Complex Anal. Oper. Theory (2012) 6:987–1010
DOI 10.1007/s11785-011-0210-x

Complex Analysis and Operator Theory

Phase Derivative of Monogenic Signals in Higher Dimensional Spaces

Yan Yang · Tao Qian · Frank Sommen

Received: 19 February 2011 / Accepted: 29 November 2011 / Published online: 21 December 2011

Abstract In the Clifford algebra setting of a Euclidean space on the boundary of a domain it is natural to define a monogenic (analytic) signal to be the boundary value of a monogenic (analytic) function inside the domain. The question is how to define a canonical phase and, correspondingly, a phase derivative. In this paper we give an answer to these questions in the unit ball and in the upper-half space. Among the possible candidates of phases and phase derivatives we decided that the right ones are those that give rise to, as in the one dimensional signal case, the equal relations between the mean of the Fourier frequency and the mean of the phase derivative, and the positivity of the phase derivative of the shifted Cauchy kernel.

Keywords Monogenic signals · Frequency · Poisson kernel · Möbius transforms

Communicated by Daniele Struppa.

This project sponsored by the National Natural Science Funds for Young Scholars (No. 10901166), Sun Yat-Sen University Operating Costs of Basic Research Projects to Cultivate Young Teachers (No. 11lgpy99) and by Research Grant of University of Macau UL017/08-Y2/MAT/QT01/FST, and the FDCT project on Clifford and Harmonic Analysis (014/2008/A1).

Y. Yang (✉)
School of Mathematics and Computational Science, Sun Yat-Sen University, Guangzhou, China
e-mail: mathyy@sina.com

T. Qian
Department of Mathematics, Faculty of Science and Technology, University of Macau, Macao, China
e-mail: fsttq@umac.mo

F. Sommen
Faculty of Engineering, Ghent University, Ghent, Belgium
e-mail: furanek@msn.com

Mathematics Subject Classification (2000) 46F15 · 30G35

1 Introduction

In time-frequency analysis the concept analytic signals is introduced (see [11,20]). It is well-known that for a real-valued square integrable signal f in the whole time range, the function $f + \mathbf{i}\mathbf{H}f$, where $\mathbf{H}$ is the Hilbert transformation on the line, is the boundary value of an analytic function in the upper-half complex plane. The function $f + \mathbf{i}\mathbf{H}f$ is called the *analytic signal associated with* f. It is, in fact, two times of the boundary value of the Cauchy integral of f in the upper-half complex plane. In the periodic signals case, for a given function $f \in L^2(T)$, written in the form

$$f(e^{\mathbf{i}t}) = \sum_{k=-\infty}^{\infty} c_k e^{\mathbf{i}kt}, \quad t \in [0, 2\pi],$$

where

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(e^{\mathbf{i}t}) e^{-\mathbf{i}kt} dt,$$

the *circular Hilbert transform* $\tilde{\mathbf{H}}$ is given by

$$\tilde{\mathbf{H}}[f](e^{\mathbf{i}t}) = -\mathbf{i} \sum_{k=-\infty}^{\infty} \mathrm{sgn(k) c_k e^{\mathbf{i}kt}}, \tag{1.1}$$

where $\mathrm{sgn(k)} = 1$, if $k > 0$; 0, if $k = 0$; and -1, if $k < 0$.

The integral form of the circular Hilbert transform is given by

$$\tilde{\mathbf{H}}(f)(e^{\mathbf{i}t}) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_{|t-x|>0} \cot\left(\frac{t-x}{2}\right) f(e^{ix}) dx.$$

The function

$$f + \mathbf{i}\tilde{\mathbf{H}}f = c_0 + 2\sum_{k=1}^{\infty} c_k e^{\mathbf{i}kt}$$

is called the *analytic signal associated with* f. It is the boundary value of an analytic function inside the unit disc. Unlike the upper-half plane case, however, it is not a constant multiple of the boundary value of the Cauchy integral of f. In the upper-half plan case the Cauchy kernel and the Schwarz kernel coincide. On the circle the circular Hilbert transform does not correspond to the imaginary part of the Cauchy integral of f. In fact, it corresponds to the imaginary part of the Schwarz integral of f, that is the conjugate Poisson integral. In general, Hilbert transformation on the boundary of a domain is defined to be the operator that maps the scalar part to its non-scalar part of the boundary limit function of the analytic functions (usually the functions in the Hardy spaces) in the domain [3]. This concept is extended to higher dimensional spaces in the Clifford algebra setting [2,27].

From the definition of circular Hilbert transform, it is easy to see that $\tilde{\mathbf{H}}(c_0) = 0$, $\tilde{\mathbf{H}}[\cos(kt)] = \sin(kt)$, $\tilde{\mathbf{H}}[\sin(kt)] = -\cos(kt)$.

In the polar coordinate representation we have

$$f + \mathbf{i}\tilde{\mathbf{H}}f = A(t)[\cos\theta(t) + \mathbf{i}\sin\theta(t)] = A(t)e^{\mathbf{i}\theta(t)},$$

where $A(t) = \sqrt{f^2 + (\tilde{\mathbf{H}}f)^2}$ is called *instantaneous amplitude*, and $\theta(t) = \operatorname{arctg}\frac{\tilde{\mathbf{H}}f}{f}$ is called the *instantaneous phase*. The derivative of the phase, $\theta'(t)$, is usually defined to be *instantaneous frequency*.

In order to make the above defined "instantaneous frequency" a qualified concept complying with its physical meaning many researchers require the additional condition $\theta'(t) \geq 0$ a.e. [22,23], while some do not [11]. In the present paper this point is not insisted.

In higher dimensional spaces, Sommer and his collaborators, including Felsberg and Bülow et al., defined monogenic signals by using the Hilbert transform in the higher dimensional spaces, being as a-valued combination of the Riesz transforms. By means of monogenic signals they defined various phases to analyze image signals [7–9,15,16] with significant applications.

Our new contribution is two fold. One is to single out a canonical scalar-valued phase function from the commonly used ones (see [16], for instance), and define two different types of phase derivatives. Some properties of the defined derivatives are proved, including the positivity of the phase derivatives of the functions in an orthonormal system generated by shifted Cauchy kernels. The second is that we prove that the average of one of the two types of the defined phase derivatives against the density function, the square of the norm of the original signal, is identical with the average of Fourier frequency against, as the density function, the square of the norm of the Fourier transform over the characteristic function inducing the Hardy spaces. This second group of results are the counterpart ones of those for the one dimensional cases [11–13], enhancing the sense of monogenicity.

Comparing with the work of Sommer et al., ours stresses on the theory aspect, but not on the specific roles of the possible phase components. We do not only concern phases, but also phase derivatives or instantaneous frequencies. We show the actual connections of the monogenic signals with the Fourier transform, as well as with the monogenic functions in the Hardy spaces of the upper half space or of the interior of the ball. Phase derivative or instantaneous frequency is a crucial concept in one dimensional signal analysis [13,20,22,23,25,26]. In higher dimensional signal analysis, including image signal analysis, phase derivatives and function decompositions have similar applications, that however, will not be concerned in the present paper.

2 Preliminary

Most of the basic knowledge and notation in relation to Clifford algebra hereby are referred to [6,14].

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be *basic elements* satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, \ldots, m$. Let

$$\mathbf{R}_1^m = \{x_0 + \underline{x}, \underline{x} \in \mathbf{R}^m\},$$

where

$$\mathbf{R}^m = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m : x_j \in \mathbf{R}, j = 1, 2, \ldots, m\}$$

be identical with the usual Euclidean space $\mathbf{R}^m$.

An element in $\mathbf{R}^m$ is called a *vector*. The real (complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, denoted by $\mathbf{R}_m$ ($\mathbf{C}_m$), is the associative algebra over the real (complex) field $\mathbf{R}$ ($\mathbf{C}$). A general element in $\mathbf{R}_m$, therefore, is of the form $x = \sum_S x_S\mathbf{e}_S$ and $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2} \ldots \mathbf{e}_{i_l}$, and S runs over all the ordered subsets of $\{1, 2, \ldots, m\}$, namely

$$S = \{1 \leq i_1 < i_2 < \cdots < i_l \leq m\}, \quad 1 \leq l \leq m.$$

The multiplication of two vectors $\underline{x} = \sum_{j=1}^m x_j e_j$ and $\underline{y} = \sum_{j=1}^m y_j e_j$ is given by

$$\underline{x}\,\underline{y} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y}$$

with

$$\underline{x} \cdot \underline{y} = -\sum_{j=1}^m x_j y_j = \frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\,\underline{x}) = -\langle \underline{x}, \underline{y} \rangle$$

$$\underline{x} \wedge \underline{y} = \sum_{i<j} e_{ij}(x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\,\underline{x})$$

being a scalar and a bi-vector, respectively. We denote by $\mathrm{Sc}(\underline{x}\,\underline{y})$ and $\mathrm{Bi}(\underline{x}\,\underline{y})$, respectively. In particular, we have $\underline{x}^2 = -\langle \underline{x}, \underline{x} \rangle = -|\underline{x}|^2 = -\sum_{j=1}^m x_j^2$.

We define the conjugation and reversion of $\mathbf{e}_S$ are $\overline{\mathbf{e}}_S = \overline{\mathbf{e}}_{il} \ldots \overline{\mathbf{e}}_{i1}, \overline{\mathbf{e}}_j = -\mathbf{e}_j$ and $\widetilde{\mathbf{e}}_S = \mathbf{e}_{il} \ldots \mathbf{e}_{i1}$. So the Clifford conjugate of a vector $\underline{x}$ is $\overline{\underline{x}} = -\underline{x}$. The Clifford reversion of a vector $\underline{x}$ is $\widetilde{\underline{x}} = \underline{x}$. It is easy to verify that $0 \neq \underline{x} \in \mathbf{R}^m$ implies

$$\underline{x}^{-1} = \frac{\overline{\underline{x}}}{|\underline{x}|^2}.$$

The *open ball with center* 0 *and radius* R in $\mathbf{R}^m$ is denoted by $B(0, R)$ and the unit sphere in $\mathbf{R}^m$ is denoted by S^{m-1}.

The natural inner product between x and y in $\mathbf{C}_m$, denoted by $< x, y >$, is the complex number $\sum_S x_S\overline{y_S}$, where $x = \sum_S x_S\mathbf{e}_S$ and $y = \sum_S y_S\mathbf{e}_S$. The norm associated with this inner product is

$$|x| = \langle x, x \rangle^{\frac{1}{2}} = \left(\sum_S |x_S|^2\right)^{\frac{1}{2}}.$$

Below we will study functions defined in $\mathbf{R}^m$ taking values in $\mathbf{C}_m$. So, they are of the form $f(\underline{x}) = \sum_S f_S(\underline{x})\mathbf{e}_S$, where f_S are complex-valued functions. We will use the *Dirac operator* $\underline{D}$, where $\underline{D} = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_m}\mathbf{e}_m$. We define the "left" and "right" roles of the operators $\underline{D}$ by

$$\underline{D}f = \sum_{i=1}^{m}\sum_{S}\frac{\partial f_S}{\partial x_i}\mathbf{e}_i\mathbf{e}_S$$

and

$$f\underline{D} = \sum_{i=1}^{m}\sum_{S}\frac{\partial f_S}{\partial x_i}\mathbf{e}_S\mathbf{e}_i.$$

If $\underline{D}f = 0$ in a domain (open and connected) Ω, then we say that f is *left-monogenic* in Ω; and, if $f\underline{D} = 0$ in Ω, we say that f is *right-monogenic* in Ω. If f is both left- and right-monogenic, then we say that f is *monogenic*.

We call

$$E(\underline{x}) = \frac{\overline{\underline{x}}}{|\underline{x}|^m}$$

the *Cauchy kernel* in $\mathbf{R}^m$. It is easy to see that $E(\underline{x})$ is a monogenic function in $\mathbf{R}^m\setminus\{0\}$.

For $\underline{x} = |\underline{x}|\underline{\xi} = r\underline{\xi}$, there is the polar form of the Dirac operator

$$\underline{D} = \underline{\xi}\partial_r - \frac{1}{r}\partial_{\underline{\xi}} = \frac{1}{r}\underline{\xi}(r\partial_r + \overline{\underline{\xi}}\partial_{\underline{\xi}}) = \frac{1}{r}\underline{\xi}(r\partial_r + \Gamma_{\underline{\xi}}),$$

where $\Gamma_{\underline{\xi}}$ is the bi-vector-valued spherical Dirac operator

$$\Gamma_{\underline{\xi}} = \overline{\underline{\xi}}\partial_{\underline{\xi}} = -\sum_{i<j}\mathbf{e}_i\mathbf{e}_j(x_j\partial_{x_j} - x_j\partial_{x_i}).$$

Denoting $\tilde{\Gamma}_{\underline{\xi}} = (m-1)I - \Gamma_{\underline{\xi}}$, where I is the identity operator, then $\tilde{\Gamma}_{\underline{\xi}} - I = (m-2)I - \Gamma_{\underline{\xi}}$.

Remark 2.1 When $m = 2$, let $\underline{\xi} = \cos t\mathbf{e}_1 + \sin t\mathbf{e}_2$, then $\overline{\underline{\xi}} = -\cos t\mathbf{e}_1 - \sin t\mathbf{e}_2$.

$$\partial_{\underline{\xi}} = [-\sin t\mathbf{e}_1 + \cos t\mathbf{e}_2]\frac{\partial}{\partial t}.$$

Therefore,

$$\begin{aligned}\Gamma_{\underline{\xi}} &= \overline{\underline{\xi}}\partial_{\underline{\xi}} \\ &= [-\cos t\mathbf{e}_1 - \sin t\mathbf{e}_2][-\sin t\mathbf{e}_1 + \cos t\mathbf{e}_2]\frac{\partial}{\partial t}\end{aligned}$$

$$= \mathbf{e}_2\mathbf{e}_1 \frac{\partial}{\partial t}$$
$$= \bar{\mathbf{e}}_1\mathbf{e}_2 \frac{\partial}{\partial t}.$$

In the complex plane, Let $\bar{\mathbf{e}}_1\mathbf{e}_2 = \mathbf{i}$, then $\underline{\xi} = \cos t\mathbf{e}_1 + \sin t\mathbf{e}_2 = \mathbf{e}_1(\cos t + \sin t\bar{\mathbf{e}}_1\mathbf{e}_2)$ is isomorphic to $\cos t + \mathbf{i}\sin t$, and $-\Gamma_{\underline{\xi}} = \frac{1}{\mathbf{i}}\frac{\partial}{\partial t}$.

Remark 2.2 When $m = 2$,

$$\tilde{\Gamma}_{\xi} - I = -\Gamma_{\xi}.$$

3 Fourier Expansion and Spherical Hilbert Transforms

It is well-known that if $f \in L^2(S^{m-1})$, then we have the Fourier–Laplace expansion

$$f(\underline{\xi}) = \sum_{k=0}^{\infty} S_k(f)(\underline{\xi}), \quad \underline{\xi} \in S^{m-1},$$

where $S_k(f) \in \mathcal{H}_k$ are k-spherical harmonics given by

$$S_k(f)(\underline{\xi}) = \dim(\mathcal{H}_k)\frac{1}{\omega_{m-1}} \int_{S^{m-1}} P_{k,m}(\langle\underline{\xi}, \underline{y}\rangle) f(\underline{y}) dS_{\underline{y}},$$

where $P_{k,m}(t)$ are Legendre polynomials of degree k in dimension m, and dS the surface area element (see [28]).

Thanks to Clifford analysis, if we denote

$$P_k(f)(\underline{\xi}) = \frac{m + k - 2 - \Gamma_{\underline{\xi}}}{2k + m - 2}[S_k(f)(\underline{\xi})]$$

and

$$Q_{k-1}(f)(\underline{\xi}) = \frac{k + \Gamma_{\underline{\xi}}}{2k + m - 2}[S_k(f)(\underline{\xi})], \quad k \geq 1,$$

then

$$f(\underline{\xi}) = \sum_{k=0}^{\infty} \Big[P_k(f)(\underline{\xi}) + Q_{k-1}(f)(\underline{\xi}) \Big], \tag{3.1}$$

where $P_0(f) = \text{constant}$ and $Q_{-1} = \{0\}$,

$$P_k(f)(\underline{\xi}) = \frac{1}{\omega_{m-1}} \int_{S^{m-1}} C^{+}_{m,k}(\underline{\xi}, \underline{y}) f(\underline{y}) dS_{\underline{y}},$$

and

$$Q_{k-1}(f)(\underline{\xi}) = \frac{1}{\omega_{m-1}} \int_{S^{m-1}} C^{-}_{m,k-1}(\underline{\xi}, \underline{y}) f(\underline{y}) dS_{\underline{y}},$$

where

$$C^{+}_{m,k}(\underline{\xi}, \underline{y}) = \frac{1}{2-m}[-(m+k-2)C_k^{(m-2)/2}(\langle \underline{\xi}, \underline{y} \rangle) \\ +(2-m)C_{k-1}^{(m/2)}(\langle \underline{\xi}, \underline{y} \rangle)(\underline{\xi} \wedge \underline{y})],$$

and

$$C^{-}_{m,k-1}(\underline{\xi}, \underline{y}) = \frac{1}{m-2}[kC_k^{(m-2)/2}(\langle \underline{\xi}, \underline{y} \rangle) \\ +(2-m)C_{k-1}^{(m/2)}(\langle \underline{\xi}, \underline{y} \rangle)(\underline{\xi} \wedge \underline{y})], \quad k \geq 1,$$

C_k^v is the Gegenbauer polynomial of degree k associated with v. We call (3.1) the *Fourier expansion* of $f \in L^2(S^{m-1})$.

The component $P_k(f)(\underline{\xi})$ is called an *inner spherical monogenics* of degree k, which is the restriction to the unit sphere of the k-homogeneous left-monogenic function $P_k(f)(r\underline{\xi})$ in the unit ball. The component $Q_{k-1}(f)(\underline{\xi})$ is called an *outer spherical monogenics* of degree $k-1$, which is the restriction to the unit sphere of the $-(m+k-2)$-homogeneous left-monogenic function $Q_{k-1}(f)(r\underline{\xi})$ in $\mathbf{R}^m \setminus \{0\}$. Therefore, $P_k(f)(\underline{\xi}) \in H_2^+(S^{m-1})$ and $Q_{k-1}(f)(\underline{\xi}) \in H_2^-(S^{m-1})$, where the Hardy spaces $H_2^+(S^{m-1})$ and $H_2^-(S^{m-1})$ are Hilbert subspaces of the L^2 on the sphere under the inner product (6.1), and they are orthogonal complements to each other.

There hold the following relations (see [14]):

$$-\Gamma_{\underline{\xi}}[P_k(\underline{\xi})] = kP_k(\underline{\xi}),$$
$$-\Gamma_{\underline{\xi}}[Q_{k-1}(\underline{\xi})] = -(k+m-2)Q_{k-1}(\underline{\xi}),$$
$$(\tilde{\Gamma}_{\underline{\xi}} - I)[P_k(\underline{\xi})] = (k+m-2)P_k(\underline{\xi}), \quad \text{and}$$
$$(\tilde{\Gamma}_{\underline{\xi}} - I)[Q_{k-1}(\underline{\xi})] = -kQ_{k-1}(\underline{\xi}).$$

For the boundary value of a left monogenic function in the ball, the non-scalar part of it is defined to be the *inner spherical Hilbert transform* of the scalar part. For the boundary value of a left monogenic function defined outside the unit ball, we similarly define *outer spherical Hilbert transform*. In this paper, we only deal with the inner spherical Hilbert transform, abbreviated as spherical Hilbert transform. The theory for outer spherical Hilbert transform is similar.

The inner and outer spherical Hilbert transform of the function $f \in L^2(S^{m-1})$ are well studied in [4,5,10,27]. The first integral representation of the spherical Hilbert transform is given by Brackx and Van Acker [10]. In [27], by using the Abelian sum of the inner spherical Hilbert transform of f, we obtain the singular integral representation, as follows.

$$\tilde{H}[f](\underline{\xi}) = \lim_{r\to 1^-} \int_{S^{m-1}} Q_r(\underline{y}, \underline{\xi}) f(\underline{y}) dS_{\underline{y}},$$

where

$$Q_r(\underline{y}, \underline{\xi}) = \frac{1}{\omega_{m-1}} \left[\frac{2}{|\underline{y} - r\underline{\xi}|^m} - \frac{m-2}{r^{m-1}} \int_0^r \frac{\rho^{m-2}}{|\underline{y} - \rho\underline{\xi}|^m} d\rho \right] r\underline{\xi} \wedge \underline{y}.$$

Denoting

$$I_m = \int_0^r \frac{\rho^{m-2}}{|\underline{y} - \rho\underline{\xi}|^m} d\rho,$$

and $t = \langle \underline{y}, \underline{\xi} \rangle$, we have

$$I_{m+2}(r, t) = \frac{1}{m} \left[\left(m - 1 + t\frac{d}{dt} \right) I_m - \frac{r^{m-1}}{|1 - 2rt + r^2|^{m/2}} \right],$$

while

$$I_2(r, t) = \frac{1}{\sqrt{1-t^2}} \operatorname{arctg} \frac{r-t}{\sqrt{1-t^2}},$$
$$I_3(r, t) = \frac{rt - 1}{(1-t^2)\sqrt{1 - 2rt + r^2}}.$$

The series form of the spherical Hilbert transform is (see [5,27]):

$$\tilde{H}[f](\underline{\xi}) = \sum_{k=1}^{\infty} \left[\frac{k}{k+m-2} P_k(f)(\underline{\xi}) - Q_{k-1}(f)(\underline{\xi}) \right].$$

From the definition of the spherical Hilbert transform, we have

$$\tilde{H}[P_0] = 0,$$
$$\tilde{H}[P_k(f) + Q_{k-1}(f)] = \frac{k}{k+m-2} P_k(f) - Q_{k-1}(f).$$

Remark 3.1 It is shown in [5] that

$$\tilde{H}[f](\underline{\xi}) = \sum_{k=1}^{\infty} \frac{-\Gamma_{\underline{\xi}}[S_k(f)(\underline{\xi})]}{k+m-2}. \tag{3.2}$$

Clearly, since f is real-valued, $\tilde{H}[f](\underline{\xi})$ is bi-vector-valued and has the form

$$\underline{\xi}\sum_{k=1}^{\infty}\frac{\partial_{\underline{\xi}}[S_k(f)(\underline{\xi})]}{k+m-2}.$$

Remark 3.2 From Remark 2.1, in the complex plane, using the isomorphic form with $m=2$, the right hand side of (3.2) becomes

$$\sum_{k=1}^{\infty}\frac{\frac{1}{\mathbf{i}}\frac{\partial}{\partial t}[c_k e^{ikt}+c_{-k}e^{-ikt}]}{k}=\sum_{k=1}^{\infty}(c_k e^{ikt}-c_{-k}e^{-ikt})$$
$$=\mathbf{i}\tilde{\mathbf{H}}[f](e^{\mathbf{i}t}).$$

That means, up to the imaginary unit $\mathbf{i}$, the spherical Hilbert transform $\tilde{\mathbf{H}}[f]$ on S^1 coincides with the circular Hilbert transform defined in (1.1).

Example 3.3 Consider the spherical Hilbert transform of the spherical harmonics $C_k^{\frac{m-2}{2}}(\langle\underline{\xi},\underline{y}\rangle)$. Using the formula (3.2), we have

$$-\Gamma_{\xi}\left[\frac{1}{k+m-2}C_k^{\frac{m-2}{2}}(\langle\underline{\xi},\underline{y}\rangle)\right]=-\frac{m-2}{k+m-2}C_k^{\frac{m}{2}}(\langle\underline{\xi},\underline{y}\rangle)\Gamma_{\xi}[\langle\underline{\xi},\underline{y}\rangle]$$
$$=-\frac{m-2}{k+m-2}C_k^{\frac{m}{2}}(\langle\underline{\xi},\underline{y}\rangle)\underline{y}\wedge\underline{\xi},$$

where we invoke the property $\frac{d}{dt}C_k^{v}(t)=2vC_{k-1}^{v+1}(t)$. This result can be found in [14]. Therefore,

$$P_k(f)(\underline{\xi})=\mathrm{Sc}[P_k(f)(\underline{\xi})]+\mathrm{Bi}[P_k(f)(\underline{\xi})]=\mathrm{Sc}[P_k(f)(\underline{\xi})]+\tilde{H}\{\mathrm{Sc}[P_k(f)(\underline{\xi})]\}.$$

4 Monogenic Signals and Phase Derivatives on the Unit Sphere

Let f be a real-valued square integrable signal on the unit sphere S^{m-1}. From the definition of the spherical Hilbert transform, we have $f(\underline{\xi})+\tilde{H}[f](\underline{\xi})\in H_2^+(S^{m-1})$, which is the boundary value of a left-monogenic function inside the unit ball. We call $f(\underline{\xi})+\tilde{H}[f](\underline{\xi})$ the *monogenic signal* associated with f.

For a monogenic signal $f^+(\underline{\xi})=f(\underline{\xi})+\tilde{H}[f](\underline{\xi})$, we can write it in the form:

$$f+\tilde{H}[f]=A(f)\left[\frac{f}{A(f)}+\frac{\tilde{H}[f]}{A(f)}\right]$$
$$=A(f)\left[\frac{f}{A(f)}+\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\frac{|\tilde{H}[f]|}{A(f)}\right]$$

$$= A(f)\left[\cos\theta(\underline{\xi}) + \frac{H[f]}{|H[f]|}\sin\theta(\underline{\xi})\right]$$
$$= A(f)e^{\frac{H[f]}{|H[f]|}\theta(\underline{\xi})},$$

where $A(f) = \sqrt{f^2 + \tilde{H}^2[f]}$ is called the *amplitude*, $\theta(\underline{\xi}) = \arctan\frac{|\tilde{H}[f]|}{f}$ the *phase* that is between 0 and $\frac{\pi}{2}$, $\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})$ the *phase vector*, and $e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]}$ the *phase direction*. We define the *directional phase derivative* by Sc $\left\{[-\Gamma_{\underline{\xi}}\theta(\underline{\xi})]\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\right\}$.

Geometrical explanation of instantaneous phase direction

When $m = 2$, by definition a monogenic signal is of the form $f_0+f_1\mathbf{e}_1\mathbf{e}_2$, the instantaneous phase is $e^{\theta\mathbf{e}_1\mathbf{e}_2}$, where $\theta = \arctan\frac{f_1}{f_0}$. The product of a vector $\underline{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 \in \mathbf{R}^2$ and the instantaneous phase $e^{\theta\mathbf{e}_1\mathbf{e}_2}$, that is

$$\underline{a}e^{\theta\mathbf{e}_1\mathbf{e}_2} = e^{-(\theta/2)\mathbf{e}_1\mathbf{e}_2}\underline{a}e^{(\theta/2)\mathbf{e}_1\mathbf{e}_2},$$

is the rotation of $\underline{a}$ anticlockwise by the angle θ.

When $m = 3$, a monogenic signal is this form $f_0 + f_1\mathbf{e}_2\mathbf{e}_3 + f_2\mathbf{e}_3\mathbf{e}_1 + f_3\mathbf{e}_1\mathbf{e}_2$. Using the Hodge dual, it can be written as $f_0-(f_1\mathbf{e}_1+f_2\mathbf{e}_2+f_3\mathbf{e}_3)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. We denote $f_1\mathbf{e}_1+f_2\mathbf{e}_2+f_3\mathbf{e}_3$ by $\underline{f}$. Then the instantaneous phase direction is $e^{-\theta\frac{\underline{f}}{|\underline{f}|}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}$, where $\theta = \arctan\frac{|\underline{f}|}{f_0}$. Let $\underline{a}$ be a vector. Then

$$e^{-(\theta/2)\frac{\underline{f}}{|\underline{f}|}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}\underline{a}e^{(\theta/2)\frac{\underline{f}}{|\underline{f}|}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}$$

is the rotation of $\underline{a}$ along the axis $\underline{f}$ by the angle θ in the clockwise looking from the arrow-head position of $\underline{f}$. For details, see [14,18].

$A(f)e^{-\theta\frac{\underline{f}}{|\underline{f}|}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}$ is a rotation of the real value $A(f)$ along the rotation axis $-\mathbf{i}\sin\phi(\underline{\xi}) + \mathbf{j}\cos\phi(\underline{\xi})$ which is orthogonal to the plane spanned by the real axis and the vector $\mathbf{i}\cos\phi(\underline{\xi}) + \mathbf{j}\sin\phi(\underline{\xi})$ by the angle θ in the clockwise looking from the arrow-head position of the rotation axis.

Remark 4.1 Particularly, for a monogenic signal of this form $f_0 + f_1\mathbf{e}_2\mathbf{e}_3 + f_2\mathbf{e}_3\mathbf{e}_1$, in [16], it is written in this form $f_0 + f_1\mathbf{i} + f_2\mathbf{j}$. Then

$$f_0 + f_1\mathbf{i} + f_2\mathbf{j} = A(f)\left[\frac{f_0}{A(f)} + \frac{f_1\mathbf{i} + f_2\mathbf{j}}{A(f)}\right]$$
$$= A(f)\left[\frac{f_0}{A(f)} + \frac{f_1\mathbf{i} + f_2\mathbf{j}}{|f_1\mathbf{i} + f_2\mathbf{j}|}\frac{|f_1\mathbf{i} + f_2\mathbf{j}|}{A(f)}\right]$$
$$= A(f)[\cos\theta(\underline{\xi}) + \mathbf{i}\sin\theta(\underline{\xi})\cos\phi(\underline{\xi}) + \mathbf{j}\sin\theta(\underline{\xi})\sin\phi(\underline{\xi})]$$
$$= A(f)e^{[\mathbf{i}\cos\phi(\underline{\xi})+\mathbf{j}\sin\phi(\underline{\xi})]\theta(\underline{\xi})},$$

where $A(f)=\sqrt{(f_0)^2+(f_1)^2+(f_2)^2}$, $\theta(\underline{\xi})=\arctan\frac{|f_1\mathbf{i}+f_2\mathbf{j}|}{f_0}=\arctan\frac{\sqrt{(f_1)^2+(f_2)^2}}{f_0}$, $\phi(\underline{\xi})=\arctan\frac{f_2}{f_1}$. In [16], the geometrical explanation of $A(f)e^{[\mathbf{i}\cos\phi(\underline{\xi})+\mathbf{j}\sin\phi(\underline{\xi})]\theta(\underline{\xi})}$ is a rotation of the real value $A(f)$ along the rotation axis $-\mathbf{i}\sin\phi(\underline{\xi})+\mathbf{j}\cos\phi(\underline{\xi})$ which is orthogonal to the plane spanned by the real axis and the vector $\mathbf{i}\cos\phi(\underline{\xi})+\mathbf{j}\sin\phi(\underline{\xi})$ by the angle θ in the clockwise looking from the arrow-head position of the rotation axis.

Next, we discuss how to define instantaneous frequency for monogenic signals. First we consider instantaneous frequency for analytic signals on the circle.

For an analytic signal $f^+=f+\mathbf{i}\tilde{\mathbf{H}}f$, we have

$$f^+(e^{\mathbf{i}t})=A(t)[\cos\theta(t)+\mathbf{i}\sin\theta(t)]=A(t)e^{\mathbf{i}\theta(t)}.$$

In the complex plane, the instantaneous frequency for an analytic signal is defined to be the derivative of the instantaneous phase $\theta'(t)$. There are various ways to compute $\theta'(t)$:

(a) $\frac{1}{\mathbf{i}}\frac{\partial}{\partial t}\,[\mathbf{i}\theta(t)]$;

(b) $\frac{1}{\mathbf{i}}\frac{\partial}{\partial t}[\cos\theta(t)]/\mathbf{i}\sin\theta(t)$;

(c) $\frac{1}{\mathbf{i}}\frac{\partial}{\partial t}[\mathbf{i}\sin\theta(t)]/\cos\theta(t)$;

(d) $\frac{1}{\mathbf{i}}\frac{\partial}{\partial t}[e^{\mathbf{i}\theta(t)}]/e^{\mathbf{i}\theta(t)}$;

(e) $\mathrm{Re}\left[\frac{\frac{1}{\mathbf{i}}\frac{\partial}{\partial t}f^+(e^{\mathbf{i}t})}{f^+(e^{\mathbf{i}t})}\right]$.

As we know, the operator $\frac{1}{\mathbf{i}}\frac{\partial}{\partial t}$ corresponds to $-\Gamma_{\underline{\xi}}=\tilde{\Gamma}_{\underline{\xi}}-I$ in the dimension $m=2$. Therefore, in higher dimensions we may have two alternative methods to define phase derivative or instantaneous frequency of a monogenic signal.

Method 1 Define instantaneous frequency by

$$\mathrm{Sc}\left\{\left[-\Gamma_{\underline{\xi}}f^+(\underline{\xi})\right]\left[f^+(\underline{\xi})\right]^{-1}\right\}. \tag{4.1}$$

We can show that

$$\begin{aligned}
&\mathrm{Sc}\left\{\left[-\Gamma_{\underline{\xi}}f^+(\underline{\xi})\right]\left[f^+(\underline{\xi})\right]^{-1}\right\}\\
&=\mathrm{Sc}\left\{\left[-\Gamma_{\underline{\xi}}e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]}\right]\left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]}\right]^{-1}\right\}\\
&=\mathrm{Sc}\left\{-\Gamma_{\underline{\xi}}\left(\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\right)\sin\theta(\underline{\xi})\left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]}\right]^{-1}\right\}+\mathrm{Sc}\left\{[-\Gamma_{\underline{\xi}}\theta(\underline{\xi})]\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\right\}.
\end{aligned} \tag{4.2}$$

In the complex plane, the first part of Eq. (4.2) reduces to zero and the second part of it is just the phase derivative.

To show (4.2), we have

$$\begin{aligned}
&-\Gamma_{\underline{\xi}} f^{+}(\underline{\xi}) \\
&= -\Gamma_{\underline{\xi}} \left\{ A(f) \left[\cos\theta(\underline{\xi}) + \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin\theta(\underline{\xi}) \right] \right\} \\
&= -\Gamma_{\underline{\xi}} A(f) \left[\cos\theta(\underline{\xi}) + \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin\theta(\underline{\xi}) \right] \\
&\quad - A(f)\Gamma_{\underline{\xi}} \left[\cos\theta(\underline{\xi}) + \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin\theta(\underline{\xi}) \right]. \\
&\quad -\Gamma_{\underline{\xi}} \left[\cos\theta(\underline{\xi}) + \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin\theta(\underline{\xi}) \right] = \sin\theta \Gamma_{\underline{\xi}}\theta - \cos\theta \Gamma_{\underline{\xi}}\theta \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \\
&\quad -\Gamma_{\underline{\xi}} \left(\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right) \sin\theta.
\end{aligned}$$

Then

$$\begin{aligned}
&[-\Gamma_{\underline{\xi}} f^{+}(\underline{\xi})][f^{+}(\underline{\xi})]^{-1} \\
&= [-\Gamma_{\underline{\xi}} f^{+}(\underline{\xi})] \left\{ \frac{1}{A(f)} \left[\cos\theta(\underline{\xi}) - \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin\theta(\underline{\xi}) \right] \right\} \\
&= \frac{-\Gamma_{\underline{\xi}} A(f)}{A(f)} + \left[-\Gamma_{\underline{\xi}} e^{[\tilde{H}[f]|\tilde{H}[f]|\theta(\underline{\xi})]} \right] \left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]} \right]^{-1} \\
&= \frac{-\Gamma_{\underline{\xi}} A(f)}{A(f)} - \Gamma_{\underline{\xi}} \left[\cos\theta(\underline{\xi}) + \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin\theta(\underline{\xi}) \right] \left[\cos\theta(\underline{\xi}) - \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin\theta(\underline{\xi}) \right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathrm{Sc}\left\{ \left[-\Gamma_{\underline{\xi}} f^{+}(\underline{\xi}) \right] \left[f^{+}(\underline{\xi}) \right]^{-1} \right\} \\
&= \mathrm{Sc}\left\{ \left[-\Gamma_{\underline{\xi}} e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]} \right] \left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]} \right]^{-1} \right\} \\
&= \mathrm{Sc}\left\{ -\Gamma_{\underline{\xi}} \left(\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right) \sin\theta(\underline{\xi}) \left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]} \right]^{-1} - \Gamma_{\underline{\xi}}\theta(\underline{\xi}) \left(\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right) \right\} \\
&= \mathrm{Sc}\left\{ -\Gamma_{\underline{\xi}} \left(\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right) \sin\theta(\underline{\xi}) \left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]} \right]^{-1} \right\} + \mathrm{Sc}\left\{ [-\Gamma_{\underline{\xi}}\theta(\underline{\xi})] \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right\}.
\end{aligned}$$

Method 2 Define instantaneous frequency by

$$\mathrm{Sc}\left\{\left[(\tilde{\Gamma}_{\underline{\xi}}-I)f^{+}(\underline{\xi})\right]\left[f^{+}(\underline{\xi})\right]^{-1}\right\}.$$

Similarly, we can show that

$$\begin{aligned}
&\mathrm{Sc}\left\{\left[(\tilde{\Gamma}_{\underline{\xi}}-I)f^{+}(\underline{\xi})\right]\left[f^{+}(\underline{\xi})\right]^{-1}\right\}\\
&\quad=(m-2)+\mathrm{Sc}\left\{\left[-\Gamma_{\underline{\xi}}f^{+}(\underline{\xi})\right]\left[f^{+}(\underline{\xi})\right]^{-1}\right\}\\
&\quad=(m-2)+\mathrm{Sc}\left\{-\Gamma_{\underline{\xi}}\left(\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\right)\sin\theta(\underline{\xi})\left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]}\right]^{-1}\right\}\\
&\quad\quad+\mathrm{Sc}\left\{[-\Gamma_{\underline{\xi}}\theta(\underline{\xi})]\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\right\}.
\end{aligned}\tag{4.3}$$

Note that in higher dimensions, phase derivative or instantaneous frequency defined through **Method 1** or **Method 2** are no longer qual to the directional derivative of the instantaneous phase.

Remark 4.2 For $P_k(\underline{\xi})$, by using the two methods, the instantaneous frequencies obtained are, respectively, k and $k+m-2$. When $m=2$, P_k reduces to e^{ikt}, and $k+m-2$ reduces to k.

Remark 4.3 If $f(\underline{x})$ is left-monogenic, then $\underline{D}f=0$. Using the polar form of the Dirac operator

$$\underline{D}=\frac{1}{r}\underline{\xi}(r\partial_r+\Gamma_{\underline{\xi}}),$$

we have $-\Gamma_{\underline{\xi}}f(\underline{x})=r\partial_r f(\underline{x})$. Therefore, $-\Gamma_{\underline{\xi}}f(\underline{\xi})=\lim_{r\to 1^-}\partial_r f(\underline{x})$. In many occasions it is easier to compute with the operator ∂_r than the operator $\Gamma_{\underline{\xi}}$.

Now we justify the above given definitions of instantaneous frequency. The following relation plays an important role in one dimensional signal analysis: let s be an analytic signal of finite energy and $s(t)=|s(t)|e^{i\varphi(t)}$. Then in both the classical sense [11] and the extended sense [12,13],

$$\langle\omega\rangle=\int_0^\infty\omega|\hat{s}(\omega)|^2d\omega=\int_{-\infty}^{\infty}\frac{d\varphi(t)}{dt}|s(t)|^2dt.\tag{4.4}$$

For periodic analytic signals there exists an analogous relation [12]

$$\langle\omega\rangle=\sum_{k=0}^{\infty}k|c_k|^2=\int_{-\infty}^{\infty}\frac{d\varphi(t)}{dt}|s(t)|^2dt,\tag{4.5}$$

where c_k's are the Fourier coefficients of f.

For a monogenic signal $f^+(\underline{\xi}) = \sum_{k=0}^{\infty} P_k(\underline{\xi})$, we have, correspondingly, two methods to define the mean of the Fourier frequency as follows.

Method 1

$$\langle k\rangle_1 = \sum_{k=0}^{\infty} k|P_k(\underline{\xi})|^2.$$

In the case we can prove, similarly to (4.5),

$$\langle k\rangle_1 = \int_{S^{m-1}} \mathrm{Sc}\left\{[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1}\right\} |f^+(\underline{\xi})|^2 dS_{\underline{\xi}}.$$

In fact,

$$\begin{aligned}
\langle k\rangle_1 &= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \sum_{k=0}^{\infty} k|P_k(\underline{\xi})|^2 dS_{\underline{\xi}} \\
&= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \sum_{k=0}^{\infty} k P_k(\underline{\xi}) \overline{P_k(\underline{\xi})} dS_{\underline{\xi}} \\
&= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \sum_{k=0}^{\infty} (-\Gamma_{\underline{\xi}}) P_k(\underline{\xi}) \overline{P_k(\underline{\xi})} dS_{\underline{\xi}} \\
&= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} (-\Gamma_{\underline{\xi}}) f^+(\underline{\xi}) \overline{f^+(\underline{\xi})} dS_{\underline{\xi}} \\
&= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} [(-\Gamma_{\underline{\xi}}) f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1} f^+(\underline{\xi}) \overline{f^+(\underline{\xi})} dS_{\underline{\xi}} \\
&= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} [(-\Gamma_{\underline{\xi}}) f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1} |f^+(\underline{\xi})|^2 dS_{\underline{\xi}} \\
&= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \mathrm{Sc}\left\{[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1}\right\} |f^+(\underline{\xi})|^2 dS_{\underline{\xi}}.
\end{aligned}$$

Method 2

$$\langle k\rangle_2 = \sum_{k=0}^{\infty} (k+m-2)|P_k(\underline{\xi})|^2.$$

Using the same technique, we can prove, similarly to (4.5),

$$\langle k\rangle_2 = \int\limits_{S^{m-1}} \mathrm{Sc}\left\{[(\tilde{\Gamma}_{\underline{\xi}} - I)f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1}\right\}|f^+(\underline{\xi})|^2 dS_{\underline{\xi}}.$$

For some physical reasons, we need to expand a signal by those of positive analytic instantaneous frequency. This aspect has recently been studied in a series papers of Qian et al. [22–26]. In this complex plane, the study is based on the analytic signals, the boundary values $e^{\mathbf{i}\theta_a(t)} = \frac{e^{\mathbf{i}t}-a}{1-\overline{a}e^{\mathbf{i}t}}$, $|a| < 1$, of the corresponding Möbius transforms from the unit disc to the unit disc, called *Fourier atoms*. Observe that [17,22]

$$\frac{1}{2\pi}\frac{d\theta_a(t)}{dt} = \frac{1}{2\pi}\frac{1-|a|^2}{|e^{\mathbf{i}t}-a|^2}$$

that is the Poisson kernel of the disc at a, and positive, we have non-trivial analytic signals of positive phase derivatives or instantaneous frequencies. Furthermore, $\left\{\frac{1}{\sqrt{2\pi}}\frac{\sqrt{1-\overline{a}e^{\mathbf{i}t}}}{|1-\overline{a}e^{\mathbf{i}t}|^2}e^{\mathbf{i}k\theta_a(t)}\right\}_{k=-\infty}^{\infty}$ is a weighed trigonometric system.

In higher dimensions, it is natural to consider the functions $P_k(\tau_{\underline{a}}(\underline{\xi}))$, where $\tau_{\underline{a}}(\underline{\xi})$ is the Möbius transformation from the unit ball to the unit ball. We first recall Möbius transforms in higher dimensional spaces.

5 Möbius Transforms in Higher Dimensional Spaces

In this section, we will recall well known results of Möbius transformation in $\mathbf{R}^m$.

Definition 5.1 [1] A Möbius transformation is a function $M : \mathbf{R}^m \cup \{\infty\} \longrightarrow \mathbf{R}^m \cup \{\infty\}$, which can be expressed as finite composition of translations, dilations, orthogonal transformations and inversions. In fact, each Möbius transformation is a homeomorphism from $\mathbf{R}^m \cup \{\infty\}$ to $\mathbf{R}^m \cup \{\infty\}$.

In $\mathbf{R}^m$, it is more difficult to describe the Möbius transformation. That is especially because that multiplication in $\mathbf{R}^m$ is not closed. For example, $\underline{ab}$ may be not in $\mathbf{R}^m$, although $\underline{a}$ and $\underline{b}$ are in $\mathbf{R}^m$. In order to solve this problem, we need the following Lemma, that is

Lemma 5.1 [1] *If $a, b, c, d \in \Gamma_m \cup \{0\}$, and satisfy*

(1) $ac^{-1}, d\tilde{c} \in \mathbf{R}^m$, *when* $c \neq 0$;
(2) $a\tilde{d} - b\tilde{c} = \pm 1$;
(3) $bd^{-1} \in \mathbf{R}^m$, *when* $c = 0$.

Then $\psi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1}$ *is a Möbius transformation from* $\mathbf{R}^m \bigcup\{\infty\}$ *to* $\mathbf{R}^m \bigcup\{\infty\}$ *and this expression factorizes to equal* $ac^{-1} \pm (c\underline{x}\tilde{c} + d\tilde{c})^{-1}$ *when* $c \neq 0$ *and* $\pm a\underline{x}\tilde{a} + bd^{-1}$ *when* $c = 0$.

This above extension was first worked out by Karl Theodor Vahlen in 1902, and was re-studied by Ahlfors over 80 years later, in 1984.

Note 5.1 The Jacobian of the transform on the sphere is given as [1]

$$J_\psi(\underline{x}) = \left(\frac{|a\tilde{d} - b\tilde{c}|}{|c\underline{x} + d|^2}\right)^m = |c\underline{x} + d|^{-2m}.$$

Example 5.1 $\underline{\omega} = \tau_{\underline{a}}(\underline{x}) = (\underline{x} - \underline{a})(1 - \bar{\underline{a}}\underline{x})^{-1}$ is a Möbius transformation which maps the unit ball onto itself and $\underline{\omega}^{-1} = \tau_{-\underline{a}}(\underline{x})$.

Unfortunately, the composition of a monogenic function and a Möbius transformation is, in general, no longer monogenic. Möbius transforms themselves, and products of monogenic functions are usually not monogenic functions. In fact, transformations of harmonic and monogenic functions are more rigid here than those in the complex case.

Let $\psi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1}$ be a Möbius transformation in $\mathbf{R}^m \cup \{\infty\}$, then (see [21], for instance)

(1) If f is monogenic, then so is

$$\frac{\widetilde{c\underline{x} + d}}{|c\underline{x} + d|^m} f(\psi(\underline{x}));$$

and

(2) If h is harmonic, then so is

$$\frac{1}{|c\underline{x} + d|^{m-2}} h(\psi(\underline{x})).$$

Next we consider the frequency of $\frac{\widetilde{1-\underline{a}\underline{\xi}}}{|\widetilde{1-\underline{a}\underline{\xi}}|^m} P_k\left((\underline{\xi} - \underline{a})(1 - \bar{\underline{a}}\underline{\xi})^{-1}\right)$.

Example 5.2 $\frac{\widetilde{1-\underline{a}\underline{\xi}}}{|\widetilde{1-\underline{a}\underline{\xi}}|^m} P_k\left((\underline{\xi} - \underline{a})(1 - \bar{\underline{a}}\underline{\xi})^{-1}\right)$ is the restriction to the unit sphere of the left-monogenic function $\frac{\widetilde{1-\bar{\underline{a}}\underline{x}}}{|\widetilde{1-\bar{\underline{a}}\underline{x}}|^m} P_k\left((\underline{x} - \underline{a})(1 - \bar{\underline{a}}\underline{x})^{-1}\right)$ in the unit ball. Using Method 1 and 2 we obtain the instantaneous frequencies $\frac{(m-1)(\langle \underline{a},\underline{\xi}\rangle - |\underline{a}|^2) + k(1-|\underline{a}|^2)}{|1-\bar{\underline{a}}\underline{\xi}|^2}$ and $\frac{(1+k)(1-|\underline{a}|^2) + (m-3)(1-\langle \underline{a},\underline{\xi}\rangle)}{|1-\bar{\underline{a}}\underline{\xi}|^2}$ (> 0 when $m > 2$), respectively. In fact, proceeding with the computation given in (4.1), we have

$$-\Gamma_{\underline{\xi}}\left[\frac{\widetilde{1 - \bar{\underline{a}}\underline{\xi}}}{|1 - \bar{\underline{a}}\underline{\xi}|^m} P_k\left((\underline{\xi} - \underline{a})(1 - \bar{\underline{a}}\underline{\xi})^{-1}\right)\right]$$

$$= \lim_{r\to 1^-} r\frac{\partial}{\partial r}\left[\frac{\widetilde{1 - \bar{\underline{a}}r\underline{\xi}}}{|1 - \bar{\underline{a}}r\underline{\xi}|^m} P_k\left((r\underline{\xi} - \underline{a})(1 - \bar{\underline{a}}r\underline{\xi})^{-1}\right)\right]$$

$$= \lim_{r\to 1^-} \frac{-\overline{\underline{\xi}}\underline{a}|1 - \bar{\underline{a}}r\underline{\xi}|^2 - m(1 - r\overline{\underline{\xi}}\underline{a})(-\langle \underline{\xi}, \underline{a}\rangle + r|\underline{a}|^2)}{|1 - \bar{\underline{a}}r\underline{\xi}|^{m+2}}$$

$$\times P_k\left((r\underline{\xi} - \underline{a})(1 - \bar{\underline{a}}r\underline{\xi})^{-1}\right)$$

$$+\lim_{r\to 1^-}\frac{\widetilde{1-\bar{\underline{a}}r\underline{\xi}}}{|1-\bar{\underline{a}}r\underline{\xi}|^m}r\frac{\partial}{\partial r}\left[P_k\left((r\underline{\xi}-\underline{a})(1-\bar{\underline{a}}r\underline{\xi})^{-1}\right)\right]$$

$$=\frac{-\overline{\underline{\xi}}\underline{a}|1-\bar{\underline{a}}\underline{\xi}|^2-m(1-\overline{\underline{\xi}}\underline{a})(-\langle\underline{\xi},\underline{a}\rangle+|\underline{a}|^2)}{|1-\bar{\underline{a}}\underline{\xi}|^{m+2}}P_k\left((\underline{\xi}-\underline{a})(1-\bar{\underline{a}}\underline{\xi})^{-1}\right)$$

$$+\frac{\widetilde{1-\bar{\underline{a}}\underline{\xi}}}{|1-\bar{\underline{a}}\underline{\xi}|^m}P_k\left((\underline{\xi}-\underline{a})(1-\bar{\underline{a}}\underline{\xi})^{-1}\right)k\frac{1-|\underline{a}|^2}{|1-\bar{\underline{a}}\underline{\xi}|^2}$$

$$=\left[\frac{-\overline{\underline{\xi}}\underline{a}+(1-m)|\underline{a}|^2+m\langle\underline{\xi},\underline{a}\rangle}{|1-\bar{\underline{a}}\underline{\xi}|^2}+k\frac{1-|\underline{a}|^2}{|1-\bar{\underline{a}}\underline{\xi}|^2}\right]\frac{\widetilde{1-\bar{\underline{a}}\underline{\xi}}}{|1-\bar{\underline{a}}\underline{\xi}|^m}$$

$$\times P_k\left((\underline{\xi}-\underline{a})(1-\bar{\underline{a}}\underline{\xi})^{-1}\right)$$

$$=Y(\underline{\xi})\frac{\widetilde{1-\bar{\underline{a}}\underline{\xi}}}{|1-\bar{\underline{a}}\underline{\xi}|^m}P_k\left((\underline{\xi}-\underline{a})(1-\bar{\underline{a}}\underline{\xi})^{-1}\right).$$

The formula (4.1) implies that the instantaneous frequency is the scalar part of $Y(\underline{\xi})$, that is $\frac{(m-1)(\langle\underline{\xi},\underline{a}\rangle-|\underline{a}|^2)+k(1-|\underline{a}|^2)}{|1-\bar{\underline{a}}\underline{\xi}|^2}$. Similarly, adopting Method 2, the instantaneous frequency is $\frac{(m-3)(1-\langle\underline{a},\underline{\xi}\rangle+(k+1)(1-|\underline{a}|^2)}{|1-\underline{a}\underline{\xi}|^2}$ (>0 when $m>2$). The Cauchy kernel $\frac{\widetilde{1-\bar{\underline{a}}\underline{\xi}}}{|\widetilde{1-\bar{\underline{a}}\underline{\xi}}|^m}$ is the special cases of the above for $k=0$ and $P_k=1$. The instantaneous frequency of it is $\frac{(m-1)(\langle\underline{\xi},\underline{a}\rangle-|\underline{a}|^2)}{|1-\bar{\underline{a}}\underline{\xi}|^2}$ and $\frac{(1-|\underline{a}|^2)+(m-3)(1-\langle\underline{a},\underline{\xi}\rangle)}{|1-\underline{a}\underline{\xi}|^2}$ (>0 when $m>2$), respectively.

Conclusion From the above two examples, we prefer to choose Method 2 to define the instantaneous frequency and the mean of the instantaneous frequency for monogenic signals.

6 A System of $L^2(S^{m-1})$

Consider $L^2(S^{m-1})$, the space of scalar-valued square integrable functions on S^{m-1}, equipped with the inner product

$$(f,g)=\int_{S^{m-1}}\bar{f}(\underline{\xi})g(\underline{\xi})dS_{\underline{\xi}}. \tag{6.1}$$

Next, we will give a monogenic orthogonal system for $L^2(S^{m-1})$.

Theorem 6.1 *Let* $\underline{a}\in B(0,1)\subseteq \mathbf{R}^m$. *Denote*

$$\mathcal{F}_{\underline{a}}=\left\{\frac{\widetilde{1+\bar{\underline{a}}\underline{\xi}}}{|1+\bar{\underline{a}}\underline{\xi}|^m}P_k(\tau_{-\underline{a}}(\underline{\xi})),\ \frac{\widetilde{1+\bar{\underline{a}}\underline{\xi}}}{|1+\bar{\underline{a}}\underline{\xi}|^m}Q_l(\tau_{-\underline{a}}(\underline{\xi})),k,l\geq 0\right\}.$$

Then $\mathcal{F}_{\underline{a}}$ *is a complete monogenic orthogonal system for* $L^2(S^{m-1})$.

Proof A shorter proof can be given but we provide a more detailed proof for the sake of completeness.

We consider three cases.

Case 1. When $k \neq l$,

$$\left(\frac{\widetilde{1+\bar{\underline{a}}\underline{\xi}}}{|1+\bar{\underline{a}}\underline{\xi}|^m} P_k(\tau_{-\underline{a}}(\underline{\xi})), \frac{\widetilde{1+\bar{\underline{a}}\underline{\xi}}}{|1+\bar{\underline{a}}\underline{\xi}|^m} P_l(\tau_{-\underline{a}}(\underline{\xi})) \right) = 0.$$

Case 2. When $k \neq l$,

$$\left(\frac{\widetilde{1+\bar{\underline{a}}\underline{\xi}}}{|1+\bar{\underline{a}}\underline{\xi}|^m} Q_k(\tau_{-\underline{a}}(\underline{\xi})), \frac{\widetilde{1+\bar{\underline{a}}\underline{\xi}}}{|1+\bar{\underline{a}}\underline{\xi}|^m} Q_l(\tau_{-\underline{a}}(\underline{\xi})) \right) = 0.$$

Case 3. For any k, l,

$$\left(\frac{\widetilde{1+\bar{\underline{a}}\underline{\xi}}}{|1+\bar{\underline{a}}\underline{\xi}|^m} P_k(\tau_{-\underline{a}}(\underline{\xi})), \frac{\widetilde{1+\bar{\underline{a}}\underline{\xi}}}{|1+\bar{\underline{a}}\underline{\xi}|^m} Q_l(\tau_{-\underline{a}}(\underline{\xi})) \right) = 0.$$

To proceed the proof of Case 1, we have

$$\begin{aligned}
&\left(\frac{\widetilde{1+\bar{\underline{a}}\underline{\xi}}}{|1+\bar{\underline{a}}\underline{\xi}|^m} P_k(\tau_{-\underline{a}}(\underline{\xi})), \frac{\widetilde{1+\bar{\underline{a}}\underline{\xi}}}{|1+\bar{\underline{a}}\underline{\xi}|^m} P_l(\tau_{-\underline{a}}(\underline{\xi})) \right) \\
&= \int_{S^{m-1}} \overline{\frac{1+\bar{\underline{\xi}}\underline{a}}{|1+\bar{\underline{a}}\underline{\xi}|^m} P_k(\tau_{-\underline{a}}(\underline{\xi}))} \frac{1+\bar{\underline{\xi}}\underline{a}}{|1+\bar{\underline{a}}\underline{\xi}|^m} P_l(\tau_{-\underline{a}}(\underline{\xi})) dS_{\underline{\xi}} \\
&= \int_{S^{m-1}} \frac{1}{|1+\bar{\underline{a}}\underline{\xi}|^{2m-2}} \overline{P}_k(\tau_{-\underline{a}}(\underline{\xi})) P_l(\tau_{-\underline{a}}(\underline{\xi})) dS_{\underline{\xi}} \\
&= \int_{S^{m-1}} \frac{|1-\bar{\underline{a}}\underline{x}|^{2m-2}}{|1-|\bar{\underline{a}}|^2|^{2m-2}} \overline{P}_k(\underline{x}) P_l(\underline{x}) \frac{dS_{\underline{\xi}}}{dS_{\underline{x}}} dS_{\underline{x}} \\
&= \int_{S^{m-1}} \frac{|1-\bar{\underline{a}}\underline{x}|^{2m-2}}{|1-|\bar{\underline{a}}|^2|^{2m-2}} \overline{P}_k(\underline{x}) P_l(\underline{x}) \frac{|1-|\bar{\underline{a}}|^2|^{m-1}}{|1-\bar{\underline{a}}\underline{x}|^{2m-2}} dS_{\underline{x}} \\
&= \frac{1}{(1-|\underline{a}|^2)^{m-1}} \int_{S^{m-1}} \overline{P}_k(\underline{x}) P_l(\underline{x}) dS_{\underline{x}} \\
&= 0.
\end{aligned}$$

Invoking the orthogonality between P_k and Q_l and that between Q_k and $Q_l, k \neq l$, we can prove Case 2 and Case 3.

Next, we will prove completeness of the system. For any $f \in L^2(S^{m-1})$, we have $\left[\frac{\widetilde{1+\bar{\underline{a}}\tau_{\underline{a}}(\underline{\xi})}}{|1+\bar{\underline{a}}\tau_{\underline{a}}(\underline{\xi})|^m}\right]^{-1} f(\tau_{\underline{a}}(\underline{\xi})) \in L^2(S^{m-1})$, then

$$\left[\frac{\widetilde{1+\bar{\underline{a}}\tau_{\underline{a}}(\underline{\xi})}}{|1+\bar{\underline{a}}\tau_{\underline{a}}(\underline{\xi})|^m}\right]^{-1} f(\tau_{\underline{a}}(\underline{\xi})) = \sum_{k=0}^{\infty}[P_k(\underline{\xi}) + Q_{k-1}(\underline{\xi})],$$

therefore,

$$f(\underline{\omega}) = \sum_{k=0}^{\infty}\left[\frac{\widetilde{1+\bar{\underline{a}}\underline{\omega}}}{|1+\bar{\underline{a}}\underline{\omega}|^m} P_k(\tau_{-\underline{a}}(\underline{\omega})) + \frac{\widetilde{1+\bar{\underline{a}}\underline{\omega}}}{|1+\bar{\underline{a}}\underline{\omega}|^m} Q_{k-1}(\tau_{-\underline{a}}(\underline{\omega}))\right].$$

We complete the proof. □

Remark 6.1 When $\underline{a} = 0$, $\tau_{\underline{a}}(\underline{x}) = \underline{x}$, so $\mathcal{F}_0$ becomes the Fourier basis of $L^2(S^{m-1})$.

Theorem 6.2 *The system*

$$\mathcal{F}_{\underline{a}} = \left\{(\sqrt{1-|\underline{a}|^2})^{m-1} \frac{\widetilde{1+\bar{\underline{a}}\underline{\xi}}}{|1+\bar{\underline{a}}\underline{\xi}|^m} P_k(\tau_{-\underline{a}}(\underline{\xi})), k \geq 0\right\}$$

is a complete orthonormal basis of $H_2^+(S^{m-1})$ and the instantaneous frequencies of the basis functions are positive.

7 Monogenic Signals and Phase Derivatives in the Upper Half Spaces

If $f \in L^2(\mathbf{R}^m)$, we define the *Fourier transform* of f by

$$\hat{f}(\underline{t}) = \int_{\mathbf{R}^m} e^{-\mathbf{i}\langle \underline{x},\underline{t}\rangle} f(\underline{x}) d\underline{x}$$

and the *inverse Fourier transform* by

$$f(\underline{x}) = \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} e^{\mathbf{i}\langle \underline{x},\underline{t}\rangle} \hat{f}(\underline{t}) d\underline{t}.$$

The Hilbert transform of $f(\underline{x})$ has the following alternative representations

$$H[f](\underline{x}) = \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} \frac{\underline{D}_{\underline{x}} e^{\mathbf{i}\langle \underline{x},\underline{t}\rangle}}{|\underline{t}|} \hat{f}(\underline{t}) d\underline{t}$$

$$= \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} \frac{\mathbf{i}\underline{t}}{|\underline{t}|} e^{\mathbf{i}\langle \underline{x}, \underline{t} \rangle} \hat{f}(\underline{t}) d\underline{t}$$

$$= \lim_{\varepsilon \to 0^+} \int_{|\underline{x}-\underline{t}|>\varepsilon} \frac{\overline{\underline{x}-\underline{t}}}{|\underline{x}-\underline{t}|^{m+1}} f(\underline{t}) d\underline{t}$$

$$= -\sum_{j=1}^{m} R_j(f)(\underline{x}) \mathbf{e}_j, \tag{7.1}$$

where $R_j(f)(\underline{x}) = \lim_{\varepsilon \to 0^+} \int_{|\underline{x}-\underline{t}|>\varepsilon} \frac{x_j - t_j}{|\underline{x}-\underline{t}|^{m+1}} f(\underline{t}) d\underline{t}$ is the jth-Riesz transform of f [28]. Clearly, if f is real-valued, then $H[f](\underline{x})$ is vector-valued.

Remark 7.1 It is easily to see that $H^2[f](\underline{x}) = f(\underline{x})$.

Remark 7.2 When $m = 1$, letting $\mathbf{e}_1 = -\mathbf{i}$, then we have $\underline{D} = \frac{1}{\mathbf{i}} \frac{\partial}{\partial x}$. The right hand side of (7.1) becomes

$$\frac{1}{(2\pi)} \int_{-\infty}^{\infty} \frac{\frac{1}{\mathbf{i}} \frac{\partial}{\partial x} e^{\mathbf{i}\langle x, t \rangle}}{|t|} \hat{f}(t) dt$$

$$= \frac{1}{(2\pi)} \int_{-\infty}^{\infty} \frac{t}{|t|} e^{\mathbf{i}\langle x, t \rangle} \hat{f}(t) dt$$

$$= \frac{\mathbf{i}}{(2\pi)} \int_{-\infty}^{\infty} -\mathbf{i}\mathrm{sgn}(t) e^{\mathbf{i}\langle x, t \rangle} \hat{f}(t) dt$$

$$= \mathbf{i}\mathbf{H}[f](x).$$

It is well known [19] that

$$\frac{1}{2} f(\underline{x}) + \frac{1}{2} H[f](\underline{x}) = \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} e^{\mathbf{i}\langle \underline{x}, \underline{t} \rangle} \frac{1}{2} (1 + \mathbf{i} \frac{\underline{t}}{|\underline{t}|}) \hat{f}(\underline{t}) d\underline{t}$$

$$= \lim_{x_0 \to 0^+} \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} e^{+}(x_0 + \underline{x}, \underline{t}) \hat{f}(\underline{t}) d\underline{t}$$

$$= f^{+}(\underline{x}),$$

where

$$e^{+}(x_0 + \underline{x}, \underline{t}) = e^{-x_0|\underline{t}|} e^{\mathbf{i}\langle \underline{x}, \underline{t} \rangle} \frac{1}{2} \left(1 + \mathbf{i} \frac{\underline{t}}{|\underline{t}|}\right)$$

is left monogenic in $\mathbf{R}_1^m$ [19]. Therefore, $f(\underline{x}) + H[f](\underline{x}) \in H_2^{+}(\mathbf{R}^m)$, being the boundary value of a left-monogenic functions in the upper half space $\mathbf{R}_1^m$.

We call $f(\underline{x}) + H[f](\underline{x})$ the *monogenic signal associated with* f, where f is a real-valued square integral signal on $\mathbf{R}^m$.

For a monogenic signal $f^+(\underline{x}) = f(\underline{x}) + H[f](\underline{x})$, we write it in the form:

$$\begin{aligned} f + \tilde{H}[f] &= A(f)\left[\frac{f}{A(f)} + \frac{H[f]}{A(f)}\right] \\ &= A(f)\left[\frac{f}{A(f)} + \frac{H[f]}{|H[f]|}\frac{|H[f]|}{A(f)}\right] \\ &= A(f)\left[\cos\theta(\underline{x}) + \frac{H[f]}{|H[f]|}\sin\theta(\underline{x})\right] \\ &= A(f)e^{[\frac{H[f]}{|H[f]|}\theta(\underline{x})]}, \end{aligned}$$

where $A(f) = \sqrt{f^2 + H^2[f]}$ is called the *amplitude*, $\theta(\underline{x}) = \arctan\frac{|H[f]|}{f}$ the *phase* that is between 0 and $\frac{\pi}{2}$, $\frac{H[f]}{|H[f]|}\theta(\underline{x})$ the *phase vector*, and $e^{\left[\frac{H[f]}{|H[f]|}\theta(\underline{x})\right]}$ the *phase direction*. We define the *directional phase derivative* by $\mathrm{Sc}\left\{[\underline{D}\theta(\underline{x})]\frac{H[f]}{|H[f]|}\right\}$ and define the *phase derivative* or *instantaneous frequency* by

$$\mathrm{Sc}\left\{[\underline{D}f^+(\underline{x})][f^+(\underline{x})]^{-1}\right\}.$$

We can prove that

$$\begin{aligned} &\mathrm{Sc}\{[\underline{D}f^+(\underline{x})][f^+(\underline{x})]^{-1}\} \\ &\quad = \mathrm{Sc}\left\{\left[\underline{D}e^{[\frac{H[f]}{|H[f]|}\theta(\underline{x})]}\right]\left[e^{[\frac{H[f]}{|H[f]|}\theta(\underline{x})]}\right]^{-1}\right\} \\ &\quad = \mathrm{Sc}\left\{[\underline{D}\frac{H[f]}{|H[f]|}]\sin\theta(\underline{x})\cos\theta(\underline{x})\right\} + \mathrm{Sc}\left\{[\underline{D}\theta(\underline{x})]\frac{H[f]}{|H[f]|}\right\}. \end{aligned}$$

In fact,

$$\begin{aligned} &\underline{D}f^+(\underline{x}) \\ &\quad = \underline{D}\left\{A(f)\left[\cos\theta(\underline{x}) + \frac{H[f]}{|H[f]|}\sin\theta(\underline{x})\right]\right\} \\ &\quad = \underline{D}A(f)\left[\cos\theta(\underline{x}) + \frac{H[f]}{|H[f]|}\sin\theta(\underline{x})\right] + A(f)\underline{D}\left[\cos\theta(\underline{x}) + \frac{H[f]}{|H[f]|}\sin\theta(\underline{x})\right]. \end{aligned}$$

$$\underline{D}\left[\cos\theta(\underline{x}) + \frac{H[f]}{|H[f]|}\sin\theta(\underline{x})\right] = -\sin\theta\,\underline{D}\theta + \cos\theta\,\underline{D}\theta\frac{H[f]}{|H[f]|} + \underline{D}\left(\frac{H[f]}{|H[f]|}\right)\sin\theta.$$

Then

$$\begin{aligned}
&\underline{D}f^{+}(\underline{x})[f^{+}(\underline{x})]^{-1}\\
&=\underline{D}f^{+}(\underline{x})\left\{\frac{1}{A(f)}\left[\cos\theta(\underline{x})-\frac{H[f]}{|H[f]|}\sin\theta(\underline{x})\right]\right\}\\
&=\frac{\underline{D}A(f)}{A(f)}+\left[\underline{D}e^{[\frac{H[f]}{|H[f]|}\theta(\underline{x})]}\right]\left[e^{[\frac{H[f]}{|H[f]|}\theta(\underline{x})]}\right]^{-1}\\
&=\frac{\underline{D}A(f)}{A(f)}+\underline{D}\left[\cos\theta(\underline{x})+\frac{H[f]}{|H[f]|}\sin\theta(\underline{x})\right]\left[\cos\theta(\underline{x})-\frac{H[f]}{|H[f]|}\sin\theta(\underline{x})\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathrm{Sc}\{[\underline{D}f^{+}(\underline{x})][f^{+}(\underline{x})]^{-1}\}\\
&=\mathrm{Sc}\left\{\left[\underline{D}e^{[\frac{H[f]}{|H[f]|}\theta(\underline{x})]}\right]\left[e^{[\frac{H[f]}{|H[f]|}\theta(\underline{x})]}\right]^{-1}\right\}\\
&=\mathrm{Sc}\left\{[\underline{D}\frac{H[f]}{|H[f]|}]\sin\theta(\underline{x})\cos\theta(\underline{x})+[\underline{D}\theta(\underline{x})]\frac{H[f]}{|H[f]|}\right\}\\
&=\mathrm{Sc}\left\{[\underline{D}\frac{H[f]}{|H[f]|}]\sin\theta(\underline{x})\cos\theta(\underline{x})\right\}+\mathrm{Sc}\left\{[\underline{D}\theta(\underline{x})]\frac{H[f]}{|H[f]|}\right\}.
\end{aligned}$$

For $m=1$ the first term of the last expression becomes zero and the second term reduces to the ordinary phase derivative. For a monogenic signal $f^{+}(\underline{x})=f(\underline{x})+H[f](\underline{x})$ we define the mean of the Fourier frequency to be

$$\langle\underline{t}\rangle=\int_{\mathbf{R}^m}|\underline{t}||\widehat{f^{+}}(\underline{t})|^2d\underline{t}.$$

Now we show, analogously with (4.4),

$$\langle\underline{t}\rangle=\int_{\mathbf{R}^m}\mathrm{Sc}\left\{[\underline{D}f^{+}(\underline{x})][f^{+}(\underline{x})]^{-1}\right\}|f^{+}(\underline{x})|^2d\underline{x}.$$

In fact,

$$\begin{aligned}
\langle\underline{t}\rangle&=\int_{\mathbf{R}^m}|\underline{t}||\widehat{f^{+}}(\underline{t})|^2d\underline{t}\\
&=\int_{\mathbf{R}^m}|\underline{t}|\widehat{f^{+}}(\underline{t})\overline{\widehat{f^{+}}(\underline{t})}d\underline{t}\\
&=\int_{\mathbf{R}^m}|\underline{t}|\left[\frac{1}{2}\left(1+\mathbf{i}\frac{\underline{t}}{|\underline{t}|}\right)\hat{f}(\underline{t})\right]\overline{\frac{1}{2}\left(1+\mathbf{i}\frac{\underline{t}}{|\underline{t}|}\right)\hat{f}(\underline{t})}d\underline{t}
\end{aligned}$$

$$= \int_{\mathbf{R}^m} \mathbf{i}\underline{t}\left[\frac{1}{2}\left(1+\mathbf{i}\frac{\underline{t}}{|\underline{t}|}\right)\hat{f}(\underline{t})\right]\overline{\frac{1}{2}\left(1+\mathbf{i}\frac{\underline{t}}{|\underline{t}|}\right)\hat{f}(\underline{t})}d\underline{t}$$

$$= \int_{\mathbf{R}^m} \widehat{\underline{D}f^{+}}(\underline{t})\overline{\widehat{f^{+}}}(\underline{t})d\underline{t}$$

$$= \int_{\mathbf{R}^m} \underline{D}f^{+}(\underline{x})\overline{f^{+}(\underline{x})}d\underline{x}$$

$$= \int_{\mathbf{R}^m} [\underline{D}f^{+}(\underline{x})][f^{+}(\underline{x})]^{-1}f^{+}(\underline{x})\overline{f^{+}(\underline{x})}d\underline{x}$$

$$= \int_{\mathbf{R}^m} [\underline{D}f^{+}(\underline{x})][f^{+}(\underline{x})]^{-1}|f^{+}(\underline{x})|^2 d\underline{x}$$

$$= \int_{\mathbf{R}^m} \mathrm{Sc}\left\{[\underline{D}f^{+}(\underline{x})][f^{+}(\underline{x})]^{-1}\right\}|f^{+}(\underline{x})|^2 d\underline{x}.$$

References

1. Ahlfors, L.V.: Clifford numbers and Möbius transformations in $\mathbf{R}^n$. In: Chisholm, J.S.R., Common, A.K. (eds.) Clifford Algebras and Their Applications in Mathematical Physics, Proceedings of the NATO and SERC Workshop Held in Canterbury, September 15–27, 1985, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, vol. 183, pp. 167–175. D. Reidel Publ., Dordrecht (1986)
2. Axelsson, A., Kou, K.I., Qian, T.: Hilbert transforms and the Cauchy integral in Euclidean space. Stud. Math. **193**(2), 161–187 (2009)
3. Bell, S.: The Cauchy Transform, Potential Theory and Conformal Mapping, Studies in Advanced Mathematics. Steven R. Bell, Purdue University, August 14 (1992)
4. Brackx, F., De Knock, B., De Schepper, H., Eelbode, D.: On the interplay between the Hilbert transform and conjugate harmonic functions. Math. Methods Appl. **(29)**(12), 1435–1450 (2006)
5. Brackx, F., De Schepper, H., Eelbode, D.: A new Hilbert transform on the unit sphere in $\mathbf{R}^m$. Complex Var. Elliptic Equ. **51**(5–6), 453–462 (2006)
6. Brackx, F., Delanghe, R., Sommen, F.: Clifford Analysis, vol. 76. Pitman, Boston (1982)
7. Bülow, T., Sommer, G.: Multi-dimensional sigal processing using an algebraically extended signal representation. In: Sommer, G., Koenderink, J.J. (eds.) Proceedings International Workshop on Algebraic Frames for the Perception-Action Cycle (AFPAC 97), Kiel. LNCS, vol. 1315, pp. 148–163. Springer, Berlin (1997)
8. Bülow, T., Sommer, G.: Algebraically extended representations of Multi-dimensional signals. In: Proceedings 10th Scandinavian Conference on Image Analysis, Lappeenranta, Finland, pp. 559–566 (1997)
9. Bülow, T.: Hypercomplex Spectral Signal Representations for the Processing and Analysis of Images. PhD thesis, University of Kiel, Germany (1999)
10. Brackx, F., Van Acker, N.: A conjugate Poisson kernel in Euclidean space. SIMON STEVIN Q. J. Pure Appl. Math. **67**(1–2), 3–14 (1993)
11. Cohen, L.: Time-Frequency Analysis: Theory and Applications. Prentice Hall, Inc., Upper Saddle River (1995)
12. Dang, P., Qian, T.: Hardy-Sobolev derivatives of phase and amplitude, and their applications. Preprint (2011)
13. Dang, P., Qian, T., You, Z.: Hardy-Sobolev spaces decomposition in signal analysis. J. Fourier Anal. Appl. **17**(1), 36–64 (2011)

14. Delanghe, R., Sommen, F., Soucek, V.: Clifford Algebra and Spinor Valued Functions. Kluwer, Dordrecht (1992)
15. Felsberg, M.: Low-Level Image Processing with the Structure Multivector. Bericht Nr. 0203. Institut für Informatik und Praktische Mathematik der Christian-Albrechts-Universität zu Kiel, Kiel (2002)
16. Felsberg, M., Sommer, G.: The monogenic signal. IEEE Trans. Signal Process. **49**(12), 3136–3144 (2001)
17. Garnett, J.B.: Bounded Analytic Functions. Academic Press, London (1981)
18. Lounesto, P.: Clifford Algebras and Spinors, 2nd edn. London Mathematical Society Lecture Note Series, vol. 286. Cambridge University Press, London (2001)
19. Li, C., Mcintosh, A., Qian, T.: Clifford algebras, Fourier transforms, and singular convolution operators on Lipschitz surfaces. Rev. Mat. Iberoamericana **10**, 665–721 (1994)
20. Picinbono, B.: On instantaneous amplitude and phase of signals. IEEE Trans. Signal Process. **45**(3), 552–560 (1997)
21. Peetre, J., Qian, T.: Moebius covariance of iterated Dirac operators. J. Aust. Math. Soc. (Ser. A) **56**, 403–414 (1994)
22. Qian, T.: Analytic signals and harmonic measure. J. Math. Anal. Appl. **314**(2), 526–536 (2006)
23. Qian, T., Chen, Q.-H., Li, L.-Q.: Analytic unit quadrature signals with non-linear phase. Phys. D Nonlinear Phenom. **203**(1–2), 80–87 (2005)
24. Qian, T.: Phase derivative of Nevanlinna functions and applications. Math. Methods Appl. Sci. **32**, 253–263 (2009)
25. Qian, T.: Intrinsic mono-component decomposition of functions: an advance of Fourier theory. Math. Methods Appl. Sci. **33**(7), 880–891 (2010)
26. Qian, T., Wang, Y.-B.: Adaptive Fourier series—a variation of greedy algorithm. Adv. Comput. Math. **34**(3), 279–293 (2011)
27. Qian, T., Yang, Y.: Hilbert transforms on the sphere with the Clifford analysis setting. J. Fourier Anal. Appl. **15**(6), 753–774 (2009)
28. Stein, E., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, NJ (1987)