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Applications of Completions of Operator Matrices to Some Properties of Operator Products on Hilbert Spaces

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Abstract In this paper, we will investigate certain properties of some operator products on Hilbert spaces, by applications of completions of operator matrices. It is shown that, quite surprisingly, the invariance properties of the operator product $T_1T_2T_2^{(1,\ldots)}T_1^{(1,\ldots)}T_1T_2$ have a neat relationship with the properties of the reverse order laws for generalized inverses of the operator product T_1T_2. That is, the mixed-type reverse order laws

$$T_2\{1,\ldots\}T_1\{1,\ldots\} \subseteq (T_1T_2)\{1\}$$

hold if and only if the operator product $T_1T_2T_2^{(1,\ldots)}T_1^{(1,\ldots)}T_1T_2$ is invariant, where $(1,\ldots)$ is taken respectively as (1), $(1,2)$, $(1,3)$, $(1,4)$, $(1,2,3)$ as well as $(1,2,4)$.

Keywords Generalized inverse · Invariance property · Mixed-type reverse order law · Operator product · Operator matrices

Mathematics Subject Classification 47A05 · 15A09 · 15A24

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1 Introduction

Throughout this paper "an operator" means "a bounded linear operator over Hilbert space". $\mathcal{H}$, $\mathcal{L}$, $\mathcal{J}$ and $\mathcal{K}$ denote arbitrary Hilbert spaces. $L(\mathcal{H},\mathcal{K})$ denotes the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. Also $L(\mathcal{H})=L(\mathcal{H},\mathcal{H})$. I denotes the unit operator over Hilbert space and O is the zero operator over Hilbert space. For an operator $T\in L(\mathcal{H},\mathcal{K})$, we use $R(T)$ to denote the range of T and $N(T)$ to denote the null-space of T. The symbol T^* denotes the adjoint of T. The operator T is a self-adjoint operator if and only if $T^*=T$.

Recall that an operator $X\in L(\mathcal{K},\mathcal{H})$ is called the Moore-Penrose inverse of $T\in L(\mathcal{H},\mathcal{K})$, if X satisfies the following four operator equations [1,25],

$$(1)\ TXT=T,\quad (2)\ XTX=X,\quad (3)\ (TX)^*=TX,\quad (4)\ (XT)^*=XT.$$

If such operator X exists then it is unique and is denoted by $T^\dagger$. It is well known that the Moore-Penrose inverse of T exists if and only if $R(T)$ is closed, see [9].

Let $\phi\neq\eta\subseteq\{1,2,3,4\}$. Then $T\eta$ denotes the set of all operators X, which satisfy (i) for all $i\in\eta$. Any $X\in T\eta$ is called an η-inverse of T and is denoted by $T^{(\eta)}$. For example, an operator X of the set $T\{1\}$ is called a $\{1\}$-inverse of T and can be written as $X=T^{(1)}\in T\{1\}$. The well-known seven common types of generalized inverses of T are respectively, the $\{1\}$-inverse (g-inverse), $\{1,2\}$-inverse (reflexive g-inverse), $\{1,3\}$-inverse (least square g-inverse), $\{1,4\}$-inverse (minimum norm g-inverse), $\{1,2,3\}$-inverse, $\{1,2,4\}$-inverse and $\{1,2,3,4\}$-inverse (Moore-Penrose inverse). We refer the reader to [1,26] for basic results on the generalized inverses of bounded linear operators.

Invariance properties of operator product involving generalized inverses are important in the theory of operators. They have attracted considerable attention and many interesting results have been obtained, see [4,16,28,29]. Let $T_i, i=1,2,3$ be three operators on Hilbert spaces. Concerning the invariance properties of $T_1T_2^{(1,\ldots)}T_3$ for various type of generalized inverses $T_2^{(1,\ldots)}$ of T_2 are well known in the literature, see [2,3,5,15,22]. It has quite important applications in operator algebra and applied fields, such as, nonlinear control theory [10,23], statistics [1,24], projection algorithms and perturbation analysis of operator [14,19]. Moreover, the invariance properties of operator product is an useful tool in many algorithms for computation of the generalized inverses of operators, see [1,14,26].

Another property of operator product is the reverse order laws for generalized inverses. Let $T_1\in L(\mathcal{H},\mathcal{L})$ and $T_1\in L(\mathcal{K},\mathcal{H})$ be two operators such that the product T_1T_2 exists. An interesting problem is, for $\phi\neq\eta\subseteq\{1,2,3,4\}$, when $T_2\eta T_1\eta\subseteq(T_1T_2)\eta$? The reverse order laws for the generalized inverses of operator products first discussed by Greville [17]. Bouldin [6] and Izumino [18] extended the results of Greville [17] to the bounded linear operators on Hilbert space, by using the gaps between subspaces. Djordjević [11] showed that the reverse order law $(T_1T_2)^\dagger=T_2^\dagger T_1^\dagger$ holds if and only if $R(T_1^*T_1T_2)\subseteq R(T_2)$ and $R(T_2T_2^*T_1^*)\subseteq R(T_1^*)$. Kohila et al. [7,20] obtained the necessary and sufficient conditions for the reverse order law of the Moore-Penrose inverse in C^*-algebra. The reader can find more results of the reverse order law for the generalized inverse of operator product in [9,12,13,18,27,30].

Recently, in [28], the authors presented invariance properties of matrix products related to the reverse order laws for generalized inverses. Liu et al. [21] extended the results of [28] to the bounded linear operators on Hilbert space. They investigated the relationship between the invariance properties of the bounded linear operator product $T_1T_2T_2^{(1,\dots)}T_1^{(1,\dots)}T_1T_2$ and the mixed-type reverse order laws for corresponding generalized inverses, by using a purely algebraic method. The drawback of their method is that they cannot be applied to some more generalized structures. In this paper, by applications of completions of operator matrices, we revisited these problem again and some more simple conditions are derived. Compared with the results given in [21], our condition can be easily checked and the proof is very simple.

We first mention the following results, which will be used in this paper.

Lemma 1.1 [8,11] *Let $T \in L(\mathcal{H},\mathcal{K})$ have a closed range. Let H_1 and H_2 be closed and mutually orthogonal subspace of $\mathcal{H}$, such that $H_1 \bigoplus H_2 = \mathcal{H}$. Let K_1 and K_2 be closed and mutually orthogonal subspace of $\mathcal{K}$, such that $\mathcal{K} = K_1 \bigoplus K_2$. Then the operator T has the following matrix representations with respect to the orthogonal sums of subspaces $\mathcal{H} = H_1 \bigoplus H_2 = R(T^*) \bigoplus N(T)$ and $\mathcal{K} = K_1 \bigoplus K_2 = R(T) \bigoplus N(T^*)$:*

1. $T = \begin{pmatrix} T_{11} & T_{12} \\ O & O \end{pmatrix} : \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \to \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix}$ *and* $T^\dagger = \begin{pmatrix} T_{11}^*E^{-1} & O \\ T_{12}^*E^{-1} & O \end{pmatrix} : \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix} \to \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$, *where $E = T_{11}T_{11}^* + T_{12}T_{12}^*$ is invertible on $R(T)$;*
2. $T = \begin{pmatrix} T_{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix} \to \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix}$ *and* $T^\dagger = \begin{pmatrix} T_{11}^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix} \to \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix}$, *where T_{11} is invertible on $R(T^*)$.*

Lemma 1.2 [1,11] *Let $T \in L(\mathcal{H},\mathcal{K})$ have a closed range and $G \in L(\mathcal{K},\mathcal{H})$. Then*

1. $TGT = T \Leftrightarrow G \in T\{1\}$ *and* $T\{1\} = \{T^\dagger + Y - T^\dagger TYTT^\dagger : Y \in L(\mathcal{K},\mathcal{H})\}$*;*
2. $TGT = T$ *and* $GTG = G \Leftrightarrow G \in T\{1,2\}$ *and* $T\{1,2\} = \{[T^\dagger + (I - T^\dagger T)Y_1]T[T^\dagger + Y_2(I - TT^\dagger)] : Y_i \in L(\mathcal{K},\mathcal{H}), i = 1,2\}$*;*
3. $TGT = T$ *and* $(TG)^* = TG \Leftrightarrow G \in T\{1,3\}$ *and* $T\{1,3\} = \{T^\dagger + (I - T^\dagger T)Y : Y \in L(\mathcal{K},\mathcal{H})\}$*;*
4. $TGT = T$ *and* $(GT)^* = GT \Leftrightarrow G \in T\{1,4\}$ *and* $T\{1,4\} = \{T^\dagger + Y(I - TT^\dagger) : Y \in L(\mathcal{K},\mathcal{H})\}$*;*
5. $TGT = T,\ GTG = G$ *and* $(TG)^* = TG \Leftrightarrow G \in T\{1,2,3\}$ *and* $T\{1,2,3\} = \{T^\dagger + (I - T^\dagger T)YTT^\dagger : Y \in L(\mathcal{K},\mathcal{H})\}$*;*
6. $TGT = T, GTG = G$ *and* $(GT)^* = GT \Leftrightarrow G \in T\{1,2,4\}$ *and* $T\{1,2,4\} = \{T^\dagger + T^\dagger TY(I - TT^\dagger) : Y \in L(\mathcal{K},\mathcal{H})\}$*;*
7. $TGT = T, (TG)^* = TG$ *and* $(GT)^* = GT \Leftrightarrow G \in T\{1,3,4\}$ *and* $T\{1,3,4\} = \{T^\dagger + (I - T^\dagger T)Y(I - TT^\dagger) : Y \in L(\mathcal{K},\mathcal{H})\}$.

Lemma 1.3 [1] *Let $T \in L(\mathcal{H},\mathcal{K})$, $E_T = I - TT^\dagger$ and $F_T = I - T^\dagger T$. Then*

$$R(F_T) = N(T) \quad and \quad N(E_T) = R(A).$$

Lemma 1.4 [1] *Let* $T \in L(\mathcal{H}, \mathcal{K})$ *and* $W \in L(\mathcal{K}, \mathcal{H})$ *have closed ranges. Then*

$$(I - TT^\dagger)(I - W^\dagger W) \Leftrightarrow N(W) \subseteq R(T).$$

Lemma 1.5 [22] *Let* $T \in L(\mathcal{H}, \mathcal{K})$ *and* $W \in L(\mathcal{I}, \mathcal{J})$ *have closed ranges. Then* $TQW = O$ *for every* $Q \in L(\mathcal{J}, \mathcal{H})$ *if and only if* $T = O$ *or* $W = O$.

2 Invariance Property of $T_1T_2T_2^{(1)}T_1^{(1)}T_1T_2$ Related to the Reverse Order Law $T_2\{1\}T_1\{1\} \subseteq (T_1T_2)\{1\}$

Given operators $T_1 \in L(\mathcal{H}, \mathcal{L})$ and $T_2 \in L(\mathcal{K}, \mathcal{H})$. In this section, we will show that the reverse order law $T_2\{1\}T_1\{1\} \subseteq (T_1T_2)\{1\}$ holds if and only if the operator product $T_1T_2T_2^{(1)}T_1^{(1)}T_1T_2$ is invariant, where $T_1^{(1)} \in T_1\{1\}$ and $T_2^{(1)} \in T_2\{1\}$ are two variant operators.

Theorem 2.1 *Let* $T_1 \in L(\mathcal{H}, \mathcal{L})$ *and* $T_2 \in L(\mathcal{K}, \mathcal{H})$ *such that* T_1, T_2 *have closed ranges and* $T_1T_2 \neq O$. *Then the following statements are equivalent:*

(1) $T_1T_2T_2^{(1)}T_1^{(1)}T_1T_2$ *is invariant with respect to the choice of* $T_1^{(1)} \in T_1\{1\}$ *and* $T_2^{(1)} \in T_2\{1\}$;

(2) $N(T_1) \subseteq R(T_2)$, *i.e,* $(I - T_2T_2^\dagger)(I - T_1^\dagger T_1) = O$;

(3) $T_2\{1\}T_1\{1\} \subseteq (T_1T_2)\{1\}$.

Proof From Lemma 1.1, we know that the operators T_1, T_2 and T_1T_2 have the following matrix forms with respect to the orthogonal sum of subspaces:

$$T_1 = \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix}, \tag{2.1}$$

$$T_1^\dagger = \begin{pmatrix} (T_1^{11})^* D^{-1} & O \\ (T_1^{12})^* D^{-1} & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}, \tag{2.2}$$

where $D = T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*$ is invertible on $R(T_1)$.

$$T_2 = \begin{pmatrix} T_2^{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}, \tag{2.3}$$

$$T_2^\dagger = \begin{pmatrix} (T_2^{11})^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}, \tag{2.4}$$

where T_2^{11} is invertible.

$$T_1T_2 = \begin{pmatrix} T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \to \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \tag{2.5}$$

and

$$(T_1T_2)^* = \begin{pmatrix} (T_2^{11})^*(T_1^{11})^* & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}. \tag{2.6}$$

Combining (2.1)–(2.6) with the results in Lemma 1.2, it follows that there exist two bounded linear operators $Y \in L(\mathcal{L}, \mathcal{H})$ and $W \in L(\mathcal{K}, \mathcal{J})$:

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix},$$
$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix},$$

such that

$$\begin{aligned} T_1^{(1)} &= T_1^\dagger + Y - T_1^\dagger T_1 Y T_1 T_1^\dagger \\ &= \begin{pmatrix} (T_1^{11})^* D^{-1} + Y_{11} - (T_1^{11})^* D^{-1} T_1^{11} Y_{11} - (T_1^{11})^* D^{-1} T_1^{12} Y_{21} & Y_{12} \\ (T_1^{12})^* D^{-1} + Y_{21} - (T_1^{12})^* D^{-1} T_1^{12} Y_{21} - (T_1^{12})^* D^{-1} T_1^{11} Y_{11} & Y_{22} \end{pmatrix}, \end{aligned} \tag{2.7}$$

$$T_2^{(1)} = T_2^\dagger + W - T_1^\dagger T_1 W T_1 T_1^\dagger = \begin{pmatrix} (T_2^{11})^{-1} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \tag{2.8}$$

where Y_{ij} and W_{ij}, $i, j = 1, 2$ are arbitrary bounded linear operators on appropriate spaces.

(1) ⇒ (2): Since the Moore-Penrose inverse is unique and belongs to the set {1}-inverses, then $T_1T_2T_2^{(1)}T_1^{(1)}T_1T_2$ is invariant with respect to the choice of $T_1^{(1)} \in T_1\{1\}$ and $T_2^{(1)} \in T_2\{1\}$ if and only if the following equation

$$T_1T_2T_2^{(1)}T_1^{(1)}T_1T_2 = T_1T_2T_2^\dagger T_1^\dagger T_1T_2, \tag{2.9}$$

holds with respect to the choice of $T_1^{(1)} \in T_1\{1\}$ and $T_2^{(1)} \in T_2\{1\}$.

Substituting $T_1^\dagger$ for $T_1^{(1)}$ in (2.9), we have

$$T_1T_2T_2^{(1)}T_1^\dagger T_1T_2 = T_1T_2T_2^\dagger T_1^\dagger T_1T_2. \tag{2.10}$$

By (2.1)–(2.5), (2.8) and (2.10), we have

$$\begin{aligned} & T_1T_2T_2^{(1)}T_1^\dagger T_1T_2 = T_1T_2T_2^\dagger T_1^\dagger T_1T_2 \\ \Leftrightarrow & \begin{pmatrix} T_1^{11}(T_1^{11})^* D^{-1} T_1^{11} T_2^{11} + T_1^{11} T_2^{11} W_{12} (T_1^{12})^* D^{-1} T_1^{11} T_2^{11} & O \\ O & O \end{pmatrix} \\ = & \begin{pmatrix} T_1^{11}(T_1^{11})^* D^{-1} T_1^{11} T_2^{11} & O \\ O & O \end{pmatrix} \\ \Leftrightarrow & \, T_1^{11} T_2^{11} W_{12} (T_1^{12})^* D^{-1} T_1^{11} T_2^{11} = O \\ \Leftrightarrow & \, T_1^{11} T_2^{11} W_{12} (T_1^{12})^* D^{-1} T_1^{11} = O. \end{aligned} \tag{2.11}$$

Since T_2^{11} is invertible, $T_1^{11}T_2^{11} \neq O$ and W_{12} is arbitrary, then from Lemma 1.5, we have

$$(T_1^{12})^* D^{-1} T_1^{11} = O. \tag{2.12}$$

Substituting (2.1)–(2.8) and (2.12) in (2.9), we have

$$\begin{aligned}
&T_1T_2T_2^{(1)}T_1^{(1)}T_1T_2 = T_1T_2T_2^{\dagger}T_1^{\dagger}T_1T_2\\
&\Leftrightarrow \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} + T_1^{11}T_2^{11}W_{12}Y_{21}T_1^{11}T_2^{11} - T_1^{11}T_2^{11}W_{12}(T_1^{12})^*D^{-1}T_1^{12}Y_{21}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix}\\
&= \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix}\\
&\Leftrightarrow T_1^{11}T_2^{11}W_{12}Y_{21}T_1^{11}T_2^{11} - T_1^{11}T_2^{11}W_{12}(T_1^{12})^*D^{-1}T_1^{12}Y_{21}T_1^{11}T_2^{11} = O\\
&\Leftrightarrow T_1^{11}T_2^{11}W_{12}(I - (T_1^{12})^*D^{-1}T_1^{12})Y_{21}T_1^{11}T_2^{11} = O.
\end{aligned} \tag{2.13}$$

Since $T_1^{11}T_2^{11} \neq O$ and W_{12}, Y_{21} are arbitrary, then from Lemma 1.5 and (2.13), we have

$$I - (T_1^{12})^* D^{-1} T_1^{12} = O. \tag{2.14}$$

By Lemma 1.3, and Lemma 1.4, we know that

$$N(T_1) \subseteq R(T_2) \Leftrightarrow (I - T_2T_2^{\dagger})(I - T_1^{\dagger}T_1) = O. \tag{2.15}$$

Substituting (2.1)–(2.4) in (2.15) yields

$$\begin{aligned}
&N(T_1) \subseteq R(T_2) \Leftrightarrow (I - T_2T_2^{\dagger})(I - T_1^{\dagger}T_1) = O\\
&\Leftrightarrow \begin{pmatrix} O & O \\ O & I \end{pmatrix}\begin{pmatrix} I - (T_1^{11})^*D^{-1}T_1^{11} & -(T_1^{11})^*D^{-1}T_1^{12} \\ -(T_1^{12})^*D^{-1}T_1^{11} & I - (T_1^{12})^*D^{-1}T_1^{12} \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix}\\
&\Leftrightarrow \begin{pmatrix} O & O \\ -(T_1^{12})^*D^{-1}T_1^{11} & I - (T_1^{12})^*D^{-1}T_1^{12} \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix}.
\end{aligned} \tag{2.16}$$

According to (2.12) and (2.14), we have (2.15) and (2.16) hold. That is (1) $\Rightarrow$ (2) holds.

(2) $\Rightarrow$ (3): If $N(T_1) \subseteq R(T_2)$, then from (2.16) we have

$$(T_1^{12})^* D^{-1} T_1^{11} = O \tag{2.17}$$

and

$$I - (T_1^{12})^* D^{-1} T_1^{12} = O. \tag{2.18}$$

On the other hand, from the formula (1) in Lemma 1.2, we know that the reverse order law $T_2\{1\}T_1\{1\} \subseteq (T_1T_2)\{1\}$ holds if and only if the equation

$$T_1T_2T_2^{(1)}T_1^{(1)}T_1T_2 = T_1T_2 \tag{2.19}$$

holds for any $T_i^{(1)} \in T_i\{1\}$, $i = 1, 2$.

Since $D = T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*$, then according to (2.1)–(2.8), (2.17) and (2.18), we get that for any $T_i^{(1)} \in T_i\{1\}$, $i = 1, 2$

$$
\begin{aligned}
&T_1T_2T_2^{(1)}T_1^{(1)}T_1T_2\\
&= \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} + T_1^{11}T_2^{11}W_{12}Y_{21}T_1^{11}T_2^{11} - T_1^{11}T_2^{11}W_{12}(T_1^{12})^*D^{-1}T_1^{12}Y_{21}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix}\\
&= \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix}\\
&= \begin{pmatrix} (D - T_1^{12}(T_1^{12})^*)D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix}\\
&= T_1T_2.
\end{aligned} \tag{2.20}
$$

Combining (2.19) with (2.20), the result (2) $\Rightarrow$ (3) is true.

(3) $\Rightarrow$ (1): If $T_2\{1\}T_1\{1\} \subseteq (T_1T_2)\{1\}$, then the result $T_2^{(1)}T_1^{(1)} \in (T_1T_2)\{1\}$ holds for any $T_1^{(1)} \in T_1\{1\}$ and $T_2^{(1)} \in T_2\{1\}$. So, from the formula (1) in Lemma 1.2, we know that the equation

$$T_1T_2T_2^{(1)}T_1^{(1)}T_1T_2 = T_1T_2 \tag{2.21}$$

holds for any $T_1^{(1)} \in T_1\{1\}$ and $T_2^{(1)} \in T_2\{1\}$. Namely, (3) $\Rightarrow$ (1) is true. □

3 Invariance Property of $T_1T_2T_2^{(1,2)}T_1^{(1,2)}T_1T_2$ Related to the Mixed-Type Reverse Order Law $T_2\{1,2\}T_1\{1,2\} \subseteq (T_1T_2)\{1\}$

Let $T_1 \in L(\mathcal{H}, \mathcal{L})$ and $T_2 \in L(\mathcal{K}, \mathcal{H})$ be two given operators, $T_1^{(1,2)} \in T_1\{1,2\}$ and $T_2^{(1,2)} \in T_2\{1,2\}$ are two variant operators. In this section, we have the following interesting results.

Theorem 3.1 *Let $T_1 \in L(\mathcal{H}, \mathcal{L})$ and $T_2 \in L(\mathcal{K}, \mathcal{H})$ such that T_1, T_2 have closed ranges and $T_1T_2 \neq O$. Then the following statements are equivalent:*

1. *$T_1T_2T_2^{(1,2)}T_1^{(1,2)}T_1T_2$ is invariant for any $T_1^{(1,2)} \in T_1\{1,2\}$ and $T_2^{(1,2)} \in T_2\{1,2\}$;*
2. *$N(T_1) \subseteq R(T_2)$;*
3. *$T_2\{1,2\}T_1\{1,2\} \subseteq (T_1T_2)\{1\}$.*

Proof Combining (2.1)–(2.6) with the results in Lemma 1.2, it follows that there exist four bounded linear operators $W \in L(\mathcal{L}, \mathcal{H})$, $M \in L(\mathcal{L}, \mathcal{H})$, $U \in L(\mathcal{H}, \mathcal{K})$ and $V \in L(\mathcal{H}, \mathcal{K})$:

$$
\begin{aligned}
W &= \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix},\\
&\quad \textit{where } W_{ij},\ i, j = 1, 2 \textit{ are arbitrary},\\
M &= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix},\\
&\quad \textit{where } M_{ij},\ i, j = 1, 2 \textit{ are arbitrary},
\end{aligned}
$$

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix},$$
$$where\ U_{ij},\ i, j = 1, 2\ are\ arbitrary,$$
$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix},$$
$$where\ V_{ij},\ i, j = 1, 2\ are\ arbitrary,$$

such that

$$T_1^{(1,2)} = [T_1^\dagger + (I - T_1^\dagger T_1)W]T_1[T_1^\dagger + M(I - T_1 T_1^\dagger)] = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix}, \tag{3.1}$$

$$T_2^{(1,2)} = [T_2^\dagger + (I - T_2^\dagger T_2)U]T_2[T_2^\dagger + V(I - T_2 T_2^\dagger)] = \begin{pmatrix} (T_2^{11})^{-1} & V_1 \\ U_1 & U_1 T_2^{11} V_1 \end{pmatrix}, \tag{3.2}$$

where U_1, V_1 are arbitrary bounded linear operators and

$$\begin{aligned} D &= T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*, \\ \mu_{11} &= (T_1^{11})^* D^{-1} + [I - (T_1^{11})^* D^{-1} T_1^{11}]W_{11} - (T_1^{11})^* D^{-1} T_1^{12} W_{21}, \\ \mu_{12} &= \mu_{11}.(T_1^{11} M_{12} + T_1^{12} M_{22}), \\ \mu_{21} &= (T_1^{12})^* D^{-1} + [I - (T_1^{12})^* D^{-1} T_1^{12}]W_{21} - (T_1^{12})^* D^{-1} T_1^{11} W_{11}, \\ \mu_{22} &= \mu_{21}.(T_1^{11} M_{12} + T_1^{12} M_{22}). \end{aligned}$$

(1) $\Rightarrow$ (2): Since the Moore–Penrose inverse is unique and belongs to the set $\{1, 2\}$-inverses, then $T_1 T_2 T_2^{(1,2)} T_1^{(1,2)} T_1 T_2$ is invariant with respect to the choice of $T_1^{(1,2)} \in T_1\{1, 2\}$ and $T_2^{(1,2)} \in T_2\{1, 2\}$ if and only if the following equation

$$T_1 T_2 T_2^{(1,2)} T_1^{(1,2)} T_1 T_2 = T_1 T_2 T_2^\dagger T_1^\dagger T_1 T_2, \tag{3.3}$$

holds with respect to the choice of $T_1^{(1,2)} \in T_1\{1, 2\}$ and $T_2^{(1,2)} \in T_2\{1, 2\}$.

Substituting $T_1^\dagger$ for $T_1^{(1,2)}$ in (3.3), we have

$$T_1 T_2 T_2^{(1,2)} T_1^\dagger T_1 T_2 = T_1 T_2 T_2^\dagger T_1^\dagger T_1 T_2. \tag{3.4}$$

By (2.1)–(2.5), (3.2) and (3.4), we have

$$\begin{aligned} & T_1 T_2 T_2^{(1,2)} T_1^\dagger T_1 T_2 = T_1 T_2 T_2^\dagger T_1^\dagger T_1 T_2 \\ \Leftrightarrow & \begin{pmatrix} T_1^{11}(T_1^{11})^* D^{-1} T_1^{11} T_2^{11} + T_1^{11} T_2^{11} V_1 (T_1^{12})^* D^{-1} T_1^{11} T_2^{11} & O \\ O & O \end{pmatrix} \\ = & \begin{pmatrix} T_1^{11}(T_1^{11})^* D^{-1} T_1^{11} T_2^{11} & O \\ O & O \end{pmatrix} \end{aligned}$$

$$\Leftrightarrow T_1^{11}T_2^{11}V_1(T_1^{12})^*D^{-1}T_1^{11}T_2^{11} = O$$
$$\Leftrightarrow T_1^{11}T_2^{11}V_1(T_1^{12})^*D^{-1}T_1^{11} = O. \tag{3.5}$$

Since T_2^{11} is invertible, $T_1^{11}T_2^{11} \neq O$ and V_1 is arbitrary, then from Lemma 1.6, we have

$$(T_1^{12})^*D^{-1}T_1^{11} = O. \tag{3.6}$$

Substituting (2.1)–(2.8), (3.1), (3.2) and (3.6) in (3.3), we have

$$\begin{aligned}
&T_1T_2T_2^{(1,2)}T_1^{(1,2)}T_1T_2 = T_1T_2T_2^{\dagger}T_1^{\dagger}T_1T_2\\
&\Leftrightarrow \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} + T_1^{11}T_2^{11}V_1[I-(T_1^{12})^*D^{-1}T_1^{12}]W_{21}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix}\\
&= \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix}\\
&\Leftrightarrow T_1^{11}T_2^{11}V_1(I-(T_1^{12})^*D^{-1}T_1^{12})W_{21}T_1^{11}T_2^{11} = O.
\end{aligned} \tag{3.7}$$

Since $T_1^{11}T_2^{11} \neq O$ and V_1, W_{21} are arbitrary, then from Lemma 1.6 and (3.7), we have

$$I - (T_1^{12})^*D^{-1}T_1^{12} = O. \tag{3.8}$$

Combining (3.6), (3.8) with (2.15) and (2.16), we have (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3): From the formulas in Lemma 1.2, we know that the reverse order law $T_2\{1,2\}T_1\{1,2\} \subseteq (T_1T_2)\{1\}$ holds if and only if

$$T_1T_2T_2^{(1,2)}T_1^{(1,2)}T_1T_2 = T_1T_2 \tag{3.9}$$

holds for any $T_i^{(1,2)} \in T_i\{1,2\}$, $i = 1,2$.

Since $D = T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*$, then according to (2.5), (3.1), (3.2), (3.6) and (3.8), we get that for any $T_i^{(1,2)} \in T_i\{1,2\}$, $i = 1,2$

$$\begin{aligned}
T_1T_2T_2^{(1,2)}T_1^{(1,2)}T_1T_2 &= \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix}\\
&= \begin{pmatrix} (D - T_1^{12}(T_1^{12})^*)D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} = \begin{pmatrix} T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix}\\
&= T_1T_2.
\end{aligned} \tag{3.10}$$

That is the results (2) $\Rightarrow$ (3) holds.

(3) $\Rightarrow$ (1): If $T_2\{1,2\}T_1\{1,2\} \subseteq (T_1T_2)\{1\}$, then the result $T_2^{(1,2)}T_1^{(1,2)} \in (T_1T_2)\{1\}$ holds for any $T_1^{(1,2)} \in T_1\{1,2\}$ and $T_2^{(1,2)} \in T_2\{1,2\}$. So from the for-

mula (1) in Lemma 1.2, we know that the equation

$$T_1T_2T_2^{(1,2)}T_1^{(1,2)}T_1T_2 = T_1T_2 \tag{3.11}$$

holds for any $T_1^{(1,2)} \in T_1\{1,2\}$ and $T_2^{(1,2)} \in T_2\{1,2\}$. Namely, (3) $\Rightarrow$ (1) is true. □

4 Invariance Property of $T_1T_2T_2^{(1,3)}T_1^{(1,3)}T_1T_2$ Related to the Mixed-Type Reverse Order Law $T_2\{1,3\}T_1\{1,3\} \subseteq (T_1T_2)\{1\}$

Given operators $T_1 \in L(\mathcal{H},\mathcal{L})$ and $T_2 \in L(\mathcal{K},\mathcal{H})$. In this section, we will show that the mixed-type reverse order law $T_2\{1,3\}T_1\{1,3\} \subseteq (T_1T_2)\{1\}$ holds if and only if the operator product $T_1T_2T_2^{(1,3)}T_1^{(1,3)}T_1T_2$ is invariant, where $T_1^{(1,3)} \in T_1\{1,3\}$ and $T_2^{(1,3)} \in T_2\{1,3\}$ are two variant operators.

Theorem 4.1 *Let $T_1 \in L(\mathcal{H},\mathcal{L})$ and $T_2 \in L(\mathcal{K},\mathcal{H})$ such that T_1, T_2 have closed ranges and $T_1T_2 \neq O$. Then the following statements are equivalent:*

(1) $T_1T_2T_2^{(1,3)}T_1^{(1,3)}T_1T_2$ *is invariant for any* $T_1^{(1,3)} \in T_1\{1,3\}$ *and* $T_2^{(1,3)} \in T_2\{1,3\}$;
(2) $N(T_1) \subseteq N(T_1T_2T_2^{\dagger})$, *i.e,* $T_1T_2T_2^{\dagger}(I - T_1^{\dagger}T_1) = O$;
(3) $T_2\{1,3\}T_1\{1,3\} \subseteq (T_1T_2)\{1\}$.

Proof Combining (2.1)–(2.6) with the results in Lemma 1.2, it follows that there exist two bounded linear operators $P \in L(\mathcal{L},\mathcal{H})$ and $Q \in L(\mathcal{H},\mathcal{K})$:

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}, \ \textit{where } P_{ij},\ i,j = 1,2 \textit{ are arbitrary},$$

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}, \ \textit{where } Q_{ij},\ i,j = 1,2 \textit{ are arbitrary},$$

such that

$$T_1^{(1,3)} = T_1^{\dagger} + (I - T_1^{\dagger}T_1)P = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}, \tag{4.1}$$

$$T_2^{(1,3)} = T_2^{\dagger} + (I - T_2^{\dagger}T_2)Q = \begin{pmatrix} (T_2^{11})^{-1} & O \\ Q_{21} & Q_{22} \end{pmatrix}, \tag{4.2}$$

where

$$\begin{aligned}
D &= T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*, \\
\tau_{11} &= (T_1^{11})^*D^{-1} + [I - (T_1^{11})^*D^{-1}T_1^{11}]P_{11} - (T_1^{11})^*D^{-1}T_1^{12}P_{21}, \\
\tau_{12} &= [I - (T_1^{11})^*D^{-1}T_1^{11}]P_{12} - (T_1^{11})^*D^{-1}T_1^{12}P_{22}, \\
\tau_{21} &= (T_1^{12})^*D^{-1} + [I - (T_1^{12})^*D^{-1}T_1^{12}]P_{21} - (T_1^{12})^*D^{-1}T_1^{11}P_{11}, \\
\tau_{22} &= [I - (T_1^{12})^*D^{-1}T_1^{12}]P_{22} - (T_1^{12})^*D^{-1}T_1^{11}P_{12}.
\end{aligned}$$

(1) $\Rightarrow$ (2): From (2.1)–(2.6), (4.1) and (4.2), we know that for any $T_1^{(1,3)} \in T_1\{1,3\}$ and $T_2^{(1,3)} \in T_2\{1,3\}$,

$$\begin{aligned} T_1T_2T_2^{(1,3)}T_1^{(1,3)}T_1T_2 &= \begin{pmatrix} T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \begin{pmatrix} (T_2^{11})^{-1} & O \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \begin{pmatrix} T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \\ &= \begin{pmatrix} (T_1^{11}\tau_{11}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix}. \end{aligned} \tag{4.3}$$

It is well known that the Moore–Penrose inverse is unique and belongs to the set $\{1,3\}$-inverses, then $T_1T_2T_2^{(1,3)}T_1^{(1,3)}T_1T_2$ is invariant with respect to the choice of $T_1^{(1,3)} \in T_1\{1,3\}$ and $T_2^{(1,3)} \in T_2\{1,3\}$ if and only if the following equation

$$T_1T_2T_2^{(1,3)}T_1^{(1,3)}T_1T_2 = T_1T_2T_2^{\dagger}T_1^{\dagger}T_1T_2, \tag{4.4}$$

holds with respect to the choice of $T_1^{(1,3)} \in T_1\{1,3\}$ and $T_2^{(1,3)} \in T_2\{1,3\}$.

Combining (2.1)–(2.6) with (4.1) and (4.2), we have

$$\begin{aligned} & T_1T_2T_2^{(1,3)}T_1^{(1,3)}T_1T_2 = T_1T_2T_2^{\dagger}T_1^{\dagger}T_1T_2 \\ \Leftrightarrow & \begin{pmatrix} (T_1^{11}\tau_{11}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} = \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \\ \Leftrightarrow & T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}P_{11}T_1^{11}T_2^{11} - T_1^{11}(T_1^{11})^*D^{-1}T_1^{12}P_{21}T_1^{11}T_2^{11} = O. \end{aligned} \tag{4.5}$$

Let $P_{11} = (T_1^{11})^*$ and $P_{21} = O$, then from (4.5), we have

$$T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}(T_1^{11})^*T_1^{11}T_2^{11} = O \Leftrightarrow (T_1^{12})^*D^{-1}T_1^{11} = O. \tag{4.6}$$

By Lemma 1.3, and Lemma 1.4, we know that

$$N(T_1) \subseteq N(T_1T_2T_2^{\dagger}) \Leftrightarrow T_1T_2T_2^{\dagger}(I - T_1^{\dagger}T_1) = O. \tag{4.7}$$

Substituting (2.1)–(2.4) in (4.7) yields

$$\begin{aligned} & N(T_1) \subseteq N(T_1T_2T_2^{\dagger}) \Leftrightarrow T_1T_2T_2^{\dagger}(I - T_1^{\dagger}T_1) = O \\ \Leftrightarrow & \begin{pmatrix} T_1^{11} & O \\ O & O \end{pmatrix} \begin{pmatrix} I - (T_1^{11})^*D^{-1}T_1^{11} & -(T_1^{11})^*D^{-1}T_1^{12} \\ -(T_1^{12})^*D^{-1}T_1^{11} & I - (T_1^{12})^*D^{-1}T_1^{12} \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \\ \Leftrightarrow & \begin{pmatrix} T_1^{12}(T_1^{12})^*D^{-1}T_1^{11} & -T_1^{11}(T_1^{11})^*D^{-1}T_1^{12} \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix}. \end{aligned} \tag{4.8}$$

From (4.6), we get that (4.8) holds. That is (1) $\Rightarrow$ (2) holds.

(2) $\Rightarrow$ (3): If $N(T_1) \subseteq N(T_1T_2T_2^{\dagger})$, then from (4.8), we have

$$(T_1^{12})^*D^{-1}T_1^{11} = O. \tag{4.9}$$

By (2.1)–(2.4), (4.1), (4.2) and (4.9), we get that for any $T_i^{(1,3)} \in T_i\{1,3\}$, $i = 1, 2$

$$\begin{aligned} T_1T_2T_2^{(1,3)}T_1^{(1,3)}T_1T_2 &= \begin{pmatrix} T_1^{11}\tau_{11}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} = \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \\ &= \begin{pmatrix} (D - T_1^{12}(T_1^{12})^*)D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} = \begin{pmatrix} T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \\ &= T_1T_2. \end{aligned} \tag{4.10}$$

According to the formulas in Lemma 1.2, we know that the mixed-type reverse order law $T_2\{1,3\}T_1\{1,3\} \subseteq (T_1T_2)\{1\}$ holds if and only if for any $T_i^{(1,3)} \in T_i\{1,3\}$, $i = 1, 2$ the equation $T_1T_2T_2^{(1,3)}T_1^{(1,3)}T_1T_2 = T_1T_2$ holds. So from (4.10), (2) $\Rightarrow$ (3) holds.

(3) $\Rightarrow$ (1): If $T_2\{1,3\}T_1\{1,3\} \subseteq (T_1T_2)\{1\}$, then the result $T_2^{(1,3)}T_1^{(1,3)} \in (T_1T_2)\{1\}$ holds for any $T_1^{(1,3)} \in T_1\{1,3\}$ and $T_2^{(1,3)} \in T_2\{1,3\}$. So from the formula (1) in Lemma 1.2, we know that the equation

$$T_1T_2T_2^{(1,3)}T_1^{(1,3)}T_1T_2 = T_1T_2 \tag{4.11}$$

holds for any $T_1^{(1,3)} \in T_1\{1,3\}$ and $T_2^{(1,3)} \in T_2\{1,3\}$. Namely, (3) $\Rightarrow$ (1) is true.

From Lemma 1.2, we know that $G \in T\{1,4\}$ if and only if $G^* \in T^*\{1,3\}$. So from the results obtained in the above section, we can get the following results without the proof. □

Theorem 4.2 *Let $T_1 \in L(\mathcal{H}, \mathcal{L})$ and $T_2 \in L(\mathcal{K}, \mathcal{H})$ such that T_1, T_2 have closed ranges and $T_1T_2 \neq O$. Then the following statements are equivalent:*

1. *$T_1T_2T_2^{(1,4)}T_1^{(1,4)}T_1T_2$ is invariant for any $T_1^{(1,4)} \in T_1\{1,4\}$ and $T_2^{(1,4)} \in T_2\{1,4\}$;*
2. *$R(T_1^{\dagger}T_1T_2) \subseteq R(T_2)$, i.e, $(I - T_2T_2^{\dagger})T_1^{\dagger}T_1T_2 = O$;*
3. *$T_2\{1,4\}T_1\{1,4\} \subseteq (T_1T_2)\{1\}$.*

5 Invariance Property of $T_1T_2T_2^{(1,2,3)}T_1^{(1,2,3)}T_1T_2$ Related to the Mixed-Type Reverse Order Law $T_2\{1,2,3\}T_1\{1,2,3\} \subseteq (T_1T_2)\{1\}$

Let $T_1 \in L(\mathcal{H}, \mathcal{L})$ and $T_2 \in L(\mathcal{K}, \mathcal{H})$ be two given operators, $T_1^{(1,2,3)} \in T_1\{1,2,3\}$ and $T_2^{(1,2,3)} \in T_2\{1,2,3\}$ are two variant operators. In this section, we will show that the mixed-type reverse order law $T_2\{1,2,3\}T_1\{1,2,3\} \subseteq (T_1T_2)\{1\}$ holds if and only if the operator product $T_1T_2T_2^{(1,2,3)}T_1^{(1,2,3)}T_1T_2$ is invariant.

Theorem 5.1 *Let $T_1 \in L(\mathcal{H}, \mathcal{L})$ and $T_2 \in L(\mathcal{K}, \mathcal{H})$ such that T_1, T_2 have closed ranges and $T_1T_2 \neq O$. Then the following statements are equivalent:*

1. *$T_1T_2T_2^{(1,2,3)}T_1^{(1,2,3)}T_1T_2$ is invariant for any $T_1^{(1,2,3)} \in T_1\{1,2,3\}$ and $T_2^{(1,2,3)} \in T_2\{1,2,3\}$;*
2. *$N(T_1) \subseteq N(T_1T_2T_2^{\dagger})$, i.e, $T_1T_2T_2^{\dagger}(I - T_1^{\dagger}T_1) = O$;*
3. *$T_2\{1,2,3\}T_1\{1,2,3\} \subseteq (T_1T_2)\{1\}$.*

Proof Combining (2.1)–(2.6) with the results in Lemma 1.2, it follows that there exist two bounded linear operators $A \in L(\mathcal{L}, \mathcal{H})$ and $B \in L(\mathcal{H}, \mathcal{K})$:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}, \text{ where } A_{ij},\ i, j = 1, 2 \text{ are arbitrary},$$

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}, \text{ where } B_{ij},\ i, j = 1, 2 \text{ are arbitrary},$$

such that

$$\begin{aligned} T_1^{(1,2,3)} &= T_1^\dagger + (I - T_1^\dagger T_1) A T_1 T_1^\dagger \\ &= \begin{pmatrix} (T_1^{11})^* D^{-1} + [I - (T_1^{11})^* D^{-1} T_1^{11}] A_{11} - (T_1^{11})^* D^{-1} T_1^{12} A_{21} & O \\ (T_1^{12})^* D^{-1} + [I - (T_1^{12})^* D^{-1} T_1^{12}] A_{21} - (T_1^{12})^* D^{-1} T_1^{11} A_{11} & O \end{pmatrix}, \end{aligned} \tag{5.1}$$

and

$$T_2^{(1,2,3)} = T_2^\dagger + (I - T_2^\dagger T_2) B T_2 T_2^\dagger = \begin{pmatrix} (T_2^{11})^{-1} & O \\ B_{21} & B_{22} \end{pmatrix}. \tag{5.2}$$

(1) $\Rightarrow$ (2): From (2.1)–(2.6), (5.1) and (5.2), we know that for any $T_1^{(1,2,3)} \in T_1\{1, 2, 3\}$ and $T_2^{(1,2,3)} \in T_2\{1, 2, 3\}$,

$$\begin{aligned} & T_1 T_2 T_2^{(1,2,3)} T_1^{(1,2,3)} T_1 T_2 \\ &= \begin{pmatrix} T_1^{11} & O \\ O & O \end{pmatrix} \begin{pmatrix} (T_1^{11})^* D^{-1} + [I - (T_1^{11})^* D^{-1} T_1^{11}] A_{11} - (T_1^{11})^* D^{-1} T_1^{12} A_{21} & O \\ (T_1^{12})^* D^{-1} + [I - (T_1^{12})^* D^{-1} T_1^{12}] A_{21} - (T_1^{12})^* D^{-1} T_1^{11} A_{11} & O \end{pmatrix} \\ &\quad \times \begin{pmatrix} T_1^{11} T_2^{11} & O \\ O & O \end{pmatrix} \\ &= \begin{pmatrix} \eta & O \\ O & O \end{pmatrix}, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} \eta = {} & T_1^{11} (T_1^{11})^* D^{-1} T_1^{11} T_2^{11} + T_1^{11} [I - (T_1^{11})^* D^{-1} T_1^{11}] A_{11} T_1^{11} T_2^{11} \\ & - T_1^{11} (T_1^{11})^* D^{-1} T_1^{12} A_{21} T_1^{11} T_2^{11}. \end{aligned}$$

It is well known that the Moore-Penrose inverse is unique and belongs to the set $\{1, 2, 3\}$-inverses, then $T_1 T_2 T_2^{(1,2,3)} T_1^{(1,2,3)} T_1 T_2$ is invariant with respect to the choice of $T_1^{(1,2,3)} \in T_1\{1, 2, 3\}$ and $T_2^{(1,2,3)} \in T_2\{1, 2, 3\}$ if and only if the following equation

$$T_1 T_2 T_2^{(1,2,3)} T_1^{(1,2,3)} T_1 T_2 = T_1 T_2 T_2^\dagger T_1^\dagger T_1 T_2, \tag{5.4}$$

holds with respect to the choice of $T_1^{(1,2,3)} \in T_1\{1, 2, 3\}$ and $T_2^{(1,2,3)} \in T_2\{1, 2, 3\}$.

Combining (2.1)–(2.6) with (5.1) and (5.2), we have

$$\begin{aligned} &T_1T_2T_2^{(1,2,3)}T_1^{(1,2,3)}T_1T_2 = T_1T_2T_2^{\dagger}T_1^{\dagger}T_1T_2 \\ &\Leftrightarrow \begin{pmatrix} \eta & O \\ O & O \end{pmatrix} = \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \\ &\Leftrightarrow T_1^{11}[I-(T_1^{11})^*D^{-1}T_1^{11}]A_{11}T_1^{11}T_2^{11} \\ &\quad -T_1^{11}(T_1^{11})^*D^{-1}T_1^{12}A_{21}T_1^{11}T_2^{11} = O. \end{aligned} \tag{5.5}$$

Let $A_{11}=(T_1^{11})^*$ and $A_{21}=O$, then from (5.5), we have

$$T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}(T_1^{11})^*T_1^{11}T_2^{11} = O \Leftrightarrow (T_1^{12})^*D^{-1}T_1^{11} = O. \tag{5.6}$$

According to the proof of Sect. 4, we know that

$$\begin{aligned} &N(T_1)\subseteq N(T_1T_2T_2^{\dagger}) \Leftrightarrow T_1T_2T_2^{\dagger}(I-T_1^{\dagger}T_1) = O \\ &\Leftrightarrow \begin{pmatrix} T_1^{11} & O \\ O & O \end{pmatrix}\begin{pmatrix} I-(T_1^{11})^*D^{-1}T_1^{11} & -(T_1^{11})^*D^{-1}T_1^{12} \\ -(T_1^{12})^*D^{-1}T_1^{11} & I-(T_1^{12})^*D^{-1}T_1^{12} \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} T_1^{12}(T_1^{12})^*D^{-1}T_1^{11} & -T_1^{11}(T_1^{11})^*D^{-1}T_1^{12} \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix}. \end{aligned} \tag{5.7}$$

From (5.6), we get that (5.7) holds. That is (1) $\Rightarrow$ (2) holds.

(2) $\Rightarrow$ (3): If $N(T_1)\subseteq N(T_1T_2T_2^{\dagger})$, then from (5.7), we have

$$(T_1^{12})^*D^{-1}T_1^{11} = O. \tag{5.8}$$

By (2.1)–(2.4), (5.1), (5.2) and (5.8), we get that for any $T_i^{(1,2,3)}\in T_i\{1,2,3\}, i=1,2$

$$\begin{aligned} T_1T_2T_2^{(1,2,3)}T_1^{(1,2,3)}T_1T_2 &= \begin{pmatrix} \eta & O \\ O & O \end{pmatrix} = \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \\ &= \begin{pmatrix} (D-T_1^{12}(T_1^{12})^*)D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \\ &= \begin{pmatrix} T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \\ &= T_1T_2. \end{aligned} \tag{5.9}$$

According to the formulas in Lemma 1.2, we know that the mixed-type reverse order law $T_2\{1,2,3\}T_1\{1,2,3\}\subseteq(T_1T_2)\{1\}$ holds if and only if for any $T_i^{(1,2,3)}\in T_i\{1,2,3\}$, $i=1,2$ the equation $T_1T_2T_2^{(1,2,3)}T_1^{(1,2,3)}T_1T_2 = T_1T_2$ holds. So from (5.9), (2) $\Rightarrow$ (3) holds.

(3) $\Rightarrow$ (1): If $T_2\{1,2,3\}T_1\{1,2,3\}\subseteq(T_1T_2)\{1\}$, then the result $T_2^{(1,2,3)}T_1^{(1,2,3)}\in(T_1T_2)\{1\}$ holds for any $T_1^{(1,2,3)}\in T_1\{1,2,3\}$ and $T_2^{(1,2,3)}\in T_2\{1,2,3\}$. So from the

formula (1) in Lemma 1.2, we know that the equation

$$T_1T_2T_2^{(1,2,3)}T_1^{(1,2,3)}T_1T_2 = T_1T_2 \tag{5.10}$$

holds for any $T_1^{(1,2,3)} \in T_1\{1,2,3\}$ and $T_2^{(1,2,3)} \in T_2\{1,2,3\}$. Namely, (3) ⇒ (1) is true. □

From Lemma 1.2, we know that $G \in T\{1,2,4\}$ if and only if $G^* \in T^*\{1,2,3\}$. So from the results obtained in the above section, we can get the following results without the proof.

Theorem 5.2 *Let $T_1 \in L(\mathcal{H},\mathcal{L})$ and $T_2 \in L(\mathcal{K},\mathcal{H})$ such that T_1, T_2 have closed ranges and $T_1T_2 \neq O$. Then the following statements are equivalent:*

1. $T_1T_2T_2^{(1,2,4)}T_1^{(1,2,4)}T_1T_2$ *is invariant for any* $T_1^{(1,2,4)} \in T_1\{1,2,4\}$ *and* $T_2^{(1,2,4)} \in T_2\{1,2,4\}$;
2. $R(T_1^\dagger T_1T_2) \subseteq R(T_2)$, *i.e,* $(I - T_2T_2^\dagger)T_1^\dagger T_1T_2 = O$;
3. $T_2\{1,2,4\}T_1\{1,2,4\} \subseteq (T_1T_2)\{1\}$.

6 The Invariance Property of $T_1T_2T_2^{(1,3,4)}T_1^{(1,3,4)}T_1T_2$ and the Mixed-Type Reverse Order Law $T_2\{1,3,4\}T_1\{1,3,4\} \subseteq (T_1T_2)\{1\}$

Given operators $T_1 \in L(\mathcal{H},\mathcal{L})$ and $T_2 \in L(\mathcal{K},\mathcal{H})$. In this section, we will study the invariance property of $T_1T_2T_2^{(1,3,4)}T_1^{(1,3,4)}T_1T_2$ and the mixed-type reverse order law $T_2\{1,3,4\}T_1\{1,3,4\} \subseteq (T_1T_2)\{1\}$.

Theorem 6.1 *Let $T_1 \in L(\mathcal{H},\mathcal{L})$ and $T_2 \in L(\mathcal{K},\mathcal{H})$ such that T_1, T_2 have closed ranges and $T_1T_2 \neq O$. Then the operator product $T_1T_2T_2^{(1,3,4)}T_1^{(1,3,4)}T_1T_2$ is invariant for any $T_1^{(1,3,4)} \in T_1\{1,3,4\}$ and $T_2^{(1,3,4)} \in T_2\{1,3,4\}$.*

Proof Combining (2.1)–(2.6) with the results in Lemma 1.2, it follows that there exist two bounded linear operators $S \in L(\mathcal{L},\mathcal{H})$ and $N \in L(\mathcal{H},\mathcal{K})$:

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}, \text{ where } S_{ij},\ i,j = 1,2 \text{ are arbitrary},$$
$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}, \text{ where } N_{ij},\ i,j = 1,2 \text{ are arbitrary},$$

such that

$$\begin{aligned} T_1^{(1,3,4)} &= T_1^\dagger + (I - T_1^\dagger T_1)S(I - T_1T_1^\dagger) \\ &= \begin{pmatrix} (T_1^{11})^*D^{-1} & [I - (T_1^{11})^*D^{-1}T_1^{11}]S_{12} - (T_1^{11})^*D^{-1}T_1^{12}S_{22} \\ (T_1^{12})^*D^{-1} & [I - (T_1^{12})^*D^{-1}T_1^{12}]S_{22} - (T_1^{12})^*D^{-1}T_1^{11}S_{12} \end{pmatrix} \end{aligned} \tag{6.1}$$

and

$$T_2^{(1,3,4)} = T_2^{\dagger} + (I - T_2^{\dagger}T_2)N(I - T_2T_2^{\dagger}) = \begin{pmatrix} (T_2^{11})^{-1} & O \\ O & N_{22} \end{pmatrix}. \tag{6.2}$$

From (2.5), (6.1) and (6.2), we know that for any $T_1^{(1,3,4)} \in T_1\{1,3,4\}$ and $T_2^{(1,3,4)} \in T_2\{1,3,4\}$,

$$\begin{aligned} &T_1T_2T_2^{(1,3,4)}T_1^{(1,3,4)}T_1T_2 \\ &= \begin{pmatrix} T_1^{11} & O \\ O & O \end{pmatrix} \begin{pmatrix} (T_1^{11})^*D^{-1} & [I - (T_1^{11})^*D^{-1}T_1^{11}]S_{12} - (T_1^{11})^*D^{-1}T_1^{12}S_{22} \\ (T_1^{12})^*D^{-1} & [I - (T_1^{12})^*D^{-1}T_1^{12}]S_{22} - (T_1^{12})^*D^{-1}T_1^{11}S_{12} \end{pmatrix} \\ &\quad \times \begin{pmatrix} T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \\ &= \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \\ &= T_1T_2T_2^{\dagger}T_1^{\dagger}T_1T_2. \end{aligned} \tag{6.3}$$

Since the Moore-Penrose inverse is unique, then from (6.3), we know that $T_1T_2T_2^{(1,3,4)}T_1^{(1,3,4)}T_1T_2$ is invariant with respect to the choice of $T_1^{(1,3,4)} \in T_1\{1,3,4\}$ and $T_2^{(1,3,4)} \in T_2\{1,3,4\}$. □

Theorem 6.2 *Let $T_1 \in L(\mathcal{H},\mathcal{L})$ and $T_2 \in L(\mathcal{K},\mathcal{H})$ such that T_1, T_2 have closed ranges and $T_1T_2 \neq O$. Then the following statements are equivalent:*

1. *$N(T_1) \subseteq N(T_1T_2T_2^{\dagger})$, i.e, $T_1T_2T_2^{\dagger}(I - T_1^{\dagger}T_1) = O$;*
2. *$T_2\{1,3,4\}T_1\{1,3,4\} \subseteq (T_1T_2)\{1\}$.*

Proof (1) ⇒ (2): If $N(T_1) \subseteq N(T_1T_2T_2^{\dagger})$, then from (5.7), we have

$$(T_1^{12})^*D^{-1}T_1^{11} = O. \tag{6.4}$$

By (2.1)–(2.4), (6.1), (6.2), (6.3) and (6.4), we get that for any $T_i^{(1,3,4)} \in T_i\{1,3,4\}$, $i = 1,2$

$$\begin{aligned} T_1T_2T_2^{(1,3,4)}T_1^{(1,3,4)}T_1T_2 &= \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \\ &= \begin{pmatrix} (D - T_1^{12}(T_1^{12})^*)D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} = \begin{pmatrix} T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} \\ &= T_1T_2. \end{aligned} \tag{6.5}$$

According to the formulas in Lemma 1.2, we know that the mixed-type reverse order law $T_2\{1,3,4\}T_1\{1,3,4\} \subseteq (T_1T_2)\{1\}$ holds if and only if for any $T_i^{(1,3,4)} \in T_i\{1,3,4\}$, $i = 1,2$ the equation $T_1T_2T_2^{(1,3,4)}T_1^{(1,3,4)}T_1T_2 = T_1T_2$ holds. So from (6.5), (1) ⇒ (2) holds.

(2) ⇒ (1): If $T_2\{1,3,4\}T_1\{1,3,4\} \subseteq (T_1T_2)\{1\}$ holds, then from (2.1)–(2.6), we have

$$T_1T_2T_2^{(1,3,4)}T_1^{(1,3,4)}T_1T_2 = T_1T_2 \Leftrightarrow \begin{pmatrix} T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix} = \begin{pmatrix} T_1^{11}T_2^{11} & O \\ O & O \end{pmatrix}$$
$$\Leftrightarrow (D - T_1^{12}(T_1^{12})^*)D^{-1}T_1^{11}T_2^{11} = T_1^{11}T_2^{11}$$
$$\Leftrightarrow (T_1^{12})^*D^{-1}T_1^{11} = O. \tag{6.6}$$

Combining (5.7) with (6.6), we get that the result (2) ⇒ (1) holds. □

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