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Complex Analysis and Operator Theory

Landau Theorem For Planar Harmonic Mappings

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Abstract Let $w = P[F]$ be a harmonic mapping of the unit disk $\mathbb{D}$ with the boundary function F. By using Poisson formula, we obtain some better estimates on Bloch constants for planar harmonic mappings.

Keywords Landau theorem · Bloch constant · Harmonic mapping · Poisson formula

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1 Introduction

Let $H(\mathbb{D})$ denote the class of holomorphic functions in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Given a function $f \in H(\mathbb{D})$ define B_f to be the least upper bound of all numbers $b > 0$ such that there exists a number $z_0 \in \mathbb{C}$ and a region $\Omega \subseteq \mathbb{D}$ which is univalently mapped onto $\{z \in \mathbb{C} : |z - z_0| < b\}$ by f. The Bloch's constant B is defined as (cf. [2])

$$B := \inf\{B_f : f \in H(\mathbb{D}) \text{ and } f'(0) = 1\}.$$

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The best known bounds for B at present are (cf. [1,3])

$$\frac{\sqrt{3}}{4}+2\times 10^{-4}\leq B\leq \frac{\sqrt{3}-1}{2}\cdot \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)},$$

where Γ is the Gamma function. However, the exact value of B is still unknown.

Let $H_M(\mathbb{D})$ denote the class of functions $f\in H(\mathbb{D})$ with $|f(z)|<M$ for $z\in\mathbb{D}$. The classical Landau theorem states that if $f\in H_M(\mathbb{D})$ with $f(0)=f'(0)-1=0$, then f is univalent in the disk $|z|<\rho_0$ with $\rho_0=\frac{1}{M+\sqrt{M^2-1}}$ and $f(|z|<\rho_0)$ contains a disk $|w|<\sigma_0$ with $\sigma_0=M\rho_0^2$. This result is sharp (cf. [8]).

Define complex derivatives of $w(z)$ as follows:

$$w_z:=\frac{1}{2}\left(w_x-iw_y\right)\quad\text{and}\quad w_{\bar{z}}:=\frac{1}{2}\left(w_x+iw_y\right), \tag{1.1}$$

where $z=x+iy$. A complex-valued function w on $\mathbb{D}$ is harmonic if it is twice continuously differentiable and satisfies Laplace's equation:

$$\Delta w=4w_{z\bar{z}}=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0\ \text{ for }\ z\in\mathbb{D}.$$

We refer to the book of Duren [5,6] for good setting of harmonic mappings. Let

$$\Lambda_w(z)=\max_{0\leq\alpha\leq\pi}|e^{i\alpha}w_z(z)+e^{-i\alpha}w_{\bar{z}}(z)|=|w_z(z)|+|w_{\bar{z}}(z)|$$

and

$$\lambda_w(z)=\min_{0\leq\alpha\leq\pi}|e^{i\alpha}w_z(z)+e^{-i\alpha}w_{\bar{z}}(z)|=||w_z(z)|-|w_{\bar{z}}(z)||.$$

It is well known that $w(z)$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if its Jacobian satisfies the following condition.

$$J_w(z)=|w_z(z)|^2-|w_{\bar{z}}(z)|^2>0\quad\text{for } z\in\mathbb{D}.$$

We know from Poisson formula that every bounded harmonic mapping w defined in $\mathbb{D}$ has the representation

$$w(z)=P[F](z)=\int_0^{2\pi}P(r,t-\varphi)F(e^{it})\,dt,\quad z=re^{i\varphi}\in\mathbb{D}, \tag{1.2}$$

where F is the boundary function defined on the unit circle $\mathbb{T}:=\{z:\ |z|=1\}$ and

$$P(r,t-\varphi)=\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(t-\varphi)+r^2},$$

denote the Poisson kernel. In what follows we write $F(t)$ instead of $F(e^{it})$ for the boundary function (cf. [9,10]).

For harmonic mappings in $\mathbb{D}$, under suitable restriction we can obtain its Bloch and Landau theorems. For $r > 0$, let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. In [4], Chen et al. proved the Landau theorem for harmonic mappings as follows.

Theorem A *Let w be a harmonic mapping of $\mathbb{D}$ satisfies $w(0) = w_{\bar{z}}(0) = w_z(0) - 1 = 0$ and $|w(z)| < M$ for $z \in \mathbb{D}$. Then w is univalent in the disk $\mathbb{D}_{r_1}$, where*

$$r_1 = \frac{\pi^2}{16mM} \approx \frac{1}{11.105M} \tag{1.3}$$

and $w(\mathbb{D}_{r_1})$ contains a schlicht disk $\mathbb{D}_{R_1}$, where

$$R_1 = \frac{r_1}{2} \approx \frac{1}{22.21M}. \tag{1.4}$$

Here $m \approx 6.85$ is the minimum of the function $(3 - r^2)/(r(1 - r^2))$ for $0 < r < 1$.

By using sharp coefficients estimate, the authors in [7] improved a version of Landau theorem for the class of bounded harmonic mappings.

Theorem B *Let w be a harmonic mapping of $\mathbb{D}$ satisfies $w(0) = w_{\bar{z}}(0) = w_z(0)-1 = 0$ and $|w(z)| < M$ for $z \in \mathbb{D}$. Then w is close-to-convex (univalent) in the disk $\mathbb{D}_{r_2}$, where*

$$r_2 = 1 - \sqrt{\frac{4M}{4M + \pi}} \tag{1.5}$$

and $w(\mathbb{D}_{r_2})$ contains a schlicht disk $\mathbb{D}_{R_2}$, where

$$R_2 = r_2 - \frac{4M}{\pi}\frac{r_2^2}{1 - r_2}. \tag{1.6}$$

By using Poisson formula, we obtain better estimates on Bloch constants for planar harmonic mappings.

2 Auxiliary Results

Lemma 2.1 *Let $z = \rho e^{i\theta} \in \mathbb{D}$. Then*

$$\frac{1}{\pi}\int_0^{2\pi} \frac{|z||2 - e^{-it}z|}{|e^{it} - z|^2} dt = \frac{2\rho\sqrt{2 - \rho^2}}{1 - \rho^2}, \tag{2.1}$$

where $0 \le \rho < 1$ and $0 \le \theta \le 2\pi$.

Proof Let $\zeta = e^{it} \in \mathbb{T}$. According to the residue theorem, we see that

$$\begin{aligned}
\frac{1}{\pi}\int_0^{2\pi} \frac{|z||2-e^{-it}z|}{|e^{it}-z|^2}dt &= \frac{1}{\pi}\int_0^{2\pi} \frac{\rho\sqrt{4-4\rho\cos(\theta-t)+\rho^2}}{1-2\rho\cos(\theta-t)+\rho^2}dt \\
&= \frac{1}{\pi i}\oint_{|\zeta|=1} \frac{\rho\sqrt{(4+\rho^2)-2\rho e^{-i\theta}(e^{2i\theta}+\zeta^2)/\zeta}}{(1+\rho^2)\zeta-\rho e^{-i\theta}(e^{2i\theta}+\zeta^2)}d\zeta \\
&= \frac{1}{\pi i}\oint_{|\zeta|=1} \frac{\rho\sqrt{(4+\rho^2)-2\rho e^{-i\theta}(e^{2i\theta}+\zeta^2)/\zeta}}{-\rho e^{-i\theta}[\zeta-\rho e^{i\theta}][\zeta-(1/\rho)e^{i\theta}]}d\zeta \\
&= \frac{2\rho\sqrt{2-\rho^2}}{1-\rho^2}.
\end{aligned}$$

This completes the proof. □

Lemma 2.2 *Let* $0 \le \alpha \le 2\pi$ *and* $w = P[F]$ *be a harmonic mapping in* $\mathbb{D}$ *with the boundary function* F. *Then*

$$\begin{aligned}
&e^{i\alpha}w_z(z) + e^{-i\alpha}w_{\bar{z}}(z) \\
&\quad = e^{i\alpha}w_z(0) + e^{-i\alpha}w_{\bar{z}}(0) + \frac{1}{\pi}\int_0^{2\pi} \mathrm{Re}\left\{e^{i\alpha}\frac{2z-e^{-it}z^2}{(e^{it}-z)^2}\right\}F(t)dt, \qquad (2.2)
\end{aligned}$$

where $z = re^{i\theta} \in \mathbb{D}$.

Proof For each $z = re^{i\theta} \in \mathbb{D}$, it follows from (1.2) that

$$w(z) = P[F](z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}F(t)}{e^{it}-z}dt + \overline{\frac{1}{2\pi}\int_0^{2\pi} \frac{z\overline{F(t)}}{e^{it}-z}dt}. \qquad (2.3)$$

Then $w_z(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}F(t)}{(e^{it}-z)^2}dt$ and $w_{\bar{z}}(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-it}F(t)}{(e^{-it}-\bar{z})^2}dt$. Let $\alpha \in [0, 2\pi]$ be an arbitrary constant. Then

$$\begin{aligned}
e^{i\alpha}w_z(z) &= e^{i\alpha}w_z(0) + \frac{e^{i\alpha}}{2\pi}\int_0^{2\pi}\left[\frac{e^{it}}{(e^{it}-z)^2} - \frac{1}{e^{it}}\right]F(t)dt \\
&= e^{i\alpha}w_z(0) + \frac{e^{i\alpha}}{2\pi}\int_0^{2\pi}\frac{2z-z^2e^{-it}}{(e^{it}-z)^2}F(t)dt.
\end{aligned}$$

Similarly, we can obtain

$$e^{-i\alpha} w_{\bar{z}}(z) = e^{-i\alpha} w_{\bar{z}}(0) + \frac{e^{-i\alpha}}{2\pi} \int_0^{2\pi} \frac{2\bar{z} - \bar{z}^2 e^{it}}{(e^{-it} - \bar{z})^2} F(t)dt.$$

Equality (2.2) holds directly from the above two equalities. □

3 Main Results

Theorem 3.1 *Let $w = P[F]$ be a harmonic mapping of $\mathbb{D}$ satisfies $w(0) = w_{\bar{z}}(0) = w_z(0) - 1 = 0$ and $|w(z)| < M$ for $z \in \mathbb{D}$, where $M > 1$ and F is the boundary function. Then w is univalent in the disk $\mathbb{D}_{r_0}$, where*

$$r_0 = \sqrt{1 - \frac{2M}{\sqrt{1 + 4M^2}}} \tag{3.1}$$

is the root of the equation $1 - \frac{2Mr\sqrt{2-r^2}}{1-r^2} = 0$ and $w(\mathbb{D}_{r_0})$ contains a schlicht disk $\mathbb{D}_{R_0}$, where

$$R_0 = r_0 - \log \frac{\sqrt{2 - r_0^2} + 1}{\sqrt{2 - r_0^2} - 1} + 2\sqrt{2 - r_0^2} + 2\left[\log(\sqrt{2} + 1) - \sqrt{2}\right]. \tag{3.2}$$

Proof For $0 < r < 1$, take $z_1, z_2 \in \mathbb{D}_r$. Let $\ell : z(x) = z_1 + (z_2 - z_1)x = \rho(x)e^{i\theta(x)}$ be the segment line of z_1 and z_2, where $0 \le x \le 1$. Then

$$\begin{aligned} |w(z_1) - w(z_2)| &= \left| \int_\ell \left[w_z(z(x))z'(x) + w_{\bar{z}}(z(x))\overline{z'(x)} \right] dx \right| \\ &= |z_1 - z_2| \left| \int_0^1 \left[e^{i\alpha} w_z(z(x)) + e^{-i\alpha} w_{\bar{z}}(z(x)) \right] dx \right|, \end{aligned}$$

where $\alpha = \arg(z_1 - z_2) \in [0, 2\pi]$. Since $0 \le \rho(x) < r < 1$ and the function $\frac{s\sqrt{2-s^2}}{1-s^2}$ is an increasing function for $0 \le s < 1$, we see that

$$\int_0^1 \frac{\rho(x)\sqrt{2 - \rho^2(x)}}{1 - \rho^2(x)} dx \le \frac{r\sqrt{2 - r^2}}{1 - r^2}.$$

Applying $w_{\bar{z}}(0) = w_z(0) - 1 = 0$ together with (2.2) and Lemma 2.1 we see that

$$\begin{aligned}
&\left| \int_0^1 \left[e^{i\alpha} w_z(z(x)) + e^{-i\alpha} w_{\bar{z}}(z(x)) \right] dx \right| \\
&= \left| e^{i\alpha} + \frac{1}{\pi} \int_0^1 \left(\int_0^{2\pi} \mathrm{Re}\left[e^{i\alpha} \frac{2z(x) - e^{-it} z^2(x)}{(e^{it} - z(x))^2} \right] F(t) dt \right) dx \right| \\
&\geq 1 - \frac{M}{\pi} \int_0^1 \left(\int_0^{2\pi} \frac{|z(x)||2 - e^{-it} z(x)|}{|e^{it} - z(x)|^2} dt \right) dx \\
&= 1 - M \int_0^1 \frac{2\rho(x)\sqrt{2 - \rho^2(x)}}{1 - \rho^2(x)} dx \\
&\geq 1 - \frac{2Mr\sqrt{2 - r^2}}{1 - r^2}.
\end{aligned}$$

Hence,

$$|w(z_1) - w(z_2)| \geq |z_1 - z_2| \left(1 - \frac{2Mr\sqrt{2 - r^2}}{1 - r^2} \right). \tag{3.3}$$

Let $1 - \frac{2Mr\sqrt{2-r^2}}{1-r^2} = 0$. Then $r_0 = \sqrt{1 - \frac{2M}{\sqrt{1+4M^2}}}$. This shows that $w(z)$ is univalent in the disk $\mathbb{D}_{r_0}$. Let $z = r_0 e^{i\varphi} \in \partial\mathbb{D}_{r_0}$. Then according to (2.2) and Lemma 2.1, we have

$$\begin{aligned}
|w(r_0 e^{i\varphi})| &= \left| \int_0^{r_0} \left[w_z(xe^{i\varphi}) e^{i\varphi} + w_{\bar{z}}(xe^{i\varphi}) e^{-i\varphi} \right] dx \right| \\
&\geq r_0 - \int_0^{r_0} \frac{2x\sqrt{2 - x^2}}{1 - x^2} dx \\
&= r_0 - \log \frac{\sqrt{2 - r_0^2} + 1}{\sqrt{2 - r_0^2} - 1} + 2\sqrt{2 - r_0^2} + 2\left[\log(\sqrt{2} + 1) - \sqrt{2} \right] \\
&= R_0.
\end{aligned}$$

We see that $w(\mathbb{D}_{r_0})$ contains the schlicht disk $\mathbb{D}_{R_0}$.

The proof is completed. □

Remark 3.2 For $1 < M \leq 5.07$, we see that $r_2 < r_0$. Let

$$r_0^* = \begin{cases} r_0 & \text{if } 1 < M \leq 5.07 \\ r_2 & \text{if } M > 5.07. \end{cases}$$

Then $w(z)$ is univalent in the disk $D_{r_0^*}$, where $r_0^* \geq r_2 > r_1$. This shows that our Theorem 3.1 has improved the former results.

Theorem 3.3 *Let $w = P[F]$ be a harmonic mapping of $\mathbb{D}$ satisfies $w(0) = w_{\bar{z}}(0) = w_z(0) - 1 = 0$ and $|w(z)| < M$ for $z \in \mathbb{D}$, where $M > 1$ and F is the boundary function. Then*

$$\Lambda_w(re^{i\theta}) \leq 1 + \frac{2Mr\sqrt{2-r^2}}{1-r^2} \quad \textit{for } z = re^{i\theta} \in \mathbb{D} \tag{3.4}$$

and

$$\lambda_w(re^{i\theta}) \geq 1 - \frac{2Mr\sqrt{2-r^2}}{1-r^2} \quad \textit{for } z = re^{i\theta} \in \mathbb{D}. \tag{3.5}$$

Proof Let $z = re^{i\theta} \in \mathbb{D}$. According to (2.2) and Lemma 2.1, we have

$$\begin{aligned} \left|e^{i\alpha} w_z(z) + e^{-i\alpha} w_{\bar{z}}(z)\right| &= \left| e^{i\alpha} + \frac{1}{\pi} \int_0^{2\pi} \mathrm{Re} \left\{ e^{i\alpha} \frac{2z - e^{-it} z^2}{(e^{it} - z)^2} \right\} F(t) dt \right| \\ &\leq 1 + \frac{M}{\pi} \int_0^{2\pi} \frac{|z||2 - e^{-it} z|}{|e^{it} - z|^2} dt \\ &= 1 + \frac{2Mr\sqrt{2-r^2}}{1-r^2}. \end{aligned}$$

This shows that $\Lambda_w \leq 1 + \frac{2Mr\sqrt{2-r^2}}{1-r^2}$. Similarly,

$$\begin{aligned} \left|e^{i\alpha} w_z(z) + e^{-i\alpha} w_{\bar{z}}(z)\right| &\geq 1 - \frac{M}{\pi} \int_0^{2\pi} \frac{|z||2 - e^{-it} z|}{|e^{it} - z|^2} dt \\ &= 1 - \frac{2Mr\sqrt{2-r^2}}{1-r^2}. \end{aligned}$$

This implies that

$$\lambda_w(z) \geq 1 - \frac{2Mr\sqrt{2-r^2}}{1-r^2}.$$

It follows from (3.3) that $1 - \frac{2Mr\sqrt{2-r^2}}{1-r^2} > 0$ holds for $0 \leq r < r_0$.

The proof is completed. □

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