\perp} &= (-1)^{1+\|F_{2a}\|+\|F_{2b}\|}\, F_1\, F_2 \ldots F_m\, e_{2a}^{\perp}\, e_{2b}^{\perp} \\ &= (-1)^{1+\|F_{2a}\|+\|F_{2b}\|}\, F_1 \ldots F_{2a-1}\, F_{2a}\, e_{2a}^{\perp}\, \widetilde{F}_{2a+1} \\ &\quad \ldots \widetilde{F}_{2b}\, e_{2b}^{\perp}\, F_{2b+1} \ldots F_m \\ &= (-1)^{|F_{2a}|+|F_{2b}|+\|F_{2a}\|+\|F_{2b}\|}\, F_1\, F_2 \ldots F_{2a-1}\, \widetilde{F}_{2a}\, \widetilde{F}_{2a+1} \\ &\quad \ldots \widetilde{F}_{2b-1}\, F_{2b}\, F_{2b+1} \ldots F_m \\ &= (-1)^{|F_{2a}|+|F_{2b}|+\|F_{2a}\|+\|F_{2b}\|}\, F^{2a,2b-1}.\end{aligned}$$

Consider the basic discrete monogenic k-homogeneous functions

$$g_{2k} = ((\xi_2 - \xi_1)\,(\xi_2 + \xi_1))^k\,[1], \qquad g_{2k+1} = (\xi_2 - \xi_1)\,((\xi_2 + \xi_1)\,(\xi_2 - \xi_1))^k\,[1].$$

From now on we denote $(k)'_+ = \left(k + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ and $(k)'_- = \left(k + \frac{1}{2}, \frac{1}{2}, \ldots, -\frac{1}{2}\right)$. We will show under which conditions on the idempotent $F = \prod_{s=1}^{m} F_s$, the space $\operatorname{span}_{\mathbb{C}}\{g_k\, F\}$ is a weight space of $\mathfrak{h}$ with weight $(k)'_+$ resp. $(k)'_-$.

Lemma 3 *The polynomial $g_k\, F \in \mathcal{M}_k$, $F = \prod_{s=1}^{m} F_s$ with $F_s \in \left\{L_s^{\pm}, M_s^{\pm}\right\}$, is a weight vector of $\mathfrak{so}(m, \mathbb{C})$ with*

- *weight $(k)'_+$ if $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even and $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even for $2 \leqslant a \leqslant n$.*
- *weight $(k)'_-$ if $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even, $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even for $2 \leqslant a \leqslant n - 1$ and $\|F_{2n-1}\| + \|F_{2n}\| + |F_{2n-1}| + |F_{2n}|$ is odd.*

Proof We consider the action of the Cartan subalgebra-elements H_s, $1 \leqslant s \leqslant n$, on $g_k\, F$. Since g_k only contains the vector variables ξ_1 and ξ_2, we will first consider H_1:

$$H_1\,(g_k\, F) = i\, V_{12} \left(L_{12} - \frac{1}{2}\right) g_k\, F\, e_1^{\perp}\, e_2^{\perp}.$$

We will also denote

$$\begin{aligned} f_{2k} &= ((\xi_2 + \xi_1)\,(\xi_2 - \xi_1))^k\,[1], \\ f_{2k+1} &= (\xi_2 + \xi_1)\,((\xi_2 - \xi_1)\,(\xi_2 + \xi_1))^k\,[1]. \end{aligned}$$

In [6] it was established that $\partial_j\, g_k = (-1)^j\, k\, f_{k-1}$ hence

$$L_{12}\, g_k = (\xi_1\,\partial_2 + \xi_2\,\partial_1)\, g_k = k\,(\xi_1 - \xi_2)\, f_{k-1} = -k\,(\xi_2 - \xi_1)\, f_{k-1} = -k\, g_k.$$

We thus get that

$$H_1\,(g_k\, F) = i \left(-k - \frac{1}{2}\right) V_{12}\, g_k\, F\, e_1^{\perp}\, e_2^{\perp}.$$

Notice that $V_{12}\, g_k = (-1)^k\, g_k\, V_{12}$ since

$$\begin{aligned} e_1^{\perp} e_1\, \xi_1 &= \left(\mathbf{e}_1^{+}\mathbf{e}_1^{-} - \mathbf{e}_1^{-}\mathbf{e}_1^{+}\right)\left(X_1^{+}\,\mathbf{e}_1^{-} + X_1^{-}\,\mathbf{e}_1^{+}\right) = \left(-X_1^{+}\,\mathbf{e}_1^{-} + X_1^{-}\,\mathbf{e}_1^{+}\right) \\ &= \left(X_1^{+}\,\mathbf{e}_1^{-} + X_1^{-}\,\mathbf{e}_1^{+}\right)\left(-\mathbf{e}_1^{+}\mathbf{e}_1^{-} + \mathbf{e}_1^{-}\mathbf{e}_1^{+}\right) = -\xi_1\, e_1^{\perp} e_1, \\ e_1^{\perp} e_1\, \xi_2 &= \xi_2\, e_1^{\perp} e_1. \end{aligned}$$

Applying this, we find that

$$\begin{aligned} H_1\,(g_k\, F) &= (-1)^{k+1}\, i \left(k + \frac{1}{2}\right) g_k\, V_{12}\, F\, e_1^{\perp}\, e_2^{\perp} \\ &= (-1)^{k+|F_1|+|F_2|+\|F_1\|+\|F_2\|} \left(k + \frac{1}{2}\right) g_k\, F. \end{aligned}$$

For $g_k\, F$ to be a weight vector with weight $(k)'_+$ it must be an eigenfunction of H_1 with eigenvalue $k + \frac{1}{2}$, hence it must hold that $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even. We find 8 possible combinations for $F_1\, F_2$:

- k even:

$$\begin{aligned} F_1\, F_2 \in \big\{ & L_1^{+}\, L_2^{+},\ L_1^{-}\, L_2^{-},\ L_1^{+}\, M_2^{+},\ L_1^{-}\, M_2^{-}, \\ & M_1^{+}\, L_2^{+},\ M_1^{+}\, M_2^{+},\ M_1^{-}\, L_2^{-},\ M_1^{-}\, M_2^{-} \big\}. \end{aligned}$$

- k odd:

$$\begin{aligned} F_1\, F_2 \in \big\{ & L_1^{+}\, L_2^{-},\ L_1^{-}\, L_2^{+},\ L_1^{+}\, M_2^{-},\ L_1^{-}\, M_2^{+},\ M_1^{+}\, L_2^{-}, \\ & M_1^{+}\, M_2^{-},\ M_1^{-}\, L_2^{+},\ M_1^{-}\, M_2^{+} \big\}. \end{aligned}$$

Next, we consider the action of H_a, $2 \leqslant a \leqslant n$, on $g_k F$ to conclude under which conditions $g_k F$ is an eigenfunction of H_a with eigenvalue $\pm\frac{1}{2}$. Since the generator g_k only contains ξ_1 and ξ_2, it vanishes under the action of $L_{2a-1,2a} = \xi_{2a-1}\, \partial_{2a} + \xi_{2a}\, \partial_{2a-1}$. Note that $V_{2a-1,2a}\, g_k = g_k\, V_{2a-1,2a}$ since g_k contains only $\mathbf{e}_1^{\pm}$ and $\mathbf{e}_2^{\pm}$. Thus

$$\begin{aligned}
H_a\,(g_k\, F) &= -\frac{i}{2}\, V_{2a-1,2a}\, g_k\, F\, e_{2a-1}^{\perp}\, e_{2a}^{\perp} = -\frac{i}{2}\, g_k\, V_{2a-1,2a}\, F\, e_{2a-1}^{\perp}\, e_{2a}^{\perp} \\
&= (-1)^{\|F_{2a-1}\|+\|F_{2a}\|+|F_{2a-1}|+|F_{2a}|+1}\, i\, \frac{i}{2}\, g_k\, F \\
&= (-1)^{\|F_{2a-1}\|+\|F_{2a}\|+|F_{2a-1}|+|F_{2a}|}\, \frac{1}{2}\, g_k\, F.
\end{aligned}$$

This equals $+\frac{1}{2}\, g_k\, F$ when $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even and $-\frac{1}{2}\, g_k\, F$ otherwise.

We find that $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even for $F_{2a-1}\, F_{2a}$ in

$$\begin{aligned}
&\{L_{2a-1}^{+}\, L_{2a}^{+}, L_{2a-1}^{-}\, L_{2a}^{-}, L_{2a-1}^{+}\, M_{2a}^{+}, L_{2a-1}^{-} M_{2a}^{-}, \\
&\quad M_{2a-1}^{+}\, L_{2a}^{+}, M_{2a-1}^{+}\, M_{2a}^{+}, M_{2a-1}^{-}\, L_{2a}^{-}, M_{2a-1}^{-}\, M_{2a}^{-}\}
\end{aligned}$$

and odd for $F_{2a-1}\, F_{2a}$ in

$$\begin{aligned}
&\{L_{2a-1}^{+}\, L_{2a}^{-}, L_{2a-1}^{-}\, L_{2a}^{+}, L_{2a-1}^{+}\, M_{2a}^{-}, L_{2a-1}^{-}\, M_{2a}^{+}, \\
&\quad M_{2a-1}^{+}\, L_{2a}^{-}, M_{2a-1}^{+}\, M_{2a}^{-}, M_{2a-1}^{-}\, L_{2a}^{+}, M_{2a-1}^{-}\, M_{2a}^{+}\}\,.
\end{aligned}$$

This proves the lemma.

Remark 1 In particular, we find that $g_{2k}\, \prod_{s=1}^{m} L_s^{+}$ respectively $g_{2k+1}\, L_1^{+}\, L_2^{-}\, \prod_{s=3}^{m} L_s^{+}$ are weight vectors of $\mathfrak{h}$ in $\mathcal{M}_{2k}$ resp. $\mathcal{M}_{2k+1}$ of weight $(2k)'_{+}$ resp. $(2k+1)'_{+}$.

Corollary 1 *There are 2^{2m-n} weight vectors $g_k\, F$, with F one of the idempotents mentioned above, of weight $(k)'_{+}$ and 2^{2m-n} weight vectors $g_k\, F$, with F one of the above mentioned idempotents, with weight $(k)'_{-}$.*

Proof To obtain weight $(k)'_{+}$, one has eight choices for each factor $F_{2s-1}\, F_{2s}$ in F, $1 \leqslant s \leqslant n$. We thus get $8^n = 2^{3n} = 2^{4n-n} = 2^{2m-n}$ choices for the idempotent F. The same reasoning can be made for the weight $(k)'_{-}$.

We will now show that the weight vectors, defined in Lemma 3 are actually *highest* weight vectors, i.e. that they vanish under the action of all positive roots.

Theorem 1 *The polynomials $g_k\, F$, with*

- $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ *even*
- $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ *even, for all* $2 \leqslant a \leqslant n-1$, *and*
- $\|F_{2n-1}\| + \|F_{2n}\| + |F_{2n-1}| + |F_{2n}|$ *even resp. odd*

are highest weight spaces with highest weight $(k)'_+$ resp. $(k)'_-$, i.e.

$$H_a\,(g_k\,F) = \left(\delta_{1a}\left(k+\frac{1}{2}\right)+\frac{1}{2}\sum_{j=2}^{n}\delta_{ja}\right) g_k\,F$$

and

$$X_{a,b}\,(g_k\,F) = Y_{a,b}\,(g_k\,F) = 0, \quad \textit{for all}\ 1 \leqslant a < b \leqslant n.$$

Proof Lemma 3 tells us that these $g_k\,F$ are weight vectors with weight $(k)'_\pm$ for the conditions stated above. The only thing that we still have to prove is that $g_k\,F$ vanishes under the action of $X_{a,b}$ and $Y_{a,b}$, $a < b$.

We first consider the action of $X_{a,b}$ on $g_k\,F$. We make a distinction between $a = 1$ and $a \neq 1$. Take $1 = a < b$, then

$$\begin{aligned}
2\,X_{1,b}\,g_k\,F &= \big(dR(e_{1,2b-1}) + i\,dR(e_{1,2b}) - i\,dR(e_{2,2b-1}) + dR(e_{2,2b})\big)\,g_k\,F\\
&= V_{1,2b-1}\left(\xi_{2b-1}\,\partial_1 - \frac{1}{2}\right) g_k\,F\,e_1^\perp\,e_{2b-1}^\perp\\
&\quad + i\,V_{1,2b}\left(\xi_{2b}\,\partial_1 - \frac{1}{2}\right) g_k\,F\,e_1^\perp\,e_{2b}^\perp\\
&\quad - i\,V_{2,2b-1}\left(\xi_{2b-1}\,\partial_2 - \frac{1}{2}\right) g_k\,F\,e_2^\perp\,e_{2b-1}^\perp\\
&\quad + V_{2,2b}\left(\xi_{2b}\,\partial_2 - \frac{1}{2}\right) g_k\,F\,e_2^\perp\,e_{2b}^\perp\\
&= V_{1,2b-1}\left(-k\,\xi_{2b-1}\,f_{k-1} - \frac{1}{2}\,g_k\right) F\,e_1^\perp\,e_{2b-1}^\perp\\
&\quad + i\,V_{1,2b}\left(-k\,\xi_{2b}\,f_{k-1} - \frac{1}{2}\,g_k\right) F\,e_1^\perp\,e_{2b}^\perp\\
&\quad - i\,V_{2,2b-1}\left(k\,\xi_{2b-1}\,f_{k-1} - \frac{1}{2}\,g_k\right) F\,e_2^\perp\,e_{2b-1}^\perp\\
&\quad + V_{2,2b}\left(k\,\xi_{2b}\,f_{k-1} - \frac{1}{2}\,g_k\right) F\,e_2^\perp\,e_{2b}^\perp.
\end{aligned}$$

Now we use

$$V_{1,2b-1}\,\xi_{2b-1} = -\xi_{2b-1}\,V_{1,2b-1}, \qquad V_{1,2b}\,\xi_{2b} = -\xi_{2b}\,V_{1,2b}.$$

Furthermore, since for $b \neq 1, 2$,

$$\begin{aligned}
V_{1,b}\,(\xi_2 \pm \xi_1) &= (\xi_2 \mp \xi_1)\,V_{1,b},\\
V_{2,b}\,(\xi_2 \pm \xi_1) &= (-\xi_2 \pm \xi_1)\,V_{2,b} = -\,(\xi_2 \mp \xi_1)\,V_{2,b},
\end{aligned}$$

we find, for $j \in \{2b-1, 2b\}$:

$$\begin{aligned}
V_{1,j}\, g_k &= V_{1,j}\,(\xi_2 - \xi_1)\,(\xi_2 + \xi_1)\,(\xi_2 - \xi_1)\dots \\
&= (\xi_2 + \xi_1)\,(\xi_2 - \xi_1)\,(\xi_2 + \xi_1)\dots V_{1,j} = f_k\, V_{1,j}, \\
V_{1,j}\, f_k &= V_{1,j}\,(\xi_2 + \xi_1)\,(\xi_2 - \xi_1)\,(\xi_2 + \xi_1)\dots \\
&= (\xi_2 - \xi_1)\,(\xi_2 + \xi_1)\,(\xi_2 - \xi_1)\dots V_{1,j} = g_k\, V_{1,j}, \\
V_{2,j}\, g_k &= V_{2,j}\,(\xi_2 - \xi_1)\,(\xi_2 + \xi_1)\,(\xi_2 - \xi_1)\dots = (-1)^k\, f_k\, V_{2,j}, \\
V_{2,j}\, f_k &= V_{2,j}\,(\xi_2 + \xi_1)\,(\xi_2 - \xi_1)\,(\xi_2 + \xi_1)\dots = (-1)^k\, g_k\, V_{2,j}.
\end{aligned}$$

We get that

$$\begin{aligned}
2\, X_{1,b}\, g_k\, F = &\left(k\, \xi_{2b-1}\, g_{k-1} - \frac{1}{2}\, f_k\right) V_{1,2b-1}\, F\, e_1^{\perp}\, e_{2b-1}^{\perp} \\
&+ i\, \left(k\, \xi_{2b}\, g_{k-1} - \frac{1}{2}\, f_k\right) V_{1,2b}\, F\, e_1^{\perp}\, e_{2b}^{\perp} \\
&- i\, \left((-1)^{1+k-1}\, k\, \xi_{2b-1}\, g_{k-1} - \frac{1}{2}\,(-1)^k\, f_k\right) V_{2,2b-1}\, F\, e_2^{\perp}\, e_{2b-1}^{\perp} \\
&+ \left((-1)^{1+k-1}\, k\, \xi_{2b}\, g_{k-1} - \frac{1}{2}\,(-1)^k\, f_k\right) V_{2,2b}\, F\, e_2^{\perp}\, e_{2b}^{\perp}.
\end{aligned}$$

Using Lemma 2, we get

$$\begin{aligned}
2\, X_{1,b}\, g_k\, F = &(-1)^{|F_1|+\|F_1\|+|F_{2b-1}|+\|F_{2b-1}\|} \\
&\left(-k\, \xi_{2b-1}\, g_{k-1} + \frac{1}{2}\, f_k + (-1)^{|F_{2b-1}|+\|F_{2b-1}\|+|F_{2b}|+\|F_{2b}\|}\right. \\
&\left(-k\, \xi_{2b}\, g_{k-1} + \frac{1}{2}\, f_k\right) \\
&+(-1)^{|F_1|+\|F_1\|+|F_2|+\|F_2\|} \left((-1)^k\, k\, \xi_{2b-1}\, g_{k-1} - (-1)^k\, \frac{1}{2}\, f_k\right) \\
&+(-1)^{|F_1|+\|F_1\|+|F_2|+\|F_2\|+|F_{2b-1}|+|F_{2b}|+\|F_{2b-1}\|+\|F_{2b}\|} \\
&\left.\left((-1)^k\, k\, \xi_{2b}\, g_{k-1} - \frac{1}{2}\,(-1)^k\, f_k\right)\right) F^{2,2b-1}.
\end{aligned}$$

We thus see that this vanishes when

$$k + |F_1| + \|F_1\| + |F_2| + \|F_2\|$$

is even.

For $1 < a < b \leqslant n$ we get that

$$\begin{aligned}
2\, X_{a,b}\, g_k\, F = -\frac{1}{2}\Big(&V_{2a-1,2b-1}\, g_k\, F\, e_{2a-1}^{\perp}\, e_{2b-1}^{\perp} + i\, V_{2a-1,2b}\, g_k\, F\, e_{2a-1}^{\perp}\, e_{2b}^{\perp} \\
&- i\, V_{2a,2b-1}\, g_k\, F\, e_{2a}^{\perp}\, e_{2b-1}^{\perp} + V_{2a,2b}\, g_k\, F\, e_{2a}^{\perp}\, e_{2b}^{\perp}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\, g_k \Big((-1)^{|F_{2a-1}|+|F_{2b-1}|+\|F_{2a-1}\|+\|F_{2b-1}\|+1} \\
&\quad +(-1)^{|F_{2a-1}|+|F_{2b}|+\|F_{2a-1}\|+\|F_{2b}\|+1} \\
&\quad +(-1)^{|F_{2a}|+|F_{2b-1}|+\|F_{2a}\|+\|F_{2b-1}\|} + (-1)^{|F_{2a}|+|F_{2b}|+\|F_{2a}\|+\|F_{2b}\|} \Big)\, F^{2a,2b-1}.
\end{aligned}
$$

This will be zero when $|F_{2a-1}| + \|F_{2a-1}\| + |F_{2a}| + \|F_{2a}\|$ is even, and this for all $2 \leqslant a \leqslant n-1$.

Note that:

$$
\begin{aligned}
X_{a,b} &= \frac{1}{2} \left(dR(e_{2a-1,2b-1}) + i\, dR(e_{2a-1,2b}) - i\, dR(e_{2a,2b-1}) + dR(e_{2a,2b}) \right), \\
Y_{a,b} &= \frac{1}{2} \left(dR(e_{2a-1,2b-1}) - i\, dR(e_{2a-1,2b}) - i\, dR(e_{2a,2b-1}) - dR(e_{2a,2b}) \right).
\end{aligned}
$$

If we apply the appropriate change of sign in the second and last term of previous calculations, we immediately get that $Y_{a,b}(g_k\, F) = 0$ for $a < b$. Since $Y_{a,b} = -Y_{b,a}$, this will also be zero for $a > b$.

Remark 2 In particular, the polynomials $g_{2k} \prod_{s=1}^{m} L_s^+$ and $g_{2k+1}\, L_1^+\, L_2^- \prod_{s=3}^{m} L_s^+$ are highest weight vectors with weight $(2k)'_+$ resp. $(2k+1)'_+$.

Remark 3 The dimension of $(k)'_\pm$ is (see e.g. [17])

$$
2^{n-1} \binom{k+m-2}{k}.
$$

As the dimension of $\mathcal{M}_k$ equals

$$
2^{2m} \binom{k+m-2}{k} = 2^{4n} \binom{k+m-2}{k}
$$

and as we found 2^{3n} disjoint isomorphic copies of $(k)'_+$ combined with 2^{3n} copies of $(k)'_-$, the space $\mathcal{M}_k$ is fully decomposed in 2^{3n} copies of $(k)'_+$ and 2^{3n} copies of $(k)'_-$.

Definition 2 We define the space of positive resp. negative spinors $\mathbb{S}_{2n}^\pm$ as the image under $\mathfrak{so}(m, \mathbb{C})$ of the idempotents $\prod_{s=1}^{m} L_s^+$, resp. $\left(\prod_{s=1}^{m-1} L_s^+\right) L_m^-$:

$$
\mathbb{S}_{2n}^+ = \mathfrak{so}(m, \mathbb{C}) \left(\mathrm{span}_{\mathbb{C}} \left\{ L_1^+\, L_2^+ \ldots L_{2n-1}^+\, L_{2n}^+ \right\} \right)
$$

and

$$
\mathbb{S}_{2n}^- = \mathfrak{so}(m, \mathbb{C}) \left(\mathrm{span}_{\mathbb{C}} \left\{ L_1^+\, L_2^+ \ldots L_{2n-1}^+\, L_{2n}^- \right\} \right).
$$

The elements $L_1^+\, L_2^+ \ldots L_{2n-1}^+\, L_{2n}^+$, resp. $L_1^+\, L_2^+ \ldots L_{2n-1}^+\, L_{2n}^-$ are highest weight vectors with weight $(0)'_+ = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ resp. $(0)'_- = \left(\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}\right)$. Let us give an example to clarify this construction.

Example 1 Let $m = 4$ (i.e. $n = 2$) and consider $L = L_1^+ L_2^+ L_3^+ L_4^+$. The Lie algebra $\mathfrak{so}(4, \mathbb{C})$ is given in this context by

$$\operatorname{span}_{\mathbb{C}} \{dR(e_{12}), dR(e_{13}), dR(e_{14}), dR(e_{23}), dR(e_{24}), dR(e_{34})\}.$$

The elements $dR(e_{12})$ and $dR(e_{34})$ return L up to complex constant. The other four rotations give us (up to a complex constant) the idempotent $L_1^+ L_2^- L_3^- L_4^+$. Hence

$$\mathbb{S}_4^+ = \operatorname{span}_{\mathbb{C}} \left\{L_1^+ L_2^+ L_3^+ L_4^+, L_1^+ L_2^- L_3^- L_4^+\right\}.$$

Starting from $L_1^+ L_2^+ L_3^+ L_4^-$, the rotations $dR(e_{13}), dR(e_{14}), dR(e_{23})$ and $dR(e_{24})$ lead us to the idempotent $L_1^+ L_2^- L_3^- L_4^-$ which shows that

$$\mathbb{S}_4^- = \operatorname{span}_{\mathbb{C}} \left\{L_1^+ L_2^+ L_3^+ L_4^-, L_1^+ L_2^- L_3^- L_4^-\right\}.$$

The space of positive/negative spinors $\mathbb{S}_4^\pm$ is 2-dimensional.

In general, the elements $dR(e_{2a-1,2a})$ acting on an idempotent return the same idempotent up to a multiplicative complex factor. Since, for $1 \leqslant a < b \leqslant n$:

$$\begin{aligned} V_{2a-1,2b-1}\, L\, e_{2a-1}^\perp\, e_{2b-1}^\perp &= -L^{2a,2b-1}, & V_{2a-1,2b}\, L\, e_{2a-1}^\perp\, e_{2b}^\perp &= i\, L^{2a,2b-1}, \\ V_{2a,2b-1}\, L\, e_{2a}^\perp\, e_{2b-1}^\perp &= i\, L^{2a,2b-1}, & V_{2a,2b}\, L\, e_{2a}^\perp\, e_{2b}^\perp &= L^{2a,2b-1}. \end{aligned}$$

with $L^{2a,2b-1} = L_1^+ L_2^+ \dots L_{2a-1}^+ L_{2a}^- \dots L_{2b-1}^- L_{2b}^+ \dots L_{2n-1}^+ L_{2n}^+$, similar to the notation used before, we see that $dR(e_{a,b})$ acting on

$$L = L_1^+ L_2^+ \dots L_{2n-1}^+ L_{2n}^+$$

changes the sign of an even number of L_a's. The operator always leaves L_1^+ and L_{2n}^+ unchanged. The resulting idempotent will always have an even number of minus-signs. Starting from the idempotent L with all plus-signs, we thus get all possible idempotents of the following form:

$$L_1^+ \underbrace{\cdot\ \cdot}\ \underbrace{\cdot\ \cdot} \cdots \underbrace{\cdot\ \cdot}\ L_{2n}^+$$

where each place $\underbrace{\cdot\ \cdot}$ consists of either $L_{2a}^+ L_{2a+1}^+$ or $L_{2a}^- L_{2a+1}^-$, $1 \leqslant a \leqslant n-1$. We get 2^{n-1} spinors belonging to the positive spinorspace and we have the following weight space decomposition

$$\mathbb{S}_{2n}^+ = \bigoplus V_{\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \dots, \pm\frac{1}{2}\right)},$$

where $V_{\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \dots, \pm\frac{1}{2}\right)}$ denotes the weight space with weight $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \dots, \pm\frac{1}{2}\right)$ and where the sum goes over all weights with an even number of minus-signs. The highest weight remains $\left(\frac{1}{2}, \dots, \frac{1}{2}\right)$ and the highest weight vector is L.

Starting from $L_1^+ L_2^+ \ldots L_{2n-1}^+ L_{2n}^-$, we will generate all possible idempotents of the following form:

$$L_1^+ \underbrace{\cdot \ \cdot}\ \underbrace{\cdot \ \cdot} \cdots \underbrace{\cdot \ \cdot}\ L_{2n}^-$$

where each place $\underbrace{\cdot \ \cdot}$ consists of either $L_{2a}^+ L_{2a+1}^+$ or $L_{2a}^- L_{2a+1}^-$, $1 \leqslant a \leqslant n-1$. We thus also get 2^{n-1} spinors belonging to the negative spinorspace and the following weight space decomposition:

$$\mathbb{S}_{2n}^- = \bigoplus V_{\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)},$$

where the sum goes over all weights with an odd number of minus-signs. The highest weight is still $\left(\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}\right)$ and the highest weight vector is $L_1^+ L_2^+ \ldots L_{2n-1}^+ L_{2n}^-$.

4.2 Odd Dimension $m = 2n + 1$

We now extend the set of generators H_a, $X_{a,b}$, $Y_{a,b}$ and $Z_{a,b}$ of $\mathfrak{so}(m, \mathbb{C})$ with $2n$ operators

$$U_a = \frac{1}{\sqrt{2}} \left(dR(e_{2a-1,m}) - i\, dR(e_{2a,m})\right),$$
$$V_a = \frac{1}{\sqrt{2}} \left(dR(e_{2a-1,m}) + i\, dR(e_{2a,m})\right),$$

where $1 \leqslant a \leqslant n$. With the addition of these $2n$ operators, we are again able to reconstruct all original $dR(e_{a,b})$'s since $\sqrt{2}\ dR(e_{2a-1,m}) = U_a + V_a$ and $-\sqrt{2}\, i\, dR(e_{2a,m}) = U_a - V_a$.

The classic commutator relations follow immediately.

Lemma 4 *For $1 \leqslant a, b \leqslant n$, it holds that*

$$[H_a, U_b] = \delta_{ab}\, U_b = L_b(H_a)\, U_b,$$
$$[H_a, V_b] = -\delta_{ab}\, V_b = -L_b(H_a)\, V_b.$$

In particular, U_b is a root vector of $\mathfrak{so}(m, \mathbb{C})$ corresponding to the positive root L_b and V_b is a root vector corresponding with the negative root $-L_b$, for all $1 \leqslant b \leqslant n$.

Lemma 5 *The operators U_c and V_d satisfy the following additional commutator relations with $X_{a,b}$, $Y_{a,b}$ and $Z_{a,b}$, $1 \leqslant a, b, c, d \leqslant n$:*

$$\left[U_c, X_{a,b}\right] = -\delta_{cb}\, U_a, \qquad \left[V_c, X_{a,b}\right] = \delta_{ca}\, V_b,$$
$$\left[U_c, Y_{a,b}\right] = 0, \qquad \left[V_c, Y_{a,b}\right] = \delta_{ca}\, U_b - \delta_{cb}\, U_a,$$
$$\left[U_c, Z_{a,b}\right] = -\delta_{cb}\, V_a + \delta_{ca}\, V_b, \qquad \left[V_c, Z_{a,b}\right] = 0,$$

$$[U_c, U_d] = -Y_{c,d},\ c \neq d, \qquad [V_c, V_d] = -Z_{c,d},\ c \neq d,$$

$$[U_c, V_d] = \begin{cases} -X_{c,d}, & c \neq d, \\ -H_c, & c = d. \end{cases}$$

Proof The statements follow immediately from the definitions of U_c and V_d and from the defining relations (2) which the operators $dR(e_{a,b})$ satisfy.

We introduce four extra idempotents

$$L_m^{\pm} = \left(\mathbf{e}_m^+ \mathbf{e}_m^- \pm i\, \mathbf{e}_m^+\right), \qquad M_m^{\pm} = \left(\mathbf{e}_m^- \mathbf{e}_m^+ \pm \mathbf{e}_m^-\right)$$

and denote

$$L = \prod_{a=1}^{n} \left(L_{2a-1}^+\, L_{2a}^+\right) L_m^+, \qquad L' = L_1^+\, L_2^- \prod_{a=2}^{n} \left(L_{2a-1}^+\, L_{2a}^+\right) L_m^+.$$

We will now show that the highest weight vectors of weight $(k)'_+$ from the even-dimensional setting are also highest weight vectors with weight $(k)'_+$ in the odd-dimensional case when we multiply them with one of the four possible extra factors $L_m^{\pm}$ and $M_m^{\pm}$.

Theorem 2 *The weight vectors $g_k\, F$, $F = \prod_{s=1}^{m} F_s$ with $F_s \in \left\{L_s^{\pm}, M_s^{\pm}\right\}$ with*

- $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ *even*
- $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ *even, for all* $2 \leqslant a \leqslant n$,

vanish under the positive root vectors U_a, $1 \leqslant a \leqslant n$, i.e. $U_a\,(g_k\, F) = 0$, for all $1 \leqslant a \leqslant n$.

Proof Consider

$$\sqrt{2}\, U_a\,(g_k\, F) = \left(dR(e_{2a-1,m}) - i\, dR(e_{2a,m})\right) g_k\, F.$$

Since g_k contains only the vector variables ξ_1 and ξ_2, we will make a distinction between $a = 1$ and $a \neq 1$. We start with assuming that $a = 1$:

$$\begin{aligned} \sqrt{2} U_1\,(g_k\, F) = {} & V_{1,m} \left(\xi_m\, \partial_1 - \frac{1}{2}\right) g_k\, F\, e_1^{\perp}\, e_m^{\perp} - i\, V_{2,m} \left(\xi_m\, \partial_2 - \frac{1}{2}\right) g_k\, F\, e_2^{\perp}\, e_m^{\perp} \\ = {} & V_{1,m} \left(-k\, \xi_m\, f_{k-1} - \frac{1}{2}\, g_k\right) F\, e_1^{\perp}\, e_m^{\perp} \\ & - i\, V_{2,m} \left(k\, \xi_m\, f_{k-1} - \frac{1}{2}\, g_k\right) F\, e_2^{\perp}\, e_m^{\perp}. \end{aligned}$$

Now we again use that, for $j \neq 1, 2$:

$$V_{1,j}\, f_k = g_k\, V_{1,j}, \quad V_{2,j}\, f_k = (-1)^k\, g_k\, V_{2,j}, \quad V_{1,j}\, g_k = f_k\, V_{1,j},$$

$$V_{2,j}\, g_k = (-1)^k\, f_k\, V_{2,j}.$$

Hence

$$\begin{aligned}\sqrt{2}\, U_1\,(g_k\, F) &= \left(k\,\xi_m\, V_{1,m}\, f_{k-1} - \frac{1}{2}\, V_{1,m}\, g_k\right) F\, e_1^{\perp}\, e_m^{\perp}\\ &\quad - i\left(-k\,\xi_m\, V_{2,m}\, f_{k-1} - \frac{1}{2}\, V_{2,m}\, g_k\right) F\, e_2^{\perp}\, e_m^{\perp}\\ &= \left(k\,\xi_m\, g_{k-1} - \frac{1}{2}\, f_k\right) V_{1,m}\, F\, e_1^{\perp}\, e_m^{\perp}\\ &\quad - i\left((-1)^k\, k\,\xi_m\, g_{k-1} - \frac{1}{2}\,(-1)^k\, f_k\right) V_{2,m}\, F\, e_2^{\perp}\, e_m^{\perp}.\end{aligned}$$

We complete the proof by noting that

$$\begin{aligned}V_{1,m}\, F\, e_1^{\perp}\, e_m^{\perp} &= (-1)^{\|F_1\|+|F_1|+\|F_m\|+|F_m|+1}\, F^{2,m},\\ V_{2,m}\, F\, e_2^{\perp}\, e_m^{\perp} &= (-1)^{\|F_2\|+|F_2|+\|F_m\|+|F_m|}\, i\, F^{2,m}.\end{aligned}$$

Thus $\sqrt{2}\, U_1(g_k\, F)$ will be zero since $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even.

When $a \neq 1$, the action of $L_{2a-1,m}$ on $g_k\, F$ results in zero hence

$$\begin{aligned}\sqrt{2}\, U_a\,(g_k\, F) &= -\frac{1}{2}\, V_{2a-1,m}\, g_k\, F\, e_{2a-1}^{\perp}\, e_m^{\perp} + \frac{i}{2}\, V_{2a,m}\, g_k\, F\, e_{2a}^{\perp}\, e_m^{\perp}\\ &= -\frac{1}{2}\, g_k\left(V_{2a-1,m}\, F\, e_{2a-1}^{\perp}\, e_m^{\perp} - i\, V_{2a,m} F\, e_{2a}^{\perp}\, e_m^{\perp}\right)\\ &= -\frac{1}{2}\, g_k\left((-1)^{\|F_{2a-1}\|+|F_{2a-1}|+\|F_m\|+|F_m|}\right.\\ &\qquad \left. +(-1)^{\|F_{2a}\|+|F_{2a}|+\|F_m\|+|F_m|+1}\right) F^{2a,m}\\ &= -\frac{1}{2}\, g_k\,(-1)^{\|F_{2a-1}\|+|F_{2a-1}|+\|F_m\|+|F_m|}\\ &\qquad \left(1 + (-1)^{\|F_{2a-1}\|+|F_{2a-1}|+\|F_{2a}\|+|F_{2a}|+1}\right) F^{2a,m}.\end{aligned}$$

This will be zero when $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even, for all $2 \leqslant a \leqslant n$.

Corollary 2 *The polynomials $g_k\, F \in \mathcal{M}_k$ with $F = \prod_{s=1}^{m} F_s$, $F_s \in \left\{L_s^{\pm}, M_s^{\pm}\right\}$ such that*

- $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ *is even*
- $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ *is even, for all* $2 \leqslant a \leqslant n$,

are highest weight vectors of weight $(k)'_+$. In particular, $g_{2k}\, L$ and $g_{2k+1}\, L'$ are highest weight vectors of weight $(2k)'_+$ resp. $(2k+1)'_+$.

Note that the particular choice for the last factor $F_m \in \left\{L_m^{\pm}, M_m^{\pm}\right\}$ does not affect the results.

We again count how many distinct highest weight vectors $g_k\, F$, $F = \prod_{s=1}^{m} F_s$, with weight $(k)'_+$ we find: for each $F_{2a-1}\, F_{2a}$, $1 \leqslant a \leqslant n$, we have 8 possible combinations, namely

$$\left\{L_{2a-1}^{+}\, L_{2a}^{+},\quad L_{2a-1}^{-}\, L_{2a}^{-},\quad L_{2a-1}^{+}\, M_{2a}^{+},\quad L_{2a-1}^{-}\, M_{2a}^{-},\right.$$
$$\left. M_{2a-1}^{+}\, L_{2a}^{+},\quad M_{2a-1}^{+}\, M_{2a}^{+},\quad M_{2a-1}^{-}\, L_{2a}^{-},\quad M_{2a-1}^{-}\, M_{2a}^{-}\right\},$$

and for F_m we have four possible choices $L_m^{\pm}$, $M_m^{\pm}$. Combining this, we find $8^n\, 2^2 = 2^{3n+2} = 2^{2m-n}$ isomorphic irreducible representations with highest weight $(k)'_+$, each of which has dimension

$$2^n \binom{k+m-2}{k}.$$

Hence the total dimension of all isomorphic irreducible representations is

$$2^{4n+2} \binom{k+m-2}{k} = \dim_{\mathbb{C}} \mathcal{M}_k,$$

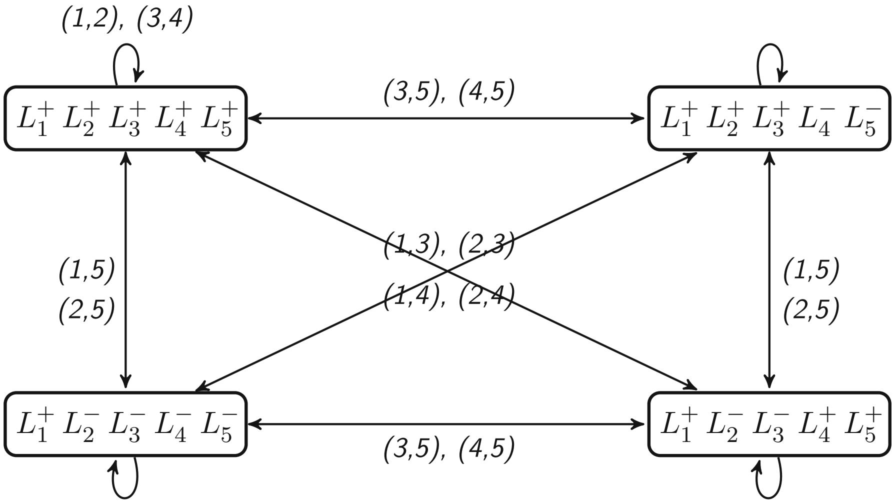
i.e. the dimensional analysis shows that $\mathcal{M}_k$ may be decomposed as 2^{3n+2} isomorphic irreducible representations with highest weight $(k)'_+$.

Definition 3 We define the spinor space $\mathbb{S}_{2n+1}$ as the image under $\mathfrak{so}(m,\mathbb{C})$ of the idempotent $L = \prod_{s=1}^{m} L_s^{+}$:

$$\mathbb{S}_{2n+1} = \mathfrak{so}(m,\mathbb{C}) \left(\mathrm{span}_{\mathbb{C}} \left\{L_1^{+}\, L_2^{+} \ldots L_{2n}^{+}\, L_{2n+1}^{+}\right\}\right).$$

The element $L_1^{+}\, L_2^{+} \ldots L_{2n}^{+}\, L_{2n+1}^{+}$ is a highest weight vector with weight $(0)'_+ = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ and thus generates an irreducible representation with the same weight.

Example 2 Let $m = 5$ (i.e. $n = 2$) and consider $L = L_1^{+}\, L_2^{+}\, L_3^{+}\, L_4^{+}\, L_5^{+}$. We denote $dR(e_{a,b})$ in short as (a, b). The Lie algebra $\mathfrak{so}(5,\mathbb{C})$ is given in this context by the span over $\mathbb{C}$ of the ten elements $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 4)$, $(3, 5)$ and $(4, 5)$. The idempotents involved interact in the following way under the action of $\mathfrak{so}(m,\mathbb{C})$:



Hence

$$\mathbb{S}_5 = \mathrm{span}_{\mathbb{C}} \left\{ L_1^+ L_2^+ L_3^+ L_4^+ L_5^+, L_1^+ L_2^- L_3^- L_4^+ L_5^+, L_1^+ L_2^- L_3^- L_4^- L_5^-, L_1^+ L_2^+ L_3^+ L_4^- L_5^- \right\}.$$

The spinorspace $\mathbb{S}_5$ is 2^2-dimensional as expected.

In general, the rotations $dR(e_{2a-1,2a}), a = 1, \ldots, n$, acting on an idempotent return the same idempotent up to a multiplicative complex number. Again, we find that $dR(e_{a,b})$ with $1 \leqslant a, b \leqslant n$, changes the sign of an even number of L_i's. The additional rotations $dR(e_{2a-1,m})$ and $dR(e_{2a,m})$, with $1 \leqslant a \leqslant n$, act as follows on L:

$$L_1^+ L_2^+ \ldots L_{2n}^+ L_{2n+1}^+ \mapsto L_1^+ L_2^+ \ldots L_{2a-1}^+ L_{2a}^- \ldots L_{2n}^- L_{2n+1}^-.$$

The rotation always leaves L_1^+ invariant. The resulting idempotent always will have an even number of minus-signs. Starting from the idempotent L with all plus-signs, we thus get all possible idempotents of the following form:

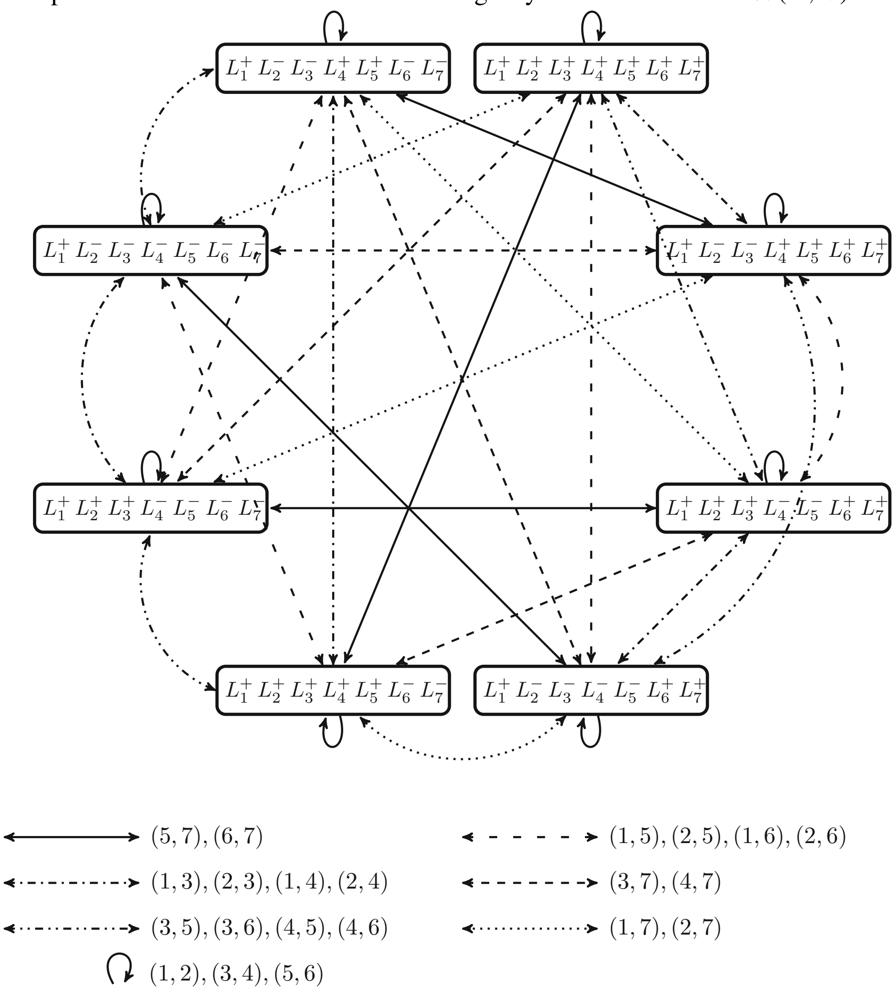
$$L_1^+ \underbrace{\cdot \ \cdot} \ \underbrace{\cdot \ \cdot} \cdots \underbrace{\cdot \ \cdot}$$

where each place $\underbrace{\cdot \ \cdot}$ consists of either $L_{2a}^+ L_{2a+1}^+$ or $L_{2a}^- L_{2a+1}^-$, $1 \leqslant a \leqslant n$. We thus get 2^n spinors and we have the following weight space decomposition

$$\mathbb{S}_{2n+1} = \bigoplus V_{\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)},$$

where the sum goes over all weights with an even number of minus-signs.

Example 3 Let $m = 7$ (i.e. $n = 3$) and consider $L = L_1^+ L_2^+ L_3^+ L_4^+ L_5^+ L_6^+ L_7^+$. We will again denote $dR(e_{a,b})$ in short as (a, b). The Lie algebra $\mathfrak{so}(5, \mathbb{C})$ is given in this setting by 21 elements and the corresponding spinorspace will be 8-dimensional. The idempotents involved interact in the following way under the action of $\mathfrak{so}(m, \mathbb{C})$:



Hence

$$\begin{aligned}\mathbb{S}_7 = \operatorname{span}_{\mathbb{C}} \{ & L_1^+ L_2^+ L_3^+ L_4^+ L_5^+ L_6^+ L_7^+,\ L_1^+ L_2^- L_3^- L_4^+ L_5^+ L_6^+ L_7^+, \\ & L_1^+ L_2^+ L_3^+ L_4^- L_5^- L_6^+ L_7^+, L_1^+ L_2^- L_3^- L_4^- L_5^- L_6^+ L_7^+, \\ & L_1^+ L_2^+ L_3^+ L_4^+ L_5^+ L_6^- L_7^-,\ L_1^+ L_2^+ L_3^+ L_4^- L_5^- L_6^- L_7^-, \\ & L_1^+ L_2^- L_3^- L_4^- L_5^- L_6^- L_7^-,\ L_1^+ L_2^- L_3^- L_4^+ L_5^+ L_6^- L_7^- \} .\end{aligned}$$

We indeed find an 8-dimensional spinorspace $\mathbb{S}_7$.

5 Conclusion and Future Research

The space $\mathcal{M}_k$ of discrete k-homogeneous monogenic polynomials is a reducible representation of $\mathfrak{so}(m, \mathbb{C})$ which can, in the odd-dimensional case $m = 2n + 1$, be decomposed into 2^{2m-n} isomorphic copies of the irreducible $\mathfrak{so}(m, \mathbb{C})$-representation with highest weight $(k)'_+$ and in the even-dimensional setting $m = 2n$, we find 2^{2m-n} isomorphic irreducible representations with highest weight $(k)'_+$ and 2^{2m-n} irreps of highest weight $(k)'_-$. This is done by means of an appropriate amount of idempotents.

Let $g_k = (\xi_2 - \xi_1)(\xi_2 + \xi_1)(\xi_2 - \xi_1)(\xi_2 + \xi_1)\ldots$ [1], (k factors), be a discrete homogeneous monogenic function of degree k and let

$$L^{\pm}_{2a-1} = \left(\mathbf{e}^{+}_{2a-1}\mathbf{e}^{-}_{2a-1} \pm i\, \mathbf{e}^{+}_{2a-1}\right), \qquad L^{\pm}_{2a} = \left(\mathbf{e}^{+}_{2a}\mathbf{e}^{-}_{2a} \pm \mathbf{e}^{+}_{2a}\right),$$
$$M^{\pm}_{2a-1} = \left(\mathbf{e}^{-}_{2a-1}\mathbf{e}^{+}_{2a-1} \pm i\, \mathbf{e}^{-}_{2a-1}\right), \qquad M^{\pm}_{2a} = \left(\mathbf{e}^{-}_{2a}\mathbf{e}^{+}_{2a} \pm \mathbf{e}^{-}_{2a}\right).$$

Denote $\|L^{\pm}_a\| = 0$, $\|M^{\pm}_a\| = 1$, $|L^{+}_a| = |M^{-}_a| = 0$ and $|L^{-}_a| = |M^{+}_a| = 1$.

In even dimension $m = 2n$, the polynomial $g_k\, F \in \mathcal{M}_k$, $F = \prod_{s=1}^{m} F_s$ with $F_s \in \left\{L^{\pm}_s, M^{\pm}_s\right\}$, is a highest weight vector of $\mathfrak{so}(m, \mathbb{C})$ with

- weight $(k)'_+$ when $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even and $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even for $2 \leqslant a \leqslant n$.
- weight $(k)'_-$ when $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even, $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even for $2 \leqslant a \leqslant n-1$ and $\|F_{2n-1}\| + \|F_{2n}\| + |F_{2n-1}| + |F_{2n}|$ is odd.

We find 2^{2m-n} highest weight vectors in $\mathcal{M}_k$ with weight $(k)'_+$ and 2^{2m-n} weight vectors, with weight $(k)'_-$, each generating an irreducible $\mathfrak{so}(m, \mathbb{C})$-representation.

In odd dimensions $m = 2n + 1$, the polynomial $g_k\, F \in \mathcal{M}_k$, $F = \prod_{s=1}^{m} F_s$ with $F_s \in \left\{L^{\pm}_s, M^{\pm}_s\right\}$, is a weight vector of $\mathfrak{so}(m, \mathbb{C})$ with weight $(k)'_+$ when $k + |F_1| + |F_2| + \|F_1\| + \|F_2\|$ is even and $\|F_{2a-1}\| + \|F_{2a}\| + |F_{2a-1}| + |F_{2a}|$ is even for $2 \leqslant a \leqslant n$. We find 2^{2m-n} highest weight vectors in $\mathcal{M}_k$ with weight $(k)'_+$ and thus as much irreps.

We have shown throughout this article how the space $\mathcal{M}_k$ of monogenic discrete k-homogeneous polynomials can be decomposed into irreducible representations of $\mathfrak{so}(m, \mathbb{C})$. However, because of the presence of the basiselements e_a and $e_a^{\perp}$ in the definition of the generators of the rotations, the spinorspace is no maximal left ideal. In future research we will investigate other possibilities to define rotations and the spinorspace in the hopes of writing the spinorspace as maximal left ideal. An equivalent description of $\mathcal{H}_k$ and $\mathcal{M}_k$ as $SO(m)/Spin(m)$-representations is also still work in progress.

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