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On q-Analogue of Modified Kantorovich-Type Discrete-Beta Operators

Preeti Sharma[1] · **Vishnu Narayan Mishra**[1,2]

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Abstract The present paper deals with the Stancu type generalization of the Kantorovich discrete q-Beta operators. We establish some direct results, which include the asymptotic formula and error estimation in terms of the modulus of continuity and weighted approximation.

Keywords Kantorovich type q-Beta operators · q-Integer · Asymptotic formula · Rate of convergence · Modulus of continuity · Stancu operator

Mathematics Subject Classification Primary 41A25 · 41A35 · 41A36

1 Introduction

In the last decade, some new generalizations of well known positive linear operators based on q-integers were introduced and studied by several authors. Stancu type generalization of positive linear operators studied by several authors [9–15,17,18] and references therein. Our aim is to investigate some approximation properties of

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✉ Preeti Sharma
preeti.iitan@gmail.com

Vishnu Narayan Mishra
vishnunarayanmishra@gmail.com

1 Applied Mathematics and Humanities Department, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Dumas Road, Surat, Gujarat 395 007, India

2 L. 1627 Awadh Puri Colony, Phase-III, Beniganj, Opposite-Industrial Training Institute (ITI.), Faizabad, Uttar Pradesh 224 001, India

a Kantorovich-Stancu type q-Beta operators. Discrete Beta operators based on q-integers was introduced by Gupta et al. in [1] and they established some approximation results. They also obtained some global direct error estimates for the operators (1.1) using the second-order Ditzian-Totik modulus of smoothness and studied the limit discrete q-Beta operator.

Gupta et al. [1] introduced discrete q-Beta operators as follows:

$$V_{n,q}(f(t);x) = V_n(f(t);q;x) = \frac{1}{[n]_q}\sum_{k=0}^{\infty} p_{n,k}(q;x) f\left(\frac{[k]_q}{[n+1]_q q^{k-1}}\right), \tag{1.1}$$

where

$$p_{n,k}(q;x) = \frac{q^{k(k-1)/2}}{B_q(k+1,n)}\frac{x^k}{(1+x)_q^{n+k+1}}. \tag{1.2}$$

Also, they gave the following equalities:

$$V_n(1;q;x) = 1, \qquad V_n(t;q;x) = x \quad \text{for every } n \in \mathbb{N} \text{ and}$$
$$V_n(t^2;q;x) = \left(\frac{1}{q[n+1]_q} + 1\right)x^2 + \frac{x}{[n+1]_q}.$$

In the recent years, applications of q-calculus in approximation theory is one of the interesting areas of research. Several authors have proposed the q-analogues of Kantorovich type modification of different linear positive operators and studied their approximation behaviors.

In 2013, Mishra et al. [2] introduced Kantorovich-type modification of discrete q-Beta operators for each positive integer n, $q \in (0,1)$ as follows:

$$V^*_{n,q}(f(t);q;x) = \frac{[n+1]_q}{[n]_q}\sum_{k=0}^{\infty}\left(\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t)d_q t\right)\frac{p_{n,k}(q;x)}{q^{2k-1}}, \tag{1.3}$$

where f is a continuous and non-decreasing function on the interval $[0,\infty)$, $x \in [0,\infty)$. The aim of this paper is to present a Kantorovich-Stancu type generalization of the operators given by (1.3) and to give some approximation properties.

Kantorovich-Stancu type generalization of the operators (1.3) is define as follows:

$$\mathcal{L}^{(\alpha,\beta)}_{n,q}(f(t);q;x) = \frac{[n+\beta+1]_q}{[n+\beta]_q}\sum_{k=0}^{\infty}\left(\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}\right)\right)d_q t\,\frac{p_{n,k}(q;x)}{q^{2k-1}}, \tag{1.4}$$

where $p_{n,k}(q;x)$ is defined as in (1.2).

2 Preliminaries

To make the article self-content, here we mention certain basic definitions of q-calculus, details can be found in [3,4] and the other recent articles. For each nonnegative integer n, the q-integer $[n]_q$ and the q-factorial $[n]_q!$ are, respectively, defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1, \end{cases}$$

and

$$[n]_q! = \begin{cases} [n]_q[n-1]_q[n-2]_q \dots [1]_q, & n = 1, 2, \dots, \\ 1, & n = 0. \end{cases}$$

Then for $q > 0$ and integers $n, k, k \geq n \geq 0$, we have

$$[n+1]_q = 1 + q[n]_q \quad \text{and} \quad [n]_q + q^n[k-n]_q = [k]_q.$$

We observe that

$$(1+x)_q^n = (-x; q)_n = \begin{cases} (1+x)(1+qx)(1+q^2x)\dots(1+q^{n-1}x), & n = 1, 2, \dots, \\ 1, & n = 0. \end{cases}$$

Also, for any real number α, we have $(1+x)_q^\alpha = \frac{(1+x)_q^\infty}{(1+q^\alpha x)_q^\infty}$. In special case, when α is a whole number, this definition coincides with the above definition.

The q-Jackson integral and q-improper integral defined as

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n$$

and

$$\int_0^{\infty/A} f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A},$$

provided sum converges absolutely.

3 Basic Results

Lemma 1 [2] *The following hold:*

(i) $V_{n,q}^*(1; q; x) = 1$,
(ii) $V_{n,q}^*(t; q; x) = x + \frac{q}{[2]_q[n+1]_q}$,

(iii) $V^*_{n,q}(t^2;q;x) = \left(\frac{q^{n-2}[n+2]_q}{[n+1]_q}\right)x^2 + \left(\frac{q^{n-1}}{[n+1]_q} + \frac{2q+1}{[n+1]_q[3]_q}\right)x + \frac{q}{[n+1]_q^2[3]_q}$.

Now we give an auxiliary lemma for the Korovkin test functions.

Lemma 2 *Let* $e_m(t) = t^m$, $m = 0, 1, 2$, *we have*

$$\begin{aligned}
\mathcal{L}_n^{(\alpha,\beta)}(1;q;x) &= 1,\\
\mathcal{L}_n^{(\alpha,\beta)}(t;q;x) &= \frac{[n]_q x}{[n]_q+\beta} + \frac{q[n]_q + \alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q},\\
\mathcal{L}_n^{(\alpha,\beta)}(t^2;q;x) &= \left(\frac{[n]_q^2}{([n]_q+\beta)^2}\frac{q^{(n-2)}[n+2]_q}{[n+1]_q}\right)x^2\\
&\quad + \left(\frac{[n]_q^2}{([n]_q+\beta)^2}\left\{\frac{q^{(n-1)}}{[n+1]_q} + \frac{(2q+1)}{[3]_q[n+1]_q}\right\}\right.\\
&\quad \left. + \frac{2[n]_q\alpha}{([n]_q+\beta)^2}\right)x + \left(\frac{[n]_q^2}{([n]_q+\beta)^2}\frac{q}{[3]_q[n+1]_q^2}\right.\\
&\quad \left. + \frac{2q[n]_q\alpha}{[2]_q[n+1]_q([n]_q+\beta)^2} + \frac{\alpha^2}{([n]_q+\beta)^2}\right).
\end{aligned}$$

Lemma 3 *For* $f \in C[0,1]$, *we have* $||\mathcal{L}_n^{(\alpha,\beta)} f|| \le ||f||$.

Lemma 4 *From Lemma 2, we have*

$$\begin{aligned}
\mathcal{L}_n^{(\alpha,\beta)}((t-x);q;x) &= \left(\frac{[n]_q}{[n]_q+\beta} - 1\right)x + \frac{q[n]_q + \alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q},\\
\mathcal{L}_n^{(\alpha,\beta)}((t-x)^2;q;x) &= \left(\frac{[n]_q^2}{([n]_q+\beta)^2}\frac{q^{(n-2)}[n+2]_q}{[n+1]_q} + 1 - \frac{2[n]_q}{[n]_q+\beta}\right)x^2\\
&\quad + \left(\frac{[n]_q^2}{([n]_q+\beta)^2}\left\{\frac{q^{(n-1)}}{[n+1]_q} + \frac{(2q+1)}{[3]_q[n+1]_q}\right\}\right.\\
&\quad \left. + \frac{2[n]_q\alpha}{([n]_q+\beta)^2} - \frac{2\left(q[n]_q + \alpha[2]_q[n+1]_q\right)}{[2]_q([n]_q+\beta)[n+1]_q}\right)x\\
&\quad + \left(\frac{[n]_q^2}{([n]_q+\beta)^2}\frac{q}{[3]_q[n+1]_q^2}\right.\\
&\quad \left. + \frac{2q[n]_q\alpha}{[2]_q[n+1]_q([n]_q+\beta)^2} + \frac{\alpha^2}{([n]_q+\beta)^2}\right).
\end{aligned}$$

Lemma 5 *For* $0 \le \alpha \le \beta$, *we have*

$$\mathcal{L}_n^{(\alpha,\beta)}((t-x)^2;q;x) \le \frac{[n+1]_q}{([n]_q+\beta)^2}\left(\phi^2(x) + \frac{q}{[3]_q[n+1]_q}\right),$$

where $\phi^2(x) = x(1+x)$.

Proposition 1 *Let f be a continuous function on $[0,\infty)$ then for $n \to \infty$, the sequence $\{\mathcal{L}_n^{(\alpha,\beta)}(f;q;x)\}$ converges uniformly to $f(x)$ in $[a,b] \subset [0,\infty)$.*

Proof For sufficiently large n, it is obvious from Lemma 2 that $\{\mathcal{L}_n^{(\alpha,\beta)}(e_0;q;x)\}$, $\{\mathcal{L}_n^{(\alpha,\beta)}(e_1;q;x)\}$, $\{\mathcal{L}_n^{(\alpha,\beta)}(e_2;q;x)\}$ converges uniformly to 1, x and x^2 respectively on every compact subset of $[0,\infty)$. Thus the required result follows from Bohman-Korovkin theorem. □

4 Some Auxiliary Results

Let the space $C_B[0,\infty)$ of all continuous and bounded functions f on $[0,\infty)$, be endowed with the norm $\|f\| = sup\{|f(x)| : x \in [0,\infty)\}$. Further let us consider the Peetre's K-functional which is defined by

$$K_2(f,\delta) = \inf_{g \in W^2}\{\|f-g\| + \delta\|g''\|\}, \tag{4.1}$$

where $\delta > 0$ and $W_\infty^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By the method as given ([6] p. 177, Theorem 2.4), there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \le C\omega_2(f,\delta), \tag{4.2}$$

where

$$\omega_2(f,\delta) = \sup_{0<h\le\delta}\ \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)| \tag{4.3}$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$. Also we set

$$\omega(f,\delta) = \sup_{0<h\le\delta}\ \sup_{x\in[0,\infty)} |f(x+h) - f(x)|. \tag{4.4}$$

We denote the usual modulus of continuity of $f \in C_B[0,\infty)$.

Theorem 1 *Let $f \in C_B[0,\infty)$, then for all $x \in [0,\infty)$, there exists an absolute constant $C > 0$ such that*

$$|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f(x)| \le C\omega_2\left(f, \sqrt{\delta_n(x) + (\alpha_n(x))^2}\right) + \omega(f,\alpha_n(x)). \tag{4.5}$$

Proof Let $g \in W_\infty^2$ and $x, t \in [0,\infty)$. By Taylor's expansion, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du. \tag{4.6}$$

Define

$$\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(f,x) = \mathcal{L}_n^{(\alpha,\beta)}(f,q;x) + f(x) - f(\eta(x,q)). \tag{4.7}$$

where $\eta(x,q)=\frac{[n]_q x}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}$.

Now, we have $\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(t-x,x)=0,\ t\in[0,\infty)$.

Applying $\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}$ on both sides of (4.6), we get

$$\begin{aligned}
\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(g,x)-g(x) &= g'(x)\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}((t-x),x)+\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du,x\right)\\
&= \mathcal{L}_n^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du,x\right)\\
&\quad+\int_x^{\frac{[n]_q x}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}}\\
&\quad\left(\frac{[n]_q x}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}-u\right)g''(u)du.\\
|\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(g,x)-g(x)| &= \mathcal{L}_n^{(\alpha,\beta)}\left(\int_x^t |(t-u)||g''(u)|du,x\right)\\
&\quad+\int_x^{\frac{[n]_q x}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}}\\
&\quad\left|\frac{[n]_q x}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}-u\right||g''(u)|du\\
&\le \mathcal{L}_n^{(\alpha,\beta)}\left((t-x)^2,x\right)||g''||\\
&\quad+\left(\frac{[n]_q x}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}-x\right)^2||g''||.
\end{aligned}$$

On the other hand from Lemma 5, we have

$$\begin{aligned}
&\mathcal{L}_n^{(\alpha,\beta)}\left((t-x)^2,x\right)+\left(\frac{[n]_q x}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}-x\right)^2\\
&\le \frac{[n+1]_q}{([n]_q+\beta)^2}\left(\phi^2(x)+\frac{q}{[n+1]_q}\right)+\left(\frac{-\beta x}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}\right)^2\\
&\le \frac{4[n+1]_q}{([n]_q+\beta)^2}\left(\phi^2(x)+\frac{q}{[3]_q[n+1]}\right).
\end{aligned}$$

Thus, one can do this

$$\begin{aligned}
\left|\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(g,x)-g(x)\right| &\le \frac{4[n+1]_q}{([n]_q+\beta)^2}\left(\phi^2(x)+\frac{q}{[3]_q[n+1]}\right)||g''||\\
&\le \frac{4[n+1]_q}{([n]_q+\beta)^2}\delta_n^2(x)||g''||, \qquad (4.8)
\end{aligned}$$

where $\delta_n^2(x) = \left(\phi^2(x) + \frac{q}{[3]_q[n+1]}\right)$, we observe that,

$$\begin{aligned}\left|\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(f,x) - f(x)\right| &\leq \left|\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(f-g,x) - (f-g)(x)\right| \\ &\quad + \left|\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(g,x) - g(x)\right| \\ &\quad + \left|f(x) - f\left(\frac{[n]_q x}{[n]_q+\beta} + \frac{q[n]_q + \alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}\right)\right| \\ &\leq \|f-g\| + \frac{4[n+1]_q}{([n]_q+\beta)^2}\delta_n^2(x)||g''|| \\ &\quad + \omega\left(f, \left|\frac{-\beta x}{[n]_q+\beta} + \frac{q[n]_q + \alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}\right|\right).\end{aligned}$$

Now, taking infimum on the right-hand side over all $g \in W^2$, we obtain

$$\begin{aligned}\left|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f(x)\right| &\leq K_2\left(f, \frac{4[n+1]_q}{([n]_q+\beta)^2}\delta_n^2(x)\right) \\ &\quad + \omega\left(f, \left|\frac{-\beta x}{[n]_q+\beta} + \frac{q[n]_q + \alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}\right|\right) \\ &\leq C\omega_2\left(f, \frac{2[n+1]_q^{(1/2)}}{([n]_q+\beta)}\delta_n(x)\right) + \omega\left(f, \frac{1}{([n]_q+\beta)}\right),\end{aligned}$$

and so the proof is completed. □

5 Weighted Approximation

In this section, we obtain the Korovkin type weighted approximation by the operators defined in (1.4). The weighted Korovkin-type theorems were proved by Gadzhiev [7]. A real function $\rho = 1 + x^2$ is called a weight function if it is continuous on $\mathbb{R}$ and $\lim_{|x|\to\infty}\rho(x) = \infty,\ \rho(x) \geq 1$ for all $x \in \mathbb{R}$.

Let $B_\rho(\mathbb{R})$ denote the weighted space of real-valued functions f defined on $\mathbb{R}$ with the property $|\ f(x)\ | \leq M_f\ \rho(x)$ for all $x \in \mathbb{R}$, where M_f is a constant depending on the function f. We also consider the weighted subspace $C_\rho(\mathbb{R})$ of $B_\rho(\mathbb{R})$ given by $C_\rho(\mathbb{R}) = \{f \in B_\rho(\mathbb{R})\colon f$ is continuous on $\mathbb{R}\}$ and $C_\rho^*[0,\infty)$ denotes the subspace of all functions $f \in C_\rho[0,\infty)$ for which $\lim_{|x|\to\infty}\frac{f(x)}{\rho(x)}$ exists finitely.

Theorem 2 (See [7] and [8])

(i) There exists a sequence of linear positive operators $A_n(C_\rho \to B_\rho)$ such that

$$\lim_{n\to\infty}\|A_n(\phi^\nu) - \phi^\nu\|_\rho = 0, \quad \nu = 0, 1, 2 \tag{5.1}$$

and a function $f^* \in C_\rho \backslash C_\rho^*$ with $\lim_{n\to\infty}\|A_n(f^*) - f^*\|_\rho \geq 1$.

(ii) If a sequence of linear positive operators $A_n(C_\rho \to B_\rho)$ satisfies conditions (5.1) then

$$\lim_{n\to\infty} \|A_n(f) - f\|_\rho = 0, \quad \textit{for every} \quad f \in C_\rho^*. \tag{5.2}$$

Throughout this paper we take the growth condition as $\rho(x) = 1 + x^2$ and $\rho_\gamma(x) = 1 + x^{2+\gamma},\ x \in [0,\infty), \gamma > 0$. Now, we are ready to prove our next result as follows:

Theorem 3 *For each $f \in C_\rho^*[0,\infty)$, we have*

$$\lim_{n\to\infty} \|\mathcal{L}_n^{(\alpha,\beta)}(f) - f\|_\rho = 0.$$

Proof Using the theorem in [7] we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \|\mathcal{L}_n^{(\alpha,\beta)}(t^r;q;x) - x^r\|_\rho = 0, \quad r = 0,1,2. \tag{5.3}$$

Since, $\mathcal{L}_n^{(\alpha,\beta)}(1;q;x) = 1$, the first condition of (5.3) is satisfied for $r = 0$. Now,

$$\begin{aligned}
\|\mathcal{L}_n^{(\alpha,\beta)}(t;q;x) - x\|_\rho &= \sup_{x\in[0,\infty)} \frac{|\ \mathcal{L}_n^{(\alpha,\beta)}(t;q;x) - x\ |}{1+x^2} \\
&\le \sup_{x\in[0,\infty)} \frac{x}{1+x^2}\left|\left(\frac{-\beta}{[n]_q+\beta}\right)\right| + \frac{q[n]_q + \alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q} \\
&\to 0 \ \ as \ \ [n]_q \to \infty.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|\mathcal{L}_n^{(\alpha,\beta)}(t^2;q;x) - x^2\|_\rho &= \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2}\left|\left(\frac{[n]_q^2}{([n]_q+\beta)^2}\frac{q^{(n-2)}[n+2]_q}{[n+1]_q}\right)\right| \\
&\quad + \sup_{x\in[0,\infty)} \frac{x}{1+x^2}\left|\left(\frac{[n]_q^2}{([n]_q+\beta)^2}\left\{\frac{q^{(n-1)}}{[n+2]_q}\right.\right.\right. \\
&\quad \left.\left.\left. + \frac{(2q+1)}{[3]_q[n+1]_q}\right\} + \frac{2[n]_q\alpha}{[n]_q+\beta}\right)\right| \\
&\quad + \left(\frac{[n]_q^2}{([n]_q+\beta)^2}\frac{q}{[3]_q[n+1]_q^2}\right. \\
&\quad \left. + \frac{2q[n]_q\alpha}{[2]_q[n+1]_q([n]_q+\beta)^2} + \frac{\alpha^2}{([n]_q+\beta)^2}\right) \\
&\to 0 \quad \text{as } [n]_q \to \infty.
\end{aligned}$$

Thus, from Gadzhievs Theorem in [7] we obtain the desired result of theorem. □

We give the following theorem to approximate all functions in $C_{x^2}[0, \infty)$.

Theorem 4 *For each* $f \in C_{x^2}[0, \infty)$ *and* $\alpha > 0$*, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\mid \mathcal{L}_n^{(\alpha,\beta)}(f; q; x) - f(x) \mid}{(1+x^2)^{1+\alpha}} = 0.$$

Proof For any fixed $x_0 > 0$,

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{\mid \mathcal{L}_n^{(\alpha,\beta)}(f; q; x) - f(x) \mid}{(1+x^2)^{1+\alpha}} &\le \sup_{x\le x_0} \frac{\mid \mathcal{L}_n^{(\alpha,\beta)}(f; q; x) - f(x) \mid}{(1+x^2)^{1+\alpha}} \\ &\quad + \sup_{x\ge x_0} \frac{\mid \mathcal{L}_n^{(\alpha,\beta)}(f; q; x) - f(x) \mid}{(1+x^2)^{1+\alpha}} \\ &\le \|\mathcal{L}_n^{(\alpha,\beta)}(f) - f\|_{C[0,x_0]} + \|f\|_{x^2} \\ &\quad \sup_{x\ge x_0} \frac{\mid \mathcal{L}_n^{(\alpha,\beta)}(1+t^2, x) \mid}{(1+x^2)^{1+\alpha}} \\ &\quad + \sup_{x\ge x_0} \frac{\mid f(x) \mid}{(1+x^2)^{1+\alpha}}.\end{aligned}$$

The first term of the above inequality tends to zero from Theorem 5. By Lemma 1(ii), for any fixed $x_0 > 0$ it is easily seen that $\sup_{x\ge x_0} \frac{|\mathcal{L}_n^{(\alpha,\beta)}(1+t^2,x)|}{(1+x^2)^{1+\alpha}}$ tends to zero as $n \to \infty$. We can choose $x_0 > 0$ so large that the last part of the above inequality can be made small enough. Thus the proof is completed. □

6 Error Estimation

The usual modulus of continuity of f on the closed interval $[0, b]$ is defined by

$$\omega_b(f, \delta) = \sup_{|t-x|\le\delta,\, x,t\in[0,b]} |f(t) - f(x)|, \;\; b > 0.$$

It is well known that, for a function $f \in E$,

$$\lim_{\delta\to 0^+} \omega_b(f, \delta) = 0,$$

where

$$E := \left\{ f \in C[0, \infty) : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \; is \;\; finite \right\}.$$

The next theorem gives the rate of convergence of the operators $\mathcal{L}_n^{(\alpha,\beta)}(f, q; x)$ to $f(x)$, for all $f \in E$.

Theorem 5 *Let $f \in E$ and $\omega_{b+1}(f,\delta)$ be its modulus of continuity on the finite interval* $[0, b+1] \subset [0,\infty)$*, where* $b > 0$*. Then we have*

$$\|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f\|_{C[0,b]} \le M_f(1+b^2)\delta_n(b) + 2\omega_{b+1}\left(f, \sqrt{\delta_n(b)}\right).$$

Proof The proof is based on the following inequality

$$\|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f\| \le M_f(1+b^2)\mathcal{L}_n^{(\alpha,\beta)}((t-x)^2, x) + \left(1 + \frac{\mathcal{L}_n^{(\alpha,\beta)}(|t-x|, x)}{\delta}\right)\omega_{b+1}(f,\delta). \tag{6.1}$$

For all $(x,t) \in [0,b] \times [0,\infty) := S$. To prove (6.1), we write

$$S = S_1 \cup S_2 := \{(x,t) : 0 \le x \le b,\ 0 \le t \le b+1\} \cup \{(x,t) : 0 \le x \le b,\ t > b+1\}.$$

If $(x,t) \in S_1$, we can write

$$|f(t) - f(x)| \le \omega_{b+1}(f, |t-x|) \le \left(1 + \frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta) \tag{6.2}$$

where $\delta > 0$. On the other hand, if $(x,t) \in S_2$, using the fact that $t - x > 1$, we have

$$\begin{aligned} |f(t) - f(x)| &\le M_f(1 + x^2 + t^2) \\ &\le M_f(1 + 3x^2 + 2(t-x)^2) \\ &\le N_f(1+b^2)(t-x)^2 \end{aligned} \tag{6.3}$$

where $N_f = 6M_f$. Combining (6.2) and (6.3), we get (6.1). Now from (6.1) it follows that

$$\begin{aligned} |\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f(x)| &\le N_f(1+b^2)\mathcal{L}_n^{(\alpha,\beta)}((t-x)^2, q; x) \\ &\quad + \left(1 + \frac{\mathcal{L}_n^{(\alpha,\beta)}(|t-x|, q; x)}{\delta}\right)\omega_{b+1}(f,\delta) \\ &\le N_f(1+b^2)\mathcal{L}_n^{(\alpha,\beta)}((t-x)^2, q; x) \\ &\quad + \left(1 + \frac{[\mathcal{L}_n^{(\alpha,\beta)}((t-x)^2, q; x)]^{1/2}}{\delta}\right)\omega_{b+1}(f,\delta). \end{aligned}$$

By Lemma 5, we have

$$\mathcal{L}_n^{(\alpha,\beta)}(t-x)^2 \le \delta_n(b).$$

$$\|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f\| \le N_f(1+b^2)\delta_n(b) + \left(1 + \frac{\sqrt{\delta_n(b)}}{\delta}\right)\omega_{b+1}(f,\delta).$$

Choosing $\delta = \sqrt{\delta_n(b)}$, we get the desired estimation. □

Now, we give some estimations of the errors $|\mathcal{L}_n^{(\alpha,\beta)}(f) - f|$, $n \in \mathbb{N}$ for unbounded functions by using a weighted modulus of smoothness associated to the space $B_{\rho_\gamma}(\mathbb{R}_+)$. The weighed modulus of continuity $\Omega_{\rho_\gamma}(f;\delta)$ was defined by López–Moreno in [16]. We consider

$$\Omega_{\rho_\gamma}(f;\delta) = \sup_{x\geq 0, 0\leq h\leq\delta} \frac{|f(x+h) - f(x)|}{1+(x+h)^{2+\gamma}}, \quad \delta > 0, \quad \gamma \geq 0. \tag{6.4}$$

It is evident that for each $f \in B_{\rho_\gamma}(\mathbb{R}_+)$, $\Omega_{\rho_\gamma}(f;\cdot)$ is well defined and

$$\Omega_{\rho_\gamma}(f;\delta) \leq 2\|f\|_{\rho_\gamma}.$$

The weighted modulus of smoothness $\Omega_{\rho_\gamma}(f;\cdot)$ possesses the following properties.

(i) $\Omega_{\rho_\gamma}(f;\lambda\delta) \leq (\lambda+1)\Omega_{\rho_\gamma}(f;\delta), \quad \delta > 0, \quad \lambda > 0$
(ii) $\Omega_{\rho_\gamma}(f;n\delta) \leq n\Omega_{\rho_\gamma}(f;\delta), \quad n \in \mathbb{N}$
(iii) $\lim_{\delta\to 0}\Omega_{\rho_\gamma}(f;\delta) = 0$.

Now, we are ready to prove our next theorem by using above properties.

Theorem 6 *For all non-decreasing $f \in B_{\rho_\gamma}(\mathbb{R}_+)$, we have*

$$|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f| \leq \sqrt{\mathcal{L}_n^{(\alpha,\beta)}(\nu_{x,\gamma}^2;q;x)}\left(1 + \frac{1}{\delta}\sqrt{\mathcal{L}_n^{(\alpha,\beta)}(\Psi_x^2;q;x)}\right)\Omega_{\rho_\gamma}(f;q;\delta),$$

$x \geq 0, \quad \delta > 0, \quad n \in \mathbb{N}$, *where*

$$\nu_{x,\gamma}(t) := 1 + (x+|t-x|)^{2+\gamma}, \qquad \Psi_x(t) := |t-x|,\ t \geq 0.$$

Proof Let $n \in \mathbb{N}$ and $f \in B_{\rho_\gamma}(\mathbb{R}_+)$. From (6.4), we can write

$$\begin{aligned}|f(t) - f(x)| &\leq \left(1 + (x+|t-x|)^{2+\gamma}\right)\left(1 + \frac{1}{\delta}|t-x|\right)\Omega_{\rho_\gamma}(f;\delta)\\ &= \nu_{x,\gamma}(t)\left(1 + \frac{1}{\delta}\Psi_x(t)\right)\Omega_{\rho_\gamma}(f;\delta).\end{aligned}$$

Now, applying operator $\mathcal{L}_n^{(\alpha,\beta)}$ on above inequality, we get

$$\begin{aligned}|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f(x)| &\leq \Omega_{\rho_\gamma}(f;\delta)\mathcal{L}_n^{(\alpha,\beta)}\left(\nu_{x,\gamma}\left(1 + \frac{1}{\delta}\Psi_x\right);q;x\right)\\ &\leq \Omega_{\rho_\gamma}(f;\delta)\left(\mathcal{L}_n^{(\alpha,\beta)}(\nu_{x,\gamma};q;x) + \mathcal{L}_n^{(\alpha,\beta)}\left(\frac{\nu_{x,\gamma}\Psi_x}{\delta};q;x\right)\right).\end{aligned} \tag{6.5}$$

By using the Cauchy-Schwartz inequality, we obtain

$$\mathcal{L}_n^{(\alpha,\beta)}\left(\frac{\nu_{x,\gamma}\Psi_x}{\delta};q;x\right) \leq \left\{\mathcal{L}_n^{(\alpha,\beta)}((\nu_{x,\gamma})^2;q;x)\right\}^{1/2}\left\{\mathcal{L}_n^{(\alpha,\beta)}\left(\left(\frac{\Psi_x}{\delta}\right)^2;q;x\right)\right\}^{1/2}$$
$$= \frac{1}{\delta}\left\{\mathcal{L}_n^{(\alpha,\beta)}(\nu_{x,\gamma}^2;q;x)\right\}^{1/2}\left\{\mathcal{L}_n^{(\alpha,\beta)}({\Psi_x}^2;q;x)\right\}^{1/2}.$$

Now, by (6.5), we get

$$|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f| \leq \sqrt{\mathcal{L}_n^{(\alpha,\beta)}(\nu_{x,\gamma}^2;q;x)}\left(1+\frac{1}{\delta}\sqrt{\mathcal{L}_n^{(\alpha,\beta)}(\Psi_x^2;q;x)}\right)\Omega_{\rho_\gamma}(f;\delta).$$

□

Theorem 7 *Let* $0 < \alpha \leq 1$ *and E be any bounded subset of the interval* $[0,\infty)$. *If* $f \in C_B[0,\infty) \bigcap Lip_L(\alpha)$, *then we have*

$$\left|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f(x)\right| \leq B\{\delta_n^{\frac{\alpha}{2}}(x) + 2(d(x,E))^\alpha\},$$

where L is a constant depending on α, $d(x;E)$ *is the distance between x and E defined as*

$$d(x;E) = inf\{|t-x|; t \in E \ and \ x \in [0,\infty)\},$$

and $\delta_n(x)$ *is as in* (4).

Proof From the properties of the infimum, there is at least one point t_o in the closure of E, that is $t_0 \in \bar{E}$, such that

$$d(x,E) = |t_0 - x|.$$

By the triangle inequality we have

$$|f(t) - f(x)| \leq |f(t) - f(t_0)| + |f(t_0) - f(x)|.$$

And

$$\begin{aligned}\left|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f(x)\right| &\leq \mathcal{L}_n^{(\alpha,\beta)}(|f(t) - f(x)|,x)\\ &\leq \mathcal{L}_n^{(\alpha,\beta)}(|f(t) - f(t_0)|,x) + \mathcal{L}_n^{(\alpha,\beta)}(|f(t_0) - f(x)|,x)\\ &\leq B\left[\mathcal{L}_n^{(\alpha,\beta)}\left(|t-t_0|^\alpha,x\right) + |t_0 - x|^\alpha\right]\\ &\leq B\left[\mathcal{L}_n^{(\alpha,\beta)}\left(|t-t_0|^\alpha,x\right) + 2|t_0 - x|^\alpha\right]\end{aligned}$$

holds. Here we choose $p_1 = \frac{2}{\alpha}$ and $p_2 = \frac{2}{2-\alpha}$, we get $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then from well-known Hölder's inequality, we have

$$\begin{aligned}\left|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f(x)\right| &\leq B\left\{\left[\mathcal{L}_n^{(\alpha,\beta)}\left(|t-x|^{\alpha p_1}, x\right)\right]^{(1/p_1)}\left[\mathcal{L}_n^{(\alpha,\beta)}\left(1^{p_2}, x\right)\right]^{(1/p_2)}\right.\\ &\quad \left. +2|t_0 - x|^{\alpha}\right\}\\ &= B\left\{\left[\mathcal{L}_n^{(\alpha,\beta)}\left(|t-x|^2, x\right)\right]^{(\alpha/2)} + 2|t_0 - x|^{\alpha}\right\}\\ &= B\{\delta_n^{\alpha/2}(x) + 2(d(x,E))^{\alpha}\}.\end{aligned}$$

This completes the proof. □

7 Global Approximation

For $f \in C[0, 1+a]$, the Ditzian–Totik moduli of smoothness of the first and second order are given by

$$\bar{\omega}_{\psi}(f,\delta) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x+h\psi(x)\in[0,1+a]} |\, f(x+h\psi(x)) - f(x)\,| \tag{7.1}$$

and

$$\omega_2^{\phi}(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\pm h\phi(x)\in[0,1+a]} |\, f(x+h\phi(x)) - 2f(x) + f(x-h\phi(x)|\,| \tag{7.2}$$

respectively and the corresponding K-functional is defined as

$$\bar{K}_{2,\phi}(f,\delta) = \inf\{||f-g|| + \delta||\psi^2 g''|| + \delta^2||g''|| : g \in W^2(\phi)\}, \tag{7.3}$$

where $\delta > 0$ and $W^2(\phi) = \{g \in C[0,1+a] : g' \in AC[0,1+a], \phi^2 g'' \in C[0,1+a]\}$ and $g' \in AC_{loc}[0, 1+a]$ means that g is differential and g' is absolutely continuous on every closed interval $[0, 1+a]$. It is well known ([5], p. 24, Theorem 1.3.1) that

$$\bar{K}_{2,\phi}(f,\delta) \leq C\omega_2^{\phi}(f,\sqrt{\delta}), \tag{7.4}$$

where ψ is being admissible step-weight function on $[0, 1+a]$.

Theorem 8 *Let $f \in C[0, 1+a]$ with $q \in (0,1)$. Then for every $x \in [0,1]$, we have*

$$\left|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x) - f(x)\right| \leq C\omega_2^{\phi}\left(f, \frac{[n+1]_q^{1/2}}{[n]_q}\right) + \bar{\omega}_{\psi}\left(f, \frac{1}{(n+\beta)}\right).$$

Proof Defining the operators $\widetilde{\mathcal{L}}^{(\alpha,\beta)}$ as in (4.7) for the function $g \in W^2(\psi)$, we have

$$\left|\widetilde{\mathcal{L}}_n^{(\alpha,\beta)}(g,q;x) - g(x)\right| \le \mathcal{L}_n^{(\alpha,\beta)}\left(\left|\int_x^t (t-v)g''(v)dv\right|, q, x\right) + \left|\int_x^{\frac{[n]_q x}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}} \left(\frac{[n]_q x}{[n]_q+\beta} + \frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q} - v\right) g''(v)dv\right|. \tag{7.5}$$

Since the function $\delta_n^2(x)$ is concave on [0,1], for $v = t + \tau(x-t), \tau \in [0,1]$ we obtain

$$\frac{|t-v|}{{\delta^2}_n(v)} = \frac{\tau|x-t|}{{\delta^2}_n(t+\tau(x-t))} \le \frac{\tau|x-t|}{{\delta_n}^2(t)+\tau({\delta_n}^2(x)-\delta_n^2(t))} \le \frac{|t-v|}{\delta_n^2(x)}.$$

Now using (7.5)

$$\begin{aligned}\left|\widetilde{\mathcal{L}}_n^{(\alpha,\beta)}(g,q;x) - g(x)\right| &\le \mathcal{L}_n^{(\alpha,\beta)}\left(\left|\int_x^t \frac{(t-v)}{\delta_n^2(v)}dv\right|, q, x\right)||\delta_n^2 g''|| \\ &+ \left|\int_x^{\frac{[n]_q x}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}} \frac{\left|\frac{[n]_q x}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q} - v\right|}{\delta_n^2(v)}dv\right| ||\delta_n^2 g''|| \\ &\le \frac{1}{\delta_n^2(x)}\mathcal{L}_n^{(\alpha,\beta)}((t-x)^2;q,x)||\delta_n^2 g''|| \\ &+ \frac{1}{\delta_n^2(x)}\left(\frac{[n]_q x}{[n]_q+\beta} + \frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q} - x\right)^2 ||\delta_n^2 g''||.\end{aligned}$$

from Lemma 5 and $||\delta_n^2 g''(x)|| \le |\phi^2 g''| + \frac{q}{[n+1]_q}||g''(x)||$, where $x \in [0,1]$, we get

$$\left|\widetilde{\mathcal{L}}_n^{(\alpha,\beta)}(g,q;x) - g(x)\right| \le \frac{[n+1]_q}{(n+\beta)^2}\left(|\phi^2 g''| + \frac{q}{(n+\beta)^2}||g''||\right), \tag{7.6}$$

using (7.6), we have for $f \in C[0, 1+a]$.

$$\begin{aligned}
&\left|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x)-f(x)\right|\\
&\quad\leq\left|\widetilde{\mathcal{L}}_n^{(\alpha,\beta)}(f-g,q,x)|+|\mathcal{L}_n^{(\alpha,\beta)}(g,q,x)-g(x)\right|+|g(x)-f(x)|\\
&\qquad+\left|f\left(\frac{[n]_qx}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}\right)-f(x)\right|\\
&\quad\leq 4\|f-g\|+\frac{[n+1]_q}{([n]_q+\beta)^2}||\phi^2g''||\\
&\qquad+\frac{q[n+1]_q}{([n]_q+\beta)^4}||g''||+\left|f\left(\frac{[n]_qx}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}\right)-f(x)\right|\\
&\quad\leq 4\|f-g\|+\frac{[n+1]_q}{[n]_q^2}||\phi^2g''||+\frac{[n+1]_q^2}{[n]_q^4}||g''||\\
&\qquad+\left|f\left(\frac{[n]_qx}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}\right)-f(x)\right|\\
&\quad\leq C\left(\|f-g\|+\delta||\phi^2g''||+\delta^2||g''||\right)\\
&\qquad+\left|f\left(\frac{[n]_qx}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}\right)-f(x)\right|
\end{aligned}$$

where $\delta=\frac{[n+1]_q}{[n]_q^2}$. Taking the infimum on the right hand side over all $g\in W^2(\phi)$, we get

$$\left|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x)-f(x)\right|\leq CK_{2,\phi}\left(f,\frac{[n+1]_q}{[n]_q^2}\right) \tag{7.7}$$

$$+\left|f\left(\frac{[n]_qx}{[n]_q+\beta}+\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}\right)-f(x)\right|. \tag{7.8}$$

Now,

$$\begin{aligned}
&\left|f\left(\frac{q[n]_q+\alpha[2]_q[n+1]_q}{[2]_q([n]_q+\beta)[n+1]_q}\right)-f(x)\right|\\
&\quad=\left|f\left(x+\psi(x).\frac{-\beta x[2]_q[n+1]_q+q[n]_q+\alpha[2]_q[n+1]_q}{\psi(x)([2]_q([n]_q+\beta)[n+1]_q))}\right)-f(x)\right|\\
&\quad\leq\sup_{t,t+\psi(t)\left(\frac{-\beta x[2]_q[n+1]_q+q[n]_q+\alpha[2]_q[n+1]_q}{\psi(x)([2]_q([n]_q+\beta)[n+1]_q))}\right)\in[0,1+p]}\\
&\qquad\left|f\left(t+\psi(t)\left(\frac{-\beta x[2]_q[n+1]_q+q[n]_q+\alpha[2]_q[n+1]_q}{\psi(x)([2]_q([n]_q+\beta)[n+1]_q))}\right)-f(t)\right|\\
&\quad\leq\omega_\phi\left(f,\frac{-\beta x[2]_q[n+1]_q+q[n]_q+\alpha[2]_q[n+1]_q}{\psi(x)([2]_q([n]_q+\beta)[n+1]_q))}\right)\\
&\quad\leq\omega_\phi\left(f,\frac{1}{(n+\beta)}\right).
\end{aligned}$$

Hence by (7.1) and (7.7), we get

$$\left|\mathcal{L}_n^{(\alpha,\beta)}(f,q;x)-f(x)\right| \leq C\omega_2^{\phi}\left(f,\frac{[n+1]_q^{1/2}}{[n]_q}\right)+\bar{\omega}_{\psi}\left(f,\frac{1}{(n+\beta)}\right),$$

which completes the proof. □

8 Motivation and Applications

In recent years, applications of q-calculus in the area of approximation theory and number theory have been an active area of research. The approximation of functions by linear positive operators is an important research topic in general mathematics and it also provides powerful tools to application areas such as computer-aided geometric design, numerical analysis, and solutions of differential equations. q-calculus is a generalization of many subjects, such as hypergeometric series, complex analysis and particle physics. Currently it continues being an important subject of study. It has been shown that linear positive operators constructed by q-numbers are quite effective as far as the rate of convergence is concerned and we can have some unexpected results, which are not observed for classical case.

9 Conclusion

By using the notion of q-integers we introduced Kantorovich-type discrete q-Beta operators and investigated some local and global approximation properties of these operators. We obtained the rate of convergence by using the modulus of continuity and also established some direct theorems. These results generalize the approximation results proved for Kantorovich-type discrete q-Beta operators which are directly obtained by our results for $q = 1$.

The results of our lemmas and theorems are more general rather than the results of any other previous proved lemmas and theorems, which will be enrich the literate of applications of quantum calculus in operator theory and convergence estimates in the theory of approximations by positive linear operators. The researchers and professionals working or intend to work in the areas of analysis and its applications will find this research article to be quite useful. Consequently, the results so established may be found useful in several interesting situation appearing in the literature on Mathematical Analysis, Applied Mathematics and Mathematical Physics.

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