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Non Commutative Functional Calculus: Bounded Operators

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Abstract In this paper we develop a functional calculus for bounded operators defined on quaternionic Banach spaces. This calculus is based on the new notion of slice-regularity, see Gentili and Struppa (Acad Sci Paris 342:741–744, 2006) and the key tools are a new resolvent operator and a new eigenvalue problem.

Keywords Functional calculus · Spectral theory · Bounded operators

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1 Introduction

Let V be a Banach space over a (possibly skew) field $\mathbb{F}$ and let E be the Banach space of linear operators acting on it. If $T \in E$, and if $\mathbb{F} = \mathbb{C}$, then the standard eigenvalue

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problem seeks the values $\lambda \in \mathbb{C}$ for which $\lambda\mathcal{I} - T$ is not invertible. This situation is well known and leads to what is known as Functional Calculus (FC). Indeed, in the complex case, one defines the spectrum of an operator as the set for which $\lambda\mathcal{I} - T$ is not invertible. The key observation for the development of the FC is the fact that the inverse of $\lambda\mathcal{I} - T$ formally coincides with a Cauchy kernel which allows integral representation for holomorphic functions. As a consequence, one is able to consider for any function f holomorphic on the spectrum of T its value $f(T)$ which is formally defined as

$$f(T) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda\mathcal{I} - T)^{-1} f(\lambda) d\lambda$$

where Γ is a closed curve surrounding the spectrum of T. The important fact is that such a definition coincides with the obvious meaning when f is a polynomial and that this definition behaves well with respect to linear combinations, product and composition of functions, i.e., $(af + bg)(T) = af(T) + bg(T)$, $a, b \in \mathbb{C}$, $(fg)(T) = f(T)g(T)$, $(f \cdot g)(T) = f(g(T))$. For the basic theory in the complex case a classical reference is [4]. Important applications of this theory to specific operators are now an important field of investigation (see for example the references in [7]).

Several new difficulties arise when one tries to deal with quaternionic linear operators. First of all, when working in a non commutative setting, it is necessary to specify that the operators are linear, for example, on the right, i.e. $T(v\alpha + w\beta) = T(v)\alpha + T(w)\beta$, for $\alpha, \beta \in \mathbb{H}$. In addition, it turns out that there are two different eigenvalue problems. The so called right eigenvalue problem, i.e. the search for non zero vectors satisfying $T(v) = v\lambda$, is widely studied by physicists. The crucial fact is that whenever there is a non real eigenvalue λ then all quaternions belonging to the sphere $r^{-1}\lambda r$, $r \in \mathbb{H} \setminus \{0\}$, are eigenvalues. This fact allows to choose the phase and to work, for example, with complex eigenvalues.

Note however that the operator of multiplication on the right is not a right linear operator, and so the operator $\mathcal{I}\lambda - T$ is not linear. On the other hand, the operator $\lambda\mathcal{I} - T$ is right linear but, in general, one cannot choose the phase as the eigenvalues do not necessarily lie on a sphere. Even more important, in the complex setting the inverse $(\lambda\mathcal{I} - T)^{-1}$ is related to a Cauchy kernel useful in the notion of Cauchy integral. In the quaternionic setting it is not even clear which type of generalized holomorphy must be used. The regularity in the sense of Fueter (see e.g. [2]) does not seem to give a good notion of exponential function and does not allow to introduce polynomials (or even powers) of operators. While many authors have looked at the quaternionic eigenvalue problem because of its applications to physics, [1], it is safe to say that until now there has been no general treatment which could allow the construction of a FC with quaternionic spectrum, as we describe in this paper.

As we will show, however, a recent new theory, the theory of slice-regular functions, see [5,6] where they are called Cullen-regular functions, allows to show that power series in the quaternionic variable are regular, so this new notion seems to be the correct setting in which a FC can be introduced and developed. The Cauchy kernel series $\sum_{n\geq 0} q^n s^{-n-1}$, $q, s \in \mathbb{H}$, which is used to write a Cauchy formula for slice-regular functions does not coincide, in general, with $(s-q)^{-1}$, thus the linear operator $S^{-1}(s,T)=\sum_{n\geq 0} T^n s^{-n-1}$ is not, in general, the inverse of $(s\mathcal{I}-T)$. The key idea is to identify the operator whose inverse is $\sum_{n\geq 0} T^n s^{-n-1}$. This new operator will give us a new notion of spectrum (the so called S-spectrum) which will allow to introduce a FC As we will see, this spectrum, as the right-spectrum, will allow the choice of the phase. In this paper we confine our attention to the case in which the operators under consideration are bounded. However, as we will show in a subsequent paper, much of this theory can be extended to the unbounded case (which of course has significant physical applications).

We close this introduction by pointing out that the spectral theory can be extended to the case of n-tuples of operators. For the complex case, the reader is referred to [10] (for the commuting case) and to [3,7,11] for the case in which the operators do not commute. We plan to come back to this issue in a subsequent paper and show how our theory can be applied to this situation as well.

2 Regular Functions and Linear Bounded Operators

2.1 Regular Functions

In this section we collect some basic results that we need in the sequel.

Let $\mathbb{H}$ be the real associative algebra of quaternions with respect to the basis $\{1, i, j, k\}$ satisfying the relations

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, ij = -ji = k, \\ &jk = -kj = i, ki = -ik = j. \end{aligned}$$

We will denote a quaternion q as $q = x_0 + ix_1 + jx_2 + k_3$, $x_i \in \mathbb{R}$, its conjugate as $\bar{q} = x_0 - ix_1 - jx_2 - kx_3$, and we will write $|q|^2 = q\overline{q}$. In the recent papers [5,6], the authors introduced a new notion of regularity for functions on $\mathbb{H}$. We will recall here both the definition and its most salient consequences.

Let $\mathbb{S}$ be the sphere of purely imaginary unit quaternions, i.e.

$$\mathbb{S} = \{q = ix_1 + jx_2 + kx_3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Definition 2.1 Let $U \subseteq \mathbb{H}$ be an open set and let $f : U \to \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $L_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I. We say that f is a left regular function if for every $I \in \mathbb{S}$

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+Iy) = 0,$$

and we say it is right regular if for every $I \in \mathbb{S}$

$$\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y} I\right) f_I(x+Iy)=0.$$

Remark 2.2 Left regular functions on $U \subseteq \mathbb{H}$ form a right vector space $\mathcal{R}(U)$ on $\mathbb{H}$ and right regular functions on $U \subseteq \mathbb{H}$ form a left vector space. It is not true, in general, that the product of two regular functions is regular.

Example 2.3 The key feature of this notion of regularity is the fact that polynomials $\sum_{n=0}^{N} q^n a_n$ in the quaternion variable q, and with quaternionic coefficients a_n are left regular (while polynomials $\sum_{n=0}^{N} a_n q^n$ are right regular). Moreover, any power series $\sum_{n=0}^{+\infty} q^n a_n$ (or more in general $\sum_{n=0}^{+\infty}(q-p_0)^n a_n$, $p_0 \in \mathbb{R}$) is left regular in its domain of convergence. As an example, the function $R(q)=(q-p_0)^{-1}$, $p_0 \in \mathbb{R}$ which corresponds to the Cauchy kernel (see below), is regular for $|q|<|p_0|$.

In fact, every regular function can be represented as a power series, [6]:

Theorem 2.4 *If $B=B(0,R)$ is the open ball centered in the origin with radius $R>0$ and $f: B \rightarrow \mathbb{H}$ is a left regular function, then f has a series expansion of the form*

$$f(q)=\sum_{n=0}^{+\infty} q^n \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0)$$

converging on B. Analogously, if f is right regular it can be expanded as

$$f(q)=\sum_{n=0}^{+\infty} \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0) q^n.$$

Remark 2.5 An analogue statement holds for regular functions in an open ball centered in $p_0 \in \mathbb{R}$.

Note that, even though the definition of regular function involves the direction of the unit quaternion I, the coefficients of the series expansion do not depend at all from the choice of I.

Remark 2.6 From now on we will not specify whether we are considering left or right regular functions, since the context will clarify it.

A main result in the theory of regular functions is the analogue of the Cauchy integral formula. In order to state the result we need some notation. Given a quaternion $q=x_0+ix_1+jx_2+kx_3$ let us denote its real part x_0 by $Re[q]$ and its imaginary part $ix_1+jx_2+kx_3$ by $Im[q]$. We set

$$I_q=\begin{cases} \frac{Im[q]}{|Im[q]|} & \text{if } Im[q] \neq 0 \\ \text{any element of } \mathbb{S} & \text{otherwise} \end{cases}$$

We have the following [6]:

Theorem 2.7 *Let $f : B(0, R) \to \mathbb{H}$ be a regular function and let $q \in B(0, R)$. Then*

$$f(q) = \frac{1}{2\pi} \int_{\partial \Delta_q(0,r)} (\zeta - q)^{-1} \, d\zeta_{I_q} \, f(\zeta)$$

where $d\zeta_{I_q} = -I_q d\zeta$ and $r > 0$ is such that

$$\overline{\Delta_q(0, r)} := \{x + I_q y \mid x^2 + y^2 \leq r^2\}$$

contains q and is contained in $B(0, R)$.

The proof of the theorem relies on the following result, which is of independent interest:

Theorem 2.8 *Let $f : B(0, R) \to \mathbb{H}$ be a real differentiable function and let $q \in B(0, R)$. If f is regular and r is such that $\overline{\Delta_q(0, r)}$ is contained in $B(0, R)$, then*

$$\int_{\partial \Delta_q(0,r)} d\zeta f(\zeta) = 0.$$

2.2 Regular Functions with Values in a Banach Space

We now turn our attention to functions $f : \mathbb{H} \to E$ where E denotes a left quaternionic Banach space. Let us revise the definition of integral in this setting.

Definition 2.9 Let E be a left quaternionic Banach space. A function $f : \mathbb{H} \to E$ is said to be regular if there exist an open ball $B = B(0, R) \subseteq \mathbb{H}$ and a sequence $\{a_n\}$ of elements of E such that, for every point $q \in B(0, R)$ the function $f(q)$ can be represented by the following series

$$f(q) = \sum_{n=0}^{+\infty} q^n a_n, \tag{1}$$

converging in the norm of E for any q such that $|q| < R$.

Consider $E' = \text{Hom}(E, \mathbb{H})$, the dual space of E, and let $x' \in E'$ be a linear continuous functional. If Γ is any compact set of B, then by the left linearity and the continuity of x' we have

$$< \int_\Gamma d\zeta \; f(\zeta), x' >= \int_\Gamma d\zeta \; < f(\zeta), x' > . \tag{2}$$

To prove the Cauchy integral formulas in the case of vector functions we need the following theorem which is a consequence of the Hahn–Banach theorem.

Theorem 2.10 *Let E be a left quaternionic Banach space and let $x \in E$. If for every linear and continuous functional x' we have $< x, x' >= 0$ then $x = 0$.*

Proof We observe that the Hahn–Banach theorem still holds true for left (or right) vector spaces on $\mathbb{H}$. Let $x \in E$, x non zero and let W be the subspace generated by x. Let $\xi' : W \to \mathbb{H}$ be the functional such that $\xi'(xq) = \|x\|q$. Obviously, ξ' is left linear continuous functional of norm 1. By the Hahn–Banach theorem there exists an extension x' of ξ' to all of E with $\|x'\| = 1$, moreover $x'(x) = \|x\|$ and the statement follows. □

Theorem 2.11 *Let E be a left quaternionic Banach space and let $f : \mathbb{H} \to E$ be a regular function on an open ball $B(0, R)$ containing $\Delta_q(0, r)$. Then*

$$\int_{\partial\Delta_q(0,r)} d\zeta \; f(\zeta) = 0. \tag{3}$$

Proof For every $x' \in E'$ and for any $q \in B(0, R)$ we have:

$$f(q) = \sum_{n=0}^{+\infty} q^n a_n, \quad (a_n \in E)$$

$$< f(q), x' >=< \sum_{n=0}^{+\infty} q^n a_n, x' >= \sum_{n=0}^{+\infty} q^n < a_n, x' > .$$

Hence, by the Abel's lemma for quaternionic power series, the function $< f(q), x' >$ is regular in $B(0, R)$. Thanks to Theorem 2.8, we have

$$\int_{\partial\Delta_q(0,r)} d\zeta \; < f(\zeta), x' >= 0.$$

By the equalities

$$0 = \int_{\partial\Delta_q(0,r)} d\zeta \; < f(\zeta), x' >=< \int_{\partial\Delta_q(0,r)} d\zeta \; f(\zeta), x' >, \quad \forall x' \in E' \tag{4}$$

and from Theorem 2.10 we obtain (3). □

Theorem 2.12 *Let E be a left Banach space and let $f : \mathbb{H} \to E$ be a regular function on $B(0, R)$. Then*

$$f(q) = \frac{1}{2\pi} \int_{\partial\Delta_q(0,r)} (\zeta - q)^{-1} \, d\zeta_{I_q} \; f(\zeta)$$

where $d\zeta_{I_q} = -I_q d\zeta$ *and* $r > 0$ *is such that*

$$\overline{\Delta_q(0,r)} = \{x + I_q y \mid x^2 + y^2 \leq r^2\} \tag{5}$$

contains q *and is contained in* $B(0, R)$.

Proof The statement follows as in the complex case. □

2.3 Linear Bounded Quaternionic Operators

We conclude this section with a quick discussion of linear operators on a right quaternionic vector space.

Definition 2.13 Let V be a right vector space on $\mathbb{H}$. A map $T : V \to V$ is said to be right linear if

$$T(u+v) = T(u) + T(v)$$
$$T(us) = T(u)s$$

for all $s \in \mathbb{H}$ and all $u, v \in V$.

Remark 2.14 Note that the set of right linear maps is not a quaternionic left or right vector space. Only if V is both left and right vector space, then the set $\mathrm{End}(V)$ of right linear maps on V is both a left and a right vector space on $\mathbb{H}$, since in that case we can define $(aT)(v) := aT(v)$ and $(Ta)(v) := T(av)$. The composition of operators can be defined in the usual way: for any two operators $T, S \in \mathrm{End}(V)$ we have

$$(TS)(u) = T(S(u)), \qquad \forall u \in V.$$

In particular, we have the identity operator $\mathcal{I}(u) = u$, for all $u \in V$ and setting $T^0 = \mathcal{I}$ we can define powers of a given operator $T \in \mathrm{End}(V)$: $T^n = T(T^{n-1})$ for any $n \in \mathbb{N}$. An operator T is said to be invertible if there exists S such that $TS = ST = \mathcal{I}$ and we will write $S = T^{-1}$.

Remark 2.15 From now on we will only consider bilateral vector spaces V. The vector space $\mathrm{End}(V)$ is not an $\mathbb{H}$-algebra with respect to the composition of operators, in fact the property $s(TS) = (sT)S = T(sS)$ is not fulfilled for any $T, S \in \mathrm{End}(V)$ and any $s \in \mathbb{H}$. In this case we have $T(sS)(u) = T(sS(u))$ while $(sT)S(u) = sT(S(u))$. Note on the other hand that $\mathrm{End}(V)$ is trivially an algebra over $\mathbb{R}$.

Definition 2.16 Let V be a bilateral quaternionic Banach space. We will denote by $\mathcal{B}(V)$ the bilateral vector space of all right linear bounded operators on V.

It is easy to verify that $\mathcal{B}(V)$ is a Banach space endowed with its natural norm.

Definition 2.17 An element $T \in \mathcal{B}(V))$ is said to be invertible if there exists $T' \in \mathcal{B}(V)$ such that $TT' = T'T = \mathcal{I}$.

It is obvious that the set of all invertible elements of $\mathcal{B}(V)$ is a group with respect to the composition of operators defined in $\mathcal{B}(V)$.

3 Algebraic Properties of the Cauchy Kernel

This section is motivated by the fact that the Cauchy kernel can be defined for non commuting variables s and q and it is the key ingredient to define a functional calculus.

Definition 3.1 Let $q, s \in \mathbb{H}$. We will call non commutative Cauchy kernel series (shortly Cauchy kernel series) the expansion

$$S^{-1}(s,q) := \sum_{n\geq 0} q^n s^{-1-n},$$

for $|q| < |s|$.

Theorem 3.2 *Let q and s be two quaternions such that $qs \neq sq$ and consider*

$$S^{-1}(s,q) := \sum_{n\geq 0} q^n s^{-1-n}.$$

Then the inverse $S(s,q)$ of $S^{-1}(s,q)$ is the non trivial solution to the equation

$$S^2 + Sq - sS = 0. \tag{6}$$

Proof Observe that

$$S^{-1}(s,q)s = \sum_{n\geq 0} q^n s^{-1-n} s = \sum_{n\geq 0} q^n s^{-n} = 1 + qs^{-1} + q^2 s^{-2} + \dots$$

and

$$qS^{-1}(s,q) = q\sum_{n\geq 0} q^n s^{-1-n} = \sum_{n\geq 0} q^{1+n} s^{-1-n} = qs^{-1} + q^2 s^{-2} + \dots$$

so that

$$S^{-1}(s,q)s - qS^{-1}(s,q) = 1$$

keeping in mind that $S^{-1}S = SS^{-1} = 1$ we get

$$S(S^{-1}s - qS^{-1})S = S^2$$

from which we obtain the proof. □

Lemma 3.3 *Let $R(s,q) := s - q$. Then $R(s,q)$ is a solution of Eq. (6) if and only if $sq = qs$.*

Proof This result follows immediately from the chain of equalities

$$(s-q)^2+(s-q)q-s(s-q)=s^2-sq-qs+q^2-s^2+sq+sq-q^2=-qs+sq$$

whose last term vanishes if and only if $sq = qs$. □

Remark 3.4 If $s = s_0 + s_1 L$, $q = q_0 + q_1 L$ for some $L \in \mathbb{S}$, then $sq = qs$.

Theorem 3.5 *Let q, $s \in \mathbb{H}$ be such that $qs \neq sq$. Then the non trivial solution of*

$$S^2 + Sq - sS = 0 \tag{7}$$

is given by

$$S(s,q)=(s+q-2\,Re[s])^{-1}(sq-|s|^2)-q=-(q-\overline{s})^{-1}(q^2-2q\,Re[s]+|s|^2). \tag{8}$$

Proof Let us begin by noticing that $S(s,q)$ is well defined: in fact, since $qs \neq sq$ by hypothesis, we have $q \neq \overline{s} = -s+2\,Re[s]$ and hence $(q-\overline{s}) = (s+q-2\,Re[s]) \neq 0$. Now, by Theorem 3.2, to prove the assertion it is enough to verify that $S(s,q) = (s+q-2\,Re[s])^{-1}(sq-|s|^2)-q$ has $S^{-1}(s,q)$ as its inverse. Consider the equality

$$S(s,q)S^{-1}(s,q) = 1 \tag{9}$$

which can be written as

$$\left[(s+q-2\,Re[s])^{-1}\left(sq-|s|^2\right)-q\right]\sum_{n\geq 0} q^n s^{-1-n} = 1$$

and also as

$$(s+q-2\,Re[s])^{-1}\left(sq-|s|^2\right)\sum_{n\geq 0} q^n s^{-1-n} - q\sum_{n\geq 0} q^n s^{-1-n} = 1. \tag{10}$$

If we multiply by $(s+q-2\,Re[s]) \neq 0$ both sides of (10), we obtain the equivalent equality

$$(sq-|s|^2)\sum_{n\geq 0} q^n s^{-1-n} - (s+q-2\,Re[s])q\sum_{n\geq 0} q^n s^{-1-n} = s+q-2\,Re[s]$$

and

$$sq\sum_{n\geq 0} q^n s^{-1-n} - |s|^2\sum_{n\geq 0} q^n s^{-1-n}$$
$$-sq\sum_{n\geq 0} q^n s^{-1-n} - q^2\sum_{n\geq 0} q^n s^{-1-n} + 2\,Re[s]q\sum_{n\geq 0} q^n s^{-1-n} = s+q-2\,Re[s].$$

We therefore obtain

$$\Big(-|s|^2 - q^2 + 2q\,Re[s]\Big)\sum_{n\geq 0} q^n s^{-1-n} = s + q - 2\;Re[s]. \tag{11}$$

Observing that $-|s|^2 - q^2 + 2q\,Re[s]$ commutes with q^n we can rewrite this last equation as

$$\sum_{n\geq 0} q^n \Big(-|s|^2 - q^2 + 2q\,Re[s]\Big)\, s^{-1-n} = s + q - 2\;Re[s]. \tag{12}$$

Now the left hand side can be written as

$$\begin{aligned}
\sum_{n\geq 0} q^n \Big(-|s|^2 - q^2 + 2q\,Re[s]\Big)\, s^{-1-n} &= (-|s|^2 - q^2 + 2q\,Re[s])s^{-1}\\
&\quad +q^1\Big(-|s|^2 - q^2 + 2q\,Re[s]\Big)\, s^{-2} + q^2\Big(-|s|^2 - q^2 + 2q\,Re[s]\Big)\, s^{-3} + \cdots\\
&= -\Big(|s|^2 s^{-1} + q(-2s\,Re[s] + |s|^2)s^{-2} + q^2(s^2 - 2s\,Re[s] + |s|^2)s^{-3}\\
&\quad +q^3(s^2 - 2s\,Re[s] + |s|^2)s^{-4} + \cdots\Big).
\end{aligned}$$

Using the identity

$$s^2 - 2s\,Re[s] + |s|^2 = 0$$

we get

$$\begin{aligned}
\sum_{n\geq 0} q^n(-|s|^2 - q^2 + 2q\,Re[s])s^{-1-n} &= -|s|^2 s^{-1} + q s^2 s^{-2} = -|s|^2 s^{-1} + q\\
&= -\bar{s} s s^{-1} + q = -\bar{s} + q = s - 2\;Re[s] + q
\end{aligned}$$

which equals the right hand side of (11), thus showing that (9) is an identity. □

Remark 3.6 It is worth noticing that the proof of this last theorem does not rely on the fact that the (real) components of q commute. In fact, the theorem would hold even if q were to be in $\mathbb{H}\otimes\mathbb{A}$ with $\mathbb{A}$ any non commutative algebra, for example an algebra of matrices on $\mathbb{H}$. This fact will be exploited in Theorem 4.2 where the variable q is formally replaced by an operator T whose components do not necessarily commute.

Remark 3.7 Equation 8 points out that $S(s,q) = -(q-\bar{s})^{-1}(q^2 - 2q\,Re[s] + |s|^2)$ will have no inverse if $(q^2 - 2q\,Re[s] + |s|^2) = 0$. Set $s = s_0 + s_1 I_s$, $(s_0, s_1 \in \mathbb{R})$. Since

$$\Big\{q\in\mathbb{H} : (q^2 - 2q\,Re[s] + |s|^2) = 0\Big\} = s_0 + s_1\mathbb{S} \tag{13}$$

then we conclude that $S(s,q)$ will have no inverse if $q \in s_0 + s_1\mathbb{S}$. A particular case appears when $q = \overline{s}$, and one could hope to be able to split in factors the polynomial $(q^2 - 2qRe[s] + |s|^2)$ and extend the inverse of $S(s,q)$ up to the case $q = \overline{s}$. However, as shown in the next results, this is not possible and the function $S^{-1}(s,q)$ cannot be extended to a continuous function in $q = \overline{s}$.

Lemma 3.8 *Let $s \in \mathbb{H}$ be a non real quaternion. Then there exists no degree-one quaternionic polynomial $Q(q)$ such that*

$$q^2 - 2qRe[s] + |s|^2 = (q - \overline{s})Q(q). \tag{14}$$

Proof If such a polynomial $Q(q)$ exists, then the right-hand side of (14) would have a finite number of roots (namely 1 or 2), while the left-hand side has infinitely many roots in view of (13). □

Theorem 3.9 *Let $q, s \in \mathbb{H}$. Then the function*

$$S^{-1}(s,q) = -(q^2 - 2qRe[s] + |s|^2)^{-1}(q - \overline{s})$$

cannot be extended continuously to any point of the set

$$\{(s,q) \in \mathbb{H} \times \mathbb{H} \ : \ (q^2 - 2qRe[s] + |s|^2) = 0\,\}.$$

In particular, if $qs \neq sq$ the limit

$$\lim_{q \to \overline{s}} S^{-1}(s,q) \tag{15}$$

does not exist.

Proof We prove that the limit (15) does not exist. Let $\varepsilon \in \mathbb{H}$ and consider

$$\begin{aligned} S^{-1}(s, \overline{s} + \varepsilon) &= ((\overline{s} + \varepsilon)^2 - 2(\overline{s} + \varepsilon)Re[s] + |s|^2)^{-1}\varepsilon \\ &= ((\overline{s} + \varepsilon)^2 - 2(\overline{s} + \varepsilon)Re[s] + |s|^2)^{-1}\varepsilon \\ &= (\overline{s}\varepsilon + \varepsilon\overline{s} + \varepsilon^2 - 2\varepsilon Re[s])^{-1}\varepsilon \\ &= (\varepsilon^{-1}(\overline{s}\varepsilon + \varepsilon\overline{s} + \varepsilon^2 - 2\varepsilon Re[s]))^{-1} = (\varepsilon^{-1}\overline{s}\varepsilon + \overline{s} + \varepsilon - 2Re[s]))^{-1}. \end{aligned}$$

If we now let $\varepsilon \to 0$, we obtain that the term $\varepsilon^{-1}\overline{s}\varepsilon$ does not have a limit because

$$\varepsilon^{-1}\overline{s}\varepsilon = \frac{\overline{\varepsilon}}{|\varepsilon|^2}\overline{s}\varepsilon$$

contains addends of type $\frac{\varepsilon_i \varepsilon_j s_\ell}{|\varepsilon|^2}$ with $i, j, \ell \in \{0, 1, 2, 3\}$ that do not have limit. □

We now define the non commutative Cauchy kernel and, with an abuse of notation, we will still denote it by $S^{-1}(s,q)$.

Definition 3.10 Let $q, s \in \mathbb{H}$ such that $sq \neq qs$. We will call non commutative Cauchy kernel (shortly Cauchy kernel) the expression

$$S^{-1}(s,q) := -\left(q^2 - 2q\,Re[s] + |s|^2\right)^{-1}(q - \overline{s}).$$

4 The *S*-Resolvent Operator and the *S*-Spectrum

Let T be a linear quaternionic operator. It is obvious that there are two natural eigenvalue problems associated to T. The first, which one could call the left eigenvalue problem consists in the solution of equation $T(v) = \lambda v$, and the second, which is called right eigenvalue problem, and consists in the solution of the equation $T(v) = v\lambda$. We will discuss the associated spectra later on, but the key observation is that none of them is useful to define a functional calculus. In this section we will identify the correct framework for the study of eigenvalues for quaternionic operators.

Definition 4.1 (*The S-resolvent operator series*) Let $T \in \mathcal{B}(V)$ and let $s \in \mathbb{H}$. We define the S-resolvent operator series as

$$S^{-1}(s,T) := \sum_{n\geq 0} T^n s^{-1-n} \tag{16}$$

for $\|T\| \leq |s|$.

If we denote by $\mathcal{I}$ the identity operator, we can state the following:

Theorem 4.2 *Let $T \in \mathcal{B}(V)$ and let $s \in \mathbb{H}$. Assume that $\overline{s}$ is such that $T - \overline{s}\mathcal{I}$ is invertible. Then*

$$S(s,T) = (T - \overline{s}\mathcal{I})^{-1}\, s\,(T - \overline{s}\mathcal{I}) - T \tag{17}$$

is the inverse of

$$S^{-1}(s,T) = \sum_{n\geq 0} T^n s^{-1-n}.$$

Moreover, we have

$$\sum_{n\geq 0} T^n s^{-1-n} = -(T^2 - 2Re[s]T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}), \tag{18}$$

for $\|T\| < |s|$.

Proof The subsequent calculations have to be intended in the norm of the bounded linear quaternionic operators, the proof mimics the one of Theorem 3.5. We verify that

$$S(s,T)S^{-1}(s,T) = \mathcal{I} \tag{19}$$

when $S(s,T)$ is given by (17). Namely (19) can be written as

$$[(T+(s-2\,Re[s])\mathcal{I})^{-1}(sT-|s|^2\mathcal{I})-T]\sum_{n\geq 0}T^n s^{-1-n}=\mathcal{I}$$

and also as

$$(T+(s-2\,Re[s])\mathcal{I})^{-1}(sT-|s|^2\mathcal{I})\sum_{n\geq 0}T^n s^{-1-n}-T\sum_{n\geq 0}T^n s^{-1-n}=\mathcal{I}.$$

By applying $T+(s-2\,Re[s])\mathcal{I}$ to both hands sides, we obtain the equality

$$\begin{aligned}(sT-|s|^2\mathcal{I})\sum_{n\geq 0}T^n s^{-1-n}-(T+(s-2\,Re[s])\mathcal{I})T\sum_{n\geq 0}T^n s^{-1-n}\\ =T+(s-2\,Re[s])\mathcal{I}\end{aligned}$$

which can be written as

$$\begin{aligned}&sT\sum_{n\geq 0}T^n s^{-1-n}-|s|^2\sum_{n\geq 0}T^n s^{-1-n}\\ &-sT\sum_{n\geq 0}T^n s^{-1-n}-T^2\sum_{n\geq 0}T^n s^{-1-n}+2\,Re[s]T\sum_{n\geq 0}T^n s^{-1-n}\\ &=T+(s-2\,Re[s])\mathcal{I}\end{aligned}$$

and then we get

$$\Big(-|s|^2\mathcal{I}-T^2+2Re[s]T\Big)\sum_{n\geq 0}T^n s^{-1-n}=T+(s-2\,Re[s])\mathcal{I}.$$

Observing that $-|s|^2\mathcal{I}-T^2+2Re[s]T$ commutes with T^n we obtain that the above identity is equivalent to

$$\sum_{n\geq 0}T^n\Big(-|s|^2-T^2+2Re[s]T\Big)s^{-1-n}=T+(s-2\,Re[s])\mathcal{I}.$$

Now we expand the series as

$$\begin{aligned}\sum_{n\geq 0}T^n(-|s|^2\mathcal{I}-T^2+2Re[s]T)s^{-1-n}&=(-|s|^2\mathcal{I}-T^2+2Re[s]T)s^{-1}\\ &+T^1(-|s|^2\mathcal{I}-T^2+2Re[s]T)s^{-2}+T^2(-|s|^2\mathcal{I}-T^2+2Re[s]T)s^{-3}+\cdots\\ &=-\Big(|s|^2s^{-1}+T(-2sRe[s]+|s|^2)s^{-2}+T^2(s^2-2sRe[s]+|s|^2)s^{-3}+\cdots\Big)\end{aligned}$$

and using the identity

$$s^2-2sRe[s]+|s|^2=0$$

we get

$$\sum_{n\geq 0} T^n(-|s|^2 - T^2 + 2Re[s]T)s^{-1-n} = -|s|^2 s^{-1}\mathcal{I} + Ts^2 s^{-2} = -|s|^2 s^{-1}\mathcal{I} + T$$
$$= -\bar{s}ss^{-1}\mathcal{I} + T = -\bar{s}\mathcal{I} + T$$
$$= (s - 2Re[s])\mathcal{I} + T.$$

The equality (18) follows directly by taking the inverse of (17). □

Theorem 4.3 *Let $T \in \mathcal{B}(V)$ and let $s \in \mathbb{H}$. Then the operator*

$$\sum_{n\geq 0}(s^{-1}T)^n s^{-1}\mathcal{I}$$

is the right and left algebraic inverse of $s\mathcal{I} - T$. Moreover, the series converges in the operator norm for $\|T\| < |s|$.

Proof Let us directly compute

$$(s\mathcal{I} - T)\sum_{n\geq 0}(s^{-1}T)^n s^{-1}\mathcal{I} = s\mathcal{I}\sum_{n\geq 0}(s^{-1}T)^n s^{-1}\mathcal{I} - T\sum_{n\geq 0}(s^{-1}T)^n s^{-1}\mathcal{I}$$
$$= s\mathcal{I}s^{-1}\mathcal{I} + Ts^{-1}\mathcal{I} + T(s^{-1}T)s^{-1}\mathcal{I} + T(s^{-1}T)^2 s^{-1}\mathcal{I} + \cdots$$
$$-Ts^{-1}\mathcal{I} - T(s^{-1}T)s^{-1}\mathcal{I} - T(s^{-1}T)^2 s^{-1}\mathcal{I} - T(s^{-1}T)^3 s^{-1}\mathcal{I} + \cdots = \mathcal{I}.$$

The same identity holds for

$$\sum_{n\geq 0}(s^{-1}T)^n s^{-1}\mathcal{I}(s\mathcal{I} - T) = \mathcal{I}.$$

Finally we consider

$$\|\sum_{n\geq 0}(s^{-1}T)^n s^{-1}\mathcal{I}\| \leq \sum_{n\geq 0}\|(s^{-1}T)^n s^{-1}\mathcal{I}\| \leq \sum_{n\geq 0}\|(s^{-1}T)\|^n |s^{-1}|$$
$$\leq \sum_{n\geq 0}\|T\|^n |s^{-1}|^{n+1}$$

which converges for $\|T\| < |s|$. □

Corollary 4.4 *When $Ts\mathcal{I} = sT$, the operator $S^{-1}(s, T)$ equals $(s\mathcal{I} - T)^{-1}$ when the series (16) converges.*

Proof It follows immediately from Theorem 4.3. □

Definition 4.5 (*The S-resolvent operator*) Let $T \in \mathcal{B}(V)$ and let $s \in \mathbb{H}$. We define the S-resolvent operator as

$$S^{-1}(s, T) := -(T^2 - 2Re[s]T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}). \quad (20)$$

Definition 4.6 (*The spectra of quaternionic operators*) Let $T : V \to V$ be a linear quaternionic operator on the Banach space V. We denote by $\sigma_L(T)$ the left spectrum of T related to the resolvent $(s\mathcal{I} - T)^{-1}$ that is

$$\sigma_L(T) = \{s \in \mathbb{H} : s\mathcal{I} - T \quad \text{is not invertible}\}.$$

We define the S-spectrum $\sigma_S(T)$ of T related to the S-resolvent operator (20) as:

$$\sigma_S(T) = \{s \in \mathbb{H} : T^2 - 2\, Re[s]T + |s|^2\mathcal{I} \quad \text{is not invertible}\}.$$

Remark 4.7 It is also possible to introduce a notion of right spectrum $\sigma_R(T)$ of T as

$$\sigma_R(T) = \{s \in \mathbb{H} : \mathcal{I} \cdot s - T \quad \text{is not invertible}\},$$

where the notation $\mathcal{I} \cdot s$ means that the multiplication by s is on the right, i.e. $\mathcal{I} \cdot s(v) = \mathcal{I}(v)s$. However, the operator $\mathcal{I} \cdot s - T$ is not linear, so we will never refer to this notion.

Theorem 4.8 *Let $T \in \mathcal{B}(V)$ and let $s \in \rho_S(T)$. Then the S-resolvent operator defined in (20) satisfies the equation*

$$S^{-1}(s, T)s - TS^{-1}(s, T) = \mathcal{I}.$$

Proof It follows by direct computation. Indeed, replacing (20) in the above equation we have

$$\begin{aligned} &-(T^2 - 2Re[s]T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I})s \\ &+T(T^2 - 2Re[s]T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}) = \mathcal{I} \end{aligned} \tag{21}$$

and applying $(T^2 - 2Re[s]T + |s|^2\mathcal{I})$ to both hands sides of (21), we get

$$\begin{aligned} &-(T - \bar{s}\mathcal{I})s + (T^2 - 2Re[s]T + |s|^2\mathcal{I})T(T^2 - 2Re[s]T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}) \\ &\quad = T^2 - 2Re[s]T + |s|^2\mathcal{I}. \end{aligned}$$

Since T and $T^2 - 2Re[s]T + |s|^2\mathcal{I}$ commute, we obtain the identity

$$-(T - \bar{s}\mathcal{I})s + T(T - \bar{s}\mathcal{I}) = T^2 - 2Re[s]T + |s|^2\mathcal{I}$$

which proves the statement. □

Definition 4.9 The equation

$$S^{-1}(s, T)s - TS^{-1}(s, T) = \mathcal{I}$$

will be called the S-resolvent equation.

Theorem 4.10 *Let $T \in \mathcal{B}(V)$. Then $\sigma_S(T)$ and $\sigma_L(T)$ are contained in the set $\{s \in \mathbb{H} : |s| \leq \|T\|\}$.*

Proof Since both the series

$$\sum_{n\geq 0}(s^{-1}T)^n s^{-1}\mathcal{I}, \quad \sum_{n\geq 0} T^n s^{-1-n}$$

converge if and only if $|s^{-1}|\|T\| < 1$ we get the statement. □

Theorem 4.11 *Let $T \in \mathcal{B}(V)$ and $s \notin \sigma_S(T)$ such that*

$$\|T - Re[s]\mathcal{I}\| < |s - Re[s]|. \tag{22}$$

Then the S-resolvent operator admits the series expansion:

$$S^{-1}(s,T) = \sum_{n\geq 0}(T - Re[s]\mathcal{I})^n (s - Re[s])^{-n-1}. \tag{23}$$

Proof Observe that

$$\begin{aligned}(T^2 - 2T Re[s] + |s|^2\mathcal{I})^{-1} &= [(T - Re[s]\mathcal{I})^2 + |s|^2\mathcal{I} - (Re[s])^2\mathcal{I}]^{-1} \\ &= (|s|^2 - (Re[s])^2)^{-1}\Big[\mathcal{I} + \frac{(T - Re[s])^2}{(|s|^2 - (Re[s])^2)}\Big]^{-1} \\ &= \sum_{n\geq 0}(-1)^n \frac{(T - Re[s])^{2n}}{(|s|^2 - (Re[s])^2)^{n+1}}.\end{aligned} \tag{24}$$

Since $|Im[s]|^2 = -(Im[s])^2$, then by replacing (24) in (20) we get

$$\begin{aligned}S^{-1}(s,T) &= \sum_{n\geq 0}(T - Re[s]\mathcal{I})^{2n+1}(s - Re[s])^{-2n-2} \\ &\quad + \sum_{n\geq 0}(T - Re[s]\mathcal{I})^{2n}(s - Re[s])^{-2n-1}\end{aligned}$$

adding the two terms we get (23) which converges when (22) holds. □

We now give a simple relation between the L-spectrum and the S- spectrum.

Proposition 4.12 *Let $T \in \mathcal{B}(V)$ and $s \in \sigma_L(T)$ and let v be the corresponding L-eigenvector. Then $s \in \sigma_S(T)$ and v is the corresponding S-eigenvector if and only if*

$$(T - s\mathcal{I})(sv) = 0.$$

Proof It follows from the relations:

$$T^2 v - 2Re[s]Tv + \bar{s}s\mathcal{I}v = T(sv) - 2Re[s](sv) + \bar{s}(sv) = (T - s\mathcal{I})(sv) = 0.$$

□

We now announce a key result, whose proof we must postpone to Sect. 5.

Theorem 4.13 (Compactness of S-spectrum) *Let $T \in \mathcal{B}(V)$. Then the S-spectrum $\sigma_S(T)$ is a compact nonempty set contained in* $\{ s \in \mathbb{H} \ : \ |s| \le \|T\| \}$.

Theorem 4.14 (Structure of the S-spectrum) *Let $T \in \mathcal{B}(V)$ and let $p = p_0 + p_1 I \in p_0 + p_1\mathbb{S} \subset \mathbb{H} \setminus \mathbb{R}$ belong to $\sigma_S(T)$. Then all the elements of the sphere $p_0 + p_1\mathbb{S}$ belong to $\sigma_S(T)$.*

Proof In the S-eigenvalue equation the coefficients depend only on the real numbers p_0, p_1 and not on $I \in \mathbb{S}$. Therefore all $s = p_0 + p_1 J$ such that $J \in \mathbb{S}$ are in the S-spectrum. □

We conclude this section with a few examples which highlight the differences between the S-spectrum and the L-spectrum.

Example 4.15 1. Consider the matrix

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}.$$

Direct computations show that $\sigma_S(T_1) = \{1\} \cup \mathbb{S}$, while $\sigma_L(T_1) = \{1, j\}$.

2. We now consider

$$T_2 = \begin{bmatrix} i & 0 \\ 0 & j \end{bmatrix}.$$

We have (as a point-set) $\sigma_S(T_2) = \mathbb{S}$ and, obviously, the L-spectrum is $\sigma_L(T_2) = \{i, \ j\}$.

3. Finally, let

$$T_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

We obtain $\sigma_S(T_3) = \{\pm 1\}$ and $\sigma_L(T_3) = \{s_0 + s_1 i + s_2 j + s_3 k \ : \ s_1 = 0, s_0^2 + s_2^2 + s_3^2 = 1\} = \{s \in \mathbb{H} : si \in \mathbb{S}\}$.

5 Functional Calculus

Definition 5.1 Let $T \in \mathcal{B}(V)$. Let $U \subset \mathbb{H}$ be an open set such that

(i) $\partial(U \cap L_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}$,

(ii) U contains the S-spectrum $\sigma_S(T)$,

(iii) $\sigma_S(T)$ is contained in a finite union of open balls $B_i \subset U$ with center in real points and of annular domains $A_j = \{x \in \mathbb{H} \mid r_j < |x - \alpha_j| < R_j,\ r_j, R_j \in \mathbb{R}^+\} \subset U$ with center in real points α_j whose boundaries do not intersect $\sigma_S(T)$.

A function f is said to be locally s-regular on $\sigma_S(T)$ if there exists an open set $U \subset \mathbb{H}$, as above, on which f is s-regular.

We will denote by $\mathcal{R}_{\sigma_S(T)}$ the set of locally s-regular functions on $\sigma_S(T)$.

Theorem 5.2 *Let $T \in \mathcal{B}(V)$ and $f \in \mathcal{R}_{\sigma_S(T)}$. Let $U \subset \mathbb{H}$ be an open set as in Definition 5.1 and let $U_I = U \cap L_I$ for $I \in \mathbb{S}$. Then the integral*

$$\frac{1}{2\pi} \int_{\partial U_I} S^{-1}(s, T)\, ds_I\ f(s) \tag{25}$$

does not depend on the choice of the imaginary unit I and on the open set U.

Proof We first note that the integral (25) does not depend on the choice of U by the Cauchy theorem applied on the plane L_I. We now show the independence of the choice of $I \in \mathbb{S}$. Note that since the S-spectrum is compact (see Theorem 4.13) we can choose a finite number of open balls $B_1, \dots, B_\nu$ and of open annular domains $A_1, \dots, A_\mu$, $\nu, \mu \in \mathbb{N}$, containing the S-spectrum of T. We observe that thanks to the Cauchy theorem we can write:

$$\frac{1}{2\pi} \int_{\partial U_I} S^{-1}(s, T) ds_I f(s)$$
$$= \frac{1}{2\pi} \sum_{i=1}^{\nu} \int_{\partial(B_i \cap L_I)} S^{-1}(s, T) ds_I f(s) + \frac{1}{2\pi} \sum_{i=1}^{\mu} \int_{\partial(A_i \cap L_I)} S^{-1}(s, T) ds_I f(s), \tag{26}$$

where the right hand side does not depend on the choice of the B_i's and A_i's. Since f admits series expansion on the B_i's for Taylor theorem and on A_i's by the Laurent theorem whose coefficients do not depend on the plane L_I (see [9]), we can integrate term by term. Let us now choose another imaginary unit $I' \in \mathbb{S}$, $I \neq I'$ and let us write the analogue of (26) on $L_{I'}$. The S-spectrum contains either real points or, by Theorem 4.14, 2-spheres of the type $\{s \in \mathbb{H} : Re[s] = s_0, |Im[s]| = r\}$. Every complex plane $L_I = \mathbb{R} + I\mathbb{R}$ contains all the real points belonging to the S-spectrum. Let $\{s \in \mathbb{H} : s = s_0 + It, |t| = r\}$ be a 2-sphere in the S-spectrum. The two points of the sphere lying on the complex plane L_I are $s_0 \pm rI$ so, on the plane, they have coordinates $(s_0, \pm r)$. The coordinates of the two intersection points on a different complex plane $L_{I'}$ are still $(s_0, \pm r)$, so the right hand side of (26) does not depend on the choice of $I \in \mathbb{S}$. Thus

$$\frac{1}{2\pi} \int_{\partial U_I} S^{-1}(s, T)\, ds_I f(s) = \frac{1}{2\pi} \int_{\partial U_{I'}} S^{-1}(s, T)\, ds_{I'} f(s).$$

□

We give a result that motivates the functional calculus.

Theorem 5.3 *Let s, $a \in \mathbb{H}$, $m \in \mathbb{N}$ and consider the monomial $s^m a$. Consider $T \in \mathcal{B}(V)$, let $U \subset \mathbb{H}$ be an open set as in Definition 5.1, and set $U_I = U \cap L_I$ for $I \in \mathbb{S}$. Then*

$$T^m a = \frac{1}{2\pi} \int_{\partial U_I} S^{-1}(s, T)\, ds_I\, s^m\, a. \tag{27}$$

Proof Let us consider the power series expansion for the operator $S^{-1}(s, T)$ and a circle C_r centered in the origin and of radius $r > \|T\|$. We have:

$$\frac{1}{2\pi} \int_{\partial U_I} S^{-1}(s, T)\, ds_I\, s^m\, a = \frac{1}{2\pi} \sum_{n \geq 0} T^n \int_{C_r} s^{-1-n+m}\, ds_I a = T^m\, a, \tag{28}$$

since

$$\int_{C_r} ds_I s^{-n-1+m} = 0 \quad if \;\; n \neq m, \quad \int_{C_r} ds_I s^{-n-1+m} = 2\pi \quad if \;\; n = m. \tag{29}$$

The Cauchy theorem shows that the above integrals are not affected if we replace C_r by ∂U_I. □

Theorem 5.4 (Compactness of S-spectrum) *Let $T \in \mathcal{B}(V)$. Then the S-spectrum $\sigma_S(T)$ is a compact nonempty set. Moreover $\sigma_S(T)$ is contained in* $\{s \in \mathbb{H} : |s| \leq \|T\|\}$.

Proof Let $U \subset \mathbb{H}$ be an open set as in Definition 5.1, and set $U_I = U \cap L_I$ for $I \in \mathbb{S}$. Then

$$\frac{1}{2\pi} \int_{\partial U_I} S^{-1}(s, T)\, ds_I\, s^m = T^m.$$

In particular, for $m = 0$, we have

$$\frac{1}{2\pi} \int_{\partial U_I} S^{-1}(s, T)\, ds_I = \mathcal{I},$$

where $\mathcal{I}$ denotes the identity operator, which shows that $\sigma_S(T)$ is a nonempty set otherwise the integral would be zero by the vector valued version of Cauchy's theorem. We now show that the S-spectrum is bounded. The series $\sum_{n \geq 0} T^n s^{-1-n}$ converges if and only if $\|T\| < |s|$ so the S-spectrum is contained in the set $\{s \in \mathbb{H} : |s| \leq \|T\|\}$, which is bounded and closed because the complement of $\sigma_S(T)$ is open. Indeed, the function $g : s \mapsto T^2 - 2Re[s]T + |s|^2\mathcal{I}$ is trivially continuous and, by Theorem 10.12 in [8], the set $\mathcal{U}(V)$ of all invertible elements of $\mathcal{B}(V)$ is an open set in $\mathcal{B}(V)$. Therefore $g^{-1}(\mathcal{U}(V)) = \rho_S(T)$ is an open set in $\mathbb{H}$. □

The preceding discussion allows to give the following definition.

Definition 5.5 Let $T \in \mathcal{B}(V)$ and $f \in \mathcal{R}_{\sigma_S(T)}$. Let $U \subset \mathbb{H}$ be an open set as in Definition 5.1, and set $U_I = U \cap L_I$ for $I \in \mathbb{S}$. We define

$$f(T) = \frac{1}{2\pi} \int_{\partial U_I} S^{-1}(s,T)\, ds_I \, f(s). \tag{30}$$

The next two theorems show that Definition 5.5 leads to a functional calculus with good properties.

Theorem 5.6 *Let $T \in \mathcal{B}(V)$ and let f and $g \in \mathcal{R}_{\sigma_S(T)}$. Then*

$$(f+g)(T) = f(T) + g(T), \quad (f\lambda)(T) = f(T)\lambda \ \ \textit{for all } \lambda \in \mathbb{H}.$$

Moreover, if $\phi(s) = \sum_{n\geq 0} s^n a_n$ and $\psi(s) = \sum_{n\geq 0} s^n b_n$, are in $\mathcal{R}_{\sigma_S(T)}$ with a_n and $b_n \in \mathbb{R}$. Then

$$(\phi\psi)(T) = \phi(T)\psi(T).$$

Proof The first part of the theorem is a direct consequence of Definition 5.5. Observe that when a_n and b_n are real the product $\phi\psi$ is regular. We have to prove that

$$\phi(T)\psi(T) = \frac{1}{2\pi} \int_{\partial U} S^{-1}(s,T)\, ds_I \, \phi(s)\psi(s),$$

where U contains $L_I \cap \sigma_S(T)$. By recurrence for $m \geq 2$ we get

$$S^{-1}(s,T)s^m - T^m S^{-1}(s,T) = \mathcal{I}s^{[m-1]_+} + Ts^{[m-2]_+} + T^2 s^{[m-3]_+} + \cdots + T^{m-1},$$

set

$$Q_m(s,T) := \mathcal{I}s^{[m-1]_+} + Ts^{[m-2]_+} + T^2 s^{[m-3]_+} + \cdots + T^{m-1}, \quad m \geq 2$$

where $s^{[n]_+} = s^n$ if $n > 0$, $s^{[n]_+} = 0$ otherwise, and

$$Q_m(s,T) := \mathcal{I} \quad \text{for} \quad m = 1 \quad \text{and} \quad Q_m(s,T) := 0 \quad \text{for} \quad m = 0,$$

which is a regular function in s, with values in the space of linear bounded quaternionic operators, so we get

$$T^m S^{-1}(s,T) = S^{-1}(s,T)s^m - Q_m(s,T), \quad m = 0, 1, 2, \ldots.$$

We have to observe that for $m = 1$ and $m = 0$ we have $TS^{-1}(s,T) = S^{-1}(s,T)s - \mathcal{I}$ and $\mathcal{I}S^{-1}(s,T) = S^{-1}(s,T)1$, respectively. Observe that

$$\psi(T) = \frac{1}{2\pi}\int_{\partial U} S^{-1}(s,T)\, ds_I\, \psi(s)$$

and

$$\begin{aligned} T^m\psi(T) &= T^m\frac{1}{2\pi}\int_{\partial U} S^{-1}(s,T)\, ds_I\, \psi(s) = \frac{1}{2\pi}\int_{\partial U} T^m S^{-1}(s,T)\, ds_I\, \psi(s) \\ &= \frac{1}{2\pi}\int_{\partial U}[S^{-1}(s,T)s^m - Q_m(s,T)]\, ds_I\, \psi(s) \\ &= \frac{1}{2\pi}\int_{\partial U} S^{-1}(s,T)s^m\, ds_I\, \psi(s) \end{aligned}$$

Now for $a_n \in \mathbb{R}$ we can write

$$T^m a_m\psi(T) = \frac{1}{2\pi}\int_{\partial U} S^{-1}(s,T)s^m a_m\, ds_I\, \psi(s)$$

summing up with respect to m

$$\sum_{m=0}^{M} T^m a_m\psi(T) = \frac{1}{2\pi}\int_{\partial U} S^{-1}(s,T)\sum_{m=0}^{M} s^m a_m\, ds_I\, \psi(s)$$

now by definition $|\sum_{m=0}^{M} s^m a_m|$ is bounded by $|\phi|$ which is continuous on the bounded set ∂U, and $\|S^{-1}(s,T)\| \le (|s| - \|T\|)^{-1}$ so we can pass to the limit for $M \to \infty$. □

Theorem 5.7 (Spectral decomposition of a quaternionic operator) *Let $T \in \mathcal{B}(V)$. Let $L_I \cap \sigma_S(T) = \sigma_{1S}(T) \cup \sigma_{2S}(T)$, with $dist(\sigma_{1S}(T), \sigma_{2S}(T)) > 0$. Let U_1 and U_2 be two open sets such that $\sigma_{1S}(T) \subset U_1$ and $\sigma_{2S}(T) \subset U_2$, on L_I, with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Set*

$$P_j := \frac{1}{2\pi}\int_{\partial U_j} S^{-1}(s,T)\, ds_I \quad T_j^m := \frac{1}{2\pi}\int_{\partial U_j} S^{-1}(s,T)\, ds_I\, s^m, \quad m \in \mathbb{N},\ j = 1, 2.$$

Then P_j are projectors and

(I) $P_1 + P_2 = \mathcal{I}$,
(II) $TP_j = T_j$,
(III) $T = T_1 + T_2$,
(IV) $T^m = T_1^m + T_2^m$, $m \ge 2$.

Proof Observe that $P_j = T_j^0$ and note that the resolvent equation for $m = 0$ is trivially $T_j^0 S^{-1}(s,T) = S^{-1}(s,T)s^0 = S^{-1}(s,T)$. We have

$$P_j^2 = P_j \frac{1}{2\pi}\int_{\partial U_j} S^{-1}(s,T)\, ds_I = \frac{1}{2\pi}\int_{\partial U_j} P_j S^{-1}(s,T)\, ds_I$$
$$= \frac{1}{2\pi}\int_{\partial U_j} S^{-1}(s,T)\, ds_I = P_j.$$

To prove (I) we use the Cauchy integral theorem, if $\overline{U}_1 \cup \overline{U}_2 \subset U$, then

$$\frac{1}{2\pi}\int_{\partial U_1} S^{-1}(s,T)\, ds_I + \frac{1}{2\pi}\int_{\partial U_2} S^{-1}(s,T)\, ds_I = \frac{1}{2\pi}\int_{\partial U} S^{-1}(s,T)\, ds_I.$$

This gives $P_1 + P_2 = \mathcal{I}$ since $\frac{1}{2\pi}\int_{\partial U} S^{-1}(s,T)\, ds_I = \mathcal{I}$.

To prove (II) we recall the resolvent relation

$$TS^{-1}(s,T) = S^{-1}(s,T)s - \mathcal{I}$$

so

$$TP_j = \frac{1}{2\pi}\int_{\partial U_j} TS^{-1}(s,T)\, ds_I = \frac{1}{2\pi}\int_{\partial U_j} [S^{-1}(s,T)s - \mathcal{I}]\, ds$$
$$= \frac{1}{2\pi}\int_{\partial U_j} S^{-1}(s,T)\, ds_I\, s = T_j.$$

Now adding the relations $T_j = TP_j$ we get

$$T_1 + T_2 = TP_1 + TP_2 = T(P_1 + P_2) = T,$$

where we have used (I).

Now we observe that

$$S^{-1}(s,T)s^2 - T^2 S^{-1}(s,T) = \mathcal{I}s + T$$

and

$$S^{-1}(s,T)s^3 - T^3 S^{-1}(s,T) = \mathcal{I}s^2 + Ts + T^2$$

and by recurrence for $m \geq 2$ we get

$$S^{-1}(s,T)s^m - T^m S^{-1}(s,T) = \mathcal{I}s^{m-1} + Ts^{m-2} + \cdots + T^{m-1}.$$

Now, for $m \geq 2$, consider

$$\begin{aligned}
T^m P_j &= \frac{1}{2\pi} \int_{\partial U_j} T^m S^{-1}(s, T)\, ds_I \\
&= \frac{1}{2\pi} \int_{\partial U_j} \left[S^{-1}(s, T) s^m - \left(\mathcal{I} s^{m-1} + T s^{m-2} + \cdots + T^{m-1} \right) \right] ds_I \\
&= \frac{1}{2\pi} \int_{\partial U_j} S^{-1}(s, T)\, ds_I\, s^m = T_j^m.
\end{aligned}$$

So adding $T^m P_1 = T_1^m$ and $T^m P_2 = T_2^m$ and recalling (I) we get (IV). □

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References

1. Adler, S.: Quaternionic Quantum Field Theory. Oxford University Press, New York (1995)
2. Colombo, F., Sabadini, I., Sommen, F., Struppa, D.C.: Analysis of Dirac Systems and Computational Algebra. Progress in Mathematical Physics, vol. 39. Birkhäuser, Boston (2004)
3. Colombo, F., Sabadini, I., Struppa, D.C.: A new functional calculus for noncommuting operators. J. Funct. Anal. **254**, 2255–2274 (2008)
4. Dunford, N., Schwartz, J.: Linear Operators, Part I: General Theory. Wiley, New York (1988)
5. Gentili, G., Struppa, D.C.: A new approach to Cullen-regular functions of a quaternionic variable. C.R. Acad. Sci. Paris **342**, 741–744 (2006)
6. Gentili, G., Struppa, D.C.: A new theory of regular functions of a quaternionic variable. Adv. Math. **216**, 279–301 (2007)
7. Jefferies, B.: Spectral Properties of Noncommuting Operators. Lecture Notes in Mathematics, 1843, Springer-Verlag, Berlin (2004)
8. Rudin, W.: Functional Analysis. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg (1973)
9. Stoppato, C.: Poles of regular quaternionic functions. Complex Var. Elliptic Equ. (2009) (in press)
10. Taylor, J.L.: The analytic-functional calculus for several commuting operators. Acta Math. **125**, 1–38 (1970)
11. Taylor, J.L.: Functions of several noncommuting variables. Bull. Amer. Math. Soc. **79**, 1–34 (1973)