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Complex Analysis and Operator Theory

Regularization by Free Additive Convolution, Square and Rectangular Cases

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Abstract. The free convolution $\boxplus$ is the binary operation on the set of probability measures on the real line which allows to deduce, from the individual spectral distributions, the spectral distribution of a sum of independent unitarily invariant square random matrices or of a sum of free operators in a non commutative probability space. In the same way, the rectangular free convolution $\boxplus_\lambda$ allows to deduce, from the individual singular distributions, the singular distribution of a sum of independent unitarily invariant rectangular random matrices. In this paper, we consider the regularization properties of these free convolutions on the whole real line. More specifically, we try to find continuous semigroups (μ_t) of probability measures such that $\mu_0 = \delta_0$ and such that for all $t > 0$ and all probability measure ν, $\mu_t \boxplus \nu$ (or, in the rectangular context, $\mu_t \boxplus_\lambda \nu$) is absolutely continuous with respect to the Lebesgue measure, with a positive analytic density on the whole real line. In the square case, for $\boxplus$, we prove that in semigroups satisfying this property, no measure can have a finite second moment, and we give a sufficient condition on semigroups to satisfy this property, with examples. In the rectangular case, we prove that in most cases, for μ in a $\boxplus_\lambda$-continuous semigroup, $\mu \boxplus_\lambda \nu$ either has an atom at the origin or doesn't put any mass in a neighborhood of the origin, and thus the expected property does not hold. However, we give sufficient conditions for analyticity of the density of $\mu \boxplus_\lambda \nu$ except on a negligible set of points, as well as existence and continuity of a density everywhere.

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1. Introduction

It is a very natural question to study the spectrum of the sum of two matrices, being given the spectrum of each of them. Such a question can of course have many different answers depending on the relations between the eigenspaces of the two matrices. If they are the same, but for instance the eigenvalues are independent and say equidistributed inside each matrix, the spectrum will simply be given by the classical convolution. If, on the contrary the eigenspaces are chosen as arbitrarily as possible with respect to each other, which corresponds to conjugating one of the matrices with an independent unitary matrix following the Haar measure, Voiculescu [24] proved that in the limit where the size of the matrices goes to infinity while the spectral measure of each of the two matrices converges weakly, the outcome only depends on these limiting measures and is given by their free convolution. More precisely, if we let A_N, B_N be two sequences of $N \times N$ Hermitian matrices with eigenvalues $(a_i^N)_{1\le i\le N}$ and $(b_i^N)_{1\le i\le N}$ respectively, such that the spectral measures $L_A^N := \frac{1}{N}\sum_{i=1}^N \delta_{a_i^N}$ and $L_B^N := \frac{1}{N}\sum_{i=1}^N \delta_{b_i^N}$ converge to probability measures μ_A and μ_B as N goes to infinity, and if U_N follows the Haar measure on the unitary group and is independent of A_N and B_N, then the spectral measure of $A_N + U_N B_N U_N^*$ converges towards the free convolution $\mu_A \boxplus \mu_B$ of μ_A and μ_B.

One of the authors, F. Benaych-Georges [7], generalized this convergence to the case of rectangular matrices. In this case, $A_{N,M}$ and $B_{N,M}$ are $N \times M$ matrices and we assume that their singular values (for $N \le M$, the singular values of an $N \times M$ matrix C are the eigenvalues of $\sqrt{CC^*}$) converge towards ν_A and ν_B. We let, for $C = A$ or B, μ_C be the symmetrization of ν_C: $\mu_C(A) = \frac{1}{2}(\nu_C(A) + \nu_C(-A))$. We consider U_N and V_M following the Haar measure on the $N \times N$ and the $M \times M$ unitary matrices respectively. Then, F. Benaych-Georges proved that, if N/M converges to some $\lambda \in [0, 1]$, then the symmetrization of the empirical measure of the singular values of $A_{N,M} + U_N B_{N,M} V_M$ converges towards a probability measure $\mu_A \boxplus_\lambda \mu_B$.

Free convolution naturally shows up in random matrix theory since important matrices such as the Gaussian ensembles are invariant under conjuguation under the unitary group and therefore can always be written as $U_N B_N U_N^*$ for some Haar distributed matrix U_N, independent of B_N, or have asymptotically the same behaviour (for instance matrices with independent equidistributed entries, see [16]).

Convolution is a standard tool in classical analysis for regularizing functions or measures. In this article, we study the regularizing properties of free (square and rectangular) additive convolution. Because we wish to be able to regularize measures by perturbations as small as desired, it is natural to regularize them by processes $\mu_t, t \ge 0$ such that μ_t tends to δ_0 as t goes to zero. To simplify, we shall consider more precisely processes corresponding to infinitely divisible laws $\mu^{\boxplus t}$, as constructed by Bercovici and Voiculescu [9] (see also Nica and Speicher [20]).

Such issues are naturally related with the possibility that the density vanishes, since the density is then likely [2] to have some infinite derivative at the boundary of the support.

Hence, we shall more precisely ask the following question: can we find μ (likely a free infinitely divisible law) such that

(H) *For any probability measure* ν, $\mu \boxplus \nu$ *(or in the rectangular case* $\mu \boxplus_\lambda \nu$*) is absolutely continuous with respect to the Lebesgue measure, with a density which is analytic and which does not vanish on* $\mathbb{R}$.

Such questions already showed up in several papers. In [25], D. Voiculescu used the regularizing properties of free convolution by semi-circular laws to study free entropy for one variable. In [13, 17], regularization by Cauchy laws is used to smooth free diffusions in one case, and to study Wasserstein metric and derive large deviations principles in the other. In fact, it was shown in [15] that free convolution is the natural concept to regularize a free diffusion since the result will still be a free diffusion, but with a different (and hopefully more regular) drift. Free convolution by Cauchy laws is well understood since it coincides with standard convolution with Cauchy laws. In particular, Cauchy laws satisfy (H). The drawback is that Cauchy laws do not possess any moments, and thus do not allow a combinatorial approach by moments. In Section 3.2, we show that it is in fact impossible to find a $\boxplus$-infinitely divisible probability measure μ satisfying (H) and with finite variance. In fact, we can then construct another probability measure ν such that the density of $\mu \boxplus \nu$ vanishes at a point where its derivative is infinite. As a positive answer, we provide in Section 3.1 sufficient conditions for a probability measure μ to satisfy (H). They require that μ has either none or infinite first moment.

In the rectangular case, we exhibit in Section 4.3 a sharp transition concerning the behaviour of the free rectangular convolution of two measures at the origin. If $\mu(\{0\}) + \nu(\{0\}) > 1$, we prove that $\mu \boxplus_\lambda \nu(\{0\})$ is positive. This generalizes a similar result of Bercovici and Voiculescu in the square case [10] . More surprisingly, when $\mu(\{0\}) + \nu(\{0\}) < 1$, we show the existence of a nonempty open neighbourhood of the origin which does not intersect the support of the density of $\mu \boxplus_\lambda \nu$, for any infinitely divisible law μ and any probability measure ν.

This phenomenon is related to the repulsion at the origin of the spectrum. Such a repulsion was also shown to hold at the finite matrices level in the square case by Haagerup [18] (by adding a form of Cauchy matrices) and by Sniady [22] (by adding Gaussian matrices). Our result is less strong since it is clear that the rectangular case carries naturally a repulsion of the origin (as can be seen on the Pastur–Marchenko laws) and it holds only asymptotically. However, we find rather amazing that it holds for any infinitely divisible law μ and any probability measure ν.

This interesting phenomenon shows that (H) cannot hold in the rectangular case. We thus show a weaker result in Corollary 4.6 and Proposition 4.10 by giving sufficient conditions for analyticity of the density of $\mu \boxplus_\lambda \nu$ except on a discrete set of points, as well as existence and continuity of a density everywhere.

2. Prerequisites in complex analysis and free probability background

2.1. Complex analysis

Let $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$. We denote by $\sphericalangle \lim_{z\to w} f(z)$, $\sphericalangle f(w)$, or

$$\lim_{\substack{z \longrightarrow w \\ \sphericalangle}} f(z)$$

the limit of f at $w \in \partial\mathbb{D}$ along points inside any angle with vertex at w and included in $\mathbb{D}$, and name it the *nontangential limit of f at w*. *Unconditionnal limit* at w corresponds to limit taken over any path in the domain D which ends at w.

We will denote $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, $\mathbb{C}^- = \{z \in \mathbb{C} : \Im(z) < 0\}$ and $\mathbb{R}^+ = [0, +\infty)$. The notion of nontangential limit extends naturally to maps defined on a half-plane. Moreover, since the rational transformation $z \mapsto \frac{z-i}{z+i}$ of the extended complex plane $\mathbb{C} \cup \{\infty\}$ carries the upper half-plane onto the unit disc, most properties of analytic functions defined in the unit disc transfer naturally to functions defined on $\mathbb{C}^+$.

We shall use in our paper several results describing the boundary behaviour of analytic functions defined in the unit disc or the upper half-plane. Let us first cite a theorem due to Lindelöf, (Theorem 2.20(i) in [14]).

Theorem 2.1 (Lindelöf). *Let f be a meromorphic function on the upper half plane such that there are at least three points of $\mathbb{C} \cup \{\infty\}$ which are not attained by f on the upper half plane. Consider $a \in \mathbb{R} \cup \{\infty\}$ such that there is a path $\gamma : [0, 1) \to \mathbb{C}^+$ with limit a at 1 such that*

$$l := \lim_{t\to 1} f\big(\gamma(t)\big)$$

exists in $\mathbb{C} \cup \{\infty\}$. Then the nontangential limit of f at a exists and equals l.

Recall that if f is a function from a subset D of $\mathbb{C} \cup \{\infty\}$ into $\mathbb{C} \cup \{\infty\}$, for all z in the boundary of D, the *cluster set* $C(f, z)$ of f at z is the set of limits in $\mathbb{C} \cup \{\infty\}$ of images, by f, of sequences of points in D which tend to z. For any subset D' of D, $C_{D'}(f, z)$ denotes the cluster set of the restriction of f to D'. For the particular case when $D = \mathbb{D}$ or $D = \mathbb{C}^+$, we define the *nontangential cluster set* $C^\Delta(f, x_0)$ of f at $x_0 \in \partial D$ in the following way: let $\Gamma(\alpha)$ be the angle with vertex at x_0 bisected by the perpendicular on ∂D at x_0, with opening $\alpha \in (0, \pi)$. Then

$$C^\Delta(f, x_0) = \overline{\cup_{\alpha\in(0,\pi)} C_{\Gamma(\alpha)}(f, x_0)}\,.$$

Thus, the existence of nontangential limit of f at x_0 means that $C^\Delta(f, x_0)$ contains only one point.

The following result (see e.g. Theorem 1.1 in [14]) concerns the connectivity of a cluster set.

Lemma 2.2. *Let D be a domain in $\mathbb{C}$ (i.e. an open connected set) and assume that D is simply connected at the point $x \in \overline{D}$ (i.e. x has a basis of neighbourhoods in $\mathbb{C}$ whose intersections with D are simply connected). If $f\colon D \to \mathbb{C} \cup \{\infty\}$ is continuous, then $C(f, x)$ is connected.*

It is known from Fatou's theorem (see [14]) that bounded analytic functions on $\mathbb{D}$ have good boundary properties, namely the nontangential limit of such a function exists at almost all points (in the sense of linear measure) of the boundary of $\mathbb{D}$. However, the set of points where the nontangential limit does not exist can be also quite rich in some situations, as the following theorem shows (see e.g. Theorem 4.8 in [14])

Recall that a subset of a metric space X is said to be *residual*, or of *second Baire category* if it isn't contained in the union of any sequence of closed subsets of X with empty interior.

Theorem 2.3. *If the real or complex function $f(z)$ is continuous in $|z| < 1$, and if for $\theta \in [0, 2\pi[$, $\{\mathcal{G}_\theta\}$ is a rotation by the angle θ of a continuum $\mathcal{G}_0$ such that $\mathcal{G}_0 \cap \{|z| = 1\} = \{1\}$, then $C_{\mathcal{G}_\theta}(f, e^{i\theta}) = C(f, e^{i\theta})$ on a residual set of points $e^{i\theta}$ on $\{|z| = 1\}$.*

This theorem says for us that for a function that has no unconditional limits at the boundary, the nontangential limit must fail to exist sometimes.

We will use this result in connection with the following theorem of Seidel (Theorem 5.4 in [14]).

Theorem 2.4. *Assume that $f : \mathbb{D} \to \mathbb{D}$ is analytic, and has nontangential limits with modulus one at almost all points θ in some given interval $(\theta_1, \theta_2) \subseteq \partial\mathbb{D}$. Then for any $\theta_0 \in (\theta_1, \theta_2)$ either f extends analytically through θ_0, or $C(f, \theta_0) = \overline{\mathbb{D}}$.*

Another useful auxiliary result is Theorem 5.2.1 from the same [14]:

Theorem 2.5. *Let f be meromorphic in the domain $\mathcal{D}$ bounded by a smooth curve γ. Consider $z_0 \in \gamma$ and suppose also that f extends in a meromorphic function in an open set containing $\gamma\backslash\{z_0\}$. Then we have $\partial C_{\mathcal{D}}(f, z_0) \subseteq C_\gamma(f, z_0)$, where ∂A denotes the boundary (in $\mathbb{C} \cup \{\infty\}$) of $A \subseteq \mathbb{C} \cup \{\infty\}$.*

We shall also use the following theorem, which can be seen as a "nontangential limit" version of the analytic continuation principle (see [14]).

Theorem 2.6 (Riesz–Privalov). *Let f be an analytic function on $\mathbb{D}$. Assume that there exists a set A of nonzero linear measure in $\partial\mathbb{D}$ such that the nontangential limit of f exists at each point of A, and equals zero. Then $f(z) = 0$ for all $z \in \mathbb{D}$.*

Consider now an analytic function $f : \mathbb{D} \longrightarrow \overline{\mathbb{D}}$. The *Denjoy–Wolff point* of f is characterized by the fact that it is the uniform limit on compact subsets of the iterates $f^{\circ n} = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$ of f. We state the following theorem of Denjoy and Wolff as it appears in Milnor's book [19], as Theorem 4.2. Recall that an hyperbolic rotation around some point $z_0 \in \mathbb{D}$ is a function of the form $z \mapsto e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z}$, $\theta \in \mathbb{R}$.

Theorem 2.7. *Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be an analytic function. Then either f is a hyperbolic rotation around some point $z_0 \in \mathbb{D}$, or the sequence of functions $f^{\circ n}$ converges uniformly on compact subsets of $\mathbb{D}$ to a unique point $w \in \overline{\mathbb{D}}$, called the Denjoy–Wolff point of f.*

Note that if the analytic function $f\colon \mathbb{D} \to \overline{\mathbb{D}}$ is not a hyperbolic rotation and has a fixed point $c \in \mathbb{D}$, then c has to be the Denjoy–Wolff point of f. In fact, the Denjoy–Wolff point $w \in \overline{\mathbb{D}}$ can be equivalently characterized (see Chapter 5 of [21]) by being the unique point that satisfies exactly one of the following two conditions:

(1) $|w| < 1$, $f(w) = w$ and $|f'(w)| < 1$;
(2) $|w| = 1$, $\sphericalangle \lim_{z\to w} f(z) = w$, and

$$\lim_{\substack{z \longrightarrow w \\ \sphericalangle}} \frac{f(z) - w}{z - w} \leq 1 \,.$$

Since the unit disc is conformally equivalent to the upper half-plane via the conformal automorphism of the extended complex plane $z \mapsto \frac{z-i}{z+i}$, the above theorem and equivalent characterization of the Denjoy–Wolff point applies to self-maps of the upper half-plane $\mathbb{C}^+ := \{z \in \mathbb{C}\colon \Im z > 0\}$, with the difference that when infinity is the Denjoy–Wolff of f, we have

$$\sphericalangle \lim_{z\to\infty} f(z)/z \geq 1 \,.$$

2.2. Free convolution, related transforms

2.2.1. Cauchy transform and Voiculescu transform. We now recall some basic notions about free convolution. Let us remind the reader that for a probability μ on $\mathbb{R}$, we denote G_μ its Cauchy–Stieljes transform

$$G_\mu(z) = \int \frac{1}{z - x}\, d\mu(x) \,, \quad z \in \mathbb{C}\backslash\mathbb{R}$$

and $F_\mu(z) = 1/G_\mu(z)$. Note that $F_\mu\colon \mathbb{C}^+ \to \mathbb{C}^+$.

The following theorem characterizes functions which appear as reciprocals (in the sense of multiplication) of Cauchy–Stieltjes transforms of probabilities on the real line. For the proof and an in-depth analysis of the subject, we refer to [1], Chapter 3.

Theorem 2.8. *Let $F\colon \mathbb{C}^+ \to \mathbb{C}^+$ be an analytic function. Then there exist $a \in \mathbb{R}$, $b \geq 0$ and a positive finite measure ρ on $\mathbb{R}$ so that*

$$F(z) = a + bz + \int_{\mathbb{R}} \frac{1 + tz}{t - z} d\rho(t) \,, \quad z \in \mathbb{C}^+ \,.$$

Moreover, F is the reciprocal of a Cauchy–Stieltjes transform of a probability measure on the real line if and only if $b = 1$. The triple (a, b, ρ) satisfies $a = \Re F(i), b = \lim_{y\to+\infty} F(iy)/iy$, and $b + \rho(\mathbb{R}) = \Im F(i)$.

Remark 2.9. An immediate consequence of Theorem 2.8 is that for any probability measure σ on $\mathbb{R}$, we have $\Im F_\sigma(z) \geq \Im z$ for all $z \in \mathbb{C}^+$, with equality for any value of z if and only if σ is a point mass. In this case, the measure ρ in the statement of Theorem 2.8 is zero.

The function F_μ can be seen to be invertible in a set of the form

$$\Gamma_{\alpha,M} = \{z \in \mathbb{C} : |z| \geq M, |\Im z| \geq \alpha |\Re z|\}$$

for some $M, \alpha > 0$.

The Voiculescu transform of μ (see paragraph 5 of [9]) is then given on $F_\mu(\Gamma_{\alpha,M})$ by $\phi_\mu(z) = F_\mu^{-1}(z) - z$.

The free convolution of two probability measures μ and ν on the real line is then characterized by the fact that

$$\phi_{\mu\boxplus\nu}(z) = \phi_\mu(z) + \phi_\nu(z) \tag{2.1}$$

on the common component of their domain that contains $i[s, +\infty)$ for some large enough $s > 0$ (for details, we refer again to [9].) Note here that Voiculescu's transform ϕ_μ and the so-called R-transform are related by $R_\mu(z) = \phi_\mu(1/z)$.

Another useful property of Cauchy–Stieltjes transforms of free convolutions of probability measures is subordination: for any μ, ν, there exist unique analytic functions $\omega_1, \omega_2 \colon \mathbb{C}^+ \to \mathbb{C}^+$ so that $G_\nu(\omega_1(z)) = G_\mu(\omega_2(z)) = G_{\mu\boxplus\nu}(z)$ for all $z \in \mathbb{C}^+$ and $\lim_{y\to+\infty} \omega_j(iy)/iy = 1$, $j = 1, 2$. This has been proved by Biane in [11].

In the following, we shall also need the following lemmas.

Lemma 2.10 (Fatou's theorem). *Let $f \colon \mathbb{C}^+ \to \mathbb{C}$ be an analytic function. If $\mathbb{C} \setminus f(\mathbb{C}^+)$ contains a half-line, then f admits finite nontangential limits at Lebesgue-almost all points of the real line.*

This lemma follows from Theorem 2.1 of [14], and conformal transformations.

Lemma 2.11. *Let ν be a probability measure on the real line.*

(i) *For almost all* (*with respect to the Lebesgue measure*) *real numbers x, the nontangential limit, at x, of $-\frac{1}{\pi}\Im G_\nu$ exists and is equal to the density, at x, of the absolutely continuous part of ν* (*with respect to the Lebesgue measure*).

(ii) *Let I be an open interval of the real line. Then we have equivalence between:*
   (a) *The restriction of G_ν to $\mathbb{C}^+$ extends analyticaly to an open set of $\mathbb{C}$ containing $\mathbb{C}^+ \cup I$.*
   (b) *The restriction of ν to I admits an analytic density.*

   Moreover, in this case, the density of the restriction of ν to I is $x \in I \mapsto -\frac{1}{\pi}\Im G_\nu(x)$, where $G_\nu(x)$ is the value, at x, of the extension mentioned in (*a*) *of the restriction of G_ν to $\mathbb{C}^+$.*

Proof. Part (i) is Theorem 3.16 from Chapter II of [23].

(ii) Suppose (a) to be true. Let us define, for $t > 0$, $\mathcal{C}_t = \frac{t dx}{\pi(t^2+x^2)}$. It is the law of tX when X is a $\mathcal{C}_1$-distributed random variable (hence a standard Cauchy variable), and thus converges weakly to δ_0 when t tends to zero. Let us also define, for $t \geq 0$,

$$\rho_t : x \in \mathbb{R} \mapsto \begin{cases} -\frac{1}{\pi}\Im G_\nu(x + it) & \text{if } t > 0 \text{ or } x \in I\,, \\ 0 & \text{in the other case}\,. \end{cases}$$

Then for all $t > 0$, ρ_t is the density of $\nu * \mathcal{C}_t$, and converges weakly (i.e. against any continuous bounded function) to ν as t tends to zero. So it suffices to prove that for all f compactly supported continuous function on I, $\int f(x)\rho_t(x)\mathrm{d}x$ tends to $\int f(x)\rho_0(x)\mathrm{d}x$ when t goes to zero, which is an easy application of dominated convergence theorem.

Suppose (b) to be true. It suffices to prove that for all $x \in I$, there is $\varepsilon_x > 0$ such that the restriction of G_ν to $\mathbb{C}^+$ admits an analytic extension g_x to $\mathbb{C}^+ \cup B(x, \varepsilon_x)$. (We denote by $B(x, \varepsilon_x)$ the open ball of center x and radius ε_x.) Indeed, in this case, since for all $x, x' \in I$, $g_x, g_{x'}$ coincide on $B(x,\varepsilon_x) \cap B(x', \varepsilon_{x'}) \cap \mathbb{C}^+$, one can define an analytic function on

$$\mathbb{C}^+ \cup \big(\cup_{x\in I} B(x,\varepsilon_x)\big)$$

which coincides with G_ν on $\mathbb{C}^+$ and with g_x on every $B(x,\varepsilon_x)$. So let us fix $x \in I$.

Without loss of generality, we may assume that (1) $x = 0$, (2) the analytic density function f of ν is defined analytic on $[-c, c]$ for some $c > 0$, (3) this function has radius of convergence $> c$ with power series $f(t) = \sum_{n=0}^\infty a_n t^n$, and (4) the support of ν is contained in $[-c, c]$ (since for all finite measures ν_1, ν_2, $G_{\nu_1+\nu_2} = G_{\nu_1} + G_{\nu_2}$ and G_{ν_1} is analytic outside of the support of ν_1). It will be enough to show that G_ν extends analytically through $(-c, c)$. Let log be the analytic function on $\mathbb{C}\backslash\{-it\,;\, t \in [0,+\infty)\}$ whose derivative is $\frac{1}{z}$. For any $|z| \le c$, $z \in \mathbb{C}^+$, we have the integral

$$\begin{aligned}
G_\nu(z) &= \int_{\mathbb{R}} \frac{f(t)}{z-t}dt = \int_{[-c,c]} \sum_{n=0}^{\infty} a_n \frac{t^n}{z-t}dt = \int_{[-c,c]} \sum_{n=0}^{\infty} a_n \frac{t^n - z^n + z^n}{z-t}dt \\
&= -\sum_{n=0}^{\infty} a_n \left(\sum_{j=0}^{n-1} z^{n-j} \frac{c^{j+1} - (-c)^{j+1}}{j+1} + z^n \big[\log(z-c) - \log(z+c)\big] \right) \\
&= \left(\sum_{n=0}^{\infty} a_n z^n \right) \big[\log(z+c) - \log(z-c)\big] - \sum_{n=0}^{\infty} a_n \left(\sum_{j=0}^{n-1} z^{n-j} \frac{c^{j+1} - (-c)^{j+1}}{j+1} \right).
\end{aligned}$$

(We can commute integral with sum because the function of $t \to |f(t)/(z-t)|$ is obviously bounded uniformly on (a neighbourhood of, even) $[-c, c]$, for any z in a compact subset of $\mathbb{C}^+ \cap\ B(0, c)$ and the sum is absolutely convergent because of the power series condition.) We claim that in fact this formula defines an extension of G_ν on $B(0, c)$. Indeed, the first sum is obviously convergent, while for the second we have $|z^{n-j} \frac{c^{j+1}-(-c)^{j+1}}{j+1}| \le c^{n-j} \cdot 2|c^{j+1}| \frac{1}{j+1} = 2\frac{c^{n+1}}{j+1}$, so, since

$$\sum_{n=0}^{\infty} a_n c^{n+1} \sum_{j=0}^{n} \frac{1}{j+1} < \infty\,,$$

(recall the radius of convergence), so must be the second term in the sum above, for all $|z| \le c$. Thus, the function G_ν admits an analytic extension on an open set containing $\mathbb{C}^+ \cup (-c, c)$, and the formula is given above. □

Lemma 2.12. *Let ν be a probability measure on the real line with support equal to $\mathbb{R}$ and which is concentrated on a set of null Lebesgue measure. Then the cluster set of the restriction of its Cauchy transform to the upper half plane at any real point is the closure, in $\mathbb{C} \cup \{\infty\}$, of the lower half plane.*

Proof. Let us apply the upper half-plane version of Theorem 2.4 to the opposite of ν's Cauchy transform. Since by (ii) of Lemma 2.11, for all real number $x, -G_\nu$ does not extend analytically to x, the theorem will imply what we want to prove. Indeed, part (i) of Lemma 2.11 implies that the imaginary part of G_ν has a null nontangential limit at Lebesgue-almost all points x of the real line. Since by Lemma 2.10 G_ν admits a finite nontangential limit at Lebesgue-almost all $x \in \mathbb{R}$, it follows that $-G_\nu$ satisfies the upper half-plane version of Theorem 2.4. This completes the proof. □

Lemma 2.13. *The set of symmetric probability measures on the real line with support $\mathbb{R}$ and which are concentrated on a set of null Lebesgue measure is dense in the set of symmetric probability measures for the topology of weak convergence.*

Proof. Let D be the set of symmetric probability measures on the real line with support $\mathbb{R}$ and which are concentrated on a set of null Lebesgue measure. Since the set of symmetric probability measures which are finite convex combination of Dirac masses is dense in the set of symmetric probability measures, it suffices to prove that for all real numbers a, $\frac{1}{2}(\delta_{-a} + \delta_a)$ is in the closure of D. This is clear, since if $\nu \in D$, then for all $\varepsilon \in (0,1)$, $\frac{1-\varepsilon}{2}(\delta_{-a} + \delta_a) + \varepsilon\nu \in D$. □

2.2.2. Free infinite divisibility. One can extend the notion of infinitely divisible law from classical convolution to free convolution: a probability measure μ is said to be $\boxplus$-infinitely divisible if for any $n \in \mathbb{N}$ there exists a probability μ_n so that $\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ times}}$. It can be shown that any such measure embeds naturally in a semigroup of measures $\{\mu^{\boxplus t} \colon t \geq 0\}$ so that $t \mapsto \mu^{\boxplus t}$ is continuous in the weak topology, $\mu^{\boxplus 1/n} = \mu_n$ for all $n \in \mathbb{N}$, $\mu_0 = \delta_0$, and $\mu^{\boxplus s+t} = \mu^{\boxplus s} \boxplus \mu^{\boxplus t}$ for all $s, t \geq 0$. It follows easily from (2.1) that for any $t \geq 0$, we have $\phi_{\mu^{\boxplus t}}(z) = t\phi_\mu(z)$ for all points z in the common domain of the two functions.

In [9], Bercovici and Voiculescu have completely described infinitely divisible probability measures with respect to free additive convolution in terms of their Voiculescu transforms.

Theorem 2.14.

(i) *A probability measure μ on $\mathbb{R}$ is $\boxplus$-infinitely divisible if and only if ϕ_μ has an analytic extension defined on $\mathbb{C}^+$ with values in $\mathbb{C}^- \cup \mathbb{R}$.*

(ii) *Let $\phi \colon \mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$ be an analytic function. Then ϕ is a continuation of ϕ_μ for some $\boxplus$-infinitely divisible measure μ if and only if*

$$\lim_{\substack{z \to \infty \\ \sphericalangle}} \frac{\phi(z)}{z} = 0\,.$$

In the following lemma, we describe some properties of free convolutions with $\boxplus$-infinitely divisible measures.

Lemma 2.15. *Let μ be a probability which is not purely atomic, and ν an arbitrary probability measure. We know that there exist two subordination functions ω_1 and ω_2 from $\mathbb{C}^+$ into $\mathbb{C}^+$ so that $F_\nu \circ \omega_1 = F_\mu \circ \omega_2 = F_{\mu\boxplus\nu}$. Moreover:*

(1) $\omega_1(z) + \omega_2(z) = F_{\mu\boxplus\nu}(z) + z$, $z \in \mathbb{C}^+$;

Assume in addition that μ is $\boxplus$-infinitely divisible. Then

(2) *ω_1 is the inverse function of $H(w) = w + \phi_\mu(F_\nu(w))$, $w \in \mathbb{C}^+$, so in particular it has a continuous extension to $\mathbb{R} \cup \{\infty\}$ and $\omega_1(x)$ is finite for all $x \in \mathbb{R}$. Moreover, $\omega_1(z)$ is the Denjoy–Wolff point of the function $g_z \colon \mathbb{C}^+ \to \mathbb{C}^+$, $g_z(w) = w + z - H(w)$, for all $z \in \mathbb{C}^+ \cup \mathbb{R}$;*

(3) *F_μ has a continuous extension to $\mathbb{R} \cup \{\infty\}$, analytic outside a discrete set in $\mathbb{R}$, and $F_\mu(x)$ is finite for all $x \in \mathbb{R}$;*

(4) *ω_2 and $F_{\mu\boxplus\nu}$ extend continuously to $\mathbb{R}$. Moreover, if $F_{\mu\boxplus\nu}(x) \in \mathbb{R}$, then so are $\omega_1(x)$ and $\omega_2(x)$;*

(5) *If all existing nontangential limits of ϕ_μ at points of $\mathbb{R}$ belong to $\mathbb{C}^-$, then $F_\mu(x) \in \mathbb{C}^+$ for all $x \in \mathbb{R}$.*

Remark 2.16. (4) already shows that at $x \in \mathbb{R}$ such that $F_{\mu\boxplus\nu}(x) \neq 0$, $\mu\boxplus\nu$ is absolutely continuous with respect to Lebesgue measure, with density $\frac{\Im(F_{\mu\boxplus\nu})}{\pi|F_{\mu\boxplus\nu}|^2}(x)$. Note also that at a point x such that $F_{\mu\boxplus\nu}(x) = 0$, Lemma 2.15 (4) implies that $\omega_2(x) \in \mathbb{R}$, and by the definition of ω_2, together with Lindelöf's Theorem 2.1,

$$0 = F_{\mu\boxplus\nu}(x) = \lim_{\substack{z \longrightarrow x \\ \sphericalangle}} F_\mu\big(\omega_2(z)\big) = \lim_{\substack{z \longrightarrow \omega_2(0) \\ \sphericalangle}} F_\mu(z)\,.$$

Since μ is infinitely divisible, Proposition 5.1 (1) of [5] then guarantees that $t\omega_2(0)$ is an atom of $\mu^{\boxplus t}$ for all $t < 1$.

Proof. Item (1) has been proved in [10], and can be easily checked from (2.1) and analytic continuation.

Item (2) is a direct consequence of the fact that H satisfies the conditions imposed on the function denoted also H in Theorem 4.6 [5] (namely it is analytic in $\mathbb{C}^+$, it decreases the imaginary part, and its derivative has strictly positive limit at infinity – in this case equal to one). Hence, by Theorem 4.6 [5] (2), H is invertible from $\omega_1(\mathbb{C}^+)$ onto $\mathbb{C}^+$ and, since by (2.1), $F_\nu = F_{\nu\boxplus\mu} \circ H$, its inverse is exactly ω_1. Moreover, by Theorem 4.6, part (2) of [5], ω_1 extends continuously to $\mathbb{R}$ with image in $\mathbb{C}^+ \cup \mathbb{R}$, while part (3) of the same theorem guarantees that $\omega_1(z)$ is the Denjoy–Wolff point of g_z for all $z \in \mathbb{C}^+ \cup \mathbb{R}$.

Item (3) follows from Proposition 2.8 (a) in [3].

We prove now (4). Assume that there exists $x_0 \in \mathbb{R}$ so that $C(F_{\mu\boxplus\nu}, x_0)$ contains more than one point (and hence, by Lemma 2.2, is a continuum). Since by (2) $\omega_1(x_0)$ is finite, Theorem 4.1 of [2] allows us to conclude that if $C(F_{\mu\boxplus\nu}, x_0) \cap \mathbb{C}^+ \neq \varnothing$, then $F_{\mu\boxplus\nu}$ extends analytically to x_0, providing a contradiction. Thus, $C(F_{\mu\boxplus\nu}, x_0)$ contains either an interval, or the complement of an interval, in $\mathbb{R}$.

By (2) $\omega_1(x_0)$ exists and then, by (1), $\omega_1(x_0) \in \mathbb{R}$. By definition, for any $c \in C(F_{\mu\boxplus\nu}, x_0)$ in such an interval there exists a sequence $\{z_n^c\}_n \subset \mathbb{C}^+$ converging to x_0 so that $\lim_{n\to\infty} F_{\mu\boxplus\nu}(z_n^c) = c$. Using (1) we obtain

$$\lim_{n\to\infty} \omega_2(z_n^c) = \lim_{n\to\infty} F_{\mu\boxplus\nu}(z_n^c) - \omega_1(z_n^c) + z_n^c = c - \omega_1(x_0) + x_0 \,.$$

Now part (3) of the lemma allows us to conclude that

$$c = \lim_{n\to\infty} F_{\mu\boxplus\nu}(z_n^c) = \lim_{n\to\infty} F_\mu\big(\omega_2(z_n^c)\big) = F_\mu\big(c - \omega_1(x_0) + x_0\big)\,,$$

for all c in an interval. The analyticity of F_μ outside a discrete subset of $\mathbb{R}$ implies (by analytic continuation) that $F_\mu(z) = z - (x_0 - \omega_1(x_0))$ for all $z \in \mathbb{C}^+$, so $\mu = \delta_{x_0-\omega_1(x_0)}$, contradicting our hypothesis. The continuity of ω_2 on the real line follows now immediately from (1) and (2).

To prove item (5) assume that $F_\mu(x) \in \mathbb{R}$. We know that equality $z = F_\mu(z) + \phi_\mu(F_\mu(z))$ extends to $\mathbb{R}$. If ϕ_μ has no nontangential limit at the point $F_\mu(x)$, then, since the continuity of F_μ guarantees the existence of $\lim_{z\to x} \phi_\mu(F_\mu(z))$, Lindelöf's Theorem 2.1 provides a contradiction (ϕ_μ has limit along the path $F_\mu(x + i\mathbb{R}^+)$ but not nontangential, at $F_\mu(x)$). If ϕ_μ has nontangential limit at $F_\mu(x)$, then, by hypothesis, it must be complex, so obviously $\Im F_\mu(x) = -\Im\sphericalangle\phi_\mu(F_\mu(x)) > 0$. Contradiction again. $\square$

2.3. Rectangular free convolution, related transforms

2.3.1. Introduction to the rectangular free convolution and to the related transforms. We recall [7] the construction of the *rectangular R-transform with ratio* λ, and of the *rectangular free convolution* $\boxplus_\lambda$ *with ratio* λ, for $\lambda \in [0,1]$: one can summarize the different steps of the construction of the rectangular R-transform with ratio λ in the following chain

$$\underset{\text{sym. prob. measure}}{\mu} \longrightarrow \underset{\text{Cauchy transf.}}{G_\mu} \longrightarrow H_\mu(z) = \lambda G_\mu\left(\frac{1}{\sqrt{z}}\right)^2 + (1-\lambda)\sqrt{z}\,G_\mu\left(\frac{1}{\sqrt{z}}\right) \longrightarrow$$
$$\underset{\text{rect. } R\text{-transf. with ratio } \lambda}{C_\mu(z) = U\left(\frac{z}{H_\mu^{-1}(z)} - 1\right)},$$

where for all $z = \rho e^{i\theta}$, with $\rho \in (0,+\infty), \theta \in [0,2\pi)$, $\sqrt{z} = \rho^{1/2}e^{i\theta/2}$ (note that $\sqrt{\cdot}$ is analytic on $\mathbb{C}\backslash\mathbb{R}^+$), and U is the inverse of $T-1$, where

$$T(z) = (\lambda z + 1)(z+1)\,,$$

$$\text{i.e.}\quad U(z) = \frac{-\lambda - 1 + \left[(\lambda+1)^2 + 4\lambda z\right]^{1/2}}{2\lambda} \quad \big(\text{when } \lambda = 0,\, U(z) = z\big),$$

where $z \mapsto z^{1/2}$ is the analytic version of the square root on the complement of the real non positive half line such that $1^{1/2} = 1$ (i.e. for all $z = \rho e^{i\theta}$, with $\rho > 0, \theta \in (-\pi,\pi)$, $z^{1/2} = \rho^{1/2}e^{i\theta/2}$).

Note that the rectangular R-transform with ratio 1 (resp. 0), for symmetric distributions, is linked to the Voiculescu transform by the relation $C_\mu(z) =$

$\sqrt{z}\varphi_\mu(1/\sqrt{z})$ (resp. $C_\mu(z) = z\varphi_\rho(z)$, where ρ is the push-forward of μ by the function $t \to t^2$).

The rectangular free convolution of two symmetric probability measures μ, ν on the real line is the unique symmetric probability measure whose rectangular R-transform is the sum of the rectangular R-transforms of μ and ν, and it is denoted by $\mu \boxplus_\lambda \nu$. Then, we have

$$C_{\mu \boxplus_\lambda \nu} = C_\mu + C_\nu \,. \tag{2.2}$$

If $\lambda = 0$, $\mu \boxplus_\lambda \nu$ is the symmetric law which push-forward by $t \to t^2$ is the free convolution of the push-forwards by $t \to t^2$ of μ and ν, and if $\lambda = 1$, it is $\mu \boxplus \nu$.

Remark 2.17. [How to compute μ when we know C_μ ?] First, we have $z/H_\mu^{-1}(z) = T(C_\mu(z))$, for $z \in \mathbb{C}\backslash\mathbb{R}^+$ small enough. From this, we can compute $H_\mu(z)$ for $z \in \mathbb{C}\backslash\mathbb{R}^+$ small enough. Then we can use the equation, for $z \in \mathbb{C}\backslash\mathbb{R}^+$,

$$\frac{1}{z}H_\mu(z) = \lambda\left(\frac{1}{\sqrt{z}}G_\mu\left(\frac{1}{\sqrt{z}}\right)\right)^2 + (1-\lambda)\frac{1}{\sqrt{z}}G_\mu\left(\frac{1}{\sqrt{z}}\right). \tag{2.3}$$

Moreover, when $z \in \mathbb{C}\backslash\mathbb{R}^+$ is small enough, $1/\sqrt{z}$ is large and in $\mathbb{C}^-$, so $\frac{1}{\sqrt{z}}G_\mu(\frac{1}{\sqrt{z}})$ is close to 1. $\frac{1}{z}H_\mu(z)$ is also close to 1, and for h, g complex numbers close to 1,

$$h = \lambda g^2 + (1-\lambda)g \Leftrightarrow g = V(h)\,,$$

$$\text{with} \quad V(z) = \frac{\lambda - 1 + \left((\lambda-1)^2 + 4\lambda z\right)^{\frac{1}{2}}}{2\lambda} = U(z-1) + 1\,.$$

So one has, for $z \in \mathbb{C}\backslash\mathbb{R}^+$ small enough,

$$\frac{1}{\sqrt{z}}G_\mu\left(\frac{1}{\sqrt{z}}\right) = V\left(\frac{H_\mu(z)}{z}\right). \tag{2.4}$$

2.3.2. A few remarks about H_μ. (a) One has, for $z \in \mathbb{C}\backslash\mathbb{R}^+$ small enough,

$$\frac{1}{\sqrt{z}}G_\mu\left(\frac{1}{\sqrt{z}}\right) = V\left(\frac{H_\mu(z)}{z}\right).$$

Note that the function $\frac{1}{\sqrt{z}}G_\mu(\frac{1}{\sqrt{z}})$ is analytic on $\mathbb{C}\backslash\mathbb{R}^+$, hence

$$U\left(\frac{H_\mu(z)}{z} - 1\right) = V\left(\frac{H_\mu(z)}{z}\right) - 1 = \frac{1}{\sqrt{z}}G_\mu\left(\frac{1}{\sqrt{z}}\right) - 1$$

admits an analytic extension to $\mathbb{C}\backslash\mathbb{R}^+$. Note that one cannot assert that this extension is given by the same formula on the whole $\mathbb{C}\backslash\mathbb{R}^+$, but we know, by analytic continuation, that if one denotes this extension by M_μ, one has, for all $z \in \mathbb{C}\backslash\mathbb{R}^+$,

$$\left[2\lambda M_\mu(z)+1+\lambda\right]^2 = (\lambda-1)^2+4\lambda\frac{H_\mu(z)}{z}\,, \quad \text{or, equivalently,} \quad H_\mu(z) = zT\big(M_\mu(z)\big)\,.$$

Let us observe that $M_\mu(z) = \psi_\mu(\sqrt{z}) = \psi_{\mu^2}(z)$, where $\psi_\mu(z) = \int \frac{zt}{1-zt}d\mu(t)$ is the so-called moment generating function of μ, and μ^2 is the probability on $[0, +\infty)$ given by $\int f(t)\,d\mu^2(t) = \int f(t^2)\,d\mu(t)$ for all Borel bounded functions f. Hence, as noticed in Proposition 6.2 of [9], M_μ maps the upper half-plane into itself and the left half-plane $i\mathbb{C}^+$ into the disc with diameter the interval $(\mu(\{0\})-1,0)$.

(b) H_μ maps $\mathbb{C}\setminus\mathbb{R}^+$ into itself and maps $i\mathbb{C}^+\cap\mathbb{C}^+$ into $i\mathbb{C}^+$. Indeed, for $z\in\mathbb{C}^+$,

$$H_\mu(z) = \underbrace{G_\mu\left(\frac{1}{\sqrt{z}}\right)}_{\in\mathbb{C}^+}\left(\underbrace{\lambda G_\mu\left(\frac{1}{\sqrt{z}}\right)+(1-\lambda)\sqrt{z}}_{\in\mathbb{C}^+}\right),$$

and the product of two elements of $\mathbb{C}^+$ cannot belong to $\mathbb{R}^+$. (As a consequence, Lemma 2.10 guarantees that the restriction of H to the upper half-plane has nontangential limits at almost all points of the positive half-line.) The second statement follows from part (a) above and the definition of T: let $z=x+iy$ be so that $x<0$ and $y>0$. Then

$$\begin{aligned}
z\big(M_\mu(z)+1\big) &= z\left(\int_{[0,+\infty)}\frac{tz}{1-tz}\,d\mu^2(t)+1\right) = \int_{[0,+\infty)}\frac{z}{1-tz}\,d\mu^2(t)\\
&= \int_{[0,+\infty)}\frac{x(1-tx)-ty^2}{(1-tx)^2+(ty)^2}\,d\mu^2(t)\\
&\quad + iy\int_{[0,+\infty)}\frac{1}{(1-tx)^2+(ty)^2}\,d\mu^2(t)\,.
\end{aligned}$$

Since $x<0$, the real part of the above expression is negative, as, since $y>0$, its imaginary part is positive. From (a) above,

$$H_\mu(z) = zT\big(M_\mu(z)\big) = z\big(M_\mu(z)+1\big)\big(\lambda M_\mu(z)+1\big)\,.$$

Since $M_\mu(z)$ belongs to the upper half of the disc of diameter $(\mu(\{0\})-1,0)$, $\lambda M_\mu(z)+1$ belongs to $\mathbb{C}^+\cap(-i\mathbb{C}^+)$, so that the product of $\lambda M_\mu(z)+1$ and $z(M_\mu(z)+1)$ must belong to $i\mathbb{C}^+$.

(c) Using the two previous remarks, we observe that if there exist $r,c\in(0,+\infty)$ and a sequence $\{z_n\}_n\subset\mathbb{C}^+$ so that $\lim_{n\to\infty} z_n = r$ and $\lim_{n\to\infty} H_\mu(z_n)=c$, then the set $\{M_\mu(z_n)\colon n\in\mathbb{N}\}$ has at most two limit points, either both negative (if $r>c$) or one negative and one non-negative (if $r\le c$). Indeed, the formula above guarantees that

$$\lim_{n\to\infty} M_\mu(z_n)\in\left\{\frac{-(1+\lambda)\sqrt{r}\pm\sqrt{r(1-\lambda)^2+4\lambda c}}{2\lambda\sqrt{r}}\right\}.$$

(d) Let us define two properties, for functions defined on $\mathbb{C}\setminus\mathbb{R}^+$.

(P1) $\qquad \forall z\in\mathbb{C}\setminus\mathbb{R}^+,\quad f(\overline{z}) = -\overline{f(z)}\,.$

(P2) $\qquad \forall z\in\mathbb{C}\setminus\mathbb{R}^+,\quad f(\overline{z}) = \overline{f(z)}\,.$

It is easy to see that $\sqrt{\cdot}$ has the property (P1) and that for μ symmetric probability measure $G_\mu(1/\sqrt{z})$ has also property (P1), hence H_μ has property (P2). As a consequence, in view of (b), $H_\mu(\mathbb{R}^-)\subset\mathbb{R}^-$ and $H_\mu(i\mathbb{C}^+)\subseteq i\mathbb{C}^+$. Similarly,

H_μ^{-1} also satisfies property (P2) and hence also C_μ satisfies property (P2), so, in particular, $C_\mu((-a,0)) \subseteq \mathbb{R}$ for any $a>0$ so that $(-a,0)$ is included in the domain of C_μ.

(e) Let us denote by x_0 the largest number in $(-\infty,0)$ so that $H'_\mu(x_0)=0$ (we do not exclude the case $x_0=-\infty$). Since $H_\mu(0)=0$ and $H'_\mu(0)=1$, H_μ^{-1} and C_μ are defined, and analytic, on the interval $(H_\mu(x_0),0)$, and moreover, $C_\mu((H_\mu(x_0),0)) \subseteq \mathbb{R}^-$. Indeed, H_μ^{-1} is obviously defined and analytic on $(H_\mu(x_0),0)$, and as $H_\mu(\mathbb{R}^-) \subseteq \mathbb{R}^-$ (by (b) and (d) above), we have $\frac{x}{H_\mu^{-1}(x)} > 0$ for all $x \in (H_\mu(x_0),0)$. Thus, $C_\mu(x) = U(\frac{x}{H_\mu^{-1}(x)} - 1)$ is defined and analytic on $(H_\mu(x_0),0)$.

To show that $C_\mu((H_\mu(x_0),0)) \subseteq \mathbb{R}^-$ it is enough to prove that $U(\frac{H_\mu(x)}{x} - 1) < 0$ for any $x \in (x_0,0)$. (As observed in Remark 2.17, the derivative of H_μ in zero is one, so that H_μ is increasing on $(x_0,0)$.) This statement is due to the inequality $\frac{H_\mu(x)}{x} < 1$, $x \in (x_0,0)$. Now, $1/\sqrt{x} \in i\mathbb{R}^-$ and μ is symmetric, so $G_\mu(1/\sqrt{x}) \in i\mathbb{R}^+$, which implies that $|G_\mu(1/\sqrt{x})| = \Im G_\mu(1/\sqrt{x})$ for all $x<0$. Remark 2.9 implies that

$$\left|\frac{1}{\sqrt{x}}\right| = -\Im\frac{1}{\sqrt{x}} < -\Im F_\mu\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\Im G_\mu\left(\frac{1}{\sqrt{x}}\right)},$$

so $|\frac{1}{\sqrt{x}}G_\mu(\frac{1}{\sqrt{x}})| < 1$, for any $\mu \neq \delta_0$, $x<0$. Thus, $0 < \frac{1}{\sqrt{x}}G_\mu(\frac{1}{\sqrt{x}}) < 1, \quad x<0$. The definition of H_μ and the fact that $0<\lambda<1$ imply now the desired result.

(f) We have

$$\begin{aligned}\lim_{x\to-\infty} H_\mu(x) &= -(1-\lambda)\int t^{-2}\,d\mu(t) \quad \big(=-\infty \text{ if } \mu(\{0\})>0\big),\\ \lim_{x\to-\infty}\frac{H_\mu(x)}{x} &= \lambda\mu(\{0\})^2 + (1-\lambda)\mu(\{0\}).\end{aligned} \tag{2.5}$$

This follows from the definition of H_μ together with the monotone convergence theorem: recall that

$$\begin{aligned}H_\mu(x) &= \lambda G_\mu(1/\sqrt{x})^2 + (1-\lambda)\sqrt{x}G_\mu\big(1/\sqrt{x}\big)\\ &= \lambda\left(\int \frac{1}{\frac{-i}{\sqrt{|x|}} - t}\,d\mu(t)\right)^2 + (1-\lambda)i\sqrt{|x|}\int\frac{1}{\frac{-i}{\sqrt{|x|}}-t}\,d\mu(t)\\ &= -\lambda\left(\int\frac{\sqrt{|x|}}{1+t^2|x|}\,d\mu(t)\right)^2 - (1-\lambda)\int\frac{|x|}{1+t^2|x|}\,d\mu(t).\end{aligned}$$

(We have used the fact that μ is symmetric in the last equality.) Since $\lim_{x\to-\infty}\frac{1}{1+t^2|x|} = \chi_{\{0\}}(t)$ and the convergence is dominated by 1, (2.5) follows.

Now observe that the functions $f_x(t) = \frac{|x|}{1+t^2|x|}$, $x<-1, t\in\mathbb{R}$, satisfy $f_{x_1}(t) > f_{x_2}(t)$ iff $|x_1|>|x_2|$, $f_x(t) < t^{-2}$ and $\lim_{x\to-\infty} f_x(t) = t^{-2}, t \in \mathbb{R}$, with the convention $1/0 = +\infty$. So by the monotone convergence theorem,

$$\lim_{x\to-\infty} \int \frac{|x|}{1+t^2|x|}\, d\mu(t) = \int t^{-2}\, d\mu(t) \in (0,+\infty]\,.$$

If $\int t^{-2}\, d\mu(t) < +\infty$, we deduce that

$$\lim_{x\to-\infty} \int \frac{\sqrt{|x|}}{1+t^2|x|}\, d\mu(t) = \lim_{x\to-\infty} |x|^{-\frac{1}{2}} \lim_{x\to-\infty} \int \frac{|x|}{1+t^2|x|}\, d\mu(t) = 0$$

so that indeed $\lim_{x\to-\infty} H_\mu(x) = -(1-\lambda)\int t^{-2}\, d\mu(t)$. This is also true when $\int t^{-2}\, d\mu(t) = +\infty$.

2.3.3. Free rectangular infinite divisibility. As for the free convolution, for any $\lambda \in [0,1]$, one can extend the notion of infinitely divisible law to rectangular free convolution with ratio λ: a symmetric probability measure μ is said to be $\boxplus_\lambda$-infinitely divisible if for any $n \in \mathbb{N}$ there exists a symmetric probability μ_n so that $\mu = \underbrace{\mu_n \boxplus_\lambda \mu_n \boxplus_\lambda \cdots \boxplus_\lambda \mu_n}_{n \text{ times}}$. It can be shown that any such measure embeds naturally in a semigroup of measures $\{\mu^{\boxplus_\lambda t} : t \geq 0\}$ so that $t \mapsto \mu^{\boxplus_\lambda t}$ is continuous in the weak topology, $\mu^{\boxplus_\lambda 1/n} = \mu_n$ for all $n \in \mathbb{N}$, $\mu_0 = \delta_0$, and $\mu^{\boxplus_\lambda (s+t)} = \mu^{\boxplus_\lambda s} \boxplus_\lambda \mu^{\boxplus_\lambda t}$ for all $s,t \geq 0$. It follows easily from (2.2) that for any $t \geq 0$, we have $C_{\mu^{\boxplus_\lambda t}}(z) = tC_\mu(z)$ for all points z in the common domain of the two functions.

In [8], the infinitely divisible probability measures with respect to $\boxplus_\lambda$ are completely described in terms of their rectangular R-transforms.

Theorem 2.18. *A symmetric probability measure on the real line is $\boxplus_\lambda$-infinitely divisible if and only if there is a symmetric positive finite measure G on the real line such that C_μ extends to $\mathbb{C}\backslash\mathbb{R}^+$ and is given by the following formula:*

$$\forall z \in \mathbb{C}\backslash\mathbb{R}^+\,, \quad C_\mu(z) = z\int_{\mathbb{R}} \frac{1+t^2}{1-zt^2}\mathrm{d}G(t)\,. \tag{2.6}$$

In this case, G is unique and is called the Lévy measure of μ.

Remark 2.19. It would be useful, in order to know if the measures to which Lemma 4.1 can be applied are all $\boxplus_\lambda$-infinitely divisible, to know if, as for the Voiculescu transform and $\boxplus$-infinite divisibility, any symmetric probability measure whose rectangular R-transform extends analytically to $\mathbb{C}\backslash\mathbb{R}^+$ is actually $\boxplus_\lambda$-infinitely divisible. Unfortunately, the proof of the analogous result in the square case involves the fact that the Voiculescu transform of any probability measure takes its values in the closure of the lower half-plane, and we still did not find the analogue of that fact in the rectangular context.

In the following we shall describe some more or less obvious consequences of Theorem 2.18. First we record for future reference the geometry of the preimage of the complex plane via $T(z) = (\lambda z+1)(z+1)$:

Remark 2.20.

(i) $T^{-1}(\{0\}) = \{-1/\lambda, -1\}$ and $T'(-(\lambda+1)/2\lambda) = 0$;

(ii) $T((-\infty, -1/\lambda]) = T([-1, +\infty)) = \mathbb{R}^+$, and T is injective on each of these two intervals;

(iii) $T((-1/\lambda, -(1+\lambda)/2\lambda]) = T([-(1+\lambda)/2\lambda, -1)) = [-(1-\lambda)^2/4\lambda, 0)$ and T is injective on each of these two intervals;

(iv) $T(-(1+\lambda)/2\lambda + i\mathbb{R}^+) = T(-(1+\lambda)/2\lambda - i\mathbb{R}^+) = (-\infty, -(1-\lambda)^2/4\lambda]$ and T is injective on each of these two sets;

(v) $\Re T(x+iy) = 0$ iff $\lambda y^2 = (\lambda x + 1)(x+1)$. In particular, the pre-image of the imaginary axis is an equilateral hyperbola whose branches go through -1 and $-1/\lambda$ and the tangents at these points to the hyperbola are vertical. More general, $\Re T(x+iy)/\Im T(x+iy) = c \geq 0$ if and only if $x^2 - y^2 - 2cxy + (1 + \frac{1}{\lambda})(x - cy) + \frac{1}{\lambda} = 0$. That is, the pre-image via T of any non-horizontal line going through the origin is an equilateral hyperbola going through -1 and $-1/\lambda$ and whose tangents at these two points are parallel to the line $cy = x$.

Let us denote $K_1 = \{z \in \mathbb{C}^+ : \Re z > -(1+\lambda)/2\lambda\}$, $K_2 = \{z \in \mathbb{C}^+ : \Re z < -(1+\lambda)/2\lambda\}$, $K_3 = \{z \in \mathbb{C}^- : \Re z > -(1+\lambda)/2\lambda\}$, $K_4 = \{z \in \mathbb{C}^- : \Re z < -(1+\lambda)/2\lambda\}$.

Note that using the formula $T(z) = \lambda[(z + \frac{\lambda+1}{2\lambda})^2 - \frac{(1-\lambda)^2}{4\lambda^2}]$, one easily sees that $K_1 = T^{-1}(\mathbb{C}^+) \cap \mathbb{C}^+, K_2 = T^{-1}(\mathbb{C}^-) \cap \mathbb{C}^+, K_3 = T^{-1}(\mathbb{C}^-) \cap \mathbb{C}^-, K_4 = T^{-1}(\mathbb{C}^+) \cap \mathbb{C}^-$.

Lemma 2.21. *Let μ be a $\boxplus_\lambda$-infinitely divisible probability measure. Then*

1. *H_μ is the right inverse of the analytic function $\mathbb{C}\setminus\mathbb{R}^+ \ni w \mapsto \frac{w}{T(C_\mu(w))}$, hence injective;*
2. *$\mu(\{0\}) > 0$ if and only if $\lim_{w\to-\infty} C_\mu(w) \in (-1, 0]$. In that case,*
$$\begin{aligned}\mu(\{0\}) &= \frac{-(1-\lambda) + \sqrt{(1-\lambda)^2 + 4\lambda T\big(\lim_{w\to-\infty} C_\mu(w)\big)}}{2\lambda} \\ &= 1 + \lim_{w\to-\infty} C_\mu(w)\,,\end{aligned} \tag{2.7}$$
or, equivalently, $\lim_{w\to-\infty} C_\mu(w) = \mu(\{0\}) - 1$.
3. *$\pi > \arg H_\mu(z) \geq \arg z$ for all $z \in \mathbb{C}^+$, with equality if and only if $\mu = \delta_0$. In particular, $C_\mu(H_\mu(\mathbb{C}^+)) \subset K_1$;*
4. *H_μ is analytic around infinity whenever $\lim_{x\to-\infty} C_\mu(x) < -1$.*

Proof. By the definition of C_μ, Theorem 2.18, parts (d) and (e) of Subsection 2.3.2, and Remark 2.17 we obtain that $T(C_\mu(H_\mu(z))) = H_\mu(z)/z$ for all $z \in (-\infty, 0)$, and, by part (b) of Subsection 2.3.2 and analytic continuation, for all $z \in \mathbb{C}\setminus\mathbb{R}^+$. This proves item 1.

We prove now item 2. Since the case $\mu = \delta_0$ is trivial, we exclude it from our analysis. This allows us to assert that the Lévy measure of μ has a positive mass, hence, by (2.6), that $C_\mu((-\infty, 0)) \subset (-\infty, 0)$. Note that (2.6) implies also that $\lim_{w\to-\infty} C_\mu(w)$ exists in $[-\infty, 0)$. As $H_\mu((-\infty, 0)) \subseteq (-\infty, 0)$, (by (b), (d) of Subsection 2.3.2) we have $H_\mu(x)/x > 0$ for all $x \in (-\infty, 0)$, and hence the relation $T(C_\mu(H_\mu(z))) = H_\mu(z)/z$ implies that $T(C_\mu(H_\mu((-\infty,0)))) \subset \mathbb{R}^{+*}$ and therefore $C_\mu(H_\mu((-\infty,0)) \subset (-\infty, -\frac{1}{\lambda}] \cup [-1,0]$. Since $\lim_{x\uparrow 0} C_\mu(H_\mu(x)) = 0$, the continuity of $x \to C_\mu(H_\mu(x))$ on $\mathbb{R}^-$ implies that $C_\mu(H_\mu((-\infty,0))) \subseteq (-1,0)$.

Using part (f) of Subsection 2.3.2 and part (1) of this lemma,

$$\lambda\mu(\{0\})^2 + (1-\lambda)\mu(\{0\}) = \lim_{x\to-\infty} H_\mu(x)/x = \lim_{x\to-\infty} T\Big(C_\mu\big(H_\mu(x)\big)\Big) .$$

If $\mu(\{0\}) > 0$, then by (f) of Subsection 2.3.2, $\lim_{x\to-\infty} H_\mu(x) = -\infty$, hence

$$\begin{aligned} T\big(\lim_{w\to-\infty} C_\mu(w)\big) &= \lim_{w\to-\infty} T\big(C_\mu(w)\big) = \lim_{x\to-\infty} T\Big(C_\mu\big(H_\mu(x)\big)\Big) \\ &= \lambda\mu(\{0\})^2 + (1-\lambda)\mu(\{0\}) \in (0,1) , \end{aligned}$$

so, since $C_\mu(H_\mu((-\infty,0))) \subseteq (-1,0)$, we have $\lim_{w\to-\infty} C_\mu(w) = \mu(\{0\}) - 1 \in (-1,0)$.

Conversely, assume that $\lim_{w\to-\infty} C_\mu(w) \in (-1,0)$. We claim that $\lim_{x\to-\infty} H_\mu(x) = -\infty$. Indeed, assume to the contrary that this limit is finite, and denote it by $c \in (-\infty, 0)$. Then we have $0 = \lim_{x\to-\infty} H_\mu(x)/x = \lim_{x\to-\infty} T(C_\mu(H_\mu(x))) = T(C_\mu(c))$, so that $C_\mu(c) = -1$ or $-1/\lambda$. But C_μ is increasing on $(-\infty,0)$ (it follows easily from the differentiation of (2.6)), and we have assumed that $\lim_{x\to-\infty} C_\mu(x) > -1$. This is a contradiction. The statement concerning the mass at the origin follows since by (2.5)

$$\begin{aligned} 0 < \lambda\mu(\{0\})^2 + (1-\lambda)\mu(\{0\}) &= \lim_{x\to-\infty} \frac{H_\mu(x)}{x} = \lim_{x\to-\infty} T\Big(C_\mu\big(H_\mu(z)\big)\Big) \\ &= \lim_{w\to-\infty} T\big(C_\mu(w)\big) . \end{aligned}$$

To conclude, one can easily deduce (2.7) from the previous equation.

To prove item 3, we claim first that $H_\mu(\mathbb{C}^+) \subseteq \mathbb{C}^+$. Assume this is not the case: there exists a point z_1 in the upper half-plane so that $H_\mu(z_1) \in \mathbb{C}^- \cup \mathbb{R}$. Observe that by the relations

$$\forall \alpha \in (0,\pi) , \qquad \lim_{z\to 0 \ |\arg z - \pi| < \alpha} H_\mu(z)/z = 1 , \tag{2.8}$$

there is a point $z_0 \in \mathbb{C}^+$ so that $H_\mu(z_0) \in \mathbb{C}^+$. Consider a segment γ uniting z_0 and z_1. Then there must be a point $z_r \in \gamma$ so that $H_\mu(z_r) \in \mathbb{R}$ and $H_\mu([z_0,z_r)) \subset \mathbb{C}^+$. Since $H_\mu(\mathbb{C}\setminus\mathbb{R}^+) \subseteq \mathbb{C}\setminus\mathbb{R}^+$, we must have $H_\mu(z_r) < 0$. But then $C_\mu(H_\mu(z_r)) < 0$, so $T(C_\mu(H_\mu(z_r))) \in \mathbb{R}$. Thus, we contradict the relation $H_\mu(z_r) = z_r T(C_\mu(H_\mu(z_r)))$. This assures us that $H_\mu(\mathbb{C}^+) \subseteq \mathbb{C}^+$.

To conclude the proof of item 3, we have to prove that $H_\mu(z)/z \in \mathbb{C}^+$ whenever $z \in \mathbb{C}^+$. This is equivalent to $T(C_\mu(H_\mu(z))) \in \mathbb{C}^+$ whenever $z \in \mathbb{C}^+$, i.e., since by (2.6), $C_\mu(\mathbb{C}^+) \subseteq \mathbb{C}^+$, to $C_\mu(H_\mu(\mathbb{C}^+)) \subseteq K_1$. Note first that by (2.8), there are some points $z \in \mathbb{C}^+$ for which $H_\mu(z)/z \in \mathbb{C}^+$, i.e. $C_\mu(H_\mu(z)) \in K_1$. Hence the inclusion $C_\mu(H_\mu(\mathbb{C}^+)) \subseteq K_1$ can fail only if $C_\mu(H_\mu(\mathbb{C}^+))$ intersects the line $-(1+\lambda)/2\lambda + i\mathbb{R}^+$, so that there exists a point $w_0 \in \mathbb{C}^+$ with the property that

$T(C_\mu(H_\mu(w_0))) < 0$. But then we obtain that $H_\mu(w_0) = w_0 T(C_\mu(H_\mu(w_0))) \in \mathbb{C}^-$, a contradiction. This proves item 3.

We proceed now with proving item 4. As observed in the beginning of the proof of item 2, we must have $C_\mu(H_\mu((-\infty,0))) \subseteq (-1,0)$, so that $\lim_{x\to-\infty} C_\mu(H_\mu(x)) \geq -1$, hence the hypothesis

$$\lim_{x\to-\infty} C_\mu(x) < -1$$

implies $\lim_{x\to-\infty} H_\mu(x) = c \in (-\infty,0)$ (note here that the limit exists by (2.5)). Thus, $H_\mu(z)$ is analytic around infinity if and only if

$$W_\mu : z \in \mathbb{C}\backslash(0,+\infty) \mapsto \begin{cases} H_\mu(1/z) & \text{if} \quad z \neq 0\,, \\ c & \text{if} \quad z = 0\,, \end{cases}$$

extends analytically around zero. The relation $H_\mu(z) = zT(C_\mu(H_\mu(z)))$ allows us to write

$$W_\mu(z) = \frac{1}{z}\Big(\lambda C_\mu\big(W_\mu(z)\big) + 1\Big)\Big(C_\mu\big(W_\mu(z)\big) + 1\Big)\,,$$

hence for $z \in \mathbb{C}\backslash\mathbb{R}^+$ small enough so that $(1-\lambda)^2 + 4\lambda z W_\mu(z) \notin \mathbb{R}^-$,

$$C_\mu\big(W_\mu(z)\big) - \frac{-\lambda - 1 + [(1-\lambda)^2 + 4\lambda z W_\mu(z)]^{1/2}}{2\lambda} = 0\,.$$

This relation holds for z in $I \cap (-\infty,0)$, where $I \subset \mathbb{R}$ is a small enough interval centered at zero, as $\mu \neq \delta_0$. Thus, let us define $f : I \times (I+c) \to \mathbb{R}$, by

$$f(z,w) = C_\mu(w) - \frac{-\lambda - 1 + [(1-\lambda)^2 + 4\lambda z w]^{1/2}}{2\lambda}$$

(recall that $c = \lim_{x\to-\infty} H_\mu(x) < 0$, hence if I is small enough, f is well defined). This function satisfies $f(z, W_\mu(z)) = 0$ for all $z \in I \cap (-\infty,0)$. Hence $f(0,c) = \lim_{z\uparrow 0} f(z, W_\mu(z)) = 0$. We observe that

$$\partial_w f(z,w) = C_\mu'(w) - \frac{z}{[(1-\lambda)^2 + 4\lambda z w]^{1/2}}\,,$$

so that $\partial_w f(0,w) = C_\mu'(w) > 0$ for all $w \in (I+c) \cap (-\infty,0)$. Thus, the conditions of the implicit function theorem are satisfied, so we conclude that there exists a unique real map g, analytic on some subinterval J of I, centered at zero, so that $g(0) = c$ and $f(x, g(x)) = 0$ for all $x \in J$. The uniqueness guarantees that $g(x) = W_\mu(x)$ on their common domain, and hence it is an analytic extension to the interval J of $W_\mu(z) = H_\mu(1/z)$. This concludes the proof. $\square$

3. The square case

Below, we prove that free convolution is regularizing, namely that we can find a set of probability measures (roughly $\boxplus$-infinitely divisible distribution whose Voiculescu transform is sufficiently nice) such that any probability measure, once convoluted by one of these measures, has a density with respect to Lebesgue measure which is analytic and positive everywhere. The fact that we require the density to be analytic everywhere or positive everywhere will be seen in Proposition 3.4 to impose that these regularizing measures have no finite second moment. We shall give also some examples of such measures after the proof of the theorem.

3.1. A result of analyticity

Theorem 3.1. *Let μ be a $\boxplus$-infinitely divisible distribution. Assume that the Voiculescu transform ϕ_μ satisfies the following conditions:*

1. *For any $x \in \mathbb{R}$, either $\sphericalangle \lim_{z\to x} \phi_\mu(z) \in \mathbb{C}^-$, or ϕ_μ has no nontangential limit at x;*
2. *Either* (i) $\sphericalangle \lim_{z\to\infty} \phi_\mu(z) = \infty$, *or* (ii) $\sphericalangle \lim_{z\to\infty} \phi_\mu(z) \in \mathbb{C}^-$, *or* (iii) $C^\Delta(\phi_\mu, \infty)$ *contains more than one point.*

Then $\mu\boxplus\nu$ has a positive, everywhere analytic density for all probability measure ν.

Proof. We have, on a neighbourhood of infinity,

$$F_\nu^{-1}(z) + \phi_\mu(z) = z + \phi_\nu(z) + \phi_\mu(z) = z + \phi_{\mu\boxplus\nu}(z) = F_{\mu\boxplus\nu}^{-1}(z)\,.$$

We replace z by $F_{\mu\boxplus\nu}(z)$, for z in an appropriate truncated cone $\Gamma_{\alpha,M}$ – see Section 2.2.1 above (possible since $F_{\mu\boxplus\nu}(z)$ is defined on the whole upper half plane and is equivalent to z as z goes to infinity in a nontangential way), and get

$$z - \phi_\mu\big(F_{\mu\boxplus\nu}(z)\big) = F_\nu^{-1}\big(F_{\mu\boxplus\nu}(z)\big)\,.$$

Applying F_ν in both sides and using analytic continuation (recall that ϕ_μ extends to $\mathbb{C}^+$ by Theorem 2.11), we obtain

$$F_{\mu\boxplus\nu}(z) = F_\nu\Big(z - \phi_\mu\big(F_{\mu\boxplus\nu}(z)\big)\Big)\,, \quad z \in \mathbb{C}^+\,. \tag{3.1}$$

We will show that $F_{\mu\boxplus\nu}$ extends analytically to $\mathbb{R}$, and $F_{\mu\boxplus\nu}(x) \in \mathbb{C}^+$ for all $x \in \mathbb{R}$. This will imply the theorem according to Lemma 2.11. Let us fix a real number x. Observe first that the existence of a continuous extension with values in $\mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$ of $F_{\mu\boxplus\nu}$ at x is guaranteed by Lemma 2.15, (4).

We are first going to prove that we do not have $\lim_{z\to x} F_{\mu\boxplus\nu}(z) = \infty$ (i.e $\lim_{z\to x} |F_{\mu\boxplus\nu}(z)| = \infty$). Suppose that this happens. Then we have

$$\begin{aligned} x - \omega_1(x) &= \lim_{z\to x} z - \omega_1(z) = \lim_{z\to x} \omega_2(z) - F_\mu\big(\omega_2(z)\big) \\ &= \lim_{z\to x} \phi_\mu\Big(F_\mu\big(\omega_2(z)\big)\Big) = \lim_{z\to x} \phi_\mu\big(F_{\mu\boxplus\nu}(z)\big) \\ &= \lim_{\substack{w \longrightarrow \infty \\ \sphericalangle}} \phi_\mu(w)\,. \end{aligned} \tag{3.2}$$

We have used part (2) of Lemma 2.15 in the first equality, (1) of Lemma 2.15 in the second equality, definition of ϕ_μ and Theorem 2.14 in the third, and Lindelöf's Theorem 2.1 in the last equality. We next show that any of the three hypotheses of Theorem 3.1 (2) are in contradiction with (3.2).

Indeed, if (i) holds, then (3.2) implies that $x - \omega_1(x) = \infty$ which contradicts part (2) of Lemma 2.15. (iii) clearly cannot hold since (3.2) implies that $C^\Delta(\phi_\mu, \infty)$

contains only one point. Finally, assume that (ii) of item 2 of our theorem holds. Then, (3.2) implies that $\omega_1(x) \in \mathbb{C}^+$. Thus, we get the contradiction

$$\infty = \lim_{z\to x} F_{\mu\boxplus\nu}(z) = \lim_{z\to x} F_\mu\big(\omega_1(z)\big) = F_\mu\big(\omega_1(x)\big) \in \mathbb{C}^+ .$$

We next prove that $c = \lim_{z\to x} F_{\mu\boxplus\nu}(z)$ cannot be real. So, we assume $c \in \mathbb{R}$ and obtain a contradiction based on the fact that we then have

$$c = \lim_{z\to x} F_{\mu\boxplus\nu}(z) = \lim_{z\to x} F_\nu\Big(z - \phi_\mu\big(F_{\mu\boxplus\nu}(z)\big)\Big) \in \mathbb{R} . \tag{3.3}$$

We observe first that $\lim_{z\to x} \phi_\mu(F_{\mu\boxplus\nu}(z))$ exists. Indeed, we see as in (3.2),

$$\lim_{z\to x} \phi_\mu\big(F_{\mu\boxplus\nu}(z)\big) = \lim_{z\to x} \omega_2(z) - F_\mu\big(\omega_2(z)\big) = \lim_{z\to x} z - \omega_1(z) = x - \omega_1(x) .$$

If ϕ_μ has nontangential limit at c, then by our assumption from item 1, it must belong to the lower half-plane. Lindelöf's Theorem 2.1 guarantees that $\lim_{z\to x} \phi_\mu(F_{\mu\boxplus\nu}(z))$ equals the nontangential limit of ϕ_μ at c, so

$$\lim_{z\to x} F_\nu\Big(z - \phi_\mu\big(F_{\mu\boxplus\nu}(z)\big)\Big) = F_\nu\left(x - \lim_{\substack{w \longrightarrow c\\ \sphericalangle}} \phi_\mu(w)\right) \in \mathbb{C}^+ ,$$

contradicting equation (3.3).

If ϕ_μ has no nontangential limit at c, then it is obvious from the existence of $\lim_{z\to x} \phi_\mu(F_{\mu\boxplus\nu}(z))$, of $c = \lim_{z\to x} F_{\mu\boxplus\nu}(z)$, finiteness of c, and from Lindelöf's Theorem 2.1 that c must belong to the upper half-plane.

Hence, we have proved that, for any $x \in \mathbb{R}, c = F_{\mu\boxplus\nu}(x) = \lim_{z\to x} F_{\mu\boxplus\nu}(z) \in \mathbb{C}^+$. We finally prove that $F_{\mu\boxplus\nu}$ extends analytically in the neighbourhood of $x \in \mathbb{R}$ by using the implicit function theorem. Note that the hypothesis that $\sphericalangle \lim_{z\to t} \phi_\mu(z) \in \mathbb{C}^-$ for all $t \in \mathbb{R}$ for which this limit exists implies that μ is not a Dirac measure, hence that $\phi_\mu(\mathbb{C}^+) \subset \mathbb{C}^-$. Let us introduce the function $f(v, w) = F_\nu(v - \phi_\mu(w))$, defined on

$$\big\{(v, w) \in \mathbb{C} \times \mathbb{C}^+ \,;\, \Im v > \Im\phi_\mu(w)\big\} ,$$

which contains $(\mathbb{C}^+ \cup \mathbb{R}) \times \mathbb{C}^+$. One has

$$f(x, c) = F_\nu\big(x - \phi_\mu(c)\big) = \lim_{z\to x} F_\nu\Big(z - \phi_\mu\big(F_{\mu\boxplus\nu}(z)\big)\Big) = \lim_{z\to x} F_{\mu\boxplus\nu}(z) = c .$$

In other words, c is the Denjoy–Wolff point of the function $f(x, \cdot)$. Since $f(x, \cdot)$ is not a conformal automorphism of $\mathbb{C}^+$, we have

$$\left|\frac{\partial}{\partial w} f(x, c)\right| < 1$$

(and in particular $\neq 1$). So with $g(v, w) = w - f(v, w)$, we have

$$\frac{\partial}{\partial w} g(x, c) \neq 0 ,$$

hence, by the implicit function theorem, there exists an analytic function L, defined in a connected neighborhood V of x and a neighborhood W of c such that for all $(v,w) \in V \times W$,

$$g(v,w) = 0 \Leftrightarrow w = L(v)\,.$$

By (3.1) the function L coincides with the function $F_{\mu\boxplus\nu}$ on $V \cap \mathbb{C}^+$, so the function $F_{\mu\boxplus\nu}$ admits an analytic extension to V, with value $c \in \mathbb{C}^+$ at x. Lemma 2.11 allows us to conclude. □

Examples. In this series of examples, we provide explicit examples of measures satisfying the hypotheses of Theorem 3.1.

1. We give here an example of a Voiculescu transform that satisfies condition (i). Let

$$\phi_\mu(z) = \frac{1}{z+i} - \sqrt{z}\,, \quad z \in \overline{\mathbb{C}^+}\,,$$

where $\sqrt{\cdot}$ is the natural continuous extension of the square root defined on $\mathbb{C} \setminus [0,+\infty)$ so that $\sqrt{-1} = i$ to $\mathbb{R} \cup \{\infty\}$. Theorem 2.14 guarantees that ϕ_μ is the Voiculescu transform of a $\boxplus$-infinitely divisible probability. Clearly $\Im\phi_\mu(z) < 0$ for all $z \in \mathbb{C}^+ \cup \mathbb{R}$, $\phi_\mu(\infty) = \infty$, and ϕ_μ is obviously continuous on $\overline{\mathbb{C}^+}$, so ϕ_μ satisfies the condition (i) in the previous theorem. Moreover, we have

$$\inf_{x\in\mathbb{R}} |\Im\phi_\mu(x)| = \lim_{x\to+\infty} \frac{1}{x^2+1} = 0\,.$$

This also shows that μ is not a convolution with a Cauchy law. It is an easy exercise to observe, based on (2.1), that $\mu = \lambda_1 \boxplus \lambda_2$, where λ_j are both infinitely divisible, λ_2 is a $\boxplus$-stable distribution (see [9]) whose density is given by $x \mapsto \frac{\sqrt{4x-1}}{2\pi x}$, $x \in [1/4,\infty)$, and $F_{\lambda_1}(z) = (z - i + \sqrt{(z+i)^2-4})/2$, $z \in \mathbb{C}^+$.

2. We observe that, if ϕ_μ extends continuously to $\mathbb{R} \cup \{\infty\}$, condition (ii) in the above theorem can be reduced to requiring that μ is the free convolution of some probability measure by a Cauchy law. Indeed, since ϕ_μ is continuous on $\mathbb{R} \cup \{\infty\}$, if $\Im\phi_\mu(x) < 0$ for all $x \in \mathbb{R}$ and $\phi_\mu(\infty) \in -\mathbb{C}^+$, then $\Im\phi_\mu(x)$ must actually be bounded away from zero (by continuity). By Theorem 2.14, there exists $c > 0$ so that $\phi_\mu(z) + ci = \phi_\mu(z) - (-ci)$ is still a Voiculescu transform of an infinitely divisible distribution, say η. Then $\phi_\mu(z) = \phi_\eta(z) + (-ci)$, and $\mathfrak{C}(z) := -ci$ is the Voiculescu transform of a Cauchy distribution.

3. The example of probability measure that satisfies condition (iii) will be constructed in terms of the Voiculescu transform, as an explicit limit of compactly supported probabilities, each whose density is an algebraic function. Specifically, we will construct two sequences $\{a_n\}_n$ and $\{t_n\}_n$ of real numbers and functions $f_n(z) = a_n \frac{1+t_n z}{t_n - z}$, $z \in \mathbb{C} \setminus \{t_n\}$, so that $g_n = \sum_{j=1}^n f_j$ converges on the upper half-plane to the nonconstant analytic function g and $C_{i\mathbb{R}_+}(g,\infty) = iC_{i\mathbb{R}_+}(\Im g,\infty) \supset i[7,+\infty]$.

Let us recall that the R-transform of the free Poisson law with parameter k is $R_p(z) = \frac{k}{1-z}$, so that its dilation by t has an R-transform given by $\frac{1}{t}R_p(\frac{z}{t}) = \frac{k}{t-z}$. Since $\phi_\mu(z) = R_\mu(1/z)$, we can write

$$a\frac{1+\frac{t}{z}}{\frac{1}{z}-t} = \frac{a}{t}\cdot\frac{z+t}{\frac{1}{t}-z} = \frac{a}{t}\left[\frac{\frac{1}{t}+t}{\frac{1}{t}-z}-1\right] = \frac{\frac{a}{t^2}+a}{\frac{1}{t}-z} - \frac{a}{t}\,,$$

so we conclude that f_n is just minus the Voiculescu transform of the translation with a_n/t_n of the dilation with $1/t_n$ of the free Poisson law of parameter $\frac{a_n}{t_n^2} + a_n$.

First let us enumerate some properties of the functions f_n.

(j) $y \mapsto \Im f_n(iy)$ is a smooth function from $[0, +\infty)$ into itself and $\Im f_n(iy) = a_n \frac{y(1+t_n^2)}{t_n^2+y^2}$;

(jj) $\max_{y\in[0,+\infty)} \Im f_n(iy) = \Im f_n(it_n) = a_n(1+t_n^2)/2t_n$. Moreover, the function $y \mapsto \Im f_n(iy)$ increases from zero to $a_n(1+t_n^2)/2t_n$ on the interval $[0, t_n]$, after which it decreases back to zero;

(jjj) There are exactly two points y_n^+ and y_n^-, right and left, respectively, from t_n, so that $\Im f_n(iy_n^+) = \Im f_n(iy_n^-) = 1$. We have

$$y_n^- = \frac{a_n(1+t_n^2) - \sqrt{a_n^2(1+t_n^2)^2 - 4t_n^2}}{2}\,,$$

so that $\lim_{|t_n|\to\infty} y_n^- = 1/a_n$. Moreover, for any $a_n > 0$, $y_n^- < 2/a_n$, and if $0 < a_n < 1$, then we also have $1/a_n < y_n^-$.

Let us observe also that if we replace f_n by the sum

$$f_n(z) = \frac{a_n}{2}\frac{1+zt_n}{t_n-z} + \frac{a_n}{2}\frac{1-zt_n}{-t_n-z}\,,$$

then we do not change the imaginary part of $f_n(iy)$, while we insure that $\Re f_n(iy) = 0$ for all $y \geq 0$, so from now on we will replace f_n with this new function. (This will correspond to the free additive convolution of two free Poisson laws as above.)

Let $a_1 = 1, t_1 = 2$. Choose $0 < a_2 < a_1/2$ so that $\Im f_1(i\frac{1}{2a_2}) < 1/10$ and $\frac{1}{2a_2} > t_1$. Item (jj) guarantees that we can make such a choice. Choose $t_2 > t_1$ so that $\Im f_2(it_2) = a_2(1+t_2^2)/2t_2 > 2$. This condition can be fulfilled because of item (jj). Observe that the monotonicity of $\Im f_n(i\,\cdot\,)$ on $[t_n + \infty)$ implies that $\Im f_1(iy_2^-) < \Im f_1(i\frac{1}{2a_2}) < 1/10$.

Assume now that we have constructed $a_j, t_j, 1 \leq j \leq n-1$ so that $0 < a_j < a_{j-1}/2, f_{j-1}(i\frac{1}{2a_j}) < 1/10^{j-1}$, $\frac{1}{2a_j} > t_{j-1}, t_j > t_{j-1}, \Im f_j(it_j) = a_j(1+t_j^2)/2t_j > j$, and $y_j^- > \frac{1}{2a_j}$, for all j between 1 and $n-1$. We choose $0 < a_n < a_{n-1}/2$ small enough so that $\Im f_{n-1}(i\frac{1}{2a_n}) < 1/10^{n-1}$, $\frac{1}{2a_n} > t_{n-1}$, (using item (jj) above), and $t_n > t_{n-1}$ large enough so that $\Im f_n(it_n) = a_n(1+t_n^2)/2t_n > n$. As before, construction is permitted by using item (jj).

Observe now that the sequence $\{a_n\}_n$ constructed this way is positive, decreasing, and satisfies $1 < \sum_{n=1}^{\infty} a_n < 2$. Moreover, $\{y_n^-\}_n$ is, by item (jjj), in its own turn an increasing sequence, and $y_n^- > t_j$ for all $j < n$ so that $\Im f_j(iy_n^-) \leq \Im f_j(iy_{j+1}^-) < \Im f_j(i\frac{1}{2a_{j+1}}) < 1/10^j$, by monotonicity of $\Im f_j$ on $[t_n, +\infty)$. Thus,

$$\Im(f_1 + f_2 + \cdots + f_{n-1} + f_n)(iy_n^-) < \frac{1}{10} + \frac{1}{10^2} + \cdots \frac{1}{10^{n-1}} + 1 < \frac{10}{9}.$$

On the other hand, for any $m > n$, we have

$$\begin{aligned}\Im f_m(iy_n^-) = a_m \frac{y_n^-(1+t_m^2)}{t_m^2 + (y_n^-)^2} &= \underbrace{a_n \frac{y_n^-(1+t_n^2)}{t_n^2 + (y_n^-)^2}}_{\Im f_n(iy_n^-)=1} \cdot \frac{a_m}{a_n} \cdot \frac{(1+t_m^2)(t_n^2 + (y_n^-)^2)}{(1+t_n^2)(t_m^2 + (y_n^-)^2)} \\ &\leq \frac{a_m}{a_n} \cdot \frac{(1+t_m^2)\left(t_n^2 + \left(\frac{2}{a_n}\right)^2\right)}{(1+t_n^2)t_m^2} \\ &< 2\frac{a_m}{a_n}\left(1 + \frac{4}{a_n^2 t_n^2}\right) \\ &< \frac{1}{2^{m-n-1}} + \frac{1}{2^{m-n-3}n^2},\end{aligned}$$

so that $\Im(f_{n+1} + f_{n+2} + \cdots + f_m)(iy_n^-) < 4$ for all $m > n$, when $n > 1$ is large enough. (We have used in the last inequality the fact that the choice of t_n so that $\Im f_n(it_n) > n$ implies that $a_n > (2nt_n)/(1+t_n^2) > n/t_n$, so that $1/n > 1/(a_n t_n)$, and our choice that $a_n < a_{n-1}/2, n \geq 1$). Also, $\Im(f_1 + f_2 + \cdots + f_{n-1} + f_n)(t_n) > \Im f_n(it_n) > n$, for all $n \in \mathbb{N}$. As seen before, $\Re f_n(iy) = 0$ for all $y \geq 0$.

Now, it is easy to verify that $(1 + t_n z)/(t_n - z)$ are uniformly bounded on, say, $i[0, 1]$, so that $\sum_{n=1}^{\infty} f_n$ is convergent and the limit g is an analytic self-map of the upper half-plane. We observe that $\Im g(iy_n^-) \leq 4 + 10/9 < 7$, $\Im g(it_n) \geq n$, and $\Re g(i[0, +\infty)) = \{0\}$ for all $n \in \mathbb{N}$, while $\lim_{n\to\infty} t_n = \lim_{n\to\infty} y_n^- = \infty$. Thus, g has no radial, hence no nontangential, limit at infinity.

At the same time, since all $-f_n$s are Voiculecu transforms, so is $-g$.

We next consider the case where we do not restrict ourselves to regularizing measures which are infinitely divisible.

Theorem 3.2. *Let μ be a Borel probability measure on $\mathbb{R}$ so that the function $h_\mu(z) = F_\mu(z) - z$ satisfies the following property:*

(H) *For any $x \in \mathbb{R} \cup \{\infty\}$, either $\sphericalangle h_\mu(x)$ does not exist, or $\sphericalangle h_\mu(x) \in \mathbb{C}^+$;*

Then for any Borel probability measure ν on $\mathbb{R}$ which is not a point mass, $\mu \boxplus \nu$ is absolutely continuous with respect to the Lebesgue measure and has a positive analytic density with respect to the Lebesgue measure.

Remark 3.3. For $x = \infty$, (H) is in fact equivalent to the fact that either $\sphericalangle \phi_\mu(\infty)$ does not exist or $\sphericalangle \phi_\mu(\infty)$ belongs to $\mathbb{C}^+$. Indeed, as observed in [9], $z + \phi_\mu(z)$ belongs to a neighbourhood of infinity for sufficiently large z in some truncated cone $\Gamma_{\alpha,M}$. Thus, by the definition of ϕ and h, $h_\mu(z+\phi_\mu(z)) = -\phi_\mu(z)$ for sufficiently large z in such a cone. Since $z+\phi_\mu(z)$ tends nontangentially to infinity when z tends nontangentially to infinity, $\sphericalangle\phi_\mu(\infty)$ exists iff $\sphericalangle h_\mu(\infty)$ exists, and they are equal. The significance of this fact for our problem will be seen in the next subsection.

Proof. We claim first that $\sphericalangle F_{\mu\boxplus\nu}(x)$ exists and belongs to the upper half-plane for all $x \in \mathbb{R}$. Indeed, with the notations from Lemma 2.15, by Theorem 3.3 of [2], the nontangential limits of ω_1 and ω_2 at x exist. As observed in part (1) of Theorem 3.3 in [2], if $\sphericalangle\omega_2(x) \in \mathbb{C}^+$, then the result is true. Assume first that $\sphericalangle\omega_2(x) \in \mathbb{R}$. Theorem 2.1 guarantees that if $\sphericalangle F_{\mu\boxplus\nu}(x) = \sphericalangle(F_\mu \circ \omega_2)(x)$ exists, it must equal the nontangential limit of F_μ in $\sphericalangle\omega_2(x)$, so the nontangential limit of h_μ in $\sphericalangle\omega_2(x)$ exists, and hence, by (H), belongs to the upper half-plane.

Suppose the nontangential limit of F_μ, and hence of h_μ, in $\sphericalangle\omega_2(x)$ does not exist. Then by Theorem 2.1, $h_\mu\circ\omega_2$ has no nontangential limit at x. But by part (1) of Lemma 2.15 and definition of h_μ, we have

$$\sphericalangle\omega_1(x) = x + \lim_{\substack{z\longrightarrow x\\ \sphericalangle}} h_\mu\big(\omega_2(z)\big)\,,$$

which implies that $\sphericalangle\omega_1(x)$ does not exist, contradicting Theorem 3.3 of [2].

The last possible case is when $\sphericalangle\omega_2(x) = \infty$. As before, if $\sphericalangle(h_\mu\circ\omega_2)(x)$ exists, then by Theorem 2.1 it must coincide with $\sphericalangle h_\mu(\sphericalangle\omega_2(x)) = \sphericalangle h_\mu(\infty)$, so, by our hypothesis (H), it must belong to the upper half-plane. Thus, $\sphericalangle\omega_1(x) \in \mathbb{C}^+$, so that, by Theorem 3.3 in [2], $\sphericalangle F_{\mu\boxplus\nu}(x) \in \mathbb{C}^+$. Assume now that $\sphericalangle(h_\mu\circ\omega_2)(x)$ does not exist, so that there exists an infinite set W of points $c \in \mathbb{C}^+ \cup\ mathbbR$ for which there is a sequence $\{z_n^c\}_n$ converging to x nontangentially so that $\lim_{n\to\infty}(h_\mu\circ\omega_2)(z_n^c) = c$. But then

$$\sphericalangle\omega_1(x) = \lim_{\substack{z\longrightarrow x\\ \sphericalangle}} \omega_1(z) = \lim_{n\to\infty} \omega_1(z_n^c) = \lim_{n\to\infty} z_n^c + h_\mu\big(\omega_2(z_n^c)\big) = x + c$$

for any $c \in W$. This contradicts the existence of the nontangential limit of ω_1 at x.

This establishes the existence of nontangential limits of $F_{\mu\boxplus\nu}$ at all points $x \in \mathbb{R}$ and the fact that $\sphericalangle F_{\mu\boxplus\nu}(x) \in \mathbb{C}^+$ for all $x \in \mathbb{R}$. We claim that $\sphericalangle\omega_2(x) \in \mathbb{C}^+$. Indeed, it is easy to see that $\sphericalangle\omega_2(x)$ is finite, since otherwise we would have, by Lemma 2.15, Theorem 2.1 and Theorem 2.8, that $\sphericalangle F_{\mu\boxplus\nu}(x) = \sphericalangle(F_\mu \circ \omega_2)(x) = \sphericalangle F_\mu(\sphericalangle\omega_2(x)) = \sphericalangle F_\mu(\infty) = \infty$, which is a contradiction. Thus, by Lemma 2.15, part (1), we have that $\sphericalangle\omega_1(x)$ is also finite. Moreover, at least one of $\sphericalangle\omega_1(x), \sphericalangle\omega_2(x)$ must then belong to the upper half-plane. Remark 2.9 guarantees that in fact both must be in the upper half-plane. Theorem 3.3 of [2] and Lemma 2.11 concludes the proof. □

3.2. A result of non existence of analytic densities

Proposition 3.4. *Assume that μ, a $\boxplus$-infinitely divisible probability so that $\mu^{\boxplus t}$ has no atoms for some $t < 1$ (the existence of $\mu^{\boxplus t}$ for $t < 1$ is guaranteed by the infinite divisibility of μ), has finite second moment. Then there exists a probability* *measure ν on $\mathbb{R}$ so that the density of $\mu \boxplus \nu$ is not analytic everywhere. Moreover, the density of $\mu \boxplus \nu$ vanishes at a point.*

Remark 3.5.

- Note that the fact that the density of $\mu \boxplus \nu$ may easily vanish inside the support of the measure was already foreseen by P. Biane in [12], Proposition 6, where he proved that if ν is a probability measure with continuous strictly positive density on $]-\epsilon, 0[\cup]0, \epsilon[$ for some $\epsilon > 0$ such that
$$\int x^{-2} d\nu(x) < \infty$$
and $\mu = \sigma_t$ is the semicircular variable with covariance t, then $\frac{d\mu \boxplus \sigma_t}{dx}(0) = 0$ for $t > 0$ small enough. Our proof extends this phenomenon to any $\boxplus$-infinitely divisible probability measure μ with finite second moment, under the (technical) hypothesis that $\mu^{\boxplus t}$ has no atoms for some $t < 1$.
- In the case where the probability measure μ has a finite second moment, by Theorem 1.3 and Remark 1.1 of [6], if m, v denote respectively the mean and the variance of μ, one has, as z goes to infinity non tangentially, $\phi_\mu(z) = m + v/z + o(1/z)$, hence the second hypothesis of Theorem 3.1 cannot be satisfied. However, the existence of a finite second moment for μ has no incidence on the first hypothesis of Theorem 3.1. Indeed, as it will be explained in the proof of Proposition 3.4, up to a translation, ϕ_μ can be expressed as the Cauchy transform of the finite positive measure $d\sigma(t) = (1 + t^2)dG(t)$, where G is the Lévy measure of μ. Hence up to the addition of a real number, the non tangential limit of ϕ_μ at any real number x only depends on the restriction of G to a neighborhood of x, which, by Proposition 2.3 of [6], is independent of the existence of a finite second moment for μ.

Proof. First note that by Theorem 3.1 of [4], the hypothesis that $\mu^{\boxplus t}$ has no atoms for some $t < 1$ implies that μ has no atoms. Hence by Remark 2.16 or Theorem 7.4 of [10], for any ν which is not a point mass, $\mu \boxplus \nu$ has a density.

Observe that if the density of $\mu \boxplus \nu$ has a hole in the support (meaning a nontrivial interval on which it is zero), it cannot be analytic on $\mathbb{R}$ by the identity principle. Similarly, the set of zeros of the density must be discrete in $\mathbb{R}$. Thus, we may assume that μ satisfies these two conditions, since otherwise we would readily obtain the probability measure ν of the proposition by taking $\nu = \delta_a$ for some $a \in \mathbb{R}$.

The strategy of the proof is as follows; we first show that $g : z \in \mathbb{C}^+ \to F_\nu(-\phi_\mu(z))$ has infinity as Denjoy–Wolff point under a certain condition. Regarding $F_{\mu \boxplus \nu}(0)$ as a fixed point of this map will guarantee that either $F_{\mu \boxplus \nu}(0)$ is infinite or belongs to $\mathbb{R}$. We will show that under some hypothesis on ν, it has to be infinite, which will prove that $\frac{d\mu \boxplus \nu}{dx}(0) = 0$ and also that $\frac{d\mu \boxplus \nu}{dx}$ is not analytic at the origin.

To study the Denjoy–Wolff point of g, we first shall write both ϕ_μ and F_ν as (roughly speaking) Cauchy–Stieljes transforms of some measures σ and ρ on $\mathbb{R}$.

Recall [9] that there exists a real number γ and a positive finite measure G on the real line, called the *Lévy measure* of μ such that for all $z \in \mathbb{C}^+$,

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t} dG(t)\,.$$

By Proposition 2.3 of [6], the finiteness of the second moment of μ is equivalent to the finiteness of the second moment of its Lévy measure G. Thus, we can represent the Voiculescu transform of μ as

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t} dG(t) = \gamma + \int_{\mathbb{R}} \left(t + \frac{1+t^2}{z-t}\right) dG(t) = \gamma' + \int_{\mathbb{R}} \frac{d\sigma(t)}{z-t}\,, \quad z \in \mathbb{C}^+\,,$$

where $\gamma' \in \mathbb{R}, d\sigma(t) = (1+t^2) \times dG(t)$. Because G has finite second moment, σ has finite mass. By a translation of μ, we may assume that $\gamma' = 0$, so that ϕ_μ is simply the Cauchy transform of the positive finite measure σ.

Let ν be a Borel probability on $\mathbb{R}$. By Theorem 2.8, we can write the reciprocal of its Cauchy transform as

$$F_\nu(z) = a + z + \int_{\mathbb{R}} \frac{1+tz}{t-z} d\rho(t) = a + z + \int_{\mathbb{R}} \frac{d\rho(t)}{t-z} + \int_{\mathbb{R}} \frac{zt}{t-z}\, d\rho(t)\,,$$

for all $z \in \mathbb{C}^+$, where $a \in \mathbb{R}$, and ρ is a positive finite measure. We will show that if ν is so that $\rho(\{0\}) \geq \sigma(\mathbb{R})$, then the density of $\mu \boxplus \nu$ vanishes at the origin. So let ν satisfy this condition. We have by Theorem 2.14(ii)

$$\begin{aligned}\lim_{y\to+\infty} \frac{F_\nu(-\phi_\mu(iy))}{iy} &= \lim_{y\to+\infty} \int_{\mathbb{R}} \frac{d\rho(t)}{iy(t+\phi_\mu(iy))} \\ &\quad + \lim_{y\to+\infty} \frac{-\phi_\mu(iy)}{iy} \int_{\mathbb{R}} \frac{t\, d\rho(t)}{t+\phi_\mu(iy)}\,.\end{aligned} \tag{3.4}$$

Observe that $\phi_\mu(iy)$ approaches zero nontangentially when y tends to infinity. Indeed, since $\lim_{y\to+\infty} iy\phi(iy) = \sigma(\mathbb{R}) > 0$, we have $\lim_{y\to+\infty} y\Re\phi_\mu(iy) = 0$, $\lim_{y\to+\infty} y\Im\phi_\mu(iy) = -\sigma(\mathbb{R})$, so, given $0 < \varepsilon < \sigma(\mathbb{R})/2$, there exists $y_\varepsilon > 1$ so that for all $y \geq y_\varepsilon$, we have $|y\Re\phi_\mu(iy)| < \varepsilon$, $|\sigma(\mathbb{R}) + y\Im\phi_\mu(iy)| < \varepsilon$. Thus,

$$\frac{|\Im\phi_\mu(iy)|}{|\Re\phi_\mu(iy)|} = \frac{|y\Im\phi_\mu(iy)|}{|y\Re\phi_\mu(iy)|} > \frac{\sigma(\mathbb{R})-\varepsilon}{\varepsilon} > 1\,,$$

for all $y \geq y_\varepsilon$. Now,

$$\lim_{y\to+\infty} \int_{\mathbb{R}} \frac{t\, d\rho(t)}{t+\phi_\mu(iy)} = \lim_{y\to+\infty} \int_{\mathbb{R}} \left(1 - \frac{\phi_\mu(iy)}{t+\phi_\mu(iy)}\right) d\rho(t) = \rho(\mathbb{R}) - \rho(\{0\}).$$

Since, by Theorem 2.14 above, $\lim_{y\to+\infty} \phi_\mu(iy)/iy = 0$, we conclude that the second limit in the equation (3.4) vanishes.

On the other hand, if we denote $f_y(t) = \frac{1}{iy(t+\phi_\mu(iy))}$, $t \in \mathbb{R}$, $y > 1$, then $\lim_{y\to+\infty} f_y(t) = \frac{1}{\sigma(\mathbb{R})}\chi_{\{0\}}(t)$ pointwise, where χ_A is the characteristic function of A. Also,

$$|f_y(t)|^2 = \frac{1}{y^2(t+\Re\phi_\mu(iy))^2 + y^2(\Im\phi_\mu(iy))^2} \leq \frac{1}{y^2(\Im\phi_\mu(iy))^2} < \frac{4}{\sigma(\mathbb{R})^2}\,,$$

for all $y > y_\varepsilon$. So by the dominated convergence theorem,

$$\lim_{y\to+\infty}\int_{\mathbb{R}}\frac{d\rho(t)}{iy(t+\phi_\mu(iy))}=\frac{\rho(\{0\})}{\sigma(\mathbb{R})}\,.$$

By (3.4), we conclude that

$$\lim_{y\to+\infty}F_\nu\big(-\phi_\mu(iy)\big)/iy=\frac{\rho(\{0\})}{\sigma(\mathbb{R})}\geq 1\,,$$

which insures that the analytic function $g\colon \mathbb{C}^+\to\mathbb{C}^+$ defined by $g(z)=F_\nu(-\phi_\mu(z))$, has infinity as its Denjoy–Wolff point.

We next show that this implies that $F_{\mu\boxplus\nu}(0)$ belongs to $\mathbb{R}\cup\{\infty\}$. So, we suppose that $F_{\mu\boxplus\nu}(0)\in\mathbb{C}^+$ to get a contradiction ($F_{\mu\boxplus\nu}$ extends continuously to $\mathbb{R}$ by Lemma 2.15 (4)). Note that by (3.1), the relation $F_\nu(z-\phi_\mu(F_{\mu\boxplus\nu}(z)))=F_{\mu\boxplus\nu}(z)$, gives, by letting z going to zero nontangentially,

$$F_\nu\Big(-\phi_\mu\big(F_{\mu\boxplus\nu}(0)\big)\Big)=F_{\mu\boxplus\nu}(0)\,. \tag{3.5}$$

Thus, $F_{\mu\boxplus\nu}(0)$ should be a fixed point of g in $\mathbb{C}^+$, and thus its Denjoy–Wolff point; this is in contradiction with the previous statment that the Denjoy–Wolff point of g is infinity. Hence one has $F_{\mu\boxplus\nu}(0)\in\mathbb{R}\cup\{\infty\}$.

Observe that $F_{\mu\boxplus\nu}(0)\neq 0$. Indeed, by Remark 2.16, this equality would imply that $t\omega_2(0)$ is an atom of $\mu^{\boxplus t}$ for all $t<1$, contradicting the hypothesis.

Thus, $\Im F_{\mu\boxplus\nu}(0)=0$ and $F_{\mu\boxplus\nu}(0)\neq 0$, so that $\Im G_{\mu\boxplus\nu}(0)=0$. Part (4) of Lemma 2.15 tells us that $F_{\mu\boxplus\nu}$ is continuous on $\mathbb{R}$. In particular, since $F_{\mu\boxplus\nu}(0)\neq 0, G_{\mu\boxplus\nu}(x)$ will be continuous and finite for x in some open interval I around zero. Lemma 2.11 (i) guarantees that $\mu\boxplus\nu$ will have a continuous density on I which vanishes at zero.

We show below a more precise statment to prove the breaking of analyticity at the origin when $\rho(\{0\})=\sigma(\mathbb{R})$, namely that $F_{\mu\boxplus\nu}(0)=\infty$.

So, assume now that $\rho(\{0\})=\sigma(\mathbb{R})$. Then we claim that in fact $F_{\mu\boxplus\nu}(0)=\infty$. Indeed, by Lemma 2.15 (2), $\omega_1(0)$ is the Denjoy–Wolff point of the function $g_0(w)=-\phi_\mu(F_\nu(w))$. We next show that this point must be the origin.

Indeed, observe that $F_\nu(iy)/iy$ goes to one as $y\in\mathbb{R}^+$ goes to infinity. Hence, $F_\nu(iy)$ approaches infinity nontangentially when $y\to+\infty$. Thus, since ϕ_μ approaches zero nontangentially at infinity, $g_0(y)=-\phi_\mu(F_\nu(iy))$ converges to zero as y goes to zero yielding $g_0(0)=0$. Also,

$$\lim_{y\to 0}\frac{g_0(y)}{iy}=\lim_{y\to 0}\frac{-\phi_\mu(F_\nu(iy))}{iy}=\frac{\lim_{y\to 0}\phi_\mu(F_\nu(iy))F_\nu(iy)}{\lim_{y\to 0}-iyF_\nu(iy)}=\frac{\sigma(\mathbb{R})}{\rho(\{0\})}=1\,.$$

The Denjoy–Wolff theorem and the remarks following it imply that zero is the Denjoy–Wolff point for g_0, so by uniqueness of the Denjoy–Wolff point, $\omega_1(0)=0$.

We know that F_ν has infinite nontangential limit at zero (because we supposed that ρ has an atom at zero), so this, coupled with the equation $F_{\mu\boxplus\nu}(z)=F_\nu(\omega_1(z))$, with the existence of the limit of $F_{\mu\boxplus\nu}$ at zero and Lindelöf's Theorem 2.1, implies that $F_{\mu\boxplus\nu}(0)=\infty$.

Observe that ω_1 is not analytic in zero. Indeed, if it were analytic, it would have a finite derivative in zero. However, with H the function given in Lemma 2.15 (2), the previous estimates show that $H(0) = 0$ and

$$H'(0) = \lim_{y\to 0} \frac{H(iy)}{iy} = 1 - \lim_{y\to 0} \frac{-\phi_\mu(F_\nu(iy))}{iy} = 0$$

implies, by Proposition 4.7 (5) in [5], that $\lim_{y\to 0} \omega_1(iy)/iy = \omega_1'(0) = 1/H'(0) = \infty$. As a consequence, $G_{\mu\boxplus\nu}$ is not differentiable at the origin, since if it were

$$\begin{aligned}\lim_{y\to 0} -\frac{G_{\mu\boxplus\nu}(iy) - G_{\mu\boxplus\nu}(0)}{iy - 0} &= -\lim_{y\to 0} \frac{1}{iyF_\nu(\omega_1(iy))} \\ &= \lim_{y\to 0} -\frac{1}{\omega_1(iy)F_\nu(\omega_1(iy))} \cdot \frac{\omega_1(iy)}{iy}\,.\end{aligned}$$

The second factor above has just been shown to converge to infinity. For the first factor, observe that $\sphericalangle \lim_{z\to 0} zF_\nu(z) = \rho(\{0\})$. Since $y \mapsto \omega_1(iy)$ is a smooth path in the upper half-plane ending at zero, Theorem 2.5 guarantees that there exists a subsequence $y_n \to 0$ so that $\lim_{n\to\infty} \omega_1(iy_n)F_\nu(\omega_1(iy_n)) = \rho(\{0\})$. So the limit above either does not exist, or is infinite. In both cases, we conclude that $G_{\mu\boxplus\nu}$ is not differentiable at zero. Thus by Lemma 2.11, the density of $\mu\boxplus\nu$ is not analytic in zero. □

The above proposition provides a large class of examples of free convolutions whose densities have cusps in their support (points where the density vanishes and is not analytic), and relates this phenomenon to the finiteness of second moments. We show below that it is possible that the density of $\mu\boxplus\nu$ vanishes at a point, but is still analytic.

Proposition 3.6. *Let μ be the semicircular distribution. Then, there exists a probability measure ν on $\mathbb{R}$ so that the density $\rho(x) = \frac{d\mu\boxplus\nu}{dx}(x)$ vanishes at the origin, is strictly positive on $]-\epsilon, 0[\cup]0, \epsilon[$ for some $\epsilon > 0$ but is analytic at the origin.*

Proof. Let μ be the semicircular distribution, so that $\phi_\mu(z) = 1/2z$, $z \in \overline{\mathbb{C}^+}$. We claim that ν given by its reciprocal Cauchy–Stieljes

$$F_\nu(z) = z + i - 1 + \frac{z-i}{z+i} - \frac{1}{z}\,, \quad z \in \mathbb{C}^+\,,$$

will satisfy the properties of the proposition.

Indeed, with the notations from the proof of the previous proposition,

$$\begin{aligned}g_0(w) = -\phi_\mu\big(F_\nu(w)\big) &= -\frac{1}{2\left(w + i - 1 + \frac{w-i}{w+i} - \frac{1}{w}\right)} \\ &= -\frac{w(w+i)}{2w^3 + 4iw^2 - 4(1+i)w - 2i}\,,\end{aligned}$$

for all $w \in \mathbb{C}^+$, so in fact g_0 extends analytically around zero, and moreover

$$g_0'(0) = \lim_{w\to 0} g_0(w)/w = 1/2 < 1 \Rightarrow |g_0'(0)| < 1 \text{ and } g_0(0) = 0\,.$$

Thus, zero is the Denjoy–Wolff point of g_0, and, by Lemma 2.15 (2), we conclude that $\omega_1(0) = 0$.

We next show that ω_1 extends analytically around the origin. In fact, the function $H(w) = w + \phi_\mu(F_\nu(w))$ has, by Lemma 2.15 (2), ω_1 as right inverse. At the same time, H extends analytically around zero, and $H'(0) = 1 - \frac{1}{2} = \frac{1}{2} \neq 0$, so H is locally invertible around zero. The analyticity of ω_1 around the origin follows from the implicit function theorem.

We now conclude that $G_{\mu\boxplus\nu}$ vanishes at the origin, is analytic in a neighborhood of the origin and has negative imaginary part in a neighborhood of the origin (except at the origin itself); this will prove the lemma according to Lemma 2.11. Now, F_ν is meromorphic around zero, with a simple pole at zero, so $G_\nu(0) = 0$ and G_ν is analytic around zero. By Lemma 2.15, we have $0 = G_\nu(0) = G_\nu(\omega_1(0)) = G_{\mu\boxplus\nu}(0)$, and $G_{\mu\boxplus\nu}(z) = G_\nu(\omega_1(z))$ is analytic on a neighbourhood of zero in $\mathbb{C}$. We finally show that $G_{\nu\boxplus\mu}$ has positive imaginary part in $]-\epsilon, 0[\cup]0, \epsilon[$ for some $\epsilon > 0$. For that, since $G_{\nu\boxplus\mu}(z) = G_\nu(\omega_1(z))$, it is enough to show that $G_\nu(z)$ has negative imaginary part for z so that $0 < |z| < \epsilon'$ (since ω_1 is analytic and null at the origin).

But, a straightforward computation gives

$$G_\nu(r) = \frac{1}{F_\nu(r)} = \frac{r(r+i)}{r^3 + 2ir^2 - 2(i+1)r - i} = \frac{r(r+i)}{(r^3 - 2r) + i(2r^2 - 2r - 1)},$$

so

$$\Im G_\nu(r) = -\frac{r^2(r-1)^2}{r^2(r^2-2)^2 + (2r^2 - 2r - 1)^2} < 0,$$

for all $r \in \mathbb{R} \setminus \{0, 1\}$.

We notice that in fact we have only used, with the notations from the previous proposition, the facts that $\rho(\{0\}) > \sigma(\mathbb{R})$, that G_ν is analytic around zero, and ϕ_μ around infinity. Thus, a much larger class of such pairs of measures μ, ν provide an analytic density around zero which is zero in zero. $\square$

Remark 3.7. Note that our construction of the example in the proposition above were based on the fact that $\rho(\{0\}) > 0$. In that case, $F_\nu(z) \approx_{z\to 0} -\frac{1}{z}\rho(\{0\})$. This is equivalent to the fact that $\int t^{-1} d\nu(t) = 0$ and

$$\int \frac{1}{t^2} d\nu(t) = \frac{1}{\rho(\{0\})}.$$

Therefore, if μ is an infinitely divisible measure with finite second moment and positive density, and if we denote again by σ the finite measure on $\mathbb{R}$ given by

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t} dG(t) = \gamma + \int_{\mathbb{R}} \left(t + \frac{1+t^2}{z-t}\right) dG(t) = \gamma' + \int_{\mathbb{R}} \frac{d\sigma(t)}{z-t}, \quad z \in \mathbb{C}^+,$$

where $\gamma' \in \mathbb{R}$, $d\sigma(t) = (1+t^2) \times dG(t)$, we have shown that we have three possibilities, which follow the intuition

1. If $\int \frac{1}{t^2} d\nu(t) < 1/\sigma(\mathbb{R})$, so ν does not put much mass in the neighborhhood of the origin, $\mu \boxplus \nu$ has a density which vanishes at the origin but has a finite derivative at the origin. The proof was detailed above in a specific example in the last proposition but could be generalized.
2. If $\int \frac{1}{t^2} d\nu(t) = 1/\sigma(\mathbb{R})$, which corresponds to a critical amount of mass around the origin, the density has a cusp at the origin (at least under the asumptions that $\mu^{\boxplus t}$ has no atoms for $t < 1$).
3. If $\nu(\{0\}) > 0$, we have an analytic strictly positive density in zero whenever $\mu^{\boxplus t}$ lacks atoms for all $t > 0$. Indeed, if one assumes $\nu(\{0\}) > 0$, then $\lim_{z \to 0} |\frac{F_\nu(z)}{z}| = \infty$ would imply $\lim_{z \to 0} z G_\nu(z) = 0$, which is obviously false, by (2) of Lemma 2.17 of [2]. Thus if $F_{\mu \boxplus \nu}(0) \in \mathbb{R}$, then $\omega_1(0) \in \mathbb{C}^+$ (by Lemma 2.15), which is impossible, and if $F_{\mu \boxplus \nu}(0) = \infty$, then $F_{\mu \boxplus \nu}(0)$ is the Denjoy–Wolff point of g (equation (3.3)), which is also impossible because $-\phi_\mu(\infty) = 0$ and $F_\nu(0) = 0$. Thus, $F_{\mu \boxplus \nu}(0)$ must belong to $\mathbb{C}^+$.

4. The rectangular case

4.1. Main result

We shall fix $\lambda \in (0, 1)$, and assume that all probability measures are symmetric. We prove here an analogue to Theorem 3.1 for the rectangular convolution, which says that the restriction to the upper half-plane of the function H extends continuously to $\mathbb{R}^+$, analytically outside a closed set of Lebesgue measure zero. We shall see that this implies that $\mu \boxplus_\lambda \nu$ admits an analytic density on the complement of that set. Unlike for the square case, we did not succeed to get rid of this closed negligible set where the density could stop being analytic. We however can give sufficient conditions so that the density is continuous everywhere (Corollary 4.6). Examples which satisfy our conditions are provided in Section 4.2. Moreover, as we shall discuss later, the density often vanishes around the origin, which is, in the rectangular setting, a very specific point. A consequence of this fact is that the full strength of Theorem 3.1 cannot be achieved in the rectangular case: given a $\boxplus_\lambda$-infinitely divisible probability μ, there exists a symmetric probability measure $\nu \neq \delta_0$ so that the density of $\mu \boxplus_\lambda \nu$ is not everywhere analytic. We shall study this phenomenon in the last paragraph.

Our main tool will be an ad-hoc subordination result for the functions H:

Lemma 4.1. *Let μ, ν be two symmetric probability measures on $\mathbb{R}$. Assume that the rectangular R-transform C_μ of μ extends analytically to $\mathbb{C} \setminus \mathbb{R}^+$ (this happens for example if μ is $\boxplus_\lambda$-infinitely divisible – see Theorem 2.18). Then there exist two unique meromorphic functions ω_1, ω_2 on $\mathbb{C} \setminus \mathbb{R}^+$ so that $H_\mu(\omega_1(z)) = H_\nu(\omega_2(z)) = H_{\mu \boxplus_\lambda \nu}(z), \omega_j(\bar{z}) = \overline{\omega_j(z)}, z \in \mathbb{C} \setminus \mathbb{R}^+$, and $\lim_{x \uparrow 0} \omega_j(x) = 0,\ j \in \{1, 2\}$. Moreover,*

- (i) *ω_2 is injective and analytic on $\mathbb{C} \setminus \mathbb{R}^+$; it is the right inverse of the meromorphic function $k(w) = \frac{H_\nu(w)}{T[C_\mu(H_\nu(w)) + M_\nu(w)]}$, $w \in \mathbb{C} \setminus \mathbb{R}^+$;*
- (ii) *$\arg z \leq \arg \omega_2(z) < \pi$, $z \in \mathbb{C}^+$.*

Proof. There exists an $\varepsilon > 0$ so that for $z \in (-\varepsilon, 0)$, by taking $w = H_{\mu\boxplus_\lambda\nu}(z)$ in relation $C_{\mu\boxplus_\lambda\nu}(w) = C_\mu(w) + C_\nu(w)$, we have

$$U\left(\frac{H_{\mu\boxplus_\lambda\nu}(z)}{z} - 1\right) - C_\mu\big(H_{\mu\boxplus_\lambda\nu}(z)\big) = U\left(\frac{H_{\mu\boxplus_\lambda\nu}(z)}{H_\nu^{-1}(H_{\mu\boxplus_\lambda\nu}(z))} - 1\right).$$

Applying T in both sides gives

$$T\left[U\left(\frac{H_{\mu\boxplus_\lambda\nu}(z)}{z} - 1\right) - C_\mu\big(H_{\mu\boxplus_\lambda\nu}(z)\big)\right] = \frac{H_{\mu\boxplus_\lambda\nu}(z)}{H_\nu^{-1}(H_{\mu\boxplus_\lambda\nu}(z))}.$$

By part (b) of Section 2.3.2, $H_{\mu\boxplus_\lambda\nu}(z)$ doesn't vanish on $\mathbb{C}\backslash\mathbb{R}^+$, hence in the interval $(-\varepsilon, 0)$ where the previous equation is valid, its left hand term doesn't vanish. So on $(-\varepsilon, 0)$ we have

$$H_{\mu\boxplus_\lambda\nu}(z) = H_\nu\left(\frac{H_{\nu\boxplus_\lambda\mu}(z)}{T\left[U\left(\frac{H_{\mu\boxplus_\lambda\nu}(z)}{z} - 1\right) - C_\mu(H_{\mu\boxplus_\lambda\nu}(z))\right]}\right). \tag{4.1}$$

This equation holds for $z \in (-\varepsilon, 0)$, and, by analytic continuity, in all points of the connected component of the domain of analyticity of the right hand term which contains $(-\varepsilon, 0)$. Thus, if we denote $f(z,w) = \frac{w}{T(U(\frac{w}{z}-1)-C_\mu(w))}$, and let $\omega_2(z) = f(z, H_{\mu\boxplus_\lambda\nu}(z))$, we have proved that $H_{\mu\boxplus_\lambda\nu}(z) = H_\nu(\omega_2(z))$ for z in some domain containing the interval $(-\varepsilon, 0)$. We shall argue in the following that this equation can be extended to all points of $\mathbb{C} \setminus \mathbb{R}^+$.

Note that since C_μ is analytic on $\mathbb{C}\backslash\mathbb{R}^+$, by (b) of the Section 2.3.2, $C_\mu(H_{\mu\boxplus_\lambda\nu}(z))$ is defined on $\mathbb{C}\backslash\mathbb{R}^+$. Moreover, by (a) of the Section 2.3.2, $z \mapsto U(\frac{H_{\mu\boxplus_\lambda\nu}(z)}{z} - 1)$ admits an analytic extension to $\mathbb{C}\backslash\mathbb{R}^+$, denoted by $M_{\mu\boxplus_\lambda\nu}$. So any point z of $\mathbb{C}\backslash\mathbb{R}^+$ which is in the boundary of the domain of the right hand term of (4.1) satisfies either

$$\omega_2(z) := \frac{H_{\nu\boxplus_\lambda\mu}(z)}{T\left[M_{\mu\boxplus_\lambda\nu}(z) - C_\mu(H_{\mu\boxplus_\lambda\nu}(z))\right]} \in \mathbb{R}^+$$

$$\text{or} \quad T\Big[M_{\mu\boxplus_\lambda\nu}(z) - C_\mu\big(H_{\mu\boxplus_\lambda\nu}(z)\big)\Big] = 0\,.$$

• We first discuss the case when ν has the property that for any x in $\mathbb{R}$, the Cauchy transform of ν does not extend continuously to x. This happens for example if ν is concentrated on a set of Lebesgue measure zero and has support equal to $\mathbb{R}$, according to Lemma 2.12.

Consider the connected component of the domain of the right hand side of (4.1) that contains $(-\varepsilon, 0)$. Assume first that $z_0 \in \mathbb{C} \setminus \mathbb{R}^+$, and yet z_0 is in the boundary of this component, which implies that either $\omega_2(z_0) \in [0, +\infty)$ (because of part (b) of Section 2.3.2), or $T\left[M_{\mu\boxplus_\lambda\nu}(z_0) - C_\mu(H_{\mu\boxplus_\lambda\nu}(z_0))\right] = 0$. The functions $H_{\mu\boxplus_\lambda\nu}$ and ω_2 are analytic and, respectively, meromorphic, in z_0 and $H_{\mu\boxplus_\lambda\nu}(z_0) \in \mathbb{C} \setminus \mathbb{R}^+$.

Observe that if the first situation occurs, there must exist a whole (1-dimensional) analytic connected variety V given by the relation $\omega_2(.) \in (0,+\infty)$ to which z_0 belongs because ω_2 is open as a meromorphic function. But now for any point $\zeta \in V$, we will have that

$$\lim_{z\to\zeta} H_{\mu\boxplus_\lambda\nu}(z) = \lim_{z\to\zeta} H_\nu\left(\frac{H_{\nu\boxplus_\lambda\mu}(z)}{T\left[M_{\mu\boxplus_\lambda\nu}(z) - C_\mu(H_{\mu\boxplus_\lambda\nu}(z))\right]}\right) = \lim_{z\to\zeta} H_\nu\big(\omega_2(z)\big)\,.$$

The left hand-side exists always and equals $H_{\nu\boxplus_\lambda\mu}(\zeta)$, while the right hand side cannot exist at least for a set of second Baire category – Theorem 2.3. Specifically, $\omega_2(V)$ must be (by the identity theorem for analytic functions) a nontrivial interval in $(0,+\infty)$; for any $r \in \omega_2(V)$, we have a $\zeta \in V \subset \mathbb{C}\setminus\mathbb{R}_+$ so that $\omega_2(\zeta) = r$, and so, by Lindelöf's Theorem 2.1, since $H_{\mu\boxplus_\lambda\nu}(\zeta) = \lim_{z\to\zeta} H_\nu(\omega_2(z))$, we have

$$H_{\mu\boxplus_\lambda\nu}(\zeta) = \lim_{z\to\zeta} H_\nu\big(\omega_2(z)\big) = \lim_{\substack{w\longrightarrow r\\ \sphericalangle}} H_\nu(w)\,.$$

Theorem 2.3, together with the equation above, implies that there is a point of $\omega_2(V)$ where the cluster set of H_ν is a single point. Hence by (2.4), we have a contradiction with the fact that for any x in $\mathbb{R}$, the Cauchy transform of ν does not extend continuously to x.

Assume now that z_0 is so that $T\left[M_{\mu\boxplus_\lambda\nu}(z_0) - C_\mu(H_{\mu\boxplus_\lambda\nu}(z_0))\right] = 0$. Observe that since z_0 is assumed to be in $\mathbb{C}\setminus\mathbb{R}^+$, $H_{\mu\boxplus_\lambda\nu}(z_0) \in \mathbb{C}\setminus\mathbb{R}^+$ by the Section 2.3.2, (b). Hence the function ω_2 is meromorphic on a neighbourhood of z_0, with a pole at z_0. We conclude that $\lim_{z\to z_0} \frac{H_{\nu\boxplus_\lambda\mu}(z)}{T[M_{\mu\boxplus_\lambda\nu}(z) - C_\mu(H_{\mu\boxplus_\lambda\nu}(z))]} = \infty$. Consider now a small enough ball $W \subset \mathbb{C}\setminus\mathbb{R}^+$ around z_0 so that z_0 is the only pole of ω_2 in W, and consider a connected component of the intersection of this ball with the domain of the function in the right hand side of (4.1). Clearly $\omega_2(W)$ is a neighbourhood of infinity and W will contain $p \geq 1$ analytic varieties that are mapped by ω_2 onto $(-\infty,-M)\cup(N,+\infty)$, for some large enough $M, N > 0$ (p is the order of the pole at z_0). The preimages of $(N,+\infty)$ divide W into p sectors. By (4.1), if a point in one of these sectors belongs to the connected component of the domain of the right hand term of (4.1) which contains $(-\varepsilon, 0)$ then all that sector will belong to it, and the two (distinct or not! – it might be a slit circle, if $p = 1$) boundaries of the sector are mapped inside $\mathbb{R}^+$. Thus, we are reduced to the previous case.

We conclude that $\omega_2(z) = f(z, H_{\mu\boxplus_\lambda\nu}(z))$ maps $\mathbb{C}\setminus\mathbb{R}^+$ into itself.

• We now generalize the previous result to any probability measure ν. To this end, we can approximate in the weak topology arbitrary symmetric probabilities ν with probabilities concentrated on a set of zero Lebesgue measure and which have total support, according to Lemma 2.13. Since $\boxplus_\lambda$ is continuous, $H_{\mu\boxplus_\lambda\nu_n}$ converges to $H_{\mu\boxplus_\lambda\nu}$, and hence $f(z, H_{\mu\boxplus_\lambda\nu_n}(z)) : \mathbb{C}\setminus\mathbb{R}^+ \to \mathbb{C}\setminus\mathbb{R}^+$ converges to $f(z, H_{\mu\boxplus_\lambda\nu}(z))$. This implies that either $f(z, H_{\mu\boxplus_\lambda\nu}(z))$ takes values also in $\mathbb{C}\setminus\mathbb{R}^+$, or it is constant. But it cannot be constant, by equation (4.1) and by the fact that $H_{\mu\boxplus_\lambda\nu}(z)$ is equivalent to z as z tends to zero in $\mathbb{C}\setminus\mathbb{R}^+$ (see [7], Proposition 4.1). Hence

$f(z, H_{\mu\boxplus_\lambda\nu}(z))$ takes values in $\mathbb{C}\setminus\mathbb{R}^+$. This proves that ω_2 maps $\mathbb{C}\setminus\mathbb{R}^+$ into itself, and thus the equation $H_\nu \circ \omega_2 = H_{\mu\boxplus_\lambda\nu}$ holds on $\mathbb{C}\setminus\mathbb{R}^+$.

• We finally show that ω_2 satisfies the announced properties.

First, it follows immediately from the definition of ω_2 and part (d) of Subsection 2.3.2 that $\omega_2(\bar{z}) = \overline{\omega_2(z)}$ for all $z \in \mathbb{C}\setminus\mathbb{R}^+$. The uniqueness of ω_2 on $(-\varepsilon, 0)$ and the analyticity of ω_2 in $\mathbb{C}\setminus\mathbb{R}^+$ proved above shows that ω_2 is uniquely defined on all $\mathbb{C}\setminus\mathbb{R}^+$.

We prove next properties (i) and (ii). As shown above, $H_\nu \circ \omega_2 = H_{\mu\boxplus_\lambda\nu}$, so that for $\varepsilon > 0$ small enough, by Subsection 2.3.2 (a), $M_\nu(\omega_2(x)) = C_\nu(H_\nu(\omega_2(x))) = C_\nu(H_{\mu\boxplus_\lambda\nu}(x))$, $x \in (-\varepsilon, 0)$. Thus, $T[C_\mu(H_\nu(\omega_2(x))) + M_\nu(\omega_2(x))] = T[C_\mu(H_{\mu\boxplus_\lambda\nu}(x)) + C_\nu(H_{\mu\boxplus_\lambda\nu}(x))] = T[C_{\mu\boxplus_\lambda\nu}(H_{\mu\boxplus_\lambda\nu}(x))] = T[M_{\mu\boxplus_\lambda\nu}(x)]$. Now it follows immediately from the definition of k in Lemma 4.1 and Subsection 2.3.2 (a) that $k(\omega_2(z)) = z$ for $z \in (-\varepsilon, 0)$, $\varepsilon > 0$ small enough, and by analytic continuation, for $z \in \mathbb{C}\setminus\mathbb{R}^+$. This proves (i).

Let us recall that $\lim_{x\uparrow 0}\omega_2(x) = 0$ and, by the definition of T and properties of the function H, $\lim_{x\uparrow 0}\frac{\omega_2(x)}{x} = [T(0)]^{-1}\lim_{x\uparrow 0}\frac{H_{\mu\boxplus_\lambda\nu}(x)}{x} = 1$. Also, for $\varepsilon > 0$ small enough, $\omega_2((-\varepsilon, 0)) \subseteq (-\infty, 0)$. Thus, the derivative of the analytic function ω_2 on the interval $(-\varepsilon, 0)$ is positive for $\varepsilon > 0$ small enough, and so there is a small enough cone $\mathfrak{K}$ with vertex at zero and bisected by the negative half-line so that $\omega_2(\mathfrak{K}\cap\mathbb{C}^+) \subseteq \mathbb{C}^+$ and $\omega_2(\mathfrak{K}\cap\mathbb{C}^-) \subseteq \mathbb{C}^-$. Clearly, since $\omega_2(\mathbb{C}\setminus\mathbb{R}^+) \subseteq \mathbb{C}\setminus\mathbb{R}^+$, $\omega_2(\mathbb{C}^+) \not\subseteq \mathbb{C}^+$ implies that there exists a point $z_0 \in \mathbb{C}^+$ with the property that $\omega_2(z_0) \in (-\infty, 0)$. Assume such a point exists. Then from the equation (4.1) and Subsection 2.3.2 (d) we obtain that $H_{\mu\boxplus_\lambda\nu}(z_0) \in (-\infty, 0)$, so that $T(M_{\mu\boxplus_\lambda\nu}(z_0) - C_\mu(H_{\mu\boxplus_\lambda\nu}(z_0))) > 0$. As observed in Remark 2.20, this requires that $M_{\mu\boxplus_\lambda\nu}(z_0) - C_\mu(H_{\mu\boxplus_\lambda\nu}(z_0)) \in \mathbb{R}$. Since, by the same Subsection 2.3.2 (d), we have $C_\mu(\mathbb{R}^-) \subseteq \mathbb{R}$, it follows that $M_{\mu\boxplus_\lambda\nu}(z_0) \in \mathbb{R}$. But then, according to Subsection 2.3.2 (a), $H_{\mu\boxplus_\lambda\nu}(z_0) = z_0 T(M_{\mu\boxplus_\lambda\nu}(z_0)) \notin \mathbb{R}$, a contradiction. We have now proved that ω_2 preserves half-planes, and thus $\arg\omega_2(z) < \pi$ for $z \in \mathbb{C}^+$.

Next we show that ω_2 increases the argument. It is known from Theorem 2.8 that

$$\omega_2(z) = a + bz + \int_{\mathbb{R}} \frac{1+tz}{t-z}\, d\rho(t)\,, \quad z \in \mathbb{C}^+$$

for some $a \in \mathbb{R}$, $b \geq 0$ and positive finite measure ρ on the real line. Since $\omega_2((-\infty, 0)) \subseteq (-\infty, 0)$ and ω_2 is analytic on the negative half-line, ρ must be supported on $\mathbb{R}^+$. Moreover, $0 = \lim_{x\uparrow 0}\omega_2(x) = a + \lim_{x\uparrow 0}\int_{\mathbb{R}}\frac{1+tx}{t-x}d\rho(t) = \int_{\mathbb{R}}\frac{1}{t}d\rho(t) + a$. Thus, $a = -\int_{\mathbb{R}}\frac{1}{t}d\rho(t)$. We conclude that

$$\begin{aligned}\omega_2(z) &= a + bz + \int_{\mathbb{R}} \frac{1+tz}{t-z}\, d\rho(t) = bz + \int_{\mathbb{R}^+}\left(\frac{1+tz}{t-z} - \frac{1}{t}\right) d\rho(t)\\ &= z\left(b + \int_{\mathbb{R}^+}\frac{t^2+1}{t(t-z)}\, d\rho(t)\right).\end{aligned}$$

It is trivial to see that the factor in the parenthesis above maps $\mathbb{C}^+$ into itself. Thus, $\arg \omega_2(z) \geq \arg z$. This proves (ii). Let us define

$$\omega_1(z) = \frac{H_{\mu \boxplus_\lambda \nu}(z)}{T[M_{\mu \boxplus_\lambda \nu}(z) - M_\nu(\omega_2(z))]}, \quad z \in \mathbb{C} \setminus \mathbb{R}^+ .$$

This function is obviously defined and meromorphic on $\mathbb{C} \setminus \mathbb{R}^+$ and analytic continuation shows immediately that for $z \in (-\varepsilon, 0)$

$$\omega_1(z) = \frac{H_{\mu \boxplus_\lambda \nu}(z)}{T[M_{\mu \boxplus_\lambda \nu}(z) - C_\nu(H_{\mu \boxplus_\lambda \nu}(z))]},$$

so that, as for ω_2, $H_\mu(\omega_1(z)) = H_{\mu \boxplus_\lambda \nu}(z)$. This equality obviously extends by analytic continuation to $\mathbb{C} \setminus \mathbb{R}^+$. However, we do not exclude the possibility that H_μ has an analytic continuation through the positive half-line that does not coincide with the one provided by the formula in the Subsection 2.3.1.

It follows easily from the definition of ω_1 and Subsection 2.3.2 that $\overline{\omega_1(z)} = \omega_1(\bar{z})$ for all $z \in \mathbb{C} \setminus \mathbb{R}^+$ and $\lim_{x \uparrow 0} \omega_1(x) = 0$. The uniqueness of ω_1 is determined by the same argument as in the case of ω_2. $\square$

Next, we study the boundary behaviour of the restriction of the subordination function ω_2 to the upper half-plane.

Lemma 4.2. *Let μ, ν and ω_2 be as in the Lemma* 4.1. *Then $\omega_2|_{\mathbb{C}^+}$ extends continuously to* $(0, +\infty)$.

Proof. Throughout the proof we will consider only $\omega_2|_{\mathbb{C}^+}$ and we will denote it as ω_2. Assume that $r \in (0, +\infty)$ is so that the cluster set $C(\omega_2, r)$ of ω_2 at r is nontrivial, and hence, by Lemma 2.2 an uncountably infinite closed connected subset of $\mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$. Consider first the case when there exists at least one element $c \in C(\omega_2, r) \cap (\mathbb{C}^+ \cup (-\infty, 0))$, and thus, by connectivity of $C(\omega_2, r)$, infinitely many. Fix such a point c, and let $\{z_n^{(c)}\}_{n \in \mathbb{N}} \subseteq \mathbb{C}^+$ be a sequence with the property that $\lim_{n \to \infty} z_n^{(c)} = r$ and $\lim_{n \to \infty} \omega_2(z_n^{(c)}) = c$. Passing to the limit in the equation $k(\omega_2(z_n^{(c)})) = z_n^{(c)}$, where k is the function from Lemma 4.1, provides $k(c) = r$ for all $c \in C(\omega_2, r) \cap (\mathbb{C}^+ \cup (-\infty, 0))$, and hence, by analytic continuation, for all $c \in \mathbb{C} \setminus \mathbb{R}^+$. This implies that k is the constant function r, an obvious contradiction to Lemma 4.1.

If $C(\omega_2, r) \subseteq \mathbb{R}^+ \cup \{\infty\}$, then $C(\omega_2, r) \cap \mathbb{R}^+$ must be a nontrivial closed interval, by Lemma 2.2. As ω_2 maps $\mathbb{C}^+$ into itself, for all $c \in C(\omega_2, r)$, with the possible exception of two points, there exists a sequence $\{z_n^{(c)}\}_{n \in \mathbb{N}} \subseteq \mathbb{C}^+$ so that $\lim_{n \to \infty} z_n^{(c)} = r$, $\lim_{n \to \infty} \omega_2(z_n^{(c)}) = c$ and $\Re \omega_2(z_n^{(c)}) = c$. As shown in Subsection 2.3.2 (b), H_ν has nontangential limits at almost all points of $\mathbb{R}^+$, and by Lemma 2.10, so do M_ν and C_μ. Thus, k must have nontangential limits at almost all points of $\mathbb{R}^+$. We have obtained that for Lebesgue-almost all points $c \in C(\omega_2, r)$,

$$r = \lim_{n \to \infty} z_n^{(c)} = \lim_{n \to \infty} k\big(\omega_2(z_n^{(c)})\big) = \lim_{\substack{w \longrightarrow c \\ \sphericalangle}} k(w),$$

so that k has constant nontangential limit r on a set of nonzero Lebesgue measure. This, according to Theorem 2.6, implies that k is the constant function r, providing the same contradiction as before.

Thus, $\omega_2|_{\mathbb{C}^+}$ extends continuously to $(0,+\infty)$. □

Now we are ready to prove our first continuity result.

Proposition 4.3. *Let μ,ν be two symmetric probability measures on $\mathbb{R}$, $\mu\neq\delta_0$. Assume that μ is $\boxplus_\lambda$-infinitely divisible. Then for any $x\in(0,+\infty)$, the limits*

$$\lim_{z\to x, z\in\mathbb{C}^+} M_{\mu\boxplus_\lambda\nu}(x) \quad \text{and} \quad \lim_{z\to x, z\in\mathbb{C}^+} H_{\mu\boxplus_\lambda\nu}(x)$$

exist in $\mathbb{C}\cup\{\infty\}$. The first limit belongs to $\mathbb{C}^+\cup\mathbb{R}\cup\{\infty\}$.

Proof. We will follow idea from the proof of the Theorem 3.1. Assume that $r\in(0,+\infty)$ is so that $C(M_{\mu\boxplus_\lambda\nu},r)$ is nontrivial. Consider first the case when $C(M_{\mu\boxplus_\lambda\nu},r)\cap\mathbb{C}^+\neq\varnothing$, and thus, by Lemma 2.2, is uncountably infinite. Let $c\in C(M_{\mu\boxplus_\lambda\nu},r)\cap\mathbb{C}^+$ and $\{z_n^{(c)}\}_{n\in\mathbb{N}}\subset\mathbb{C}^+$ be so that $\lim_{n\to\infty} z_n^{(c)}=r$ and $\lim_{n\to\infty} M_{\mu\boxplus_\lambda\nu}(z_n^{(c)})=c$. We know from Lemma 4.2 that $\omega_2(r):=\lim_{z\to r}\omega_2|_{\mathbb{C}^+}(z)$ exists in $\overline{\mathbb{C}^+}$. Using the definition of ω_2 and Subsection 2.3.2 (a), we have:

$$\begin{aligned}
\omega_2(r)=\lim_{n\to\infty}\omega_2(z_n^{(c)}) &= \lim_{n\to\infty}\frac{H_{\mu\boxplus_\lambda\nu}(z_n^{(c)})}{T\left[M_{\mu\boxplus_\lambda\nu}(z_n^{(c)})-C_\mu(H_{\mu\boxplus_\lambda\nu}(z_n^{(c)}))\right]}\\
&= \lim_{n\to\infty}\frac{z_n^{(c)}T(M_{\mu\boxplus_\lambda\nu}(z_n^{(c)}))}{T\left[M_{\mu\boxplus_\lambda\nu}(z_n^{(c)})-C_\mu(z_n^{(c)}T(M_{\mu\boxplus_\lambda\nu}(z_n^{(c)})))\right]}\\
&= \frac{rT(c)}{T[c-C_\mu(rT(c))]}\,,\quad c\in C(M_{\mu\boxplus_\lambda\nu},r)\cap\mathbb{C}^+\,.
\end{aligned}$$

Thus, the meromorphic function $g_r\colon\mathbb{C}^+\cup(-1/\lambda,-1)\cup\mathbb{C}^-\to\mathbb{C}\cup\{\infty\}$, given by $g_r(c)=\frac{T(c)}{T[c-C_\mu(rT(c))]}$ is, by analytic continuation, constant, equal to $\omega_2(r)/r$. It is trivial to observe that this implies $\omega_2(r)\notin\{0,\infty\}$.

We shall express C_μ as a function of $s=T(c)$ to obtain a contradiction. Indeed, consider $c\in(-\frac{1}{2\lambda}-\frac{1}{2},-1)$. Then $s=T(c)$ if and only if $c=\frac{-1-\lambda+\left[(1-\lambda)^2+4\lambda s\right]^{1/2}}{2\lambda}$, $s\in(\frac{1}{4}(2-\lambda-\lambda^{-1}),0)$ (recall the notations from Section 2.3.1.) Thus,

$$\frac{rs}{\omega_2(r)}=T\left[\frac{-1-\lambda+\left[(1-\lambda)^2+4\lambda s\right]^{1/2}}{2\lambda}-C_\mu(rs)\right]. \tag{4.2}$$

As it is known that $C_\mu((-\infty,0))\subseteq(-\infty,0)$ and $\lim_{x\uparrow 0}C_\mu(x)=0$, we conclude that for $s\in(-\frac{(1-\lambda)^2}{4\lambda},0)$ close enough to zero,

$$T\left[\frac{-1-\lambda+\left[(1-\lambda)^2+4\lambda s\right]^{1/2}}{2\lambda}-C_\mu(rs)\right]\in\mathbb{R}\,,$$

so that $\omega_2(r) \in \mathbb{R}\backslash\{0\}$. Thus, since $\lim_{x\uparrow 0} C_\mu(x) = 0$, (4.2) is equivalent to

$$C_\mu(rs) = \frac{-1-\lambda+\left[(1-\lambda)^2+4\lambda s\right]^{1/2}}{2\lambda} - \frac{-1-\lambda+\left[(1-\lambda)^2+4\lambda\frac{rs}{\omega_2(r)}\right]^{1/2}}{2\lambda}.$$

But this implies either that $\omega_2(r) = r$, so that $C_\mu(s) = 0$ and thus $\mu = \delta_0$, or that C_μ is not analytic in the point $-\frac{r(1-\lambda)^2}{4\lambda} \in (-\infty, 0)$, an obvious contradiction.

Now consider the case when $C(M_{\mu\boxplus_\lambda\nu}, r) \subseteq \mathbb{R}\cup\{\infty\}$. By Subsection 2.3.2 (a) and Remark 2.20, in this case $C(H_{\mu\boxplus_\lambda\nu}, r) \subseteq [-\frac{r(1-\lambda)^2}{4\lambda}, +\infty]$ is a nontrivial interval. As in the proof of Lemma 4.2, for any $d \in C(H_{\mu\boxplus_\lambda\nu}, r) \setminus \{-\frac{r(1-\lambda)^2}{4\lambda}, \infty\}$, with the possible exception of two points, there exists a sequence $\{z_n^{(d)}\}_{n\in\mathbb{N}} \subset \mathbb{C}^+$ so that $\lim_{n\to\infty} z_n^{(d)} = r, \lim_{n\to\infty} H_{\mu\boxplus_\lambda\nu}(z_n^{(d)}) = d$ and $\Re H_{\mu\boxplus_\lambda\nu}(z_n^{(d)}) = d, n \in \mathbb{N}$.

Let us observe that, by Subsection 2.3.2 (a) and (c), we have

$$\lim_{n\to\infty} M_{\mu\boxplus_\lambda\nu}(z_n^{(d)}) \in \left\{\frac{-1-\lambda\pm\left[(1-\lambda)^2+4\lambda\frac{d}{r}\right]^{1/2}}{2\lambda}\right\},$$

where we have the sign plus when $H_{\mu\boxplus_\lambda\nu}(z_n^{(d)})$ tends to d from $\mathbb{C}^+$, and the sign minus when $H_{\mu\boxplus_\lambda\nu}(z_n^{(d)})$ tends to d from $\mathbb{C}^-$. By dropping if necessary to a subsequence, we may assume that $\lim_{n\to\infty} M_{\mu\boxplus_\lambda\nu}(z_n^{(d)})$ exists. It is clear from the definition of the cluster set and the above considerations that $C(H_{\mu\boxplus_\lambda\nu}, r) \setminus \{-\frac{r(1-\lambda)^2}{4\lambda}, \infty\} \subseteq A_+ \cup A_-$, where

$$A_+ = \left\{ d \in \mathbb{R} \setminus \left\{\frac{r(1-\lambda)^2}{-4\lambda}, \infty\right\} : \begin{array}{l} \exists\{z_n^{(d)}\}_{n\in\mathbb{N}} \subseteq \mathbb{C}^+ \text{ so that} \\ \lim\limits_{n\to\infty} z_n^{(d)} = r, \Re H_{\mu\boxplus_\lambda\nu}(z_n^{(d)}) = d, \\ \lim\limits_{n\to\infty} H_{\mu\boxplus_\lambda\nu}(z_n^{(d)}) = d, H_{\mu\boxplus_\lambda\nu}(z_n^{(d)}) \in \mathbb{C}^+ \end{array} \right\},$$

$$A_- = \left\{ d \in \mathbb{R} \setminus \left\{\frac{r(1-\lambda)^2}{-4\lambda}, \infty\right\} : \begin{array}{l} \exists\{z_n^{(d)}\}_{n\in\mathbb{N}} \subseteq \mathbb{C}^+ \text{ so that} \\ \lim\limits_{n\to\infty} z_n^{(d)} = r, \Re H_{\mu\boxplus_\lambda\nu}(z_n^{(d)}) = d, \\ \lim\limits_{n\to\infty} H_{\mu\boxplus_\lambda\nu}(z_n^{(d)}) = d, H_{\mu\boxplus_\lambda\nu}(z_n^{(d)}) \in \mathbb{C}^- \end{array} \right\}.$$

are two (not necessarily disjoint) sets. Thus, at least one of A_+, A_- has nonzero Lebesgue measure. Denote C_μ^+ the restriction of C_μ to the upper half-plane and C_μ^- the restriction of C_μ to the lower half-plane.

Assume first that A_+ has nonzero Lebesgue measure. Then again

$$\omega_2(r) = \lim_{n\to\infty} \omega_2(z_n^{(d)}) = \lim_{n\to\infty} \frac{H_{\mu\boxplus_\lambda\nu}(z_n^{(d)})}{T\left[M_{\mu\boxplus_\lambda\nu}(z_n^{(d)}) - C_\mu(H_{\mu\boxplus_\lambda\nu}(z_n^{(d)}))\right]}$$
$$= \frac{d}{T\left[\frac{-1-\lambda+\left[(1-\lambda)^2+4\lambda\frac{d}{r}\right]^{1/2}}{2\lambda} - \lim_{w \underset{\sphericalangle}{\longrightarrow} d} C_\mu^+(w)\right]}, \quad d \in A_+ .$$

By the Riesz–Privalov theorem we obtain again that

$$\omega_2(r)T\left[\frac{-1-\lambda+\left[(1-\lambda)^2+4\lambda\frac{d}{r}\right]^{1/2}}{2\lambda} - C_\mu^+(d)\right] = d\,, \quad d \in \mathbb{C}^+ .$$

Recalling that C_μ extends analytically to the negative half-line and considering values of $d \in (-\infty, 0)$ close enough to zero, we observe as before that $\omega_2(r) \in \mathbb{R}\backslash\{0\}$ and by analytic continuation

$$C_\mu(d) = \frac{\left[(1-\lambda)^2+4\lambda\frac{d}{r}\right]^{1/2}}{2\lambda} - \frac{\left[(1-\lambda)^2+4\lambda\frac{d}{\omega_2(r)}\right]^{1/2}}{2\lambda}\,,$$

providing the same contradiction as in the previous case.

Assume next that A_- has nonzero Lebesgue measure, so that

$$\omega_2(r) = \lim_{n\to\infty} \omega_2(z_n^{(d)}) = \lim_{n\to\infty} \frac{H_{\mu\boxplus_\lambda\nu}(z_n^{(d)})}{T\left[M_{\mu\boxplus_\lambda\nu}(z_n^{(d)}) - C_\mu(H_{\mu\boxplus_\lambda\nu}(z_n^{(d)}))\right]}$$
$$= \frac{d}{T\left[\frac{-1-\lambda-\left[(1-\lambda)^2+4\lambda\frac{d}{r}\right]^{1/2}}{2\lambda} - \lim_{w \underset{\sphericalangle}{\longrightarrow} d} C_\mu^-(w)\right]}, \quad d \in A_- .$$

Exactly as for A_+, we obtain that $\omega_2(r) \in \mathbb{R}\backslash\{0\}$, and, from the Riesz–Privalov theorem, the formula $C_\mu(d) = \frac{-[(1-\lambda)^2+4\frac{\lambda}{r}]^{1/2}+[(1-\lambda)^2+4\frac{\lambda}{\omega_2(r)}]^{1/2}}{2\lambda}$, which provides again the same contradiction.

Thus, we have established that the limit

$$M_{\mu\boxplus_\lambda\nu}(x) = \lim_{z\to x, z\in\mathbb{C}^+} M_{\mu\boxplus_\lambda\nu}(z)$$

exists for any $x \in (0, +\infty)$. The existence of the similar limit for $H_{\mu\boxplus_\lambda\nu}$ follows immediately from Subsection 2.3.2 (a), and since $M_{\mu\boxplus_\lambda\nu}(\mathbb{C}^+) \subseteq \mathbb{C}^+\cup\{0\}$, it follows that $M_{\mu\boxplus_\lambda\nu}(x) \in \mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$. □

Corollary 4.4. *Under the assumptions of Proposition* 4.3*, the absolutely continuous part* (*with respect to the Lebesgue measure*) *of* $\mu\boxplus_\lambda\nu$ *is continuous outside a closed set of zero Lebesgue measure, and its singular part, if it exists, is supported on a closed subset of* $\mathbb{R}$ *of zero Lebesgue measure.*

Proof. Recall that, by part (1) of Lemma 2.17 in [2], the nontangential limit of the Cauchy transform of $\mu\boxplus_\lambda\nu$ is infinite for almost all points in the support of the singular part of $\mu\boxplus_\lambda\nu$. Thus, using Proposition 4.3 and the equality $M_{\mu\boxplus_\lambda\nu}(z) = \frac{1}{\sqrt{z}}G_{\mu\boxplus_\lambda\nu}(\frac{1}{\sqrt{z}}) - 1$ from Subsection 2.3.2 (a), we can state that the support of the singular part of $\mu\boxplus_\lambda\nu$ is concentrated on $S = S^+ \cup S^- \cup \{0\}$, where $S^+ = \{x \in (0, +\infty) \colon M_{\mu\boxplus_\lambda\nu}(1/x^2) = \infty\}$ and $S^- = -S^+$.

By the Riesz–Privalov theorem (Theorem 2.6) it follows that the set S^+ must be of zero Lebesgue measure, and by Proposition 4.3, it follows that S^+, being the preimage of a point via a continuous map, must be closed in $(0, +\infty)$. This proves the second statement of the corollary.

The first statement follows from Lemma 2.11 (i): the Cauchy transform $G_{\mu\boxplus_\lambda\nu}$ extends continuously and with finite values to $\mathbb{R} \setminus S$, so that the density of $\mu\boxplus_\lambda\nu$ with respect to the Lebesgue measure is continuous on this set. □

Next, we show that, under some stricter conditions imposed on the rectangular R-transform of μ, we can guarantee that $\mu\boxplus_\lambda\nu$ has much better regularity properties. This will follow as a corollary of the proposition below.

Proposition 4.5. *We assume, in addition to the hypotheses of Proposition* 4.3*, that*

$$\lim_{x\to-\infty} \left[C_\mu(x)\right]^2/x \neq 0 \,.$$

Then for any $x \in (0, +\infty)$*,*

$$M_{\mu\boxplus_\lambda\nu}(x) = \lim_{z\to x, z\in\mathbb{C}^+} M_{\mu\boxplus_\lambda\nu}(x) \quad \text{and} \quad H_{\mu\boxplus_\lambda\nu}(x) = \lim_{z\to x, z\in\mathbb{C}^+} H_{\mu\boxplus_\lambda\nu}(z)$$

are finite.

Proof. Fix $x \in (0, +\infty)$. The existence of the limits has been established in Proposition 4.3. We shall prove the statement for $H_{\mu\boxplus_\lambda\nu}$, and the statement for $M_{\mu\boxplus_\lambda\nu}$ will follow from Subsection 2.3.2 (a). We shall prove that this limit is finite by exploiting the asymptotic behaviour of $C_\mu \circ H_{\mu\boxplus_\lambda\nu}$ and $M_{\mu\boxplus_\lambda\nu}$ as $H_{\mu\boxplus_\lambda\nu}$ tends to infinity in order to obtain a contradiction.

Let $c = H_{\mu\boxplus_\lambda\nu}(x)$. Assume towards contradiction that $c = \infty$. Let $\ell = \lim_{z\to x} \omega_2(z)$, where the limit is considered from the upper half-plane (the limit exists by Lemma 4.2). By Theorem 2.1, together with the above, this implies that

$$\lim_{\substack{z\longrightarrow \ell \\ \sphericalangle}} H_\nu(z) = \infty \,.$$

Subsection 2.3.2 (a) guarantees that if $H_{\mu\boxplus_\lambda\nu}(z)$ tends to infinity as z tends to x, then so does $M_{\mu\boxplus_\lambda\nu}(z)$ and moreover $H_{\mu\boxplus_\lambda\nu}(z)/M_{\mu\boxplus_\lambda\nu}(z)^2$ tends to λx as $z \to x$. Also, since $T(M_{\mu\boxplus_\lambda\nu}(z))$ and $H_{\mu\boxplus_\lambda\nu}(z) = zT(M_{\mu\boxplus_\lambda\nu}(z))$ belong to $\mathbb{C} \setminus \mathbb{R}^+$ for $z \in \mathbb{C}^+$, we have $\lim_{z\to x} \frac{\sqrt{H_{\mu\boxplus_\lambda\nu}(z)}}{M_{\mu\boxplus_\lambda\nu}(z)} = \sqrt{\lambda x}$. We will use this fact to determine the possible values of ℓ.

Let us observe that the existence of ℓ guarantees the existence of $k := \lim_{t\to-\infty} C_\mu(t)/\sqrt{t}$. Indeed, as the limit of $H_{\mu\boxplus_\lambda\nu}$ at t is infinite, and

$$\begin{aligned}\ell &= \lim_{z\to x, z\in\mathbb{C}^+} \frac{H_{\mu\boxplus_\lambda\nu}(z)}{\lambda(M_{\mu\boxplus_\lambda\nu}(z)-C_\mu(H_{\mu\boxplus_\lambda\nu}(z)))^2+(1+\lambda)(M_{\mu\boxplus_\lambda\nu}(z)-C_\mu(H_{\mu\boxplus_\lambda\nu}(z)))+1}\\ &= \frac{1}{\lambda}\left(\frac{1}{\sqrt{\lambda x}} - \lim_{z\to x, z\in\mathbb{C}^+} \frac{C_\mu(H_{\mu\boxplus_\lambda\nu}(z))}{\sqrt{H_{\mu\boxplus_\lambda\nu}(z)}}\right)^{-2}. \end{aligned} \tag{4.3}$$

On the other hand, as C_μ satisfies $\arg C_\mu(z) \in (\arg z, \pi)$, (by (2.6)) $C_\mu(\bar{z}) = \overline{C_\mu(z)}$ for $z \in \mathbb{C}^+$, and $C_\mu(\mathbb{R}^-) \subseteq \mathbb{R}^-$, it follows that Theorem 2.1 applies to the map $w \mapsto \frac{C_\mu(w)}{\sqrt{w}}$. Thus, $k := \lim_{x\to-\infty} \frac{C_\mu(x)}{\sqrt{x}}$ exists and is purely imaginary ($k = i|k|$) since $C_\mu(x)$ is negative for x negative. Thus, (4.3) gives

$$\lambda\ell = \frac{1}{((\lambda x)^{-\frac{1}{2}} - i|k|)^2}\,.$$

It follows immediately from equation (4.1) and analyticity of H_ν on $\mathbb{C}^+$ that when $|k| \in (0, +\infty)$, $\ell \in \mathbb{C}^+$ and we obtain the contradiction with the fact that $\infty = H_\nu(\ell)$.

Assume that k is infinite. Then $\ell = 0$. But this contradicts Theorem 2.1 and Remark 2.17: indeed, we obtain that the limit at zero of H_ν along $\omega_2(z) = \frac{H_{\mu\boxplus_\lambda\nu}(z)}{T[M_{\mu\boxplus_\lambda\nu}(z)-C_\mu(H_{\mu\boxplus_\lambda\nu}(z))]}$ (as $z \to x$ from the upper half-plane) is infinite (by (4.1)), while the limit at zero of H_ν along the negative half-line is zero.

This completes the proof of the proposition since k is not zero if and only if we have $\lim_{x\to-\infty}[C_\mu(x)]^2/x \neq 0$. □

Corollary 4.6. *Under the assumptions of Proposition* 4.5*, $\mu\boxplus_\lambda\nu$ is absolutely continuous with respect to the Lebesgue measure and its density is continuous.*

Proof. Since by Proposition 4.5 $M_{\mu\boxplus_\lambda\nu}(x)$ exists and is finite for all $x \in (0, +\infty)$, the corollary is a consequence of a variant of Lemma 2.11 (which states the existence of a continuous density of a measure with Cauchy transform which extends continuously to the real line) and the following Propositions 4.12 and 4.13 (which, with Lemma 2.21 (b), allow us to claim that $\mu\boxplus_\lambda\nu$ has no atom at the origin). □

In the following we discuss the issue of analyticity for the density of $\mu\boxplus_\lambda\nu$.

Lemma 4.7. *Under the hypotheses of Proposition* 4.3*, if $\omega_2(x) \in \mathbb{C}^+$, then there exists an $\varepsilon > 0$ so that $M_{\mu\boxplus_\lambda\nu}$ extends analytically to $(x-\varepsilon, x)\cup(x, x+\varepsilon)$.*

Proof. By continuity of ω_2, guaranteed in Lemma 4.2, there exists $\eta > 0$ so that $\omega_2([x-\eta, x+\eta]) \subseteq \mathbb{C}^+$ is a nontrivial curve in the upper half-plane. We claim that in fact ω_2 is injective on $[x-\eta, x+\eta]$. Indeed, if we assume that $v_1, v_2 \in [x-\eta, x+\eta]$ satisfy $\omega_2(v_1) = \omega_2(v_2)$, then, since k is meromorphic on $\mathbb{C}\setminus\mathbb{R}^+$, we obtain $v_1 = k(\omega_2(v_1)) = k(\omega_2(v_2)) = v_2$.

Let us observe that again since $k(\omega_2(z)) = z$ and k is meromorphic on $\mathbb{C}\setminus\mathbb{R}^+$, the set $\{w \in \omega_2([x-\eta, x+\eta])\colon k(w) = \infty \text{ or } k'(w) = 0\}$ is discrete in $\omega_2([x-$

$\eta, x+\eta]$). If $\omega_2(x)$ belongs to this set, then there exists an $0<\varepsilon\leq\eta$ so that $\omega_2((x-\varepsilon,x))\cup\omega_2((x,x+\varepsilon))$ does not intersect this set. Thus by the inverse function theorem ω_2 extends analytically through $(x-\varepsilon,x)\cup(x,x+\varepsilon)$. Otherwise, we apply the inverse function theorem to a neighbourhood of $\omega_2(x)$ to obtain the same result.

Since $H_{\mu\boxplus_\lambda\nu}=H_\nu\circ\omega_2$ and H_ν is analytic on $\mathbb{C}\setminus\mathbb{R}^+$, the statement of the lemma follows directly from Subsection 2.3.2 (a). $\square$

Lemma 4.8. *Under the hypotheses of Proposition* 4.3, *assume that*

$$H_{\mu\boxplus_\lambda\nu}(x)=\lim_{z\to x,z\in\mathbb{C}^+}H_{\mu\boxplus_\lambda\nu}(z)\in\mathbb{C}\setminus\mathbb{R}$$

for some $x\in(0,+\infty)$. *Then* $\omega_2(x):=\lim_{z\to x}\omega_2|_{\mathbb{C}^+}(z)\in\mathbb{C}^+$.

Proof. The equality $\omega_2(x)=\frac{H_{\mu\boxplus_\lambda\nu}(x)}{T[M_{\mu\boxplus_\lambda\nu}(x)-C_\mu(H_{\mu\boxplus_\lambda\nu}(x))]}$ assures us that $\omega_2(x)$ cannot be infinite. Indeed, assume to the contrary that $\omega_2(x)=\infty$. Then, by Theorem 2.1 and Subsection 2.3.2 (f)

$$H_{\mu\boxplus_\lambda\nu}(x)=\lim_{z\to x}H_{\mu\boxplus_\lambda\nu}(z)=\lim_{z\to x}H_\nu\big(\omega_2(z)\big)=\lim_{x\to-\infty}H_\nu(x)\in[-\infty,0)\,,$$

a contradiction to our assumption on $H_{\mu\boxplus_\lambda\nu}(x)$.

Assume first that $H_{\mu\boxplus_\lambda\nu}(x)\in\mathbb{C}^+$. We show next that $\omega_2(x)\notin\mathbb{R}$. Clearly by Subsection 2.3.2 (d), $\omega_2(x)\notin(-\infty,0]$. Assume again towards contradiction that $\omega_2(x)\in\mathbb{R}^+$.

Then $\frac{H_{\mu\boxplus_\lambda\nu}(x)}{x}=T(M_{\mu\boxplus_\lambda\nu}(x))$ and $T\ [M_{\mu\boxplus_\lambda\nu}(x)-C_\mu(H_{\mu\boxplus_\lambda\nu}(x))]$ belong to the same half-line χ originating at zero and passing through the point $H_{\mu\boxplus_\lambda\nu}(x)\in\mathbb{C}^+$. Thus both points $M_{\mu\boxplus_\lambda\nu}(x)$ and $M_\nu(\omega_2(x))=M_{\mu\boxplus_\lambda\nu}(x)-C_\mu(H_{\mu\boxplus_\lambda\nu}(x))$ belong to the same hyperbola $\mathcal{H}$ described in Remark 2.20 (v) whose tangents at the intersection with $-1/\lambda$ and -1 are parallel to χ. Call these tangents $\mathcal{T}_{1/\lambda}$ and $\mathcal{T}_1$. In particular, since $H_{\mu\boxplus_\lambda\nu}(x)\in\mathbb{C}^+$, we have that $M_{\mu\boxplus_\lambda\nu}(x)\in\mathcal{H}\cap K_1$ (recall the notations from Remark 2.20), and since $\omega_2(\mathbb{C}^+)\subseteq\mathbb{C}^+$, we have $M_\nu(\omega_2(x))\in\mathbb{C}^+\cup\mathbb{R}$, and so $M_\nu(\omega_2(x))=M_{\mu\boxplus_\lambda\nu}(x)-C_\mu(H_{\mu\boxplus_\lambda\nu}(x))\in\mathcal{H}\cap K_1$.

Now let us recall that $\pi>\arg C_\mu(H_{\mu\boxplus_\lambda\nu}(x))\geq\arg H_{\mu\boxplus_\lambda\nu}(x)>0$, so that $M_{\mu\boxplus_\lambda\nu}(x)-C_\mu(H_{\mu\boxplus_\lambda\nu}(x))$ has imaginary part strictly less that the imaginary part of $M_{\mu\boxplus_\lambda\nu}(x)$, so on $\mathcal{H}\cap K_1$ it must be below $M_{\mu\boxplus_\lambda\nu}(x)$. But at the same time $-C_\mu(H_{\mu\boxplus_\lambda\nu}(x))$ is in $\mathbb{C}^-$ and to the right of the line $\chi\cup-\chi$. Since the tangent $\mathcal{T}_1$ is parallel to $\chi\cup-\chi$, adding this number to $M_{\mu\boxplus_\lambda\nu}(x)$ will give a point in the upper half-plane that is necessarily at a greater distance from $\mathcal{T}_1$ than $M_{\mu\boxplus_\lambda\nu}(x)$, and thus it cannot be on $\mathcal{H}\cap K_1$ in between -1 and $M_{\mu\boxplus_\lambda\nu}(x)$ (as this part of the hyperbola is closer to $\mathcal{T}_1$ than $M_{\mu\boxplus_\lambda\nu}(x)$ is), which provides a contradiction. Thus, if $H_{\mu\boxplus_\lambda\nu}(x)\in\mathbb{C}^+$, then $\omega_2(x)\in\mathbb{C}^+$.

The case when $H_{\mu\boxplus_\lambda\nu}(x)\in\mathbb{C}^-$ is similar, and we will only sketch the proof. Indeed, then it is clear that, since $M_{\mu\boxplus_\lambda\nu}(x)\in\mathbb{C}^+$ (it cannot be in $\mathbb{R}$ because of Section 2.3.2 (a)), we must have, with the notations from the previous case, $M_{\mu\boxplus_\lambda\nu}(x)\in\mathcal{H}\cap K_2$. Now, $-\pi<\arg C_\mu(H_{\mu\boxplus_\lambda\nu}(x))\leq\arg H_{\mu\boxplus_\lambda\nu}(x)<0$, and so, as above, $\Im M_{\mu\boxplus_\lambda\nu}(x) < \Im[M_{\mu\boxplus_\lambda\nu}(x) - C_\mu(H_{\mu\boxplus_\lambda\nu}(x))]$. This time, however, we obtain that $-C_\mu(H_{\mu\boxplus_\lambda\nu}(x)) \in \mathbb{C}^+$, and to the *right* of $\chi \cup -\chi$. Thus, since $\mathcal{T}_{1/\lambda}$ is parallel to $\chi \cup -\chi$, the point $M_{\mu\boxplus_\lambda\nu}(x) - C_\mu(H_{\mu\boxplus_\lambda\nu}(x))$ will either be closer to $\mathcal{T}_{1/\lambda}$ and on its left side, or it will be on its right side. But the part of $\mathcal{H} \cap K_2$ which has an imaginary part greater than the imaginary part of $M_{\mu\boxplus_\lambda\nu}(x)$ is on the left side of $\mathcal{T}_{1/\lambda}$ *and* farther away from $\mathcal{T}_{1/\lambda}$ than $M_{\mu\boxplus_\lambda\nu}(x)$ is. Contradiction again. Thus, if $H_{\mu\boxplus_\lambda\nu}(x) \in \mathbb{C}^-$, then $\omega_2(x) \in \mathbb{C}^+$. $\square$

Lemma 4.9. *Under the hypotheses of Proposition* 4.3*, assume that* $x \in (0, +\infty)$ *is so that*

$$M_{\mu\boxplus_\lambda\nu}(x) := \lim_{z\to x, z\in\mathbb{C}^+} M_{\mu\boxplus_\lambda\nu}(z) \in i\big(\mathbb{R}^+ \setminus \{0\}\big) - \frac{1+\lambda}{2\lambda}\,.$$

Then $\omega_2(x) \in \mathbb{C}^+$.

Proof. The proof of this lemma is immediate. Indeed, by Remark 2.20 and Subsection 2.3.2 (a), we have $H_{\mu\boxplus_\lambda\nu}(x) \in (-\infty, -(1-\lambda)^2/4\lambda]$, so that $C_\mu(H_{\mu\boxplus_\lambda\nu}(x)) < 0$. But

$$\omega_2(x) = \frac{H_{\mu\boxplus_\lambda\nu}(x)}{T\left[M_{\mu\boxplus_\lambda\nu}(x) - C_\mu(H_{\mu\boxplus_\lambda\nu}(x))\right]}\,,$$

so by Remark 2.20, $\omega_2(x) \in \mathbb{C}^+$. $\square$

We can now prove the analogue of Theorem 3.1 for the rectangular case.

Proposition 4.10. *Let* μ *and* ν *be as in Proposition* 4.5*. Assume in addition that the restriction of* C_μ *to the upper half-plane extends continuously to* $(0, +\infty)$ *and* $C_\mu|_{\mathbb{C}^+}(x) \in \mathbb{C}^+$ *for all* $x \in (0, +\infty)$*. Then* $\mu\boxplus_\lambda\nu$ *is absolutely continuous with respect to the Lebesgue measure, and there exists an open set* $U \subset \mathbb{R}$ *so that* $(\mu\boxplus_\lambda\nu)(U) = 1$ *and the density* $h(x) = \frac{d(\mu\boxplus_\lambda\nu)(x)}{dx}$ *is analytic on* U*.*

Proof. We shall use the notations from Proposition 4.5. We know from Proposition 4.5 that $H_{\mu\boxplus_\lambda\nu}(x)$ is finite for any $x \in (0, +\infty)$. Fix such an x. We show first that $H_{\mu\boxplus_\lambda\nu}(x) \notin (0, +\infty)$. Assume towards contradiction that $H_{\mu\boxplus_\lambda\nu}(x) > 0$. By Proposition 4.5 $m := M_{\mu\boxplus_\lambda\nu}(x)$ exists and by Subsection 2.3.2 (a), is real. We know that $C_\mu(H_{\mu\boxplus_\lambda\nu}(x)) \in \mathbb{C}^+$ by hypothesis. Thus, using (4.1), we get that

$$\begin{aligned} H_{\mu\boxplus_\lambda\nu}(x) = \lim_{z\to x} H_\nu\big(\omega_2(z)\big) &= \lim_{z\to x} H_\nu\left(\frac{H_{\nu\boxplus_\lambda\mu}(z)}{T\left[M_{\mu\boxplus_\lambda\nu}(z) - C_\mu(H_{\mu\boxplus_\lambda\nu}(z))\right]}\right) \\ &= H_\nu\left(\frac{H_{\mu\boxplus_\lambda\nu}(x)}{T\left[m - C_\mu(H_{\mu\boxplus_\lambda\nu}(x))\right]}\right). \end{aligned}$$

Now, by our hypothesis on C_μ, we have $m - C_\mu(H_{\mu\boxplus_\lambda\nu}(x)) \in \mathbb{C}^-$. Thus, from the definition of T, $T(m - C_\mu(H_{\mu\boxplus_\lambda\nu}(x))) \notin [-(1-\lambda)^2/4\lambda, +\infty) \supset [0, +\infty)$. We have reached a contradiction since $H_\nu(\mathbb{C}\backslash\mathbb{R}^+) \subset \mathbb{C}\backslash\mathbb{R}^+$ by Section 2.3.2, (b).

Thus, $H_{\mu\boxplus_\lambda\nu}((0, +\infty)) \subset \mathbb{C} \setminus (0, +\infty)$. In particular, $M_{\mu\boxplus_\lambda\nu}(x) \in \mathbb{C}^+ \cup [-1/\lambda, -1]$.

Since $[-1/\lambda, -1]$ is a closed set and $M_{\mu\boxplus_\lambda\nu}$ is continuous on $(0, +\infty)$, the set $S = \{x \in (0, +\infty)\colon M_{\mu\boxplus_\lambda\nu}(1/x^2) \in [-1/\lambda, -1]\}$ is closed in $(0, +\infty)$. We claim that the set S satisfies $(\mu\boxplus_\lambda\nu)(S) = 0$. Indeed, the equality $M_{\mu\boxplus_\lambda\nu}(1/x^2) = xG_{\mu\boxplus_\lambda\nu}(x) - 1$, Lemma 2.11 (i) and the closeness of S make the claim obvious.

We claim next that for any $x \notin S$, $\omega_2(1/x^2) \in \mathbb{C}^+$. Indeed, $x \notin S$ implies that either $M_{\mu\boxplus_\lambda\nu}(1/x^2) \in i(\mathbb{R}^+ \setminus \{0\}) - \frac{1+\lambda}{2\lambda}$, and then the statement follows from Lemma 4.9, or $M_{\mu\boxplus_\lambda\nu}(1/x^2) \in K_1 \cup K_2$ and then the statement follows from Remark 2.20 and Lemma 4.8.

Now by Lemma 2.11 (ii), Proposition 4.5 and Lemma 4.7 the statement of the proposition follows. The possibility of the existence of an atom at zero will be discarded in the next section. □

4.2. Examples

We have a whole family of measures satisfying the previous proposition and corollary. In particular, we are going to see that all $\boxplus_\lambda$-stable distributions with index strictly smaller than 2 work. Recall that $\boxplus_\lambda$-infinitely divisible measures and their Lévy measures where introduced in Section 2.3.3.

Proposition 4.11. *Let $G \neq \delta_0$ be a symmetric positive finite measure on the real line, whose restriction to $(0, +\infty)$ admits an analytic positive density. Let μ be the $\boxplus_\lambda$-infinitely divisible measure μ with Lévy measure G.*

Then the restriction of C_μ to the upper half-plane extends analytically at any point x of $(0, +\infty)$ and satisfies $\Im(C_\mu(x)) > 0$.

Proof. Let ρ be the density of the restriction of the positive measure G to $(0, +\infty)$. By Theorem 2.18, C_μ extends analytically to $\mathbb{C}\backslash\mathbb{R}^+$ by the formula

$$C_\mu(z) = z\left(G(\{0\}) + 2\int_0^{+\infty} \frac{(1+t^2)\rho(t)}{1 - zt^2}\mathrm{d}t\right) = G_\tau(1/z)\,,$$

where τ is push-forward of the measure $(1 + t^2)\mathrm{d}G(t)$ by the function $t \to t^2$, and G_τ denotes the Cauchy transform of τ. Note that τ is a positive Radon measure, and that its restriction to $(0, +\infty)$ admits the density $u \to \frac{(1+u)\rho(u^{\frac12})}{u^{\frac12}}$ on $(0, +\infty)$. This density is analytic, hence by (ii) of Lemma 2.11 (which extends easily to positive measures on the real line which integrate $\frac{1}{1+|u|}$, as τ does) and by the fact that the behavior of G_τ on the lower half-plane can be deduced from its behaviour on the upper half-plane by the formula $G_\tau(\bar{\cdot}) = \overline{G_\tau(\,\cdot\,)}$, the restriction of C_μ to the upper half-plane extends analytically at any point x of $(0, +\infty)$ and satisfies, by (i) of Lemma 2.11,

$$\Im\big(C_\mu(x)\big) = \pi\frac{(1+x)\rho(x^{\frac12})}{x^{\frac12}} > 0\,.$$

□

It is proved in [8] that there is a bijection between the set of symmetric $*$-infinitely divisible distributions and the set of $\boxplus_\lambda$-infinitely divisible distributions, which preserves many properties, as limit theorems and the fact of being stable.

Hence, for all $\alpha \in (0,2)$, the set of $\boxplus_\lambda$-stable laws μ_α with index α is the set of $\boxplus_\lambda$-infinitely divisible laws which Lévy measure is of the type $t\frac{|x|^{1-\alpha}}{1+x^2}\mathrm{d}x$, where t can be any positive constant, so Proposition 4.11 can be applied to them. In fact, an application of the residue formula gives the rectangular R-transform with ratio λ of the $\boxplus_\lambda$-infinitely divisible law μ_α with Lévy measure $\frac{|x|^{1-\alpha}}{1+x^2}\mathrm{d}x$: for all $z \in \mathbb{C}\backslash\mathbb{R}^+$,

$$C_{\mu_\alpha}(z) = -\frac{\pi}{2\sin(\pi\alpha/2)}(-z)^{\alpha/2}\,,$$

where the power is defined on $\mathbb{C}\backslash\mathbb{R}^-$ in relation with the argument with value 0 on the positive half line. For $\alpha \in [1,2)$, μ_α satisfies the hypothesis of Proposition 4.10. As a consequence, for any positive number t, the same holds for the t-th power, with respect to $\boxplus_\lambda$, of μ_α. In deed, if one denotes this measure by μ_α^t (it should be denoted by $\mu_\alpha^{\boxplus_\lambda t}$, but this notation is a bit hard to swallow), one has $C_{\mu_\alpha^t}(z) = tC_{\mu_\alpha}(z)$.

A matricial model for the measures μ_α^t was given in [8], Section 5.

Moreover, for any positive t, the density of μ_1^t has been computed in Section 4.2 of [8]:

$$\frac{d\mu_1^t}{dx}(x) = \frac{t}{\pi(\lambda t^2 + x^2)}\left(1 - \frac{t^2(\lambda-1)^2}{4x^2}\right)^{\frac{1}{2}}$$

on its support $\mathrm{Supp}(\mu_1^t) = \mathbb{R}\backslash(-\frac{t(1-\lambda)}{2}, \frac{t(1-\lambda)}{2})$.

Remark however that the "rectangular Gaussian laws", i.e. the $\boxplus_\lambda$-stable laws with index 2, which are symmetric square roots of dilations of Pastur–Marchenko laws, which are the laws $\mu_2^t, t > 0$, satisfying $C_{\mu_2^t}(z) = tz$, do not satisfy the hypotheses of Proposition 4.3 since $C_{\mu_2^t}((0,\infty)) \subset (0,\infty)$.

4.3. Study of the density around the origin

In this section, we study the existence of a hole around the origin in the support of the free convolution $\mu\boxplus_\lambda\nu$. Since in our approach the origin itself is a very specific point, we shall study separately the existence of an atom at the origin and then existence of a set $[-\epsilon,\epsilon]$ which does not intersect the support of $\mu\boxplus_\lambda\nu$.

Some of the considerations of this section do not require the assumptions of Proposition 4.3.

Proposition 4.12.

1. *For all symmetric probability measures* μ,ν, $(\mu\boxplus_\lambda\nu)(\{0\}) \geq \mu(\{0\}) + \nu(\{0\}) - 1$.
2. *Assume that* μ *is* $\boxplus_\lambda$*-infinitely divisible* (ν *is still an arbitrary symmetric probability measure*). *If* $(\mu\boxplus_\lambda\nu)(\{0\}) > 0$, *then* $\mu(\{0\}) + \nu(\{0\}) > 1$ *and* $(\mu\boxplus_\lambda\nu)(\{0\}) = \mu(\{0\}) + \nu(\{0\}) - 1$.

Proof. We prove item 1. Consider a sequence $p_n \geq n$ of positive integers such that

$$n/p_n \underset{n\to\infty}{\longrightarrow} \lambda$$

and, on a probability space Ω, an independent set of random variables

$$(X_i)_{i\geq 1}, (Y_i)_{i\geq 1}, (U_n)_{n\geq 1}, (V_n)_{n\geq 1}$$

such that

- each X_i is distributed according to μ,
- each Y_i is distributed according to ν,
- for all n, U_n is an n by n Haar-distributed unitary random matrix,
- for all n, V_n is a p_n by p_n Haar-distributed unitary random matrix.

Let us define, for all n,

a) M_n to be the n by p_n random matrix with $|X_1|, \ldots, |X_n|$ on the diagonal, and zeros everywhere else,

b) N_n to be U_n times the n by p_n random matrix with $|Y_1|, \ldots, |Y_n|$ on the diagonal, and zeros everywhere else times V_n.

Let, for all n, d_n (resp. d'_n, d''_n) be the random variable equal to the number of null singular values of M_n (resp. $N_n, M_n + N_n$). Note that $d_n + (p_n - n) = \dim\ker M_n$, $d'_n + (p_n - n) = \dim\ker N_n$, $d''_n + (p_n - n) = \dim\ker M_n + N_n$. Note also that since $\ker M_n \cap \ker N_n \subset \ker(M_n + N_n)$, one has

$$\dim\ker M_n + N_n \geq \dim\ker M_n \cap \ker N_n \geq \dim\ker M_n + \dim\ker N_n - p_n\,,$$

hence

$$d''_n + (p_n - n) \geq d_n + (p_n - n) + d'_n + (p_n - n) - p_n\,,$$

i.e.

$$d''_n \geq d_n + d'_n - n\,. \tag{4.4}$$

Note that the singular values of M_n (resp. of N_n) are $|X_1|, \ldots, |X_n|$ (resp. $|Y_1|, \ldots, |Y_n|$), hence by the law of large numbers, the symmetrization of the singular law of M_n (resp. of N_n) converges almost surely weakly to μ (resp. ν). Thus by Theorem 4.8 of [7], the singular law $\mathrm{SL}(M_n + N_n)$ of $M_n + N_n$ converges in probability to $\mu\boxplus_\lambda\nu$ in the metric space of the set of probability measures on the real line endowed with a distance which defines the weak convergence. So for almost all $\omega \in \Omega$, there is a subsequence $\varphi(n)$ of the sequence $\mathrm{SL}(M_n + N_n)(\omega)$ which converges weakly to $\mu\boxplus_\lambda\nu$. For such an ω, one has

$$(\mu\boxplus_\lambda\nu)(\{0\}) \geq \limsup_{n\to\infty} \mathrm{SL}(M_{\varphi(n)} + N_{\varphi(n)})(\omega)(\{0\}) = \limsup_{n\to\infty} \frac{d''_{\varphi(n)}}{\varphi(n)}\,. \tag{4.5}$$

Note moreover that for all n, d_n (resp. d'_n) is the number of i's in $\{1, \ldots, n\}$ such that $X_i = 0$ (resp. $Y_i = 0$), hence the law of large numbers implies also that for almost all $\omega \in \Omega$

$$\frac{d_n(\omega)}{n} \underset{n\to\infty}{\longrightarrow} \mu(\{0\})\,, \qquad \frac{d'_n(\omega)}{n} \underset{n\to\infty}{\longrightarrow} \nu(\{0\})\,. \tag{4.6}$$

Putting together (4.4), (4.5), (4.6), one gets $(\mu\boxplus_\lambda\nu)(\{0\}) \geq \mu(\{0\}) + \nu(\{0\}) - 1$.

Let us now prove item 2. First of all, we exclude the case $\nu = \delta_0$, which is trivial. The strategy will be to use Lemma 2.21 and the description of atoms given in 2.3.2 (f) together with the formula (4.1) to prove the equality $(\mu\boxplus_\lambda\nu)(\{0\}) = \mu(\{0\}) + \nu(\{0\}) - 1$. We shall first prove that $\lim_{x\to-\infty} C_\mu(x) > -1$. Note that by 2.3.2 (f), our hypothesis implies that $\lim_{x\to-\infty} H_{\mu\boxplus_\lambda\nu}(x) = -\infty$. So we will prove that if $\lim_{x\to-\infty} H_{\mu\boxplus_\lambda\nu}(x) = -\infty$, then $\lim_{x\to-\infty} C_\mu(x) > -1$. For future use, we prefer to prove now that $\lim_{x\to-\infty} C_\mu(x) > -1$ under the hypothesis $\lim_{x\to-\infty} H_{\mu\boxplus_\lambda\nu}(x) = -\infty$ than under the stronger one of the proposition.

Assume thus that $\lim_{x\to-\infty} H_{\mu\boxplus_\lambda\nu}(x) = -\infty$. Recall first equality (4.1):

$$\forall z \in (-\infty, 0)\,, \quad H_{\mu\boxplus_\lambda\nu}(z) = H_\nu\left(\frac{H_{\nu\boxplus_\lambda\mu}(z)}{T\left[M_{\mu\boxplus_\lambda\nu}(z) - C_\mu(H_{\mu\boxplus_\lambda\nu}(z))\right]}\right), \tag{4.7}$$

where $M_{\mu\boxplus_\lambda\nu}(z)$ is the analytic extension of $U(\frac{H_{\mu\boxplus_\lambda\nu}(z)}{z} - 1)$ which can be found in 2.3.2 (a)(and which allows us to claim that $\lim_{x\to-\infty} M_{\mu\boxplus_\lambda\nu}(x)$ exists and is equal to $\mu\boxplus_\lambda\nu(\{0\}) - 1$).

Equality (4.7) together with the continuity of H_ν on $(-\infty, 0]$ and the hypothesis $\lim_{x\to-\infty} H_{\mu\boxplus_\lambda\nu}(x) = -\infty$ imply that

$$\lim_{x\to-\infty} \frac{H_{\mu\boxplus_\lambda\nu}(x)}{T\left[M_{\mu\boxplus_\lambda\nu}(x) - C_\mu(H_{\mu\boxplus_\lambda\nu}(x))\right]} = -\infty$$

(indeed, for any sequence x_n of negative numbers which tends to $-\infty$, since by 2.3.2, for all n, $y_n := \frac{H_{\mu\boxplus_\lambda\nu}(x_n)}{T[M_{\mu\boxplus_\lambda\nu}(x_n) - C_\mu(H_{\mu\boxplus_\lambda\nu}(x_n))]} \in (-\infty, 0)$, if y_n doesn't tend to $-\infty$, a subsequence of $H_{\mu\boxplus_\lambda\nu}(x_n)$ will have a finite limit, which is impossible). So by 2.3.2 (f) we obtain

$$\begin{aligned}
T(\nu(\{0\}) - 1) &= \lambda\nu(\{0\})^2 + (1-\lambda)\nu(\{0\}) \\
&= \lim_{x\to-\infty} \frac{H_\nu(x)}{x} \\
&= \lim_{x\to-\infty} \frac{T(M_{\mu\boxplus_\lambda\nu}(x) - C_\mu(H_{\mu\boxplus_\lambda\nu}(x)))}{H_{\mu\boxplus_\lambda\nu}(x)} \\
&\quad \times H_\nu\left(\frac{H_{\mu\boxplus_\lambda\nu}(x)}{T(M_{\mu\boxplus_\lambda\nu}(x) - C_\mu(H_{\mu\boxplus_\lambda\nu}(x)))}\right) \\
&= \lim_{x\to-\infty} \frac{T(M_{\mu\boxplus_\lambda\nu}(x) - C_\mu(H_{\mu\boxplus_\lambda\nu}(x)))}{H_{\mu\boxplus_\lambda\nu}(x)} H_{\mu\boxplus_\lambda\nu}(x) \\
&= \lim_{x\to-\infty} T(M_{\mu\boxplus_\lambda\nu}(x) - C_\mu(H_{\mu\boxplus_\lambda\nu}(x)))\,.
\end{aligned}$$

Thus, $\lim_{x\to-\infty} T(M_{\mu\boxplus_\lambda\nu}(x) - C_\mu(H_{\mu\boxplus_\lambda\nu}(x))) \in [0,1]$. Note that (2.6) allows us to claim that $\lim_{x\to-\infty} C_\mu(x)$ exists in $[-\infty, 0]$, hence by above, $M_{\mu\boxplus_\lambda\nu}(x) - C_\mu(H_{\mu\boxplus_\lambda\nu}(x))$ has also a limit $l = (\mu\boxplus_\lambda\nu)(\{0\}) - 1 - \lim_{w\to-\infty} C_\mu(w) \geq -1$ as x goes to $-\infty$. Since $T^{-1}([0,1]) = [-\frac{1}{\lambda} - 1, -\frac{1}{\lambda}] \cup [-1, 0]$, one has $l \in [-1, 0]$, hence $l = \nu(\{0\}) - 1$. We conclude that

$$\lim_{w\to-\infty} C_\mu(w) = \lim_{x\to-\infty} C_\mu\big(H_{\mu\boxplus_\lambda\nu}(x)\big) = (\mu\boxplus_\lambda\nu)(\{0\}) - 1 - \big(\nu(\{0\}) - 1\big) \in (-1, 0]\,,$$

as claimed. Moreover, this equality together with Lemma 2.21 implies that $\mu(\{0\}) - 1 = (\mu\boxplus_\lambda\nu)(\{0\}) - \nu(\{0\})$, which is equivalent to item 2. $\square$

Proposition 4.13. *Let μ be $\boxplus_\lambda$-infinitely divisible and ν be arbitrary. Assume that $\nu(\{0\}) + \mu(\{0\}) < 1$. Then* $\operatorname{supp}(\mu\boxplus_\lambda\nu)$ *has a hole around the origin.*

Proof. Let us denote $r := \lim_{x\to-\infty} H_{\mu\boxplus_\lambda\nu}(x)$ (which exists and belongs to $[-\infty, 0)$ by 2.3.2 (f)). The first step in our proof will be to show that under our hypothesis, $r > -\infty$. Then we will view r as the Denjoy–Wolff point of a certain self-map of the left half-plane $i\mathbb{C}^+$, and use this fact to see $W_{\mu\boxplus_\lambda\nu}(x) = H_{\mu\boxplus_\lambda\nu}(1/x)$ as an implicit function which is defined on a neighbourhood of zero and extends $H_{\mu\boxplus_\lambda\nu}(1/x)$ from the left half-line. Finally, we will argue that on a small enough interval, $H_{\mu\boxplus_\lambda\nu}(1/x) \in [-(1-\lambda)^2/4\lambda x, 0]$ for all $x > 0$ small enough, which is equivalent to the existence of an open neighborhood of the origin which does not intersect the support of $\mu\boxplus_\lambda\nu$ (as can be checked by using Lemma 2.11 and Remark 2.17).

We shall prove that r is the Denjoy–Wolff point of f_1 if

$$f_t(z) = H_\nu\left(\frac{z}{T[-t - C_\mu(z)]}\right), \quad t \in [0,1]\,.$$

First, we claim that f_t is defined on the left half-plane $i\mathbb{C}^+$, and moreover that $f_t(i\mathbb{C}^+) \subseteq i\mathbb{C}^+$ for all $t \in [0,1]$. Indeed, from Theorem 2.18 it follows that $C_\mu(\mathbb{C}^+) \subseteq \mathbb{C}^+$ and $\arg C_\mu(z) > \arg z$ for any $z \in \mathbb{C}^+$. Thus, since $0 < \lambda < 1$, and $0 \le t \le 1$, $\pi > \arg(C_\mu(z)+t-1) \ge \arg C_\mu(z) > \arg z$ and $\pi > \arg(C_\mu(z)+t-\frac{1}{\lambda}) > \arg C_\mu(z) > \arg z$, so that $\arg T[-t - C_\mu(z)] \in (2\arg z, 2\pi)$ for all $z \in \mathbb{C}^+ \cap i\mathbb{C}^+$. We conclude that $\arg(\frac{T[-t-C_\mu(z)]}{z}) \in (\arg z, 2\pi - \arg z) \subset (\pi/2, 3\pi/2)$ for any $z \in \mathbb{C}^+ \cap i\mathbb{C}^+$, so that $\frac{T[-t-C_\mu(z)]}{z}$ maps $\mathbb{C}^+ \cap i\mathbb{C}^+$ in $i\mathbb{C}^+$. Since $i\mathbb{C}^+$ is invariant under the maps $z \mapsto 1/z$ and $z \mapsto \bar{z}$, and $\overline{(\frac{T[-t-C_\mu(z)]}{z})} = \frac{T[-t-C_\mu(\bar{z})]}{\bar{z}}$, we conclude that $z \mapsto \frac{z}{T[-t-C_\mu(z)]}$ maps $i\mathbb{C}^+$ into itself. Since, by Subsection 2.3.2 (d), $H_\nu(i\mathbb{C}^+) \subseteq i\mathbb{C}^+$, our claim is proved.

By the last remark, we also have that $f_t(\bar{z}) = \overline{f_t(z)}$, for all $t \in [0,1]$, and hence in particular $f_t((-\infty, 0]) \subset (-\infty, 0]$.

We next show the existence and uniqueness of the Denjoy–Wolff point of f_1 as a consequence of Theorem 2.7. In fact, f_1 is not a conformal automorphism of $i\mathbb{C}^+$. Indeed, there are only two conformal automorphisms of $i\mathbb{C}^+$ which fix $(-\infty, 0]$ up to multiplication by positive scalar; the identity and $z \to 1/z$. The case $az = H_\nu(\frac{z}{T[-1-C_\mu(z)]})$ can be discarded since as z goes to zero along the negative half-line, $C_\mu(z)/z$ converges to $\int(1+t^2)dG(t)$ by monotone convergence theorem (with G the Lévy measure of μ) and so $y(z) := \frac{z}{T[-1-C_\mu(z)]} \in (-\infty, 0]$ goes to the constant

$$\begin{aligned}\lim_{x\uparrow 0} \frac{x}{T[-1 - C_\mu(x)]} &= \lim_{x\uparrow 0} \frac{1}{\lambda\frac{C_\mu(x)^2}{x} + (\lambda-1)\frac{C_\mu(x)}{x}} \\ &= \left((\lambda - 1)\int (t^2+1)\, dG(t)\right)^{-1},\end{aligned} \tag{4.8}$$

which is null only if $\int(1+t^2)dG(t)$ is infinite. If this constant does not vanish, we obtain a contradiction since H_ν does not vanish on $(-\infty, 0)$. If it vanishes, we write $aT[-1 - C_\mu(z)] = H_\nu(y(z))/y(z)$ with $y(z)$ negative going to zero as z

goes to zero. This is in contradition with the fact that $H_\nu(z)/z$ goes to one (see Remark 2.17). The case $a/z = H_\nu(\frac{z}{T[-1-C_\mu(z)]})$ leads also to a contradiction by letting z going to zero.

The uniqueness of the Denjoy–Wolff point given by Theorem 2.7 implies that this point can only belong to $[-\infty, 0]$ since $f_t(\bar{z}) = \bar{f}_t(z)$. We shall first show that zero cannot be the Denjoy–Wolff point of f_1, and secondly we show that infinity can be the Denjoy–Wolff point of f_1 only when $\mu(\{0\}) + \nu(\{0\}) \geq 1$.

For zero to be the Denjoy–Wolff point of f_1, we would first need to have $\lim_{x\uparrow 0} f_1(x) = 0$. Since H_ν vanishes on $(-\infty, 0]$ only at the origin, we must have by (4.8) that the Lévy measure of μ has infinite second moment. The second requirement for zero to be the Denjoy–Wolff point of f_1 is that $\lim_{x\uparrow 0} f_1(x)/x \in (0, 1]$. But

$$\begin{aligned}
\lim_{x\uparrow 0} \frac{f_1(x)}{x} &= \lim_{x\uparrow 0} \frac{H_\nu\left(\frac{x}{T[-1-C_\mu(x)]}\right)}{\frac{x}{T[-1-C_\mu(x)]}} \cdot \frac{1}{T[-1-C_\mu(x)]} \\
&= \lim_{x\uparrow 0} \frac{H_\nu(x)}{x} \cdot \lim_{x\uparrow 0} \frac{1}{T[-1-C_\mu(x)]} = \infty \,,
\end{aligned}$$

since $\lim_{x\uparrow 0} \frac{H_\nu(x)}{x} = 1$ and $\lim_{x\uparrow 0} C_\mu(x) = 0$. We conclude that zero cannot be the Denjoy–Wolff point of f_1.

Now we show under that under our condition $\nu(\{0\}) + \mu(\{0\}) < 1$, f_1 cannot have infinity as Denjoy–Wolff point (recall that $(\mu \boxplus_\lambda \nu)(\{0\}) = 0$ by Proposition 4.12 under this assumption.) The two requirements that f_1 must verify to have infinity as Denjoy–Wolff point are $\lim_{x\to-\infty} f_1(x) = -\infty$ and $\lim_{x\to-\infty} f_1(x)/x \in [1, +\infty)$. The continuity of H_ν on $(-\infty, 0]$ translates the first requirement into $\lim_{x\to-\infty} \frac{x}{T[-1-C_\mu(x)]} = -\infty$ and $\lim_{x\to-\infty} H_\nu(x) = -\infty$. Applying 2.3.2 (f) and the above, we obtain:

$$\begin{aligned}
\lim_{x\to-\infty} \frac{f_1(x)}{x} &= \lim_{x\to-\infty} \frac{H_\nu\left(\frac{x}{T[-1-C_\mu(x)]}\right)}{\frac{x}{T[-1-C_\mu(x)]}} \cdot \frac{1}{T[-1-C_\mu(x)]} \\
&= \lim_{x\to-\infty} \frac{H_\nu(x)}{x} \cdot \lim_{x\to-\infty} \frac{1}{T[-1-C_\mu(x)]} \\
&= \big(\lambda\nu(\{0\})^2 + (1-\lambda)\nu(\{0\})\big) \cdot \frac{1}{\lambda c^2 + (\lambda-1)c} \\
&= \frac{T(\nu(\{0\})-1)}{T(-c-1)} \,,
\end{aligned}$$

where $c := \lim_{x\to-\infty} C_\mu(x) \in [-\infty, 0)$. To have $\lim_{x\to-\infty} f_1(x)/x \geq 1$, we must have $\nu(\{0\}) > 0$ and $c > -\infty$. Thus, we may write

$$T\big(\nu(\{0\}) - 1\big) \geq T(-c-1) \,,$$

which implies, since T is increasing on $[-1, +\infty)$, that $1 \geq \nu(\{0\}) \geq -c$. Using Lemma 2.21 (2) we conclude that $\nu(\{0\}) + \mu(\{0\}) \geq 1$.

Thus, $\nu(\{0\}) + \mu(\{0\}) < 1$ implies that f_1 has a Denjoy–Wolff point $s \in (-\infty, 0)$. We claim that $s = r$. Indeed, by taking limit when $z \to -\infty$ in equation (4.1), using Proposition 4.12 and the fact that $\lim_{x\to-\infty} M_{\mu\boxplus_\lambda\nu}(x) = (\mu\boxplus_\lambda\nu)(\{0\}) - 1 = -1$, we obtain that

$$r = H_\nu\left(\frac{r}{T[-1 - C_\mu(r)]}\right) = f_1(r)\,, \tag{4.9}$$

where, if $r = -\infty$, the second term must be also understood as a limit.

We finally show that r cannot be infinite, which will imply with (4.9) that $r = s$. By Theorem 2.7, $f_1'(s) \in (-1, 1)$. By continuity of f_1', there exists $\delta > 0$ so that if $D = \{y : |x - s| < \delta\}$, $\rho := \sup_{\bar{D}} |f_1'(x)| < 1$ and therefore $f_1(\bar{D}) \subset \{y : |y - s| \le \rho\delta\} \subset D$. Since f_t converges to f_1 as $t \to 1$ uniformly on compact subsets of $i\mathbb{C}^+$, there exists $\varepsilon > 0$ so that $f_t(\bar{D}) \subseteq D$ and moreover f_t is not an hyperbolic rotation for all $1 - \varepsilon \le t \le 1$. Thus, by Theorem 2.7, f_t has a unique Denjoy–Wolff point and it must be in $\bar{D}$ (has can be seen by iterating f from a point in $\bar{D}$). Thus, the Denjoy–Wolff points of f_t converge to s as $t \to 1$.

Now, since $\lim_{x\to-\infty} M_{\mu\boxplus_\lambda\nu}(x) = -1$ as $\mu\boxplus_\lambda\nu(\{0\}) = 0$, for x large enough we have $M_{\mu\boxplus_\lambda\nu}(x) \in (-1, \varepsilon - 1)$. From equation (4.1) it follows that

$$H_{\mu\boxplus_\lambda\nu}(x) = f_{-M_{\mu\boxplus_\lambda\nu}(x)}\big(H_{\mu\boxplus_\lambda\nu}(x)\big)$$

and therefore $H_{\mu\boxplus_\lambda\nu}(x)$ is the Denjoy–Wolff point of $f_{-M_{\mu\boxplus_\lambda\nu}(x)}$. Thus we conclude that $r = \lim_{x\to-\infty} H_{\mu\boxplus_\lambda\nu}(x) = s$, which proves our claim.

Let us define $W_{\mu\boxplus_\lambda\nu}(z) = H_{\mu\boxplus_\lambda\nu}(1/z)$ and

$$g(x, w) = H_\nu\left(\frac{w}{T[U(xw - 1) - C_\mu(w)]}\right) - w\,.$$

It is easy to observe that for $x < 0$ close to zero, we have $g(x, W_{\mu\boxplus_\lambda\nu}(x)) = 0$, as the formula $M_{\mu\boxplus_\lambda\nu}(z) = U(H_{\mu\boxplus_\lambda\nu}(z)/z - 1)$ must hold for all $z \in \mathbb{R}^-$. Moreover, there obviously exists a small enough interval I centered at zero so that g is actually defined on $I \times (I + r)$, and of course, by equation (4.9), $g(0, r) = 0$. Let us differentiate g with respect to w:

$$\begin{aligned}&\partial_w g(x, w) = H_\nu'\left(\frac{w}{T[U(xw - 1) - C_\mu(w)]}\right)\\ &\times \frac{T[U(xw - 1) - C_\mu(w)] - wT'[U(xw - 1) - C_\mu(w)][xU'(xw - 1) - C_\mu'(w)]}{T[U(xw - 1) - C_\mu(w)]^2} - 1\,.\end{aligned}$$

Since U is differentiable in -1, and $T[-1 - C_\mu(r)] \neq 0$, we have

$$\begin{aligned}\partial_w g(0, r) &= H_\nu'\left(\frac{r}{T[-1 - C_\mu(r)]}\right)\frac{T[-1 - C_\mu(r)] - rT'[-1 - C_\mu(r)][-C_\mu'(r)]}{T[-1 - C_\mu(r)]^2} - 1\\ &= f_1'(r) - 1.\end{aligned}$$

Since we have shown that $|f_1'(r)| < 1$, we conclude that $\partial_w g(0, r) \neq 0$, so we can apply the implicit function theorem to it in the point $(0, r)$ to extend $W_{\mu\boxplus_\lambda\nu}$ to a small neighborhood of the origin. Then, $W_{\mu\boxplus_\lambda\nu}(x) = H_{\mu\boxplus_\lambda\nu}(1/x)$ takes its values in a finite neighborhood of $r \in (-\infty, 0)$ which is included into $[-(1-\lambda)^2/4\lambda x, 0]$ for sufficiently small x, and hence $\mu \boxplus_\lambda \nu$ put no mass in a open neighborhood of the origin. $\square$

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