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Complex Analysis and Operator Theory

Toeplitz Operators with Vertical Symbols Acting on the Poly-Bergman Spaces of the Upper Half-Plane

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Abstract We describe the C^*-algebra generated by the Toeplitz operators acting on each poly-Bergman space of the upper half-plane $\Pi \subset \mathbb{C}$. We consider bounded symbols depending only on $y = \operatorname{Im} z$ and having limit values at $y = 0$ and $y = \infty$. This C^* algebra is isomorphic to the C^*-algebra of all matrices of dimension $n \times n$ whose entries are continuous functions over the positive reals, and are scalar multiples of the identity matrix at $y = 0$ and $y = \infty$.

Keywords Algebras of Toeplitz operators · Poly-Bergman space · Poly-Bergman projection

Mathematics Subject Classification Primary 47L80; Secondary 30H99, 32A25

1 Introduction

Recall that the space $\mathcal{A}_n^2(D)$ of n-analytic functions is the subspace of $L_2(D)$ consisting of all functions $\varphi = \varphi(z, \overline{z}) = \varphi(x, y)$ that satisfy the equation

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$$\left(\frac{\partial}{\partial \overline{z}}\right)^n \varphi = \frac{1}{2^n}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^n \varphi = 0,$$

where $D \subset \mathbb{C}$ is a bounded domain with smooth boundary. We denote by $\mathcal{A}^2_{(n)}(\Pi)$ the space of all true-n-analytic functions, that is,

$$\mathcal{A}^2_{(n)}(D) = \mathcal{A}^2_n(D) \ominus \mathcal{A}^2_{n-1}(D),$$

for $n \geq 1$, and $\mathcal{A}^2_{(0)}(D) = \{0\}$. Of course, $\mathcal{A}^2_1(D)$ is the usual Bergman space of D, which is simply denoted by $\mathcal{A}^2(D)$. Similarly, we introduce the spaces $\widetilde{\mathcal{A}}^2_n(D)$ and $\widetilde{\mathcal{A}}^2_{(n)}(D)$ of all n-anti-analytic and true-n-anti-analytic functions, respectively. Actually, each n-anti-analytic function is just the complex conjugation of a n-analytic function.

For the upper half-plane Π, Vasilevski [10] proved that $L_2(\Pi)$ has a decomposition as a direct sum of the n-true-analytic and n-true-anti-analytic function spaces:

$$L_2(\Pi) = \bigoplus_{k=1}^{\infty} \mathcal{A}^2_{(k)}(\Pi) \oplus \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}^2_{(k)}(\Pi).$$

The spaces $\mathcal{A}^2_{(n)}(\Pi)$ and $\tilde{\mathcal{A}}^2_{(n)}(\Pi)$ are isomorphic and isometric to

$$L_2(\mathbb{R}_+) \otimes L_{n-1} \text{ and } L_2(\mathbb{R}_-) \otimes L_{n-1},$$

respectively, where L_{n-1} is the one-dimensional space generated by Laguerre function of order $n-1$. Moreover, N. Vasilevski found the explicit expressions for the reproduction kernels of all these function spaces.

We introduce as well the following bounded singular integral operators on $L_2(D)$:

$$(S_D\varphi)(z) = -\frac{1}{\pi}\int_D \frac{\varphi(\zeta)}{(\zeta - z)^2} d\nu(\zeta),$$
$$(S^*_D\varphi)(z) = -\frac{1}{\pi}\int_D \frac{\varphi(\zeta)}{(\overline{\zeta} - \overline{z})^2} d\nu(\zeta),$$

where $d\nu = dxdy$ is the usual Lebesgue measure on D. Dzhuraev [1] showed that the orthogonal projections $B_{D,n}$ and $\widetilde{B_{D,n}}$ of $L_2(D)$ onto the spaces $\mathcal{A}^2_{(n)}(D)$ and $\widetilde{\mathcal{A}}^2_{(n)}(D)$, respectively, can be expressed in the form

$$B_{D,n} = I - (S_D)^n(S^*_D)^n + K_n$$

and

$$\widetilde{B_{D,n}} = I - (S^*_D)^n(S_D)^n + \widetilde{K_n},$$

where K_n and $\widetilde{K_n}$ are compact operators. Moreover Ramírez and Spitkovsky [8] proved that the compact summands K_n and $\widetilde{K_n}$ are equal to zero for $D = \Pi$. Vasilevski [11] described a direct connection between the poly-Bergman type spaces on the upper half-plane and the operators S_Π and S^*_Π, each of them is unitary equivalent to the direct sum of two unilateral shift operators with infinite multiplicity.

On the other hand, consider the algebra of pseudodifferential operators $\mathcal{R}(C(\overline{\mathbb{D}}); S_\mathbb{D}, S^*_\mathbb{D})$, which is generated by $S_\mathbb{D}$, $S^*_\mathbb{D}$ and the multiplication operators $a(z)I$, where $a(z) \in C(\overline{\mathbb{D}})$ and $\mathbb{D}$ is the unit disk $\{z : |z| < 1\}$. Sánchez-Nungaray and Vasilevski [9] studied the C^* algebra $\mathcal{T}_n(\mathcal{R}(C(\overline{\mathbb{D}}), S_\mathbb{D}, S^*_\mathbb{D}))$ generated by the Toeplitz operators over the poly-Bergman spaces of $\mathbb{D}$ with defining symbols from the algebra $\mathcal{R}(C(\overline{\mathbb{D}}); S_\mathbb{D}, S^*_\mathbb{D})$. They proved that the algebra $\mathcal{T}_n(\mathcal{R}(C(\overline{\mathbb{D}}), S_\mathbb{D}, S^*_\mathbb{D}))$ is unitary equivalent to the matrix algebra

$$\mathcal{T}(C(\overline{\mathbb{D}})) \otimes M_n(\mathbb{C}),$$

where $\mathcal{T}(C(\overline{\mathbb{D}}))$ is the algebra generated by the Toeplitz operators over the Bergman space with symbols in $C(\overline{\mathbb{D}})$. The Fredholm symbol algebra of $\mathcal{T}_n(C(\overline{\mathbb{D}}))$ is isomorphic and isometric to $C(S^1)$, where S^1 is the unit circle $\{z : |z| = 1\}$; while the Fredholm symbol algebra of $\mathcal{T}_n(\mathcal{R}(C(\overline{\mathbb{D}}); S_\mathbb{D}, S^*_\mathbb{D}))$ is isomorphic and isometric to the matrix algebra

$$M_n(C(S^1)) = C(S^1) \otimes M_n(\mathbb{C}).$$

Grudsky et al. [3] characterized all the commutative C*-algebras of Toeplitz operators acting on the Bergman space of the unit disk. Every commutative C^*-algebra of Toeplitz operators arises from a class of symbols invariant under the action of a maximal abelian group of Möbius transformations on the unit disk. There exist three types of such maximal abelian groups: 1) the group of elliptic transformations, 2) the group of parabolic transformations, 3) the group of hyperbolic transformations. Since $\mathbb{D}$ and Π are diffeomorphic to each other, all the commutative C*-algebras of Toeplitz operators on the Bergman space of Π are automatically classified.

Lozano and Loaiza [6,7] used the three classes of symbols described in the previous paragraph and studied the corresponding Toeplitz operator algebras acting on the harmonic Bergman space. An interesting and unexpected result is that two such operator algebras are commutative whereas the last one (hyperbolic case) is not.

The main result of this work is the isomorphic description of the C^*-algebra generated by the Toeplitz operators with bounded vertical symbols and acting over each poly-Bergman space $\mathcal{A}^2_n(\Pi)$. This paper is organized as follows. In Sect. 2 we introduce preliminary results about the n-polyanalytic function spaces and their relationship with the Laguerre polynomials. In Sect. 3 we prove that every Toeplitz operator, with bounded vertical symbol $a(z)$ and acting on $\mathcal{A}^2_n(\Pi)$, is unitary equivalent to a multiplication operator $\gamma^{n,a}(x)I$ acting on $(L_2(\mathbb{R}_+))^n$, where $\gamma^{n,a}(x)$ is a continuous matrix-valued function on $(0, \infty)$.

Finally, in Sect. 4 we consider bounded vertical symbols having limit values at $y = 0, \infty$ and prove that the C^* algebra $\mathcal{T}^{(n)}_{0\infty}$ generated by all the Toeplitz operator acting on $\mathcal{A}^2_n(\Pi)$ is isomorphic and isometric to the C^*-algebra

$$\mathfrak{D} = \{M \in M_n(\mathbb{C}) \otimes C[0, \infty] : M(0), M(\infty) \in \mathbb{C}I\}.$$

To prove the above statement, we will use the non-commutative Stone-Weierstrass conjecture: *Let $\mathcal{B}$ be a C*-subalgebra of a C*-algebra $\mathcal{A}$, and suppose that $\mathcal{B}$ separates all the pure states of $\mathcal{A}$ (and 0 if $\mathcal{A}$ is non-unital). Then $\mathcal{A} = \mathcal{B}$.* For type I C*-algebras this conjecture was proved by Kaplansky [4]. In our case, we have that $\mathfrak{D}$ is a type I C*-algebra, and we prove that the C*-algebra $\mathcal{T}_{0\infty}^{(n)}$ separates the pure states of $\mathfrak{D}$.

2 Bergman and Poly-Bergman Spaces

Let Π be the upper half-plane in $\mathbb{C}$, and consider the space $L_2(\Pi, d\nu)$, where $d\nu(z) = dxdy$ is the usual Lebesgue measure and $z = x+iy$. Let $\mathcal{A}^2(\Pi)$ be the Bergman space of Π, and B_Π be the Bergman projection of $\mathcal{A}^2(\Pi)$. The Bergman space $\mathcal{A}^2(\Pi)$ is the closed subspace of $L_2(\Pi)$, which consists of all functions satisfying the equation

$$2\frac{\partial}{\partial \bar{z}}\varphi = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\varphi = 0.$$

Introduce the unitary operator

$$U_1 = F \otimes I : L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \to L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+), \tag{2.1}$$

where F is the Fourier transform given by

$$(Fh)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(t)e^{-ixt}dt.$$

The image space $\mathcal{A}_1^2 = U_1(\mathcal{A}^2(\Pi))$ is the subspace of $L_2(\Pi)$ which consists of all functions $\varphi(x, y) = \sqrt{2x}\, f(x)\, e^{-xy}$, where $f \in L_2(\mathbb{R}_+)$ and $\mathbb{R}_+$ is the set of the positive reals. Let χ_+ be the characteristic function of $\mathbb{R}_+$. Then the orthogonal projection B_1 from $L_2(\Pi)$ onto $\mathcal{A}_1^2$ is given by $B_1 = U_1 B_\Pi U_1^{-1}$, and

$$(B_1\varphi)(x, y) = \chi_+(x)\, 2xe^{-xy}\int_{\mathbb{R}_+}\varphi(x, t)\, e^{-xt}dt.$$

Introduce the unitary operator U_2 on $L_2(\Pi)$ by the rule

$$(U_2\varphi)(x, y) = \frac{1}{\sqrt{2|x|}}\varphi\left(x, \frac{y}{2|x|}\right). \tag{2.2}$$

Then $B_2 = U_2 B_1 U_2^{-1}$ is the orthogonal projection from $L_2(\Pi)$ onto $\mathcal{A}_2^2 = U_2(\mathcal{A}_1^2)$, and is given by

$$(B_2\varphi)(x, y) = \chi_+(x)e^{-y/2}\int_{\mathbb{R}_+}\varphi(x, t)e^{-t/2}dt.$$

Introducing $l_0(y) = e^{-y/2}$, we have $l_0 \in L_2(\mathbb{R}_+)$ and $||l_0|| = 1$. Denote by L_0 the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $l_0(y)$. Then the one-dimensional projection P_0 from $L_2(\mathbb{R}_+)$ onto L_0 has the form

$$(P_0\phi)(y) = \langle \phi, l_0 \rangle \cdot l_0 = e^{-y/2} \int_{\mathbb{R}_+} \phi(v) e^{-v/2} dv.$$

Theorem 2.1 (Vasilevski [10]) *The unitary operator $U = U_2 U_1$ gives an isometric isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$, under which*

1. *The Bergman space $\mathcal{A}^2(\Pi)$ is mapped onto $L_2(\mathbb{R}_+) \otimes L_0$.*
2. *The Bergman projection B_Π is unitary equivalent to the following one:*

$$B_2 := U B_\Pi U^{-1} = \chi_+(x) I \otimes P_0.$$

The poly-Bergman space $\mathcal{A}_n^2(\Pi)$ consists of all n-analytic functions in $L_2(\Pi)$, that is, it is the closed subspace of $L_2(\Pi)$ consisting of all functions satisfying the equation

$$\left(\frac{\partial}{\partial \bar{z}}\right)^n \varphi = \frac{1}{2^n} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)^n \varphi = 0.$$

Similarly, the anti-poly-Bergman space $\tilde{\mathcal{A}}_n^2(\Pi)$ consists of all functions in $L_2(\Pi)$ satisfying the equation $(\partial/\partial z)^n \varphi = 0$. Introduce the true-poly-Bergman and true-anti-poly-Bergman spaces as follows:

$$\mathcal{A}_{(n)}^2(\Pi) = \mathcal{A}_n^2(\Pi) \ominus \mathcal{A}_{n-1}^2(\Pi),$$
$$\tilde{\mathcal{A}}_{(n)}^2(\Pi) = \tilde{\mathcal{A}}_n^2(\Pi) \ominus \tilde{\mathcal{A}}_{n-1}^2(\Pi),$$

where $\mathcal{A}_0^2(\Pi) = \tilde{\mathcal{A}}_0^2(\Pi) = \{0\}$. Of course $\mathcal{A}_1^2(\Pi) = \mathcal{A}_{(1)}^2(\Pi)$ is the usual Bergman space.

Poly-Bergman spaces are related to Laguerre functions as shown below. Recall that the Laguerre polynomial $L_n(y)$ of degree n and type 0 is defined by

$$L_n(y) = \frac{e^y}{n!} \frac{d^n}{dy^n}(e^{-y} y^n), \quad n = 0, 1, 2, \ldots$$

The system of Laguerre functions

$$l_n(y) = (-1)^n e^{-y/2} L_n(y), \quad n = 0, 1, 2, \ldots$$

form an orthonormal basis for $L_2(\mathbb{R}_+)$. For $n = 0, 1, \ldots$, denote by L_n the one-dimensional space generated by $l_n(y)$. Further, define

$$L_n^{\oplus} = \bigoplus_{k=0}^{n} L_k.$$

The one-dimensional projection $P_{(n)}$ from $L_2(\mathbb{R}_+)$ onto L_n is given by $(P_{(n)}\phi)(y) = \langle \phi, l_n\rangle \cdot l_n(y)$. Thus, $P_n = P_{(0)} \oplus \cdots \oplus P_{(n)}$ is the orthogonal projection from $L_2(\mathbb{R}_+)$ onto $L_n^{\oplus}$, and

$$(P_n\phi)(y) = \sum_{k=0}^{n} \langle \phi, l_k\rangle \cdot l_k(y) = \sum_{k=0}^{n} l_k(y) \int_{\mathbb{R}_+} \phi(v) l_k(v) dv.$$

Let $B_{\Pi,(n)}$ and $B_{\Pi,n}$ be the orthogonal projections from $L_2(\Pi)$ onto $\mathcal{A}^2_{(n)}(\Pi)$ and $\mathcal{A}^2_n(\Pi)$, respectively.

Theorem 2.2 (Vasilevski [10]) *The unitary operator $U = U_2 U_1$ gives an isometric isomorphism of the space $L_2(\Pi)$, under which*

1. *The true-poly-Bergman space $\mathcal{A}^2_{(n)}(\Pi)$ is mapped onto $L_2(\mathbb{R}_+) \otimes L_{n-1}$.*
2. *The true-poly-Bergman projection $B_{\Pi,(n)}$ is unitary equivalent to the following one:*

$$U B_{\Pi,(n)} U^{-1} = \chi_+(x) I \otimes P_{(n-1)}.$$

3. *The poly-Bergman space $\mathcal{A}^2_n(\Pi)$ is mapped onto $L_2(\mathbb{R}_+) \otimes L^{\oplus}_{n-1}$.*
4. *The poly-Bergman projection $B_{\Pi,n}$ is unitary equivalent to the following one:*

$$U B_{\Pi,n} U^{-1} = \chi_+(x) I \otimes P_{n-1}.$$

Introduce the isometric embedding

$$R_{0,(n)} : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

by the rule

$$(R_{0,(n)} f)(x, y) = \chi_+(x) f(x) l_{n-1}(y).$$

Of course the adjoint operator $R^*_{0,(n)} : L_2(\Pi) \to L_2(\mathbb{R}_+)$ is given by

$$(R^*_{0,(n)}\varphi)(x) = \chi_+(x) \int_{\mathbb{R}_+} \varphi(x, v) l_{n-1}(v) dv.$$

Since the image of $R_{0,(n)}$ is the space $U(\mathcal{A}^2_{(n)}(\Pi)) = L_2(\mathbb{R}_+) \otimes L_{n-1}$, we have

$$R^*_{0,(n)} R_{0,(n)} = I : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$$

and

$$R_{0,(n)} R^*_{0,(n)} = \chi_+(x) I \otimes P_{(n-1)} : L_2(\Pi) \to L_2(\mathbb{R}_+) \otimes L_{n-1}.$$

On the other hand, we introduce the operator

$$R_{(n)} = R^*_{0,(n)} U$$

which maps $L_2(\Pi)$ onto $L_2(\mathbb{R}_+)$, and its restriction to $\mathcal{A}^2_{(n)}(\Pi)$ is an isometric isomorphism. Thus, the adjoint operator $R^*_{(n)} = U^* R_{0,(n)}$ is an isometric isomorphism from $L_2(\mathbb{R}_+)$ onto the subspace $\mathcal{A}^2_{(n)}(\Pi)$. The operator $R^*_{(n)}$ plays the same role as the Bargmann transform does in the Segal–Bargmann space [10]. Thus we have

$$R^*_{(n)} R_{(n)} = B_{\Pi,(n)} : L_2(\Pi) \to \mathcal{A}^2_{(n)}(\Pi)$$

and

$$R_{(n)} R^*_{(n)} = I : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+).$$

Similarly, introduce the isometric embedding

$$R_{0,n} : (L_2(\mathbb{R}_+))^n \to L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

by the rule

$$\begin{aligned}(R_{0,n} f)(x,y) &= \sum_{k=1}^{n} \chi_+(x) f_k(x) l_{k-1}(y) \\ &= \chi_+(x)[N_n(y)]^T f(x),\end{aligned}$$

where $f = (f_1, \dots, f_n)^T$,

$$N_n(y) = (l_0(y), \dots, l_{n-1}(y))^T,$$

and the super-script T means that we are taking the transpose matrix. Further, the adjoint operator $R^*_{0,n} : L_2(\Pi) \to (L_2(\mathbb{R}_+))^n$ is given by

$$\begin{aligned}(R^*_{0,n}\varphi)(x) &= \left(\chi_+(x) \int_{\mathbb{R}_+} \varphi(x,y) l_0(y) dy, \dots, \chi_+(x) \int_{\mathbb{R}_+} \varphi(x,y) l_{n-1}(y) dy\right)^T \\ &= \chi_+(x) \int_{\mathbb{R}_+} \varphi(x,y) N_n(y) dy.\end{aligned}$$

Since the image of $R_{0,n}$ is the space $U(\mathcal{A}^2_n(\Pi)) = L_2(\mathbb{R}_+) \otimes L^{\oplus}_{n-1}$, we have

$$R^*_{0,n} R_{0,n} = I : (L_2(\mathbb{R}_+))^n \to (L_2(\mathbb{R}_+))^n$$

and

$$R_{0,n} R^*_{0,n} = \chi_+ I \otimes P_{n-1} : L_2(\Pi) \to L_2(\mathbb{R}_+) \otimes L^{\oplus}_{n-1}.$$

Now the operator

$$R_n := R_{0,n}^* U \tag{2.3}$$

maps $L_2(\Pi)$ onto $(L_2(\mathbb{R}_+))^n$, and its restriction to $\mathcal{A}_n^2(\Pi)$ is an isometric isomorphism. Furthermore, the adjoint operator $R_n^* = U^* R_{0,n}$ is an isometric isomorphism from $(L_2(\mathbb{R}_+))^n$ onto the space $\mathcal{A}_n^2(\Pi)$. Thus

$$R_n^* R_n = B_{\Pi,n} : L_2(\Pi) \to \mathcal{A}_n^2(\Pi)$$

and

$$R_n R_n^* = I : (L_2(\mathbb{R}_+))^n \to (L_2(\mathbb{R}_+))^n.$$

3 Toeplitz Operators with Vertical Symbol

In this section we introduce a certain class of Toeplitz operators acting on the poly-Bergman spaces, and we prove that they are unitarily equivalent to multiplication operators by continuous matrix-valued functions on $(0, \infty)$. Let $a(z) = a(y)$ be a function in $L_\infty(\Pi)$ depending only on $y = \operatorname{Im} z$. We shall say that $a(z) = a(y)$ is a vertical symbol. The Toeplitz operator acting on $\mathcal{A}^2(\Pi)$ with symbol $a(y)$ is the operator defined by

$$T_a : \varphi \in \mathcal{A}^2(\Pi) \longmapsto B_\Pi(a\varphi) \in \mathcal{A}^2(\Pi).$$

Theorem 3.1 (Vasilevski [12]) *For any $a(y) \in L_\infty(\Pi)$, the Toeplitz operator T_a acting on $\mathcal{A}^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_a(x)I = R_0 T_a R_0^*$ acting on $L_2(\mathbb{R}_+)$, where R_0 is defined in (2.3). The function γ_a is given by*

$$\gamma_a(x) = \int_{\mathbb{R}_+} a\left(\frac{y}{2|x|}\right) e^{-y} dy.$$

Our aim is to generalize this known result for Toeplitz operators acting on the poly-Bergman spaces. The Toeplitz operator acting on $\mathcal{A}_{(n)}^2(\Pi)$ with symbol $a(z) = a(y)$ is the operator

$$T_{(n),a} : \varphi \in \mathcal{A}_{(n)}^2(\Pi) \longmapsto B_{\Pi,(n)}(a\varphi) \in \mathcal{A}_{(n)}^2(\Pi).$$

Theorem 3.2 *For any $a(y) \in L_\infty(\Pi)$, the Toeplitz operator $T_{(n),a}$ acting on $\mathcal{A}_{(n)}^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_{(n),a}I = R_{(n)} T_{(n),a} R_{(n)}^*$ acting on $L_2(\mathbb{R}_+)$, where the function $\gamma_{(n),a}$ is given by*

$$\gamma_{(n),a}(x) = \int_{\mathbb{R}_+} a\left(\frac{y}{2|x|}\right) (l_{n-1}(y))^2 dy. \tag{3.1}$$

Proof We have

$$
\begin{aligned}
R_{(n)} T_{(n),a} R_{(n)}^{*} &= R_{(n)} B_{\Pi,(n)} a B_{\Pi,(n)} R_{(n)}^{*} \\
&= R_{(n)} R_{(n)}^{*} R_{(n)} a R_{(n)}^{*} R_{(n)} R_{(n)}^{*} \\
&= (R_{(n)} R_{(n)}^{*}) R_{(n)} a R_{(n)}^{*} (R_{(n)} R_{(n)}^{*}) \\
&= R_{(n)} a R_{(n)}^{*} \\
&= R_{0,(n)}^{*} U_2 U_1 a(y) U_1^{-1} U_2^{-1} R_{0,(n)} \\
&= R_{0,(n)}^{*} U_2 a(y) U_2^{-1} R_{0,(n)} \\
&= R_{0,(n)}^{*} a\left(\frac{y}{2|x|}\right) R_{0,(n)}.
\end{aligned}
$$

Finally

$$\left(R_{0,(n)}^{*} a\left(\frac{y}{2|x|}\right) R_{0,(n)} f\right)(x) = \int_{\mathbb{R}_+} a\left(\frac{y}{2|x|}\right) f(x)(l_{n-1}(y))^2 dy = \gamma_{(n),a}(x) \cdot f(x).$$

□

The Toeplitz operator acting on $\mathcal{A}_n^2(\Pi)$ with vertical symbol $a(z) = a(y)$ is the operator

$$T_{n,a} : \varphi \in \mathcal{A}_n^2(\Pi) \longmapsto B_{\Pi,n}(a\varphi) \in \mathcal{A}_n^2(\Pi).$$

Theorem 3.3 *For any $a(y) \in L_\infty(\Pi)$, the Toeplitz operator $T_{n,a}$ acting on $\mathcal{A}_n^2(\Pi)$ is unitary equivalent to the matrix multiplication operator $\gamma^{n,a}(x)I = R_n T_{n,a} R_n^{*}$ acting on $(L_2(\mathbb{R}_+))^n$, where the matrix-valued function $\gamma^{n,a} = (\gamma_{ij}^{n,a})$ is given by*

$$\gamma^{n,a}(x) := \int_{\mathbb{R}_+} a\left(\frac{y}{2|x|}\right) N_n(y)[N_n(y)]^T \, dy, \tag{3.2}$$

that is,

$$\gamma_{ij}^{n,a}(x) = \int_{\mathbb{R}_+} a\left(\frac{y}{2|x|}\right) l_{i-1}(y)\, l_{j-1}(y)\, dy, \tag{3.3}$$

for $i, j = 1, \ldots, n$.

Proof We have

$$
\begin{aligned}
R_n T_{n,a} R_n^{*} &= R_n B_{\Pi,n} a B_{\Pi,n} R_n^{*} \\
&= R_n R_n^{*} R_n a R_n^{*} R_n R_n^{*} \\
&= (R_n R_n^{*}) R_n a R_n^{*} (R_n R_n^{*}) \\
&= R_n a R_n^{*} \\
&= R_{0,n}^{*} U_2 U_1 a(y) U_1^{-1} U_2^{-1} R_{0,n} \\
&= R_{0,n}^{*} U_2 a(y) U_2^{-1} R_{0,n} \\
&= R_{0,n}^{*} a\left(\frac{y}{2|x|}\right) R_{0,n}.
\end{aligned}
$$

For $f = (f_1, \ldots, f_n)^T \in (L_2(\mathbb{R}_+))^n$,

$$\begin{aligned}
\left[R_{0,n}^* a\left(\frac{y}{2|x|}\right) R_{0,n} f \right](x) &= R_{0,n}^* \left(a\left(\frac{y}{2|x|}\right) \chi_+(x)[N_n(y)]^T f(x) \right) \\
&= \chi_+(x) \int_{\mathbb{R}_+} \left[a\left(\frac{y}{2|x|}\right) [N_n(y)]^T f(x) \right] N_n(y)\, dy \\
&= \chi_+(x) \int_{\mathbb{R}_+} a\left(\frac{y}{2|x|}\right) N_n(y)[N_n(y)]^T f(x)\, dy \\
&= \chi_+(x)\, \gamma^{n,a}(x) f(x).
\end{aligned}$$

Finally, it is easy to see that each component of $\gamma^{n,a}$ is given by (3.3). □

The component function (3.3) is bounded because of the Cauchy–Schwarz inequality. Further

$$\gamma_{ij}^{n,a}(x) = 2|x| \int_{\mathbb{R}_+} a(y)\, l_{i-1}(2|x|y)\, l_{j-1}(2|x|y)\, dy.$$

Thus, the continuity of $l_{i-1}(2xy)l_{j-1}(2xy)$ implies the continuity of $\gamma_{ij}^{n,a}(x)$ on $(0, \infty)$.

4 C^*-Algebra Generated by Toeplitz Operators

Denote by $L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$ the closed subspace of $L_\infty(\mathbb{R}_+)$ which consists of all functions having limit values at the "endpoints" 0 and $+\infty$, i.e., for each $a \in L_\infty^{\{0,\infty\}}(\mathbb{R}_+)$ the following limits exist

$$\lim_{y \to 0} a(y) = a(0) \quad \text{and} \quad \lim_{y \to +\infty} a(y) = a(+\infty).$$

We will identify the functions $a \in L_\infty^{\{0,\infty\}}(\mathbb{R}_+)$ with their extensions $a(z) = a(y)$ to the upper half-plane Π, where $y = \operatorname{Im} z$. We shall say that $a \in L_\infty^{\{0,\infty\}}(\mathbb{R}_+)$ is a vertical symbol.

In this section we study the C^*-algebra generated by all the Toeplitz operators on $\mathcal{A}_n^2(\Pi)$ with such vertical symbols.

Lemma 4.1 *Take a vertical function $a(y) \in L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$, and let*

$$\begin{aligned}
a_0 &= \lim_{y \to 0^+} a(y), \\
a_\infty &= \lim_{y \to +\infty} a(y).
\end{aligned}$$

Then the matrix-valued function $\gamma^{n,a}(x)$ *satisfies*

$$a_\infty I = \lim_{x\to 0^+} \gamma^{n,a}(x),$$
$$a_0 I = \lim_{x\to +\infty} \gamma^{n,a}(x).$$

Proof We will calculate the limit value of each entry of $\gamma^{n,a}$. Consider $C_{ij} = \int_0^\infty |l_{i-1}(y)l_{j-1}(y)|dy$. Take $\epsilon > 0$. Then there exists $y_0 > 0$ such that $\int_{y_0}^\infty |l_{i-1}(y) l_{j-1}(y)|dy < \epsilon$. Assume that $a_0 = 0$. Let δ be a positive number such that $|a(t)| < \epsilon$ for $0 < t < \delta$. Then

$$\begin{aligned} |\gamma_{ij}^{n,a}(x)| &\le \int_0^{y_0} \left| a\left(\frac{y}{2x}\right) l_{i-1}(y)l_{j-1}(y)\right| dy + \int_{y_0}^\infty \left| a\left(\frac{y}{2x}\right) l_{i-1}(y)l_{j-1}(y)\right| dy \\ &\le C_{ij} \max_{0<y<y_0} \left| a\left(\frac{y}{2x}\right)\right| + \|a\|_\infty \epsilon. \end{aligned}$$

Let $N = y_0/(2\delta)$. We have $|y/(2x)| < \delta$ for $x > N$ and $y \in (0, y_0)$. Thus $|\gamma_{ij}^{n,a}(x)| \le (C_{ij} + \|a\|_\infty)\epsilon$ for $x > N$. We have proved that $\lim_{x\to+\infty} \gamma_{ij}^{na}(x) = 0$. If $a_0 \neq 0$, take $b(y) = a(y) - a_0$. Then

$$\begin{aligned} \lim_{x\to+\infty} \gamma_{ij}^{n,a}(x) &= \lim_{x\to+\infty} \gamma_{ij}^{n,b}(x) + \lim_{x\to+\infty} \gamma_{ij}^{n,a_0}(x) \\ &= a_0 \int_0^\infty l_{i-1}(y)l_{j-1}(y)dy \\ &= a_0\delta_{ij}. \end{aligned}$$

The proof of the equality $\lim_{x\to 0} \gamma_{ij}^{n,a}(x) = a_\infty\delta_{ij}$ is similar. □

Let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ matrices with complex entries. Let $\mathfrak{C} = M_n(\mathbb{C}) \otimes C[0, \infty]$, and let $\mathfrak{D}$ be the C^*-subalgebra of $\mathfrak{C}$ given by

$$\mathfrak{D} = \{M \in \mathfrak{C} :\ M(0), M(\infty) \in \mathbb{C}I\}.$$

Let $\mathfrak{B}$ be the C^*-subalgebra of $\mathfrak{D}$ generated by all the matrix-valued functions $\gamma^{n,a}(x)$, with $a \in L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$. Obviously $\mathfrak{B}$ is isomorphic to the C^*-algebra generated by all the Toeplitz operators $T_{n,a}$. We will prove that $\mathfrak{B} = \mathfrak{D}$ by using a Stone–Weierstrass theorem [4]. Actually, we are going to prove that $\mathfrak{B}$ separates all the pure states of $\mathfrak{D}$. We know that $\mathfrak{D}$ is a C*-bundle, and the set of its pure states is given by the pure states on the fibers $\mathfrak{D}(x_0) = \{M(x_0) :\ M \in \mathfrak{D}\}$, where $x_0 \in [0, \infty]$. See [2] for more details. Thus, each pure state of $\mathfrak{D}$ has the form

$$f(M) = f_{x_0}(M(x_0)), \quad M \in \mathfrak{D},$$

where $x_0 \in [0, \infty]$, and f_{x_0} is a pure state of $\mathfrak{D}(x_0)$. Of course $\mathfrak{D}(x_0) = M_n(\mathbb{C})$ for $x_0 \in (0, \infty)$, whereas $\mathfrak{D}(x_0) = \mathbb{C}I$ for $x_0 = 0, \infty$.

For a matrix $A = (a_{ij}) \in M_n(\mathbb{C})$, let $\operatorname{tr} A$ denote the trace of A, that is, $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$. Further, consider the following linear functional on $M_n(\mathbb{C})$ associated to A:

$$f_A(Q) = \sum_{i,j=1}^{n} a_{ij} q_{ij}, \quad Q = (q_{ij}) \in M_n(\mathbb{C}).$$

Consider the set of all positive matrices of trace 1:

$$M_+^{tr=1} = \{A = (a_{ij}) \in M_n(\mathbb{C}) |\ A \geq 0 \text{ and } \operatorname{tr} A = 1\}.$$

Theorem 4.2 (Lee [5]) *Let St_n denote the set of all states of the matrix algebra $M_n(\mathbb{C})$. We have*

1. *The functional f_A belongs to St_n if and only if A belongs to $M_+^{tr=1}$. The mapping $A \mapsto f_A$ is a one-to-one correspondence from $M_+^{tr=1}$ onto $St(M_n)$.*
2. *The functional f_A is a pure state of $M_n(\mathbb{C})$ if and only if $A \in M_+^{tr=1}$ is an orthogonal projection with rank* 1. *In such a case, there exists an unit vector $v \in \mathbb{C}^n$ such that $A = \overline{v} v^T$, and*

$$f_A(Q) = \langle Qv, v \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{C}^n$.

Thus, the set of pure states of $\mathfrak{D}$ consists of all functionals having the form

$$f_{x_0,v}(M) = \langle M(x_0)v, v \rangle, \quad M \in \mathfrak{D},$$

where $x_0 \in (0, \infty)$ and v is a unit vector in $\mathbb{C}^n$, or $x_0 = 0, \infty$ and $v = (1, 0, \dots, 0)^T$. That is, $\mathfrak{D}(x_0)$ has only one state for $x_0 = 0, \infty$.

Let F_0, F_∞ be the (pure) states of $\mathfrak{D}(0)$ and $\mathfrak{D}(\infty)$, respectively. If $a(z)$ is any vertical symbol in $L_\infty^{\{0,+\infty\}}(\mathbb{R}_+)$ satisfying $a_0 \neq a_\infty$, then $F_0(\gamma^{n,a}(x)) = a_\infty$ and $F_\infty(\gamma^{n,a}(x)) = a_0$. Thus, F_0 and F_∞ are separated by $\gamma^{n,a}(x)$. In the general case, we will separate pure states using vertical symbols of the form $c(z) = \chi_{[\alpha,\beta]}(y)$, for which

$$\begin{aligned} \gamma^{n,c}(x) &= \int_{\mathbb{R}_+} c\left(\frac{y}{2|x|}\right) N_n(y)[N_n(y)]^T \, dy \\ &= \int_{2\alpha x}^{2\beta x} N_n(y)[N_n(y)]^T \, dy. \end{aligned}$$

Let $v \in \mathbb{C}^n$ be a unit vector. Consider the function $h_v(y) = |\langle v, N_n(y) \rangle|^2$. We have $h_v(y) = q_v(y)e^{-y}$, where

$$q_v(y) = |v_0 L_0(y) + \cdots + v_{n-1}(-1)^{n-1} L_{n-1}(y)|^2$$

is a polynomial of degree at most $2n-2$ taking non-negative values. Of course, there exists $K_v > 0$ such that $h_v(y)$ is decreasing on $[K_v, \infty)$.

Lemma 4.3 *Let $v \in \mathbb{C}^n$ be a unit vector, and $\gamma^{n,c}(x)$ be the symbol of the Toeplitz operator $T_{n,c}$, where $c = \chi_{[\alpha,\beta]}$. Let K_v be a positive real number such that $h_v(y)$ is decreasing on $[K_v, \infty)$. If $0 < x_0 < x_1 < \infty$ and α, β satisfy $K_v < 2\alpha x_0$ and $\frac{x_0}{x_1} < \frac{\alpha}{\beta} < 1$, then $\gamma^{n,c}(x)$ separates the pure state $f_{x_0,v}$ from $f_{x_1,v}$.*

Proof We have

$$\begin{aligned} f_{x_0,v}(\gamma^{n,c}(x)) &= \langle \gamma^{n,c}(x_0)v, v\rangle, \\ &= \left\langle \int_{2\alpha x_0}^{2\beta x_0} N_n(y)[N_n(y)]^T\, dy\ v, v \right\rangle \\ &= \int_{2\alpha x_0}^{2\beta x_0} \langle N_n(y)[N_n(y)]^T v, v\rangle\, dy \\ &= \int_{2\alpha x_0}^{2\beta x_0} \langle v, N_n(y)\rangle \langle N_n(y), v\rangle\, dy \\ &= \int_{2\alpha x_0}^{2\beta x_0} |\langle v, N_n(y)\rangle|^2\, dy. \end{aligned}$$

From $\frac{x_0}{x_1} < \frac{\alpha}{\beta} < 1$ we get $K_v < 2\alpha x_0 < 2\beta x_0 < 2\alpha x_1 < 2\beta x_1$. Thus, the matrix-valued function $\gamma^{n,c}(x)$ separates the pure states $f_{x_0,v}$, $f_{x_1,v}$:

$$\begin{aligned} f_{x_0,v}(\gamma^{n,c}(x)) &= \int_{2\alpha x_0}^{2\beta x_0} h_v(y)\, dy \\ &> \int_{2\alpha x_1}^{2\beta x_1} h_v(y)\, dy \\ &= f_{x_1,v}(\gamma^{n,c}(x)). \end{aligned}$$

□

Consider $c(z) = \chi_{[\alpha,\beta]}(y)$ and $x_0 \in (0, \infty)$. Then the function $\gamma^{n,c}(x)$ separates $f_{x_0,v}$ from F_0 and F_∞ because $f_{x_0,v}(\gamma^{n,c}(x)) > 0$ and $F_0(\gamma^{n,c}(x)) = F_\infty(\gamma^{n,c}(x)) = 0$.

Lemma 4.4 below says that the functions $\gamma^{n,c}(x)$ separates the pure states $f_{x_0,v}$, $f_{x_1,w}$ from each other if $x_0 \neq x_1$, with $x_0, x_1 \in (0, \infty)$.

Lemma 4.4 *Let $v, w \in \mathbb{C}^n$ be unit vectors, and $x_0, x_1 \in (0, \infty)$. Let $\gamma^{n,c}(x)$ be the symbol of the Toeplitz operator $T_{c,n}$, where $c = \chi_{[\alpha,\beta]}$. Suppose that*

$$f_{x_0,v}(\gamma^{n,c}(x)) = f_{x_1,w}(\gamma^{n,c}(x)), \quad for\ all\ 0 < \alpha < \beta < \infty.$$

Then $x_0 = x_1$ and

$$|\langle v, N_n(y)\rangle|^2 = |\langle w, N_n(y)\rangle|^2 \quad for\ all\ y \in (0, \infty).$$

Proof The assumption $f_{x_0,v}(\gamma^{n,c}(x)) = f_{x_1,w}(\gamma^{n,c}(x))$ means

$$\int_{2\alpha x_0}^{2\beta x_0} h_v(y)\,dy = \int_{2\alpha x_1}^{2\beta x_1} h_w(y)\,dy.$$

Take the derivative with respect to β in both sides of this equation:

$$\begin{aligned} 2x_0\, h_v(2\beta x_0) &= 2x_1\, h_w(2\beta x_1), \quad \forall\beta \\ x_0\, q_v(2\beta x_0)e^{-2\beta x_0} &= x_1\, q_w(2\beta x_1)e^{-2\beta x_1}, \quad \forall\beta \\ x_0\, q_v(2\beta x_0) &= x_1\, q_w(2\beta x_1)e^{2\beta(x_0-x_1)}, \quad \forall\beta. \end{aligned}$$

Since q_v, q_w are polynomials, the exponential growth behavior in both sides of this equation implies that $x_0 = x_1$. Therefore $q_v(2\beta x_0) = q_w(2\beta x_0)$ for all $\beta > 0$, which means that the polynomials q_v, q_w are equal to each other. Thus $h_v(y) = h_w(y)$, that is, $|\langle v, N_n(y)\rangle|^2 = |\langle w, N_n(y)\rangle|^2 \quad \forall y$. □

Lemma 4.5 *Let $y_1, \dots, y_n$ be positive real numbers different from each other. Then the set $\{N_n(y_k)\}_{k=1}^n$ is a basis for $\mathbb{C}^n$. Actually, the determinant of the matrix $N = [N_n(y_1), \dots, N_n(y_n)]^T$ is given by*

$$\det N = \frac{e^{-(y_1+\cdots+y_n)/2}}{\prod_{k=1}^{n-1} k!} \prod_{1\le i<j\le n} (y_j - y_i).$$

Proof Since $N_n(y) = e^{-y/2}(L_0(y), -L_1(y), \dots, (-1)^{n-1}L_{n-1}(y))^T$ we have

$$N = D \begin{pmatrix} L_0(y_1) & -L_1(y_1) & \cdots & (-1)^{n-1}L_{n-1}(y_1) \\ L_0(y_2) & -L_1(y_2) & \cdots & (-1)^{n-1}L_{n-1}(y_2) \\ \vdots & \vdots & & \vdots \\ L_0(y_n) & -L_1(y_n) & \cdots & (-1)^{n-1}L_{n-1}(y_n) \end{pmatrix},$$

where $D = \operatorname{diag}\{e^{-y_1/2}, \dots, e^{-y_n/2}\}$. Since $1/k!$ is the leading coefficient of $(-1)^k L_k(y)$,

$$(-1)^k L_k(y) = \frac{1}{k!}(y^k + \text{lower degree terms})$$

Therefore the determinant of N has the form

$$\det N = \frac{\det D}{\prod_{k=0}^{n-1} k!} \begin{vmatrix} 1 & y_1 + a_0 & y_1^2 + b_1 y_1 + b_0 & \cdots & y_1^{n-1} + \cdots \\ 1 & y_2 + a_0 & y_2^2 + b_1 y_2 + b_0 & \cdots & y_2^{n-1} + \cdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & y_n + a_0 & y_n^2 + b_1 y_n + b_0 & \cdots & y_n^{n-1} + \cdots \end{vmatrix}.$$

Applying the linearity of the determinant with respect to the second column, and then with respect to the third column, and so on, we get

$$\det N = \frac{\det D}{\prod_{k=0}^{n-1} k!} \begin{vmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{n-1} \\ 1 & y_2 & y_2^2 & \cdots & y_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & y_n & y_n^2 & \cdots & y_n^{n-1} \end{vmatrix}.$$

We have here the Vandermonde determinant, it is well known its value. $\square$

Next lemma completes the separation of all pure states.

Lemma 4.6 *Let $v, w \in \mathbb{C}^n$ be unit vectors, and $x_0 \in (0, \infty)$. Let $\gamma^{n,a}(x)$ and $\gamma^{n,b}(x)$ be the symbols of the Toeplitz operators $T_{n,a}$ and $T_{n,b}$, respectively, where $a = \chi_{[0,\alpha]}$ and $b = \chi_{[0,\beta]}$. Suppose that*

$$f_{x_0,v}(\gamma^{n,a}(x)\gamma^{n,b}(x)) = f_{x_0,w}(\gamma^{n,a}(x)\gamma^{n,b}(x)), \quad \forall \alpha, \beta \in [0, \infty]. \tag{4.1}$$

Then $v = \lambda w$, where λ is a uni-modular complex number, that is, $f_{x_0,v} = f_{x_0,w}$.

Proof Note that $\gamma^{n,b}(x) = I$ if $\beta = \infty$. Thus $f_{x_0,v}(\gamma^{n,a}(x)) = f_{x_0,w}(\gamma^{n,a}(x))$ $\forall \alpha \in (0, \infty)$. Since $c = \chi_{[\alpha,\beta]} = \chi_{[0,\beta]} - \chi_{[0,\alpha]}$ for $\alpha < \beta$ we have $f_{x_0,v}(\gamma^{n,c}(x)) = f_{x_0,w}(\gamma^{n,c}(x))$. According to Lemma 4.4, we have $|\langle v, N_n(y)\rangle|^2 = |\langle w, N_n(y)\rangle|^2 \quad \forall y$. Therefore

$$\langle v, N_n(y)\rangle = e^{i\theta(y)}\langle w, N_n(y)\rangle \quad \forall\, y,$$

where $\theta(y)$ is a certain function. On the other hand, the assumption (4.1) means that

$$\langle \gamma^{n,a}(x_0)\gamma^{n,b}(x_0)v, v\rangle = \langle \gamma^{n,a}(x_0)\gamma^{n,b}(x_0)w, w\rangle. \tag{4.2}$$

Since

$$\gamma^{n,a}(x_0) = \int_0^{2\alpha x_0} N_n(y)[N_n(y)]^T\, dy,$$

the left-hand side of (4.2) equals

$$\overline{v}^T \left(\int_0^{2\alpha x_0} N_n(y)[N_n(y)]^T\, dy\right) \left(\int_0^{2\beta x_0} N_n(t)[N_n(t)]^T\, dt\right) v.$$

Without lost of generality we can assume that $x_0 = 1/2$. By taking partial derivatives with respect to α and β in the left-hand side of (4.2), we get

$$\overline{v}^T \left(N_n(\alpha)[N_n(\alpha)]^T\right) \left(N_n(\beta)[N_n(\beta)]^T\right) v.$$

We have $N_n(\beta) \neq 0$ for every positive number β according to Lemma 4.5. Thus, $e^{\alpha/2} N_n(\alpha)^T N_n(\beta)$ is a nonzero polynomial with respect α, and it could be the zero scalar for at most $n-1$ values of α. Therefore, if we take partial derivatives in both sides of (4.2), we obtain

$$\overline{v}^T N_n(\alpha)[N_n(\beta)]^T v = \overline{w}^T N_n(\alpha)[N_n(\beta)]^T w$$

or

$$\begin{aligned}
\overline{\langle v, N_n(\alpha)\rangle}\langle v, N_n(\beta)\rangle &= \overline{\langle w, N_n(\alpha)\rangle}\langle w, N_n(\beta)\rangle \\
\overline{\langle w, N_n(\alpha)\rangle}\langle w, N_n(\beta)\rangle e^{i\theta(\beta)-i\theta(\alpha)} &= \overline{\langle w, N_n(\alpha)\rangle}\langle w, N_n(\beta)\rangle
\end{aligned}$$

Thus $e^{i\theta(\beta)-i\theta(\alpha)} = 1$ for all α, β, which means $\langle v, N_n(y)\rangle = e^{i\theta_0}\langle w, N_n(y)\rangle$ for all $y \in (0, \infty)$ and some constant θ_0. If $u = v - e^{i\theta_0} w$, then $\langle u, N_n(y)\rangle = 0$. Take now n values for y in the last equation, then

$$\langle u, N_n(y_k)\rangle = 0, \quad k = 1, \dots, n.$$

Thus, u must be the zero vector because of $\{N_n(y_k)\}_{k=1}^n$ is a basis for $\mathbb{C}^n$. $\square$

We are going to describe now the C*-algebra generated by the Toeplitz operators. *The non-commutative Stone-Weierstrass conjecture* Let $\mathcal{B}$ be a C*-subalgebra of a C*-algebra $\mathcal{A}$, and suppose that $\mathcal{B}$ separates all the pure states of $\mathcal{A}$ (and 0 if $\mathcal{A}$ is non-unital). Then $\mathcal{A} = \mathcal{B}$.

Kaplansky [4] proved this conjecture for type I (or GCR) C*-algebras. Recall that $\mathcal{A}$ is a type I C*-algebra if $K \subset \pi(\mathcal{A})$ for every irreducible representation π of $\mathcal{A}$ on a Hilbert space H, where K is the ideal of all compact operators. In our case, $\mathfrak{D}$ is a type I C*-algebra, and $\mathfrak{B}$ separates the pure states of $\mathfrak{D}$.

Let $\mathcal{T}_{0\infty}^n$ be the C*-algebra generated by all the Toeplitz operators $T_{n,a}$ acting on the poly-Bergman space $\mathcal{A}_n^2(\Pi)$, with $a \in L_\infty^{\{0,\infty\}}(\mathbb{R}_+)$.

Theorem 4.7 *The C*-algebra $\mathcal{T}_{0\infty}^n$ is isomorphic and isometric to the C*-algebra $\mathfrak{D}$. The isomorphism is given by*

$$\mathcal{T}_{0\infty}^n : T_{n,a} \longmapsto (Sym\ T_{n,a})(x) = \gamma^{n,a}(x),$$

where $\gamma^{n,a}(x)$ is given in (3.2).

Let $\mathcal{T}_{0\infty}^{(n)}$ be the C*-algebra generated by all the Toeplitz operators $T_{(n),a}$ acting on the true-poly-Bergman space $\mathcal{A}_{(n)}^2(\Pi)$, with $a \in L_\infty^{\{0,\infty\}}(\mathbb{R}_+)$.

Theorem 4.8 *The C*-algebra $\mathcal{T}_{0\infty}^{(n)}$ is isomorphic and isometric to the commutative C*-algebra $C[0, \infty]$. The isomorphism is given by*

$$\mathcal{T}_{0\infty}^{(n)} : T_{(n),a} \longmapsto (Sym\ T_{(n),a})(x) = \gamma_{(n),a}(x),$$

where $\gamma_{(n),a}(x)$ is given in (3.1).

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