Complex Anal. Oper. Theory (2011) 5:313–323
DOI 10.1007/s11785-009-0043-z

Complex Analysis and Operator Theory

Nonlinear Equations with Infinitely many Derivatives

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Received: 8 October 2009 / Accepted: 27 October 2009 / Published online: 17 November 2009

Abstract We study the *generalized bosonic string equation*

$$\Delta e^{-c\,\Delta}\phi = U(x,\phi), \quad c > 0$$

on Euclidean space $\mathbb{R}^n$. First, we interpret the nonlocal operator $\Delta e^{-c\,\Delta}$ using entire vectors of Δ in $L^2(\mathbb{R}^n)$, and we show that if $U(x,\phi) = \phi(x) + f(x)$, in which $f \in L^2(\mathbb{R}^n)$, then there exists a unique *real-analytic* solution to the Euclidean bosonic string in a Hilbert space $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$ we define precisely below. Second, we consider the case in which the potential $U(x,\phi)$ in the generalized bosonic string equation depends nonlinearly on ϕ, and we show that this equation admits *real-analytic* solutions in $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$ under some symmetry and growth assumptions on U. Finally, we show that the above given equation admits real-analytic solutions in $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$ if $U(x,\phi)$ is suitably near $U_0(x,\phi) = \phi$, even if no symmetry assumptions are imposed.

Communicated by Palle Jorgensen.

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1 Introduction

In this paper we begin our study of linear and nonlinear equations with an infinite number of derivatives. Our motivation comes from string theory and cosmology [3,4,6–8,17,21,23–26]. Specifically, in bosonic string theory on Euclidean space $\mathbb{R}^n$ appears the nonlocal Lagrangian density [7]

$$\mathcal{L}(\phi) = \phi\, \Delta e^{-c\,\Delta}\phi - \mathcal{U}(x,\phi), \quad c > 0, \tag{1}$$

in which $x \in \mathbb{R}^n$. Applying formally the variational principle we obtain the nonlocal equation of motion

$$\Delta e^{-c\,\Delta}\phi - U(x,\phi) = 0, \quad c > 0, \tag{2}$$

where $U = \partial\mathcal{U}/\partial\phi$. Equation (2) encompasses (in hyperbolic signature) the bosonic string [15], and a simplified case of the supersymmetric string [7]. Other nonlocal equations similar to (2) also arise as important toy models of string theory; for example, the equation

$$p^{(1/2)\,\partial_t^2}\phi = \phi^p \tag{3}$$

describes the dynamics of the open p-adic string for the scalar tachyon field (see [4,8,17,23–25] and references therein). This equation has been studied rigorously via integral equations of convolution type in [23,24]. It has been also noted that in the limit $p \to 1$, Eq. (3) becomes the *local* logarithmic Klein–Gordon equation [5,10].

Our goal is to study global existence, uniqueness and regularity of solutions to the nonlocal equation (2). *Linear* nonlocal equations appeared in mathematics already in the 1930's, as mentioned in [3], and more recently they have been studied in connection with the modern theory of pseudo-differential operators, see for instance [9,22]. It is interesting, however, that the full machinery of pseudo-differential operators is not necessary for the study of Eq. (2). Indeed, we attach a meaning to the formal nonlocal operator $\Delta\, e^{-c\,\Delta}$ directly by using the theory of entire vectors (see [11] and the recent [12]) and the fact that the Laplacian operator Δ in $L^2(\mathbb{R}^n)$ is the generator of an analytic semigroup with angle $\pi/2$, see [2]. Then we solve Eq. (2) in the Hilbert space $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$, to be defined precisely below, of C^∞ functions on which the operator $\Delta\, e^{-c\,\Delta} - I$ acts naturally. This approach allows us to study existence, regularity and analyticity of solutions not only in the linear case, but also for a large class of nonlinear problems (2).

We organize this work as follows. In Sect. 2 we define the Hilbert space $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$ for each $c \in (0,\infty)$. In this space we can understand the operator $\Delta\, e^{-c\,\Delta}$ using convergent series. We stress that $e^{-c\,\Delta}$, $c \in (0,\infty)$, *is not* the standard diffusion semigroup, and therefore interpreting $\Delta\, e^{-c\,\Delta}$ as a *bona fide* linear operator does require some extra work, as we show in Sect. 2. In Sect. 3 we show existence, uniqueness and *analyticity* of solutions to (2) if $U(x,\phi) = \phi(x) + f(x)$, $f \in L^2(\mathbb{R}^n)$. Actually, once we define correctly the spaces on which $\Delta\, e^{-c\,\Delta}$ acts, solving the linear equation $\Delta e^{-c\,\Delta}\phi - \phi = f(x)$ is straightforward, but proving analyticity directly is a more delicate matter: it appears this is the first time such a result is formally stated. In Sect. 4 we consider the case in which $U(x, \phi)$ is a nonlinear function of ϕ and a radial function of $x \in \mathbb{R}^n$. We show the existence of analytic solutions to (2). Thus, in our setting we can prove rigorously—Euclidean versions of—some analyticity claims frequently made in physical literature. For example, it is stated in [17] that solutions to the p-adic string equation (3) should be analytic. Finally, also in Sect. 4, we study Eq. (2) on $\mathbb{R}^n$ without symmetry assumptions, and we obtain existence of analytic solutions to this equation for functions $U(x, \phi)$ close to the "identity" $U_0(x, \phi) = \phi$. Our main tool here is simply the inverse function theorem on Banach spaces [1]. We interpret this result as an Euclidean analog of the interesting perturbation analysis of equations such as (3) carried out in the recent paper [4].

In this short paper we state our theorems and mostly sketch their proofs. Full details will be included elsewhere [13].

2 Preliminaries

First of all we introduce appropriate spaces of functions so as to interpret the nonlocal terms appearing in Eq. (2), and also in Lagrangian densities such as (1). Let us denote by L_c the formal operator

$$L_c = \Delta e^{-c\Delta} - Id.$$

Definition 2.1 For $c > 0$, the space $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$ is the subspace of $L^2(\mathbb{R}^n)$ given by:

$$\mathcal{H}^{c,\infty}(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left(1 + |\xi|^2 e^{c|\xi|^2}\right)^2 |\mathcal{F}(f)|^2 \, d\xi < \infty \right\},$$

in which $\mathcal{F}$ stands for the Fourier transform in $L^2(\mathbb{R}^n)$.

We remark that $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$ strictly contains Dubinskii's space of test functions [9] and in fact, as we show in Proposition 2.1 below, it is quite "large" in a specific sense. Let us consider the function

$$\widehat{K}(\xi) = \frac{1}{1 + |\xi|^2 e^{c|\xi|^2}}.$$

Clearly this function belongs to the Schwarz space $\mathcal{S}(\mathbb{R}^n)$, and therefore [19] there exists a unique function $K \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transform $\mathcal{F}(K)$ is precisely $\widehat{K}$. We have

Proposition 2.1 *A function $h \in L^2(\mathbb{R}^n)$ belongs to the space $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$ if and only if $h = K * f$ for some $f \in L^2(\mathbb{R}^n)$. Furthermore, if we endow $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$ with the inner product*

$$\langle f, g \rangle = \int_{\mathbb{R}^n} \left(1 + |\xi|^2 e^{c|\xi|^2}\right)^2 \mathcal{F}(f)(\xi)\overline{\mathcal{F}(g)(\xi)} \, d\xi,$$

then $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$ *is a Hilbert space and the transformation* $\mathcal{K} : L^2(\mathbb{R}^n) \to \mathcal{H}^{c,\infty}(\mathbb{R}^n)$ *given by*

$$\mathcal{K}(f) = K * f$$

is an isometric isomorphism.

The proof uses properties of the Fourier transform, see [19]. We will need below the following corollary of Proposition 2.1: If $\mathcal{D}$ is a dense subset of $L^2(\mathbb{R}^n)$, then the set of functions

$$\left\{u_f = K * f : f \in \mathcal{D}\right\}$$

is dense in $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$.

We now state two important technical observations:

Lemma 2.1 *For each non-negative number s, the embedding* $\mathcal{H}^{c,\infty}(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n)$ *holds. In other words,* $\|f\|_{H^s(\mathbb{R}^n)} \le C(s)\|f\|_{\mathcal{H}^{c,\infty}(\mathbb{R}^n)}$ *for some constant* $C(s) > 0$.

Proof Our claim follows immediately from the elementary properties of the map $x \mapsto e^x$. □

Lemma 2.2 *For each* $m \ge 0$ *the embedding* $\mathcal{H}^{c,\infty}(\mathbb{R}^n) \hookrightarrow C^m(\mathbb{R}^n)$ *holds.*

Proof Our claim follows from Lemma 2.1 and the Sobolev embedding theorem as it appears for example in [20]. □

Now we can interpret the operator $L_c = \Delta\, e^{-c\,\Delta} - Id$ as an operator on $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$. We recall that if A is a closed operator on a Banach space X, a vector $u \in C^\infty(A) = \cap_{n\ge 0} D(A^n)$, where $D(A)$ is the domain of A, is called an *entire vector of A* [11] if the series

$$\sum_{k=0}^{\infty} \frac{z^k}{k!}\, A^k\, u$$

converges for all $z \in \mathbb{C}$. We have the following proposition:

Proposition 2.2 *If* $u \in \mathcal{H}^{c,\infty}(\mathbb{R}^n)$ *then we have:*

(i) $\Delta^k u \in L^2(\mathbb{R}^n)$ *for each* $k \in \mathbb{N}$.
(ii) $|\xi|^2 e^{c|\xi|^2}\mathcal{F}(u) \in L^2(\mathbb{R}^n)$.
(iii) *Let* $\mathcal{E}(\Delta)$ *be the set of all entire vectors of* Δ *in* $L^2(\mathbb{R}^n)$*, let* $f \in \mathcal{E}(\Delta)$ *and set* $u = K * f$*. Then, the expression*

$$\Delta \sum_{k=0}^{\infty} \frac{(-c)^k}{k!}\, \Delta^k u$$

defines a vector in $L^2(\mathbb{R}^n)$.

Claim (i) follows from Lemma 2.2, Proposition 2.1 and the Plancherel theorem, and Claim (ii) is elementary. Claim (iii) is more delicate: using the hypothesis that $f \in \mathcal{E}(\Delta)$, and the technical observation that if $K \in C^2(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$ then

$$(\Delta\, K) * f = \Delta\,(K * f),$$

we show that

$$\mathcal{F}(\Delta K)\, e^{c|\xi|^2}\mathcal{F}(f) = \mathcal{F}\left(\Delta \sum_{k=0}^{\infty} \frac{(-c)^k}{k!}\, \Delta^k (K * f)\right), \tag{4}$$

and that the left hand side of (4) is in $L^2(\mathbb{R}^n)$. Therefore (iii) follows by applying inverse Fourier transform.

Proposition 2.2 allows us to interpret the operator $L_c = \Delta e^{-c\Delta} - Id$ on $u = K * f$, $f \in \mathcal{E}(\Delta)$, as

$$L_c u = \Delta \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} \underbrace{\Delta \circ \ldots \circ \Delta}_{n-\text{times}} u - u, \tag{5}$$

since the series in the right hand side of (5) converges in $L^2(\mathbb{R}^n)$. For *any* $u \in \mathcal{H}^{c,\infty}(\mathbb{R}^n)$ we now prove:

Theorem 2.1 *The operator* $L_c = \Delta e^{-c\Delta} - Id$ *can be represented as*

$$L_c u = -\mathcal{F}^{-1}\left(\mathcal{F}(u) + |\,\xi\,|^2 e^{c|\,\xi\,|^2}\mathcal{F}(u)\right) \tag{6}$$

for any $u \in \mathcal{H}^{c,\infty}(\mathbb{R}^n)$. *Furthermore,* L_c *is an isometry from* $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$ *into* $L^2(\mathbb{R}^n)$.

The proof of Theorem 2.1 begins by taking Fourier transform in Equation (5). We get

$$L_c u = -\mathcal{F}^{-1}\left(\mathcal{F}(u) + |\,\xi\,|^2 e^{c|\,\xi\,|^2}\mathcal{F}(u)\right) \quad \text{for } \ u = K * f\ , \quad f \in \mathcal{E}(\Delta), \tag{7}$$

where by $\mathcal{F}^{-1}$ we have denoted the inverse to the Fourier transform. We now notice that the set $\mathcal{E}(\Delta)$ is *dense* in $L^2(\mathbb{R}^n)$, as it follows from [2] and [12], and therefore (see our remarks after Proposition 2.1) the set of all the $u = K * f \in \mathcal{H}^{c,\infty}(\mathbb{R}^n)$, $f \in \mathcal{E}(\Delta)$, is dense in $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$. Equation (7) for L_c is then valid *for all* $u \in \mathcal{H}^{c,\infty}(\mathbb{R}^n)$. Expression (7) also tells us that the operator L_c is an isometry from $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

3 The Linear Equation $L_c u = f$

The linear problem $L_c u = f$ on $\mathbb{R}^n$ can be solved completely using the foregoing set-up:

Theorem 3.1 *For each $c > 0$ and each $f \in L^2(\mathbb{R}^n)$, there exists a unique solution $u_f \in \mathcal{H}^{c,\infty}(\mathbb{R}^n)$ to the linear problem*

$$L_c u = f. \tag{8}$$

Moreover, the equation $\|u_f\|_{\mathcal{H}^{c,\infty}(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$ holds.

The proof consists in checking that

$$u_f = -\mathcal{F}^{-1}\left(\frac{\mathcal{F}(f)}{1 + |\xi|^2 e^{c|\xi|^2}}\right) = -K * f \tag{9}$$

is a solution to problem (8). Proposition 2.1 implies that u_f is in the domain of L_c.

Now, as stated in Sect. 1, a non-trivial assumption appearing frequently in physical literature is that solutions to nonlocal equations should be analytic, see for example [17] respect to the p-adic string equation (3). In the linear case we can prove the following rigorous result:

Theorem 3.2 *Let us assume $f \in L^2(\mathbb{R}^n)$. Then, the solution u to problem (8) is analytic, i.e., $u \in C^\omega(\mathbb{R}^n)$.*

Proof Let us fix $z \in \mathbb{R}^n$. We represent the solution to problem (8) in the form

$$u(z) = -\int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{e^{2\pi i \xi (z-y)} f(y)}{1 + |\xi|^2 e^{c|\xi|^2}} d\xi dy,$$

and we apply dominated convergence in $L^1(\mathbb{R}^n)$ in order to compute derivatives of u. We get the estimate

$$\begin{aligned}\left|D^k u(z)\right| &\leq (2\pi)^{|k|} \int\limits_{\mathbb{R}^n} \left(\left|\frac{|\xi|^{|k|} e^{2\pi i \xi z}}{1 + |\xi|^2 e^{c|\xi|^2}}\right| \left|\int\limits_{\mathbb{R}^n} e^{-2\pi i \xi y} f(y) dy\right|\right) d\xi \\ &= (2\pi)^{|k|} \int\limits_{\mathbb{R}^n} \left(\frac{|\xi|^{|k|}}{1 + |\xi|^2 e^{c|\xi|^2}} \left|\mathcal{F}(f(\xi))\right|\right) d\xi.\end{aligned}$$

Now, using Schwarz inequality and Plancherel theorem we get

$$|D^k u(z)| \leq (2\pi)^{|k|} \|f\|_{L^2} \left(\int\limits_{\mathbb{R}^n} \left(\frac{|\xi|^{|k|}}{1 + |\xi|^2 e^{c|\xi|^2}}\right)^2 d\xi\right)^{\frac{1}{2}}.$$

We are interested in estimating the last term in the above inequality. Introducing polar coordinates we can write

$$\int_{\mathbb{R}^n} \left(\frac{|\xi|^{|k|}}{1+|\xi|^2 e^{c|\xi|^2}} \right)^2 d\xi \le C \int_0^\infty \frac{r^{2k+n-1}}{1+r^4 e^{2cr^2}} dr.$$

Next, using the inequality

$$r^{2|k|+n-3} \le \frac{(2|k|+n-3)!}{(2c)^{\frac{2|k|+n-3}{2}}} e^{2cr^2}$$

we get

$$|D^k u(z)| \le \frac{A\|f\|_{L^2}}{(2c)^{\frac{n-3}{4}}} \left(\frac{2\pi}{\sqrt{2c}} \right)^{|k|} ((2|k|+n-3)!)^{\frac{1}{2}},$$

where

$$A = \sqrt{C} \left(\int_0^\infty \frac{e^{2cr^2}}{1+r^4 e^{2cr^2}} dr \right)^{\frac{1}{2}}.$$

The Taylor series for u at z is

$$\sum_k \frac{D^k u(z)}{k!} (x-z)^k,$$

where the sum is taken over all multi-indices k. We show that this power series converges, provided

$$|z-x| < R := \frac{c}{6\sqrt{2}\pi n^2}.$$

In order to show this claim, we have to estimate the remainder term:

$$R_m(x) = \sum_{|k|=m} \frac{D^k u(z+t(x-z))(x-z)^k}{k!},$$

where $t \in [0, 1]$. Hence, thanks to the elementary inequality $\frac{1}{k!} \le \frac{n^{|k|}}{|k|!}$ we get

$$|R_m(x)| \le \frac{A\|f\|_{L^2}}{(2c)^{\frac{n-3}{4}}} \left(\frac{2\pi}{\sqrt{2c}} n^2 R \right)^m \frac{((2m+n-3)!)^{\frac{1}{2}}}{m!}.$$

Now, using Stirling's formula ($n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \delta_n$, where $\delta_n \to 1$) we can write

$$|R_m(x)| \leq \frac{A\|f\|_{L^2} 3^{\frac{1}{4}+n}}{(2ce)^{\frac{n-3}{4}}} \left(\frac{2\pi}{\sqrt{2c}} n^2 3R\right)^m m^{n-\frac{1}{4}} A_m,$$

where $A_m \to 1$. Since $R = \frac{c}{6\sqrt{2}\pi n^2}$ we get that $|R_m(x)| \to 0$. □

This theorem has the following consequence on the structure of our Hilbert space $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$:

Corollary 3.1 *If u belongs to the Hilbert space $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$, then u is real-analytic.*

Proof If u belongs to $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$, Proposition 2.1 tells us that we can write it as $u = K * f$ for some f in $L^2(\mathbb{R}^n)$. This means that u is the unique solution to the linear problem $L_c u = -f$, and so by the previous theorem, u is real-analytic. □

4 The Nonlinear Equation $\Delta e^{-c\,\Delta}\phi - U(x, \phi) = 0$

4.1 The Radial Case

We show existence of solutions to the nonlocal equation (2) if the nonlinearity U is spherically symmetric with respect to x, that is, if it satisfies $U(x, y) = U(|x|, y)$. Instead of the spaces used in the previous section, we use Banach spaces of radial functions:

Definition 4.1

$$L^2_r(\mathbb{R}^n) = \overline{C^\infty_{0,r}(\mathbb{R}^n)}^{L^2(\mathbb{R}^n)}, \quad H^s_r(\mathbb{R}^n) = \overline{C^\infty_{0,r}(\mathbb{R}^n)}^{H^s(\mathbb{R}^n)},$$

in which $C^\infty_{0,r}(\mathbb{R}^n)$ denotes the space of smooth radial functions with compact support on $\mathbb{R}^n$. In addition, we denote by $\mathcal{H}^{c,\infty}_r(\mathbb{R}^n)$ the closed subspace of all radial functions in $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$.

We can check, by using Proposition 2.1, that if the non-homogeneity f appearing in problem (8) is a radial function, so is its unique solution:

Theorem 4.1 *If $f \in L^2_r(\mathbb{R}^n)$, then the unique solution $u \in \mathcal{H}^{c,\infty}(\mathbb{R}^n)$ to problem (8) is a radial function. Moreover, the equality $\|u\|_{\mathcal{H}^{c,\infty}_r(\mathbb{R}^n)} = \|f\|_{L^2_r(\mathbb{R}^n)}$ holds.*

Our first existence and regularity theorem is:

Theorem 4.2 *Let us assume that U is spherically symmetric with respect to x. Furthermore, we assume that there exist constants $\alpha, \beta \geq 1$, a constant $C > 0$ and a function $h \in L^2(\mathbb{R}^n)$ such that the following two inequalities hold:*

$$|U(x, y)| \leq C\left(h(x) + |y|^\alpha\right), \quad \left|\frac{\partial}{\partial y} U(x, y)\right| \leq C\left(1 + |y|^\beta\right). \tag{10}$$

Then, there exists a C^∞ *radial solution to the problem*

$$\Delta e^{-c\Delta}\phi - U(x,\phi) = 0. \tag{11}$$

The proof rests on a Schauder fixed point type argument. The crucial technical issue we need to take care of is that we cannot use standard results such as the Rellich–Kondrakov theorem [20], since we are working on the whole of $\mathbb{R}^n$. Instead, we need to apply Lions' compactness theorem for radial spaces (see [14,16]).

Remark If the function h appearing in the enunciate of the theorem is the zero function, then the solution ϕ to (11) obtained via Theorem 4.2 may be simply the trivial solution $\phi(x) = 0$. On the other hand, if $h \neq 0$, then it is possible to show that the solution ϕ cannot be trivial.

We now state an analyticity result in the nonlinear case which can be proved using our Theorems 3.2 and 4.2:

Theorem 4.3 *Let us assume that* U *is spherically symmetric with respect to* x *and that it satisfies the hypothesis of Theorem* 4.2*. Then, there exists a* C^ω*- solution to problem* (11)*.*

4.2 The General Case

We now consider the nonlinear equation

$$\Delta e^{-c\Delta}\phi = U(x,\phi) \tag{12}$$

without assuming that U is spherically symmetric. We use the fact that we have already proven the existence of a unique solution ϕ_{linear} to the linear problem

$$\Delta e^{-c\Delta}\phi - \phi = f \tag{13}$$

for any $f \in L^2(\mathbb{R}^n)$, so that, in particular, the homogeneous problem $\Delta e^{-c\Delta}\phi - \phi = 0$ admits the unique solution $\phi_0 \equiv 0$. We then find solutions to (12) for $U(x,\phi)$ near $U(x,\phi) = \phi$. Our set-up is the following:

We consider functions ϕ in the space $\mathcal{H}^{c,\infty}(\mathbb{R}^n)$ and complex-valued functions $U(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^2$. We also need the normed space of all functions U satisfying the conditions

1. $U \in C^1(\mathbb{R}^n \times \mathbb{R}^2)$, and
2. There exist $\alpha \geq 1$, a constant C and an L^2 function h such that

$$|U(x,y)| \leq C\left(h(x) + |y|^\alpha\right),$$

equipped with the Wiener norm $\|U\| = \int_{\mathbb{R}^n\times\mathbb{R}^2} |\, U(x,y)\,|^2\, e^{-(\|x\|^2+\|y\|^2)}dxdy$. We then define the Banach space $\mathcal{C}$ as the completion of this normed space, and we solve

Eq. (12) via the implicit function theorem for Banach spaces [1]. More precisely, we define the C^1 function

$$g(U, \phi) = \Delta e^{-c\Delta}\phi - U(\,\cdot\,, \phi) \tag{14}$$

on $\mathcal{C} \times \mathcal{H}^{c,\infty}(\mathbb{R}^n)$ with values in $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^2, e^{-\|y\|^2}dy)$, and we take $U_0(x, y) = y$ in $\mathcal{C}$ and $\phi_0 = 0$. Obviously we have

$$g(U_0, \phi_0) = \Delta e^{-c\Delta}\phi_0 - \phi_0 = 0,$$

and we can prove

Theorem 4.4 *There is a neighborhood $\mathcal{U} \subseteq \mathcal{C}$ of U_0, a neighborhood $\mathcal{W}$ of $0 \in L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^2, e^{-\|y\|^2}dy)$, and a unique function $\Phi : \mathcal{U} \times \mathcal{W} \to \mathcal{V} \subseteq \mathcal{H}^{c,\infty}(\mathbb{R}^n)$, in which $\mathcal{V}$ is a neighborhood of ϕ_0, such that for all $(U, w) \in \mathcal{U} \times \mathcal{W}$, we have*

$$g(U, \Phi(U, w)) = w. \tag{15}$$

This theorem, together with Corollary 3.1, allows us to obtain existence of analytic solutions to Eq. (12):

Corollary 4.1 *For any function $U(x, y)$ suitably near $U_0(x, y) = y$, there exists a solution $\phi \in \mathcal{H}^{c,\infty}(\mathbb{R}^n)$ to Eq. (12).*

We finish this paper with the remark that we can also solve nonlocal equations such as (12) on compact Riemannian manifolds (M, g) via Hodge's theorem [18] on the spectral properties of the Laplacian operator Δ_g on (M, g). We interpret these results as rigorous analogs of the constructions appearing in [4] for linear nonlocal equations. We once more refer the reader to [13] for details.

Acknowledgments We are very grateful to Profs. Niky Kamran and Jacek Szmigielski, and also to an anonymous referee, for their constructive criticism and interesting comments. Our research has been partially supported by the project "Redes de Anillos R04", CONICYT (P. Górka), FONDECYT grant #3100019 (P. Górka), FONDECYT grant #1070191 (E.G. Reyes), and FONDECYT grant #1070127 (H. Prado).

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