Complex Anal. Oper. Theory (2017) 11:109–117
DOI 10.1007/s11785-016-0575-y

Complex Analysis and Operator Theory

Nonlinear Maps Preserving the Jordan Triple 1-∗-Product on Von Neumann Algebras

Changjing Li[1] · Fangyan Lu[2]

Received: 24 February 2016 / Accepted: 20 June 2016 / Published online: 23 June 2016

Abstract In this paper, we investigate a bijective map Φ between two von Neumann algebras, one of which has no central abelian projections, satisfying $\Phi(A \bullet B \bullet C) = \Phi(A) \bullet \Phi(B) \bullet \Phi(C)$ for all A, B, C in the domain, where $A \bullet B = AB + BA^*$ is the Jordan 1-$*$-product of A and B. It is showed that the map $\Phi(I)\Phi$ is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism, where $\Phi(I)$ is a self-adjoint central element in the range with $\Phi(I)^2 = I$.

Keywords Jordan triple $*$-product · Isomorphism · Von Neumann algebras

Mathematics Subject Classification 47B48 · 46L10

1 Introduction

Let $\mathcal{A}$ be a $*$-algebra and η be a non-zero scalar. For $A, B \in \mathcal{A}$, define the Jordan η-$*$-product of A and B by $A \diamondsuit_\eta B = AB + \eta BA^*$. The Jordan η-$*$-product, particularly the

Communicated by Daniel Aron Alpay.

The first author is supported by the National Natural Science Foundation of China (Grant No. 11526123) and the Natural Science Foundation of Shandong Province, China (Grant No. ZR2015PA010). The second author is supported by the National Natural Science Foundation of China (Grant No. 11571247).

✉ Changjing Li
lcjbxh@163.com

Fangyan Lu
fylu@suda.edu.cn

1 School of Mathematical Sciences, Shandong Normal University, Jinan 250014, People's Republic of China

2 Department of Mathematics, Soochow University, Suzhou 215006, People's Republic of China

Jordan (-1)-$*$-product and the Jordan 1-$*$-product, is very meaningful and important in some research topics (see, for example, [1,3,8–11]). A map Φ between $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to preserve the Jordan η-$*$-product if $\Phi(A\diamondsuit_\eta B) = \Phi(A)\diamondsuit_\eta\Phi(B)$ for all $A, B \in \mathcal{A}$. Recently, many authors pay more attention to maps preserving the Jordan η-$*$-product between $*$-algebra (see, for example, [2,6]). In [6], Li et al. considered maps which preserve the Jordan 1-$*$-product and proved that such a map between factor von Neumann algebras is a $*$-ring isomorphism. In [2], Dai and Lu completely described maps preserving the Jordan η-$*$-product between von Neumann algebras without central abelian projections for all non-zero scalars η. They proved that if Φ is a bijective map preserving the Jordan η-$*$-product between two von Neumann algebras, one of which has no central abelian projections, then Φ is a linear $*$-isomorphism if η is not real and Φ is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism if η is real.

Recently, Huo et al. [4] studied a more general problem. They considered the Jordan triple η-$*$-product of three elements A, B and C in a $*$-algebra $\mathcal{A}$ defined by $A\diamondsuit_\eta B\diamondsuit_\eta C = (A\diamondsuit_\eta B)\diamondsuit_\eta C$ (we should be aware that $\diamondsuit_\eta$ is not necessarily associative). A map Φ between $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to preserve the Jordan triple η-$*$-product if $\Phi(A\diamondsuit_\eta B\diamondsuit_\eta C) = \Phi(A)\diamondsuit_\eta\Phi(B)\diamondsuit_\eta\Phi(C)$ for all $A, B, C \in \mathcal{A}$. Clearly a map between $*$-algebras preserving the Jordan η-$*$-product also preserves the Jordan triple η-$*$-product, but not conversely. For example, for $\alpha, \beta \in \mathbb{R}$, define $\Phi(\alpha + \beta i) = -4(\alpha^3 + \beta^3 i)$. Then the map $\Phi : \mathbb{C} \to \mathbb{C}$ is a bijection. It is not difficult to verify that Φ preserves the Jordan triple (-1)-$*$-product and Jordan triple 1-$*$-product, but it does not preserve the Jordan (-1)-$*$-product or Jordan 1-$*$-product. So, the class of those maps preserving the Jordan triple η-$*$-product is, in principle wider than the class of maps preserving the Jordan η-$*$-product.

Let $\eta \neq -1$ be a non-zero complex number, and let Φ be a bijection between two von Neumann algebras, one of which has no central abelian projections, satisfying $\Phi(I) = I$ and preserving the Jordan triple η-$*$-product. Huo et al. [4] showed that Φ is a linear $*$-isomorphism if η is not real and Φ is the sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism if η is real. It is easy to see that a map Φ preserving the Jordan triple η-$*$-product does not need satisfy $\Phi(I) = I$. Indeed, let $\Phi(A) = -A$ for all $A \in \mathcal{A}$. Then Φ preserves the Jordan triple η-$*$-product but $\Phi(I) = -I$. In this paper, we will discuss maps preserving the Jordan triple 1-$*$-product without the assumption $\Phi(I) = I$. We prove that if Φ is a bijective map preserving the Jordan triple 1-$*$-product between two von Neumann algebras, one of which has no central abelian projections, then the map $\Phi(I)\Phi$ is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism, where $\Phi(I)$ is a self-adjoint central element in the range with $\Phi(I)^2 = I$. We mention that the methods in [4] do not fit for solving our problem since their proofs heavily depend on the assumption $\Phi(I) = I$.

2 Proof of Main Result

Before embarking on the proof, we need some notations and preliminaries. In this section, we often write the Jordan 1-$*$-product by $A \bullet B$, that is $A \bullet B = AB + BA^*$. Algebras and spaces are over the complex number field $\mathbb{C}$. A von Neumann algebra $\mathcal{A}$

is a weakly closed, self-adjoint algebra of operators on a Hilbert space H containing the identity operator I. The set $\mathcal{Z}(\mathcal{A}) = \{S \in \mathcal{A} : ST = TS \text{ for all } T \in \mathcal{A}\}$ is called the center of $\mathcal{A}$. A projection P is called a central abelian projection if $P \in \mathcal{Z}(\mathcal{A})$ and $P\mathcal{A}P$ is abelian. Recall that the central carrier of A, denoted by $\overline{A}$, is the smallest central projection P satisfying $PA = A$. It is not difficult that the central carrier of A is the projection onto the closed subspace spanned by $\{BA(x) : B \in \mathcal{A}, x \in H\}$. If A is self-adjoint, then the core of A, denoted by $\underline{A}$, is $\sup\{S \in \mathcal{Z}(\mathcal{A}) : S = S^*, S \leq A\}$. If P is a projection, it is clear that $\underline{P}$ is the largest central projection Q satisfying $Q \leq P$. A projection P is said to be core-free if $\underline{P} = 0$. It is easy to see that $\underline{P} = 0$ if and only if $\overline{I - P} = I$.

Lemma 2.1 ([7, Lemma 4]) *Let $\mathcal{A}$ be a von Neumann algebra with no central abelian projections. Then there exists a projection $P \in \mathcal{A}$ such that $\underline{P} = 0$ and $\overline{P} = I$.*

Lemma 2.2 *Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space H. Let A be an operator in $\mathcal{A}$ and $P \in \mathcal{A}$ is a projection with $\overline{P} = I$. If $ABP = 0$ for all $B \in \mathcal{A}$, then $A = 0$. Consequently, if $Z \in \mathcal{Z}(\mathcal{A})$, then $ZP = 0$ implies $Z = 0$.*

Proof From $\overline{P} = I$, it follows that the linear span of $\{BP(x) : B \in \mathcal{A}, x \in H\}$ is dense in H. So $ABP = 0$ for all $B \in \mathcal{A}$, then $A = 0$. If $Z \in \mathcal{Z}(\mathcal{A})$ and $ZP = 0$, then $ZBP = 0$ for all $B \in \mathcal{A}$, hence $Z = 0$. □

Lemma 2.3 *Let $\mathcal{A}$ be a von Neumann algebra and $A \in \mathcal{A}$. Then $AB + BA^* = 0$ for all $B \in \mathcal{A}$ implies that $A = -A^* \in \mathcal{Z}(\mathcal{A})$.*

Proof We take $B = I$, then $A = -A^*$. Therefore $AB = BA$ for all $B \in \mathcal{A}$, which implies A belongs to the center of $\mathcal{A}$. □

Theorem 2.4 ([4, Theorem 2.1]) *Let $\mathcal{A}$ be a von Neumann algebra with no central abelian projections and $\mathcal{B}$ be a $*$-algebra. Suppose that a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi(A \bullet B \bullet C) = \Phi(A) \bullet \Phi(B) \bullet \Phi(C)$ for all $A, B, C \in \mathcal{A}$. Then Φ is additive.*

Our main result in this paper reads as follows.

Theorem 2.5 *Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras, one of which has no central abelian projections. Suppose that a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi(A \bullet B \bullet C) = \Phi(A) \bullet \Phi(B) \bullet \Phi(C)$ for all $A, B, C \in \mathcal{A}$. Then the following statements hold:*

(1) *$\Phi(I)$ is a self-adjoint central element in $\mathcal{B}$ with $\Phi(I)^2 = I$.*
(2) *Defining a map $\phi : \mathcal{A} \to \mathcal{B}$ by $\phi(A) = \Phi(I)\Phi(A)$ for all $A \in \mathcal{A}$. Then there exsits a central projection $E \in \mathcal{A}$ such that the restriction of ϕ to $\mathcal{A}E$ is a linear $*$-isomorphism and the restriction of ϕ to $\mathcal{A}(I - E)$ is a conjugate linear $*$-isomorphism.*

The proof will be organized in some lemas. First note that Φ is additive. Indeed, if $\mathcal{A}$ has no central abelian projections, Lemma 2.4 assures that Φ is additive. If $\mathcal{B}$ has no central abelian projections, observe that $\Phi^{-1} : \mathcal{B} \to \mathcal{A}$ is a bijection and preserves the Jordan triple 1-$*$-product. Applying Lemma 2.4 to Φ^{-1}, we know that Φ^{-1} and hence Φ is additive. In what follows, without loss of generality, we assume that $\mathcal{B}$ has no central abelian projections.

Lemma 2.6 (1) *For each* $A \in \mathcal{A}$, $A = -A^*$ *if and only if* $\Phi(A) = -\Phi(A)^*$;
(2) $\Phi(\mathcal{Z}(\mathcal{A})) = \mathcal{Z}(\mathcal{B})$;
(3) $(\Phi(I) + \Phi(I)^*)^2 = 4I$.

Proof Let $A \in \mathcal{A}$ be arbitrary. Since Φ is surjective, there exists $B \in \mathcal{A}$ such that $\Phi(B) = I$. Then

$$\begin{aligned} 0 &= \Phi(iI \bullet A \bullet B) \\ &= \Phi(iI) \bullet \Phi(A) \bullet I \\ &= \Phi(iI)\Phi(A) + \Phi(A)\Phi(iI)^* + \Phi(A)^*\Phi(iI)^* + \Phi(iI)\Phi(A)^* \end{aligned}$$

holds true for all $A \in \mathcal{A}$. That is,

$$\Phi(iI)(\Phi(A) + \Phi(A)^*) + (\Phi(A) + \Phi(A)^*)\Phi(iI)^* = 0$$

holds true for all $A \in \mathcal{A}$. So $\Phi(iI)B + B\Phi(iI)^* = 0$ holds true for all $B = B^* \in \mathcal{B}$. Since for every $B \in \mathcal{B}$, $B = B_1 + iB_2$ with $B_1 = \frac{B+B^*}{2}$ and $B_2 = \frac{B-B^*}{2i}$, it follows that $\Phi(iI)B + B\Phi(iI)^* = 0$ holds true for all $B \in \mathcal{B}$. It follows from Lemma 2.3 that $\Phi(iI) = -\Phi(iI)^* \in \mathcal{Z}(\mathcal{B})$. Similarly, $\Phi^{-1}(iI) \in \mathcal{Z}(\mathcal{A})$.

Let $A = -A^* \in \mathcal{A}$ and $\Phi(B) = I$. Since $0 = B \bullet A \bullet \Phi^{-1}(iI)$, it follows that

$$0 = \Phi(B \bullet A \bullet \Phi^{-1}(iI)) = I \bullet \Phi(A) \bullet (iI) = 2i(\Phi(A) + \Phi(A)^*).$$

This implies that $\Phi(A) = -\Phi(A)^*$. Similarly, we note that Φ^{-1} also preserves the Jordan triple 1-$*$-product. If $\Phi(A) = -\Phi(A)^*$, then

$$0 = \Phi^{-1}(\Phi(I) \bullet \Phi(A) \bullet \Phi(iI)) = I \bullet A \bullet (iI) = 2i(A + A^*),$$

and so $A = -A^*$. Now we have proved that $A = -A^*$ if and only if $\Phi(A) = -\Phi(A)^*$ for each $A \in \mathcal{A}$.

Let $Z \in \mathcal{Z}(\mathcal{A})$ be arbitrary and $\Phi(B) = I$. For every $A = -A^* \in \mathcal{A}$, we have

$$0 = \Phi(B \bullet A \bullet Z) = I \bullet \Phi(A) \bullet \Phi(Z) = 2(\Phi(A)\Phi(Z) + \Phi(Z)\Phi(A)^*).$$

That is $\Phi(A)\Phi(Z) = -\Phi(Z)\Phi(A)^*$ holds true for all $A = -A^* \in \mathcal{A}$. Since Φ preservers conjugate self-adjoint elements, it follows that $C\Phi(Z) = \Phi(Z)C$ holds true for all $C = -C^* \in \mathcal{B}$. Since for every $C \in \mathcal{B}$, we have $C = C_1 + iC_2$, where $C_1 = \frac{C-C^*}{2}$ and $C_2 = \frac{C+C^*}{2i}$ are conjugate self-adjoint elements. Hence $C\Phi(Z) = \Phi(Z)C$ holds true for all $C \in \mathcal{A}$. Then $\Phi(Z) \in \mathcal{Z}(\mathcal{B})$, which implies that $\Phi(\mathcal{Z}(\mathcal{A})) \subseteq \mathcal{Z}(\mathcal{B})$. Thus $\Phi(\mathcal{Z}(\mathcal{A})) = \mathcal{Z}(\mathcal{B})$ by considering Φ^{-1}.

Let $\Phi(B) = I$. Since $\Phi(I) \in \mathcal{Z}(\mathcal{B})$, then

$$4I = 4\Phi(B) = \Phi(I \bullet I \bullet B) = \Phi(I) \bullet \Phi(I) \bullet I = (\Phi(I) + \Phi(I)^*)^2.$$

□

Lemma 2.7 *Let P be a projection in $\mathcal{A}$ and set $Q_P = \frac{1}{4}(\Phi(I) + \Phi(I)^*)(\Phi(P) + \Phi(P)^*)$. Then the following statements hold:*

(1) *Q_P is a projection and $\Phi(P) = \Phi(I)Q_P$;*
(2) *Suppose that A in $\mathcal{A}$ such that $A = PA(I - P)$. Then $\Phi(A) = Q_P\Phi(A) + \Phi(A)Q_P$.*

Proof Let P be a projection in $\mathcal{A}$. Since $\Phi(I) \in \mathcal{Z}(\mathcal{B})$, then

$$\begin{aligned} 4\Phi(P) &= \Phi(I \bullet P \bullet I) = \Phi(I) \bullet \Phi(P) \bullet \Phi(I) \\ &= \Phi(I)(\Phi(I) + \Phi(I)^*)(\Phi(P) + \Phi(P)^*) \\ &= 4\Phi(I)Q_P. \end{aligned}$$

Hence

$$\begin{aligned} 4\Phi(P) &= \Phi(I \bullet P \bullet P) = \Phi(I) \bullet \Phi(P) \bullet \Phi(P) \\ &= (\Phi(I) + \Phi(I)^*)\Phi(P)(\Phi(P) + \Phi(P)^*) \\ &= 4\Phi(P)Q_P = 4\Phi(I)Q_P^2. \end{aligned}$$

This implies that $\Phi(P) = \Phi(I)Q_P^2$. Taking the adjoint and noting that Q_P is self-adjoint, $\Phi(P)^* = \Phi(I)^*Q_P^2$. Summing the last two equations, we get $\Phi(P) + \Phi(P)^* = (\Phi(I) + \Phi(I)^*)Q_P^2$. Hence $(\Phi(I) + \Phi(I)^*)(\Phi(P) + \Phi(P)^*) = (\Phi(I) + \Phi(I)^*)^2 Q_P^2$. By Lemma 2.6 (3), we obtain $Q_P = Q_P^2$. So Q_P is a projection.

Let A in $\mathcal{A}$ such that $A = PA(I - P)$. Noticing that $\Phi(P) = \Phi(I)Q_P$, we have

$$\begin{aligned} 2\Phi(A) &= \Phi(I \bullet P \bullet A) = \Phi(I) \bullet \Phi(P) \bullet \Phi(A) \\ &= (\Phi(I) + \Phi(I)^*)(\Phi(P)\Phi(A) + \Phi(A)\Phi(P)^*) \\ &= (\Phi(I) + \Phi(I)^*)(\Phi(I)Q_P\Phi(A) + \Phi(I)^*\Phi(A)Q_P). \end{aligned}$$

Since $(\Phi(I) + \Phi(I)^*)^2 = 4I$ and $\Phi(I), \Phi(I)^* \in \mathcal{Z}(\mathcal{B})$, multiplying both sides of the above equation by Q_P from the left and right respectively, we get that $Q_P\Phi(A)Q_P = 0$. Multiplying both sides of the above equation by $I - Q_P$ from the left and right respectively, we get that $(I - Q_P)\Phi(A)(I - Q_P) = 0$, which implies that $\Phi(A) = Q_P\Phi(A) + \Phi(A)Q_P$. □

Lemma 2.8 $\Phi(I)$ *is a self-adjoint central element in $\mathcal{B}$ with $\Phi(I)^2 = I$.*

Proof Since $\mathcal{B}$ has no central abelian projections, by Lemma 2.1, we can choose a projection $Q \in \mathcal{B}$ satisfying $\underline{Q} = 0$ and $\overline{Q} = I$. Let B be in $\mathcal{B}$ such that $B = QB(I - Q)$. Let $P = \frac{1}{4}(\Phi^{-1}(I) + \Phi^{-1}(I)^*)(\Phi^{-1}(Q) + \Phi^{-1}(Q)^*)$. Applying Lemma 2.7 to Φ^{-1}, we know that P is a projection and $\Phi^{-1}(B) = P\Phi^{-1}(B) + \Phi^{-1}(B)P$.

Moreover

$$\begin{aligned}\Phi(P) &= \frac{1}{4}\Phi((\Phi^{-1}(I) + \Phi^{-1}(I)^*)(\Phi^{-1}(Q) + \Phi^{-1}(Q)^*)) \\ &= \frac{1}{4}\Phi(\Phi^{-1}(I) \bullet \Phi^{-1}(Q) \bullet I) \\ &= \frac{1}{4}(I \bullet Q \bullet \Phi(I)) = \Phi(I)Q.\end{aligned}$$

Hence

$$\begin{aligned}B &= \Phi(P\Phi^{-1}(B) + \Phi^{-1}(B)P) \\ &= \frac{1}{2}\Phi(I \bullet P \bullet \Phi^{-1}(B)) \\ &= \frac{1}{2}(\Phi(I) \bullet \Phi(P) \bullet B) \\ &= \frac{1}{2}((\Phi(I) + \Phi(I)^*)(\Phi(P)B + B\Phi(P)^*)) \\ &= \frac{1}{2}((\Phi(I) + \Phi(I)^*)(\Phi(I)QB + \Phi(I)^*BQ)) \\ &= \frac{1}{2}(\Phi(I) + \Phi(I)^*)\Phi(I)B.\end{aligned}$$

This implies that $(2I - (\Phi(I) + \Phi(I)^*)\Phi(I))B = 0$. For arbitrary B we have $(2I - (\Phi(I) + \Phi(I)^*)\Phi(I))QB(I - Q) = 0$ and since $\overline{I - Q} = I$, it follows from Lemma 2.2 that $(2I - (\Phi(I) + \Phi(I)^*)\Phi(I))Q = 0$. Since $2I - (\Phi(I) + \Phi(I)^*)\Phi(I) \in \mathcal{Z}(\mathcal{B})$ and $\overline{Q} = I$, by Lemma 2.2 , we obtain that $2I - (\Phi(I) + \Phi(I)^*)\Phi(I) = 0$. This together with Lemma 2.6 (3) implies that $\Phi(I) = \Phi(I)^*$ and $\Phi(I)^2 = I$. □

Now, defining a map $\phi : \mathcal{A} \to \mathcal{B}$ by $\phi(A) = \Phi(I)\Phi(A)$ for all $A \in \mathcal{A}$. Then ϕ has the following properties.

Lemma 2.9 (1) *ϕ is an additive bijection and satisfies*

$$\phi(A \bullet B \bullet C) = \phi(A) \bullet \phi(B) \bullet \phi(C)$$

for all $A, B, C \in \mathcal{A}$;
(2) *$\phi(I) = I$ and $\phi(\mathcal{Z}(\mathcal{A})) = \mathcal{Z}(\mathcal{B})$;*
(3) *$\phi(A^*) = \phi(A)^*$ for all $A \in \mathcal{A}$;*
(4) *P is a projection in $\mathcal{A}$ if and only if $\phi(P)$ is a projection in $\mathcal{B}$.*

Proof (1) follows from Theorem 2.4 and Lemma 2.8 and (2) follows from Lemmas 2.8 and 2.6 (2). (3) For all $A \in \mathcal{A}$, since

$$\begin{aligned}2(\phi(A) + \phi(A^*)) = 2\phi(A + A^*) = \phi(A \bullet I \bullet I) = \phi(A) \bullet I \bullet I \\ = 2(\phi(A) + \phi(A)^*),\end{aligned}$$

we have $\phi(A^*) = \phi(A)^*$. (4) If P be a projection in $\mathcal{A}$, then by Lemma 2.7 (1), we see that $\phi(P) = \Phi(I)\Phi(P) = \Phi(I)^2 Q_P = Q_P$. So $\phi(P)$ is a projection in $\mathcal{B}$. Conversely, if $\phi(P)$ is a projection in $\mathcal{B}$, applying Lemma 2.7 to ϕ^{-1}, we know that $P = \frac{1}{4}\phi^{-1}(I)(\phi^{-1}(I) + \phi^{-1}(I)^*)(P + P^*)$ and $\frac{1}{4}(\phi^{-1}(I) + \phi^{-1}(I)^*)(P + P^*)$ is a projection. But $\phi^{-1}(I) = I$ by (2), it follows that P is a projection. □

Since $\mathcal{B}$ has no central abelian projections, by Lemma 2.1, there exists a projection Q_1 in $\mathcal{B}$ such that $\underline{Q_1} = 0$ and $\overline{Q_1} = I$. Then by Lemma 2.9 (4), $P_1 = \phi^{-1}(Q_1)$ is a projection in $\mathcal{A}$. Set $P_2 = I - P_1$ and $Q_2 = I - Q_1$. Denote $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$ and $\mathcal{B}_{ij} = Q_i \mathcal{B} Q_j$. Then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$ and $\mathcal{B} = \sum_{i,j=1}^{2} \mathcal{B}_{ij}$.

Lemma 2.10 $\phi(\mathcal{A}_{ij}) = \mathcal{B}_{ij}, \phi(\mathcal{A}_{ii}) \subseteq \mathcal{B}_{ii}, 1 \leq i \neq j \leq 2$.

Proof Let A_{12} be an arbitrary element in $\mathcal{A}_{12}$. Then

$$\begin{aligned} 2\phi(A_{12}) &= \phi(I \bullet P_1 \bullet A_{12}) \\ &= I \bullet Q_1 \bullet \phi(A_{12}) \\ &= 2Q_1\phi(A_{12}) + 2\phi(A_{12})Q_1, \end{aligned}$$

we get that $Q_1\phi(A_{12})Q_1 = Q_2\phi(A_{12})Q_2 = 0$. Hence $\phi(A_{12}) = B_{12} + B_{21}$ for some $B_{12} \in \mathcal{B}_{12}$ and $B_{21} \in \mathcal{B}_{21}$.

Now to show that $\phi(A_{12}) \subseteq \mathcal{B}_{12}$, we have to show that $B_{21} = 0$. This can be seen from

$$\begin{aligned} 0 &= \phi(I \bullet A_{12} \bullet P_1) \\ &= I \bullet \Phi(A_{12}) \bullet Q_1 \\ &= 2(B_{21} + B_{21}^*). \end{aligned}$$

So $B_{21} = 0$, which implies that $\phi(A_{12}) \subseteq \mathcal{B}_{12}$. Hence by considering ϕ^{-1}, we have $\phi(\mathcal{A}_{12}) = \mathcal{B}_{12}$. Similarly, we have $\phi(\mathcal{A}_{21}) = \mathcal{B}_{21}$.

Let A_{ii} be an arbitrary element in $\mathcal{A}_{ii}$. Then for $j \neq i$, we have

$$0 = \phi(I \bullet P_j \bullet A_{ii}) = I \bullet Q_j \bullet \phi(A_{ii}) = 2(Q_j\phi(A_{ii}) + \phi(A_{ii})Q_j),$$

which implies that $Q_i\phi(A_{ii})Q_j = Q_j\phi(A_{ii})Q_i = Q_j\phi(A_{ii})Q_j = 0$. So $\phi(A_{ii}) = Q_i\phi(A_{ii})Q_i \subseteq \mathcal{B}_{ii}$. □

Lemma 2.11 *ϕ is multiplicative.*

Proof Let A and B be in $\mathcal{A}$. Write $A = \sum_{i,j=1}^{2} A_{ij}$ and $B = \sum_{i,j=1}^{2} B_{ij}$, where $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$. To show $\phi(AB) = \phi(A)\phi(B)$, by the additivity of ϕ, it suffices to show that $\phi(A_{ij}B_{kl}) = \phi(A_{ij})\phi(B_{kj})$ for all $i, j, k, l \in \{1, 2\}$. Since if $j \neq k$ then $\phi(A_{ij}B_{kl}) = \phi(A_{ij})\phi(B_{kj}) = 0$ by Lemma 2.10, we only need to consider the cases with $j = k$.

First of all, since $\phi(B_{12})\phi(A_{11})^* = 0$, which implies that

$$\begin{aligned}\phi(A_{11}B_{12}) + \phi(B_{12}^*A_{11}^*) &= \phi(A_{11} \bullet B_{12} \bullet I)\\ &= \phi(A_{11}) \bullet \phi(B_{12}) \bullet I\\ &= \phi(A_{11})\phi(B_{12}) + \phi(B_{12})^*\phi(A_{11})^*.\end{aligned}$$

Thus we have $\phi(A_{11}B_{12}) = \phi(A_{11})\phi(B_{12})$ by Lemma 2.10. Similarly, we can prove that $\phi(A_{22}B_{21}) = \phi(A_{22})\phi(B_{21})$.

It is easy to compute that

$$\begin{aligned}\phi(A_{12}B_{21}) + \phi(B_{21}A_{12}) &= \phi(A_{12} \bullet I \bullet B_{21})\\ &= \phi(A_{12}) \bullet I \bullet \phi(B_{21})\\ &= \phi(A_{12})\phi(B_{21}) + \phi(B_{21})\phi(A_{12}).\end{aligned}$$

Thus $\phi(A_{12}B_{21}) = \phi(A_{12})\phi(B_{21})$ and $\phi(B_{21}A_{12}) = \phi(B_{21})\phi(A_{12})$ by Lemma 2.10.

For $D_{12} \in \mathcal{B}_{12}$, we have $C_{12} = \phi^{-1}(D_{12}) \in \mathcal{A}_{12}$ by Lemma 2.10. Thus

$$\phi(A_{11}B_{11})D_{12} = \phi(A_{11}B_{11}C_{12}) = \phi(A_{11})\phi(B_{11}C_{12}) = \phi(A_{11})\phi(B_{11})D_{12}$$

for all $D_{12} \in \mathcal{B}_{12}$. Since $\overline{Q_2} = I$, by Lemma 2.2 and 2.10, $\phi(A_{11}B_{11}) = \phi(A_{11})\phi(B_{11})$. Similarly, we can prove that $\phi(A_{22}B_{22}) = \phi(A_{22})\phi(B_{22})$.

For $D_{21} \in \mathcal{B}_{21}$, we have $C_{21} = \phi^{-1}(D_{21}) \in \mathcal{A}_{12}$ by Lemma 2.10. Thus

$$\phi(A_{12}B_{22})D_{21} = \phi(A_{12}B_{22}C_{21}) = \phi(A_{12})\phi(B_{22}C_{21}) = \phi(A_{12})\phi(B_{22})D_{21}$$

for all $D_{21} \in \mathcal{B}_{21}$. Since $\overline{Q_1} = I$, by Lemmas 2.2 and 2.10, $\phi(A_{12}B_{22}) = \phi(A_{12})\phi(B_{22})$. Similarly, we can prove that $\phi(A_{21}B_{11}) = \phi(A_{21})\phi(B_{11})$. □

Now we come to the position to show Theorem 2.5.

Proof of Theorem 2.5. For every rational number q, we have $\phi(qI) = qI$. Indeed, since q is rational number, there exist two integers r and s such that $q = \frac{r}{s}$. Since $\phi(I) = I$ and ϕ is additive, we get that

$$\phi(qI) = \phi\left(\frac{r}{s}I\right) = r\phi\left(\frac{1}{s}I\right) = \frac{r}{s}\phi(I) = qI.$$

Now we show that ϕ is real linear. Let A be a positive element in $\mathcal{A}$. Then $A = B^2$ for some self-adjoint element $B \in \mathcal{A}$. It follows from Lemma 2.11 that $\phi(A) = \phi(B)^2$. By Lemma 2.9 (3), we get that $\phi(B)$ is self-adjoint. So $\phi(A)$ is positive. This shows that ϕ preserves positive elements. Let $\lambda \in \mathbb{R}$. Choose sequence $\{a_n\}$ and $\{b_n\}$ of rational numbers such that $a_n \leq \lambda \leq b_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lambda$. It follows from

$$a_nI \leq \lambda I \leq b_nI$$

that

$$a_n I \leq \phi(\lambda I) \leq b_n I.$$

Taking the limit, we get that $\phi(\lambda I) = \lambda I$. Hence for all $A \in \mathcal{A}$,

$$\phi(\lambda A) = \phi((\lambda I)A) = \phi(\lambda I)\phi(A) = \lambda\phi(A).$$

Hence ϕ is real linear.

By Lemma 2.11, $\phi(iI)^2 = \phi((iI)^2) = -\phi(I) = -I$. By Lemma 2.9 (3), $\phi(iI)^* = \phi((iI)^*) = -\phi(iI)$. Let $F = \frac{I - i\phi(iI)}{2}$. Then it is easy to verify that F is a central projection in $\mathcal{B}$. Let $E = \phi^{-1}(F)$. Then by Lemma 2.9, E is a central projection in $\mathcal{A}$. Moreover, for $A \in \mathcal{A}$, there hold

$$\phi(iAE) = \phi(A)\phi(E)\phi(iI) = \phi(A)\phi(E)i(2F - I) = i\phi(A)F = i\phi(AE),$$

and

$$\phi(iA(I - E)) = \phi(A)\phi(I - E)\phi(iI) = -i\phi(A)(I - F) = -i\phi(A(I - E)).$$

That is, the restriction of ϕ to $\mathcal{A}E$ is linear and the restriction of ϕ to $\mathcal{A}(I - E)$ is conjugate linear. This together with Lemmas 2.9 and 2.11 shows Theorem 2.5. □

Acknowledgements The authors would like to thank the referee for the very thorough reading of the paper and many helpful comments.

References

1. Brešar, M., Fošner, A.: On ring with involution equipped with some new product. Publ. Math. Debrecen **57**, 121–134 (2000)
2. Dai, L., Lu, F.: Nonlinear maps preserving Jordan $*$-products. J. Math. Anal. Appl. **409**, 180–188 (2014)
3. Fošner, M.: Prime rings with involution equipped with some new product. Southeast Asian Bull. Math. **26**, 27–31 (2002)
4. Huo, D., Zheng, B., Liu, H.: Nonlinear maps preserving Jordan triple η-$*$-products. J. Math. Anal. Appl. **430**, 830–844 (2015)
5. Kleinecke, D.C.: On operator commutators. Proc. Am. Math. Soc. **8**, 535–536 (1957)
6. Li, C., Lu, F., Fang, X.: Mappings preserving new product $XY + YX^*$ on factor von Neumann algebras. Linear Algebra Appl. **438**, 2339–2345 (2013)
7. Miers, C.R.: Lie homomorphisms of operator algebras. Pac. J. Math. **38**, 717–735 (1971)
8. Molnár, L.: A condition for a subspace of $\mathcal{B}$(H) to be an ideal. Linear Algebra Appl. **235**, 229–234 (1996)
9. Šemrl, P.: Quadratic functionals and Jordan $*$-derivations. Studia Math. **97**, 157–165 (1991)
10. Šemrl, P.: Quadratic and quasi-quadratic functionals. Proc. Am. Math. Soc **119**, 1105–1113 (1993)
11. Šemrl, P.: On Jordan $*$-derivations and an application. Colloq. Math **59**, 241–251 (1990)