Compl. anal. oper. theory 3 (2009), 739–758

1661-8254/040739-20, *published online* June 20, 2008
DOI 10.1007/s11785-008-0077-7

**Complex Analysis and Operator Theory**

# On Complex Infinite-Dimensional Grassmann Manifolds

Daniel Beltiţă and José E. Galé

**Abstract.** We investigate geometric properties of Grassmann manifolds and their complexifications in an infinite-dimensional setting. Specific structures of quaternionic type are constructed on these complexifications by a direct method that does not require any use of the cotangent bundles.

**Mathematics Subject Classification (2000).** Primary 46L05; Secondary 58B12; 53C15; 43A85; 22E65.

**Keywords.** Grassmann manifold, homogeneous vector bundle, complexification, hypercomplex structure, $C^*$-algebra.

## 1. Introduction

In this paper we discuss differential geometric properties of Grassmann manifolds in a $C^*$-algebraic setting. Background information for the circle of ideas centered on such manifolds here can be found in the works [12,16–18,23–25,27,31], and [4, 30]. Our approach to the subject originated in some specific questions concerning holomorphy, which in a natural way led us to investigate complex structures as well as complexifications of these manifolds.

Let $\mathcal{H}$ be a *complex* Hilbert space. Recall that the Grassmannian $\mathrm{Gr}(\mathcal{H})$ associated with $\mathcal{H}$, or, alternatively, with the Banach algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$, is formed by all closed linear subspaces of $\mathcal{H}$ and it is a complex Banach manifold (see [30]). For more general $C^*$-algebras the definition of the Grassmann manifold is a bit more involved, and not always standard. For instance, if $A$ is a unital $C^*$-algebra, the Grassmannian of $A$ is sometimes assumed to be the set $\mathcal{P}_{sa}(A)$ of selfadjoint idempotents of $A$, endowed with the relative topology. This is so considered for example in [27], where such Grassmannians are studied in connection with Hermitian holomorphic bundles as motivated by certain aspects of the Cowen–Douglas theory. The manifold $\mathcal{P}_{sa}(A)$ is in principle *real*-analytic, so that one must be careful in order to deal with holomorphy, see [27], p. 278. In other settings it is suitable to deal with explicit complex Grassmann manifolds, as for example in [25] or [16, 17]. This is accomplished in the following way.

Recall that, for a given unital, complex, associative ($C^*$-) algebra $A$, two idempotents $p, q \in A$ are said to be equivalent if $pq = q$ and $qp = p$. The equivalence class of an idempotent $p$ is denoted here by $[p]$, and the quotient set formed by all the classes $[p]$ is denoted by $\mathrm{Gr}(A)$ and called the Grassmannian of $A$. Let $\mathrm{G}_A$ be the group of invertible elements of $A$ and let $\mathrm{U}_A$ be the subgroup of $\mathrm{G}_A$ of unitary elements. Then, see Section 2 below, $\mathrm{G}_A$ has a natural action on $\mathrm{Gr}(A)$. The isotropy subgroup of $\mathrm{G}_A$ at $[p]$ for such an action is denoted by $\mathrm{G}_A([p])$. Then the $\mathrm{G}_A$-orbits in $\mathrm{Gr}(A)$ coincide with the quotients $\mathrm{G}_A/\mathrm{G}_A([p])$, which, endowed with their respective quotient topologies, are holomorphic Banach manifolds [16]. In analogy, put $\mathrm{G}_A(p) := \{u^{-1}pu : u \in \mathrm{G}_A\}$ and $\mathrm{U}_A(p) := \{u^{-1}pu : u \in \mathrm{U}_A\}$. As above, $\mathrm{G}_A/\mathrm{G}_A(p)$ is a complex Banach manifold and $\mathrm{U}_A/\mathrm{U}_A(p)$ is, for $p \in \mathcal{P}_{sa}(A)$, a real-analytic Banach manifold, in their respective quotient topologies. Let us remark that $\mathrm{U}_A/\mathrm{U}_A(p) = \mathrm{U}_A/\mathrm{U}_A([p])$ for every $p \in \mathcal{P}_{sa}(A)$ (see Remark 2.5) and that $\mathcal{P}_{sa}(A) \equiv \bigcup_{p\in\mathcal{P}_{sa}(A)} \mathrm{U}_A/\mathrm{U}_A([p])$. The complex structures of the Grassmannian $\mathrm{Gr}(A) \equiv \bigcup \mathrm{G}_A/\mathrm{U}_G([p])$ and of its associated (holomorphic) Stiefel bundle have been plainly used in [16] and [17] in order to obtain holomorphic parametrizations of framings for projections on a fixed Banach space. It sounds sensible to analyze the relationship between the differentiable structures of the two types of Grassmannians $\mathrm{U}_A/\mathrm{U}_A(p)$ and $\mathrm{G}_A/\mathrm{G}_A([p])$ when $p = p^*$.

Another motivation in writing the present paper came from representation theory. To explain this, recall the classical setting of the Bott–Borel–Weil theorem of [11] involving the flag manifolds and realizations of representations on holomorphic sections spaces of holomorphic bundles. The notion of *complexification* plays a central role in this area, inasmuch as one of the ways to describe the complex structure of the flag manifolds is to view the latter as homogeneous spaces of complexifications of compact Lie groups.

A similar circle of ideas can be found in infinite dimensions for a class of manifolds which play a central role in many areas of functional analysis and operator theory. Specifically, let $\mathbf{1} \in B \subseteq A$ be unital $C^*$-algebras such that there exists a conditional expectation $E\colon A \to B$. Let $\varphi$ be a state of $A$ such that $\varphi \circ E = \varphi$, and let $\pi_\varphi$ denote the Gelfand–Naimark–Segal (GNS, for short) representation obtained out of $\varphi$. Then it is possible to realize $\pi_\varphi$ as acting on spaces of real-analytic sections of a certain homogeneous Hermitian vector bundle on $\mathrm{U}_A/\mathrm{U}_B$. Further, in some cases involving finiteness properties of spectra and traces of elements in $A$, one can prove that the homogeneous space $\mathrm{U}_A/\mathrm{U}_B$ is a *complex* manifold as well and that the realization space is formed by *holomorphic* sections (see Theorem 5.4 and Theorem 5.8 in [5]).

Apart from the above result, the holomorphic character of the manifolds $\mathrm{U}_A/\mathrm{U}_B$ and associated bundles is far from being clear in general. Since the Grassmannians $\mathrm{U}_A/\mathrm{U}_A(p)$ and tautological bundles over $\mathrm{U}_A/\mathrm{U}_A(p)$ are universal objects (see for example [2, 17] in connection with ideas here), we search for related results of holomorphy in this special case. Using a charaterization (and labelling) of invariant complex structures on infinite-dimensional homogeneous manifolds (Theorem 6.1 in [4]), we prove in Section 3 below that $\mathrm{U}_A/\mathrm{U}_A(p)$ and $\mathrm{G}_A/\mathrm{G}_A([p])$ are locally biholomorphic complex manifolds. Moreover, it is also shown that $\mathrm{G}_A/\mathrm{G}_A(p)$ is a complexification of $\mathrm{U}_A/\mathrm{U}_A(p)$, and then these results are translated in terms of homogeneous vector bundles, in the spirit of Theorem 5.8 in [5]. (A complementary perspective on these manifolds can be found in [6].) Section 4 is devoted to exhibit how the above results look like in the (on the other hand well known) case of the algebra $A = \mathcal{B}(\mathcal{H})$ and corresponding universal, tautological vector bundles.

Finally, the manifold $\mathrm{G}_A/\mathrm{G}_A(p)$ being a complexification of the complex manifold $\mathrm{U}_A/\mathrm{U}_A(p)$, we point out the existence of quaternionic structures for the above complexifications in Section 5. The occurrence of quaternionic structures on this level is a fairly known phenomenon in finite dimensions; see for instance the complexifications of Hermitian symmetric spaces of compact type studied in [9] and [10]. An infinite-dimensional version of this phenomenon was discussed in [29] in the special case of the restricted Grassmann manifold. The latter manifold is modeled on a Hilbert space and is endowed with a Riemannian structure which allows one to construct almost complex structures on the tangent bundle by identifying it with the cotangent bundle.

Nothing of this kind is available in the case of the $C^*$-algebraic Grassmann manifolds investigated in the present paper. Instead, we have to construct the almost complex structures in a direct manner inspired by some of the earliest insights into the geometry of the tangent bundles; see [19] and [14]. This approach leads to almost hypercomplex structures on the complexifications of the $C^*$-algebraic Grassmann manifolds provided by the tangent bundles and is related to the theory of adapted complex structures developed in finite dimensions in papers like [22,28], and [8].

## 2. Grassmann manifolds in an algebraic setting

We begin with several elementary considerations about idempotents in complex associative algebras.

*Notation* 2.1. We are going to use the following notation: $A$ is a unital associative algebra over $\mathbb{C}$ with unit $\mathbf{1}$ and set of idempotents $\mathcal{P}(A) = \{p \in A \mid p^2 = p\}$; for $p_1, p_2 \in \mathcal{P}(A)$ the notation $p_1 \sim p_2$ means that we have both $p_1 p_2 = p_2$ and $p_2 p_1 = p_1$. For each $p \in \mathcal{P}(A)$ we denote its equivalence class by

$$[p] := \{q \in \mathcal{P}(A) \mid q \sim p\}.$$

The quotient set is denoted by $\mathrm{Gr}(A) = \mathcal{P}(A)/\sim$ (the *Grassmannian* of $A$) and the quotient map by $\pi\colon\ p \mapsto [p], \mathcal{P}(A) \to \mathrm{Gr}(A)$.

The group of invertible elements of $A$ is denoted by $\mathrm{G}_A$, and it has a natural action on $\mathcal{P}(A)$ by

$$\alpha\colon\ (u, q) \mapsto uqu^{-1}, \quad \mathrm{G}_A \times \mathcal{P}(A) \to \mathcal{P}(A).$$

The corresponding isotropy group at $p \in \mathcal{P}(A)$ is

$$\big\{u \in \mathrm{G}_A \mid \alpha(u,p) = p\big\} = \mathrm{G}_A \cap \{p\}' = \mathrm{G}_{\{p\}'} =: \mathrm{G}(p)\,,$$

where we denote by $\{p\}'$ the commutant subalgebra of $p$ in $A$ (see p. 484 in [17]).

**Lemma 2.2.** *There exists a well-defined action of the group* $\mathrm{G}_A$ *upon* $\mathrm{Gr}(A)$ *like this:*

$$\beta\colon\ \big(u,[p]\big) \mapsto [upu^{-1}]\,,\quad \mathrm{G}_A \times \mathrm{Gr}(A) \to \mathrm{Gr}(A)\,,$$

*and the diagram*

$$\begin{array}{ccc} \mathrm{G}_A \times \mathcal{P}(A) & \xrightarrow{\ \alpha\ } & \mathcal{P}(A) \\ {\scriptstyle \mathrm{id}_{\mathrm{G}}\times\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathrm{G}_A \times \mathrm{Gr}(A) & \xrightarrow{\ \beta\ } & \mathrm{Gr}(A) \end{array}$$

*is commutative.*

*Proof.* See for instance the end of Section 3 in [16]. □

**Definition 2.3.** For every idempotent $p \in \mathcal{P}(A)$ we denote by $\mathrm{G}_A([p])$ the isotropy group of the action $\beta\colon \mathrm{G}_A \times \mathrm{Gr}(A) \to \mathrm{Gr}(A)$ at the point $[p] \in \mathrm{Gr}(A)$, that is, $\mathrm{G}_A([p]) = \{u \in \mathrm{G}_A \mid [upu^{-1}] = [p]\}$.

The following statement concerns the relationship between isotropy groups for the actions $\alpha$ and $\beta$ of $\mathrm{G}_A$ upon $\mathcal{P}(A)$ and $\mathrm{Gr}(A)$, respectively.

**Proposition 2.4.** *The following assertions hold.*

(i) *For every* $p \in \mathcal{P}(A)$ *we have* $\mathrm{G}_A([p]) \cap \mathrm{G}_A([\mathbf{1}-p]) = \mathrm{G}(p)$.

(ii) *If* $\mathrm{U}$ *is a subgroup of* $\mathrm{G}_A$ *and* $p \in \mathcal{P}(A)$ *with* $\mathrm{U} \cap \mathrm{G}_A([p]) = \mathrm{U} \cap \mathrm{G}_A([\mathbf{1}-p])$, *then* $\mathrm{U} \cap \mathrm{G}_A([p]) = \mathrm{U} \cap \{p\}' =: \mathrm{U}(p)$.

*Proof.* (i) We have

$$\mathrm{G}_A([p]) = \big\{u \in \mathrm{G}_A \mid [upu^{-1}] = [p]\big\}$$

and

$$\mathrm{G}_A([\mathbf{1}-p]) = \Big\{u \in \mathrm{G}_A \Big| \big[u(\mathbf{1}-p)u^{-1}\big] = [\mathbf{1}-p]\Big\}\,,$$

so that clearly $\mathrm{G}_A([p]) \cap \mathrm{G}_A([\mathbf{1}-p]) \supseteq \mathrm{G}_A \cap \{p\}'$. For the converse inclusion let $u \in \mathrm{G}_A([p]) \cap \mathrm{G}_A([\mathbf{1}-p])$ arbitrary. In particular $u \in \mathrm{G}_A([p])$, so $upu^{-1} \sim p$, which is equivalent to the fact that $(upu^{-1})p = p$ and $p(upu^{-1}) = upu^{-1}$. Consequently we have both

$$pu^{-1}p = u^{-1}p \tag{2.1}$$

and

$$pup = up\,. \tag{2.2}$$

On the other hand, since $u \in \mathrm{G}_A([\mathbf{1}-p])$ as well, it follows that $(\mathbf{1}-p)u^{-1}(\mathbf{1}-p) = u^{-1}(\mathbf{1}-p)$ and $(\mathbf{1}-p)u(\mathbf{1}-p) = u(\mathbf{1}-p)$. The latter equation is equivalent to $u - up - pu + pup = u - up$, that is, $pup = pu$. Then (2.2) implies that $up = pu$, that is, $u \in \mathrm{G}(p)$.

(ii) This follows at once from part (i). □

*Remark* 2.5. For instance, Proposition 2.4 (ii) can be applied if the algebra $A$ is equipped with an involution $a \mapsto a^*$ such that $p = p^*$, and

$$\mathrm{U} = \mathrm{U}_A := \{u \in \mathrm{G}_A \mid u^{-1} = u^*\}$$

is the corresponding unitary group. In this case, it follows by (2.1) and (2.2) that $up = pu$ whenever $u \in \mathrm{U}_A \cap \mathrm{G}_A([p])$, hence $\mathrm{U}_A \cap \mathrm{G}_A([p]) = \mathrm{U}_A \cap \mathrm{G}_A([\mathbf{1} - p]) = \mathrm{U}_A \cap \{p\}' =: \mathrm{U}_A(p)$.

For $q \in \mathcal{P}(A)$, put $\hat{q} := \mathbf{1} - q$ and $A^q := \{a \in A \mid \hat{q}aq = 0\}$. The following result is partly a counterpart, for algebras, of Proposition 2.4.

**Proposition 2.6.** *Assume that $A$ is equipped with an involution and let $p \in \mathcal{P}(A)$ such that $p = p^*$. Then the following assertions hold:*

(i) $uA^pu^{-1} = A^p$, *for every* $u \in U_A(p)$;
(ii) $A^p \cap A^{\hat{p}} = \{p\}'$;
(iii) $A^p + A^{\hat{p}} = A$;
(iv) $(A^p)^* = A^{\hat{p}}$.

*Proof.* (i) This is readily seen.

(ii) Firstly, note that, for $a \in A$, we have $\hat{p}ap = pa\hat{p}$ if and only if $ap = pa$. Moreover, if $ap = pa$ then $\hat{p}ap = ap - pap = ap - ap = 0$ and analogously $pa\hat{p} = 0$. From this, the equality of the statement follows.

(iii) For every $a \in A$ and $q \in \mathcal{P}(A)$ we have $qa \in A^q$. Hence $a = pa + \hat{p}a \in A^p + A^{\hat{p}}$, as we wanted to show.

(iv) Take $a \in A^p$. Then $pap = ap$, that is, $pa^*p = pa^*$. Hence, $\hat{p}a^*\hat{p} = (a^* - pa^*)(\mathbf{1} - p) = a^* - pa^* - a^*p + a^*p = a^* - a^*p = a^*\hat{p}$. This means that $a^* \in A^{\hat{p}}$. Conversely, if $a \in A^{\hat{p}}$ then, as above, $pa^*p = a^*p$; that is, $a = (a^*)^*$ with $a^* \in A^p$. □

## 3. Homogeneous complex structures and complexifications

**Definition 3.1.** Let $X$ be a Banach manifold. A *complexification* of $X$ is a complex Banach manifold $Y$ endowed with an anti-holomorphic involutive diffeomorphism $y \mapsto y^{-*}$ such that the fixed point submanifold $Y_0 = \{y \in Y \mid y = y^{-*}\}$ is diffeomorphic to $X$.

Assume from now on that $A$ is a unital $C^*$-algebra. Then $\mathrm{G}_A$ is a Banach–Lie group whose Lie algebra coincides with $A$. The $\mathrm{G}_A$-orbits in $\mathrm{Gr}(A)$, obtained by the action $\beta$ and equipped with the topology inherited from $\mathrm{Gr}(A)$, are holomorphic Banach manifolds diffeomorphic to $\mathrm{G}_A/G_A([p])$ (endowed with its quotient topology), see Theorem 2.2 in [17]. Also, the Grassmannian $\mathrm{Gr}(A)$ can be described as the discrete union of these $\mathrm{G}_A$-orbits, see [16] and Theorem 2.3 in [17]. Moreover, $\mathrm{U}_A$ is a Banach–Lie subgroup of $\mathrm{G}_A$ with the Lie algebra $\mathfrak{u}_A := \{a \in A \mid a^* = -a\}$. As it is well known, the complexification of $\mathfrak{u}_A$ is $A$, via the decomposition $a = \{(a - a^*)/2\} + \mathrm{i}\{(a + a^*)/2\mathrm{i}\}$, $(a \in A)$. Thus the conjugation of $A$ is given by $a \mapsto \overline{a} := \{(a - a^*)/2\} - \mathrm{i}\{(a + a^*)/2\mathrm{i}\} = -a^*$. We seek for possible topological and/or differentiable relationships between the $\mathrm{G}_A$-orbits and the $\mathrm{U}_A$-orbits $\mathrm{U}_A/\mathrm{U}_A(p)$ in $\mathrm{Gr}(A)$.

The above observations lead to the following result.

**Theorem 3.2.** *Assume that $A$ is a unital $C^*$-algebra, $p = p^* \in \mathcal{P}(A)$ and $\mathfrak{u}_A(p) := \mathfrak{u}_A \cap \{p\}'$. Let $\mathrm{Ad}_{\mathrm{U}}$ denote the adjoint representation of $\mathrm{U}_A$. Then the following assertions hold:*

(i) $\mathrm{Ad}_{\mathrm{U}}(u)A^p \subset A^p$, $(u \in \mathrm{U}_A(p))$; $A^p \cap \overline{A^p} = \mathfrak{u}_A(p) + \mathrm{i}\mathfrak{u}_A(p)$; $A^p + \overline{A^p} = A$.

(ii) *The manifold $\mathrm{U}_A/\mathrm{U}_A(p)$ has a $\mathrm{U}_A$-invariant complex structure and is locally biholomorphic to $\mathrm{G}_A/\mathrm{G}_A([p])$.*

(iii) *The manifold $\mathrm{G}_A/\mathrm{G}_A(p)$ endowed with the involutive diffeomorphism given by $a\mathrm{G}_A(p) \mapsto (a^*)^{-1}\mathrm{G}_A(p)$ is a complexification of $\mathrm{U}_A/\mathrm{U}_A(p)$.*

*Proof.* It is clear that $\mathfrak{u}_A(p) + \mathrm{i}\mathfrak{u}_A(p) = \{p\}'$. Now the first part of the statement follows by Proposition 2.6.

Also, there is a natural identification between $\mathfrak{u}_A/\mathfrak{u}_A(p)$ and the tangent space $T_{[p]}(\mathrm{U}_A/\mathrm{U}_A(p))$. Then assertion (ii) follows from Theorem 6.1 in [4]. In fact, by assertion (i) it is readily seen that $\mathfrak{u}_A/\mathfrak{u}_A(p) \simeq A/A^p$ whence we obtain that $\mathrm{U}_A/\mathrm{U}_A(p)$ and $\mathrm{G}_A/\mathrm{G}_A([p])$ are locally diffeomorphic, and so $\mathrm{U}_A/\mathrm{U}_A(p)$ inherits the complex structure induced by $\mathrm{G}(A)/\mathrm{G}_A([p])$.

For assertion (iii), it is easy to see that the mapping $a\mathrm{G}_A(p) \mapsto (a^*)^{-1}\mathrm{G}_A(p)$ is an anti-holomorphic diffeomorphism (which in terms of orbits corresponds to the mapping $apa^{-1} \mapsto (a^*)^{-1}pa^*$). Then $a\mathrm{G}_A(p) = (a^*)^{-1}\mathrm{G}_A(p)$ if and only if $(a^*a)\mathrm{G}_A(p) = \mathrm{G}_A(p)$, that is, $(a^*a)p = p(a^*a)$. Using the functional calculus for $C^*$-algebras, we can pick $b := \sqrt{a^*a}$ in $A$ and obtain $bp = pb$. Since $a^*a = b^2 = b^*b$ we have $(ab^{-1})^* = (b^{-1})^*a^* = (b^*)^{-1}a^* = ba^{-1} = (ab^{-1})^{-1}$ and therefore $u := ab^{-1} \in \mathrm{U}_A$. Finally, $a\mathrm{G}_A(p) = ub\mathrm{G}_A(p) = u\mathrm{G}_A(p) \equiv u\mathrm{U}_A(p) \in \mathrm{U}_A/\mathrm{U}_A(p)$. □

*Remark* 3.3. Since $\mathrm{G}_A(p) \subset \mathrm{G}_A([p])$, there exists the canonical projection mapping $\mathrm{G}_A/\mathrm{G}_A(p) \to \mathrm{G}_A/\mathrm{G}_A([p])$. It is clear that its restriction to $\mathrm{U}_A/\mathrm{U}_A(p)$ becomes the identity map $\mathrm{U}_A/\mathrm{U}_A(p) \to \mathrm{U}_A/\mathrm{U}_A([p])$.

According to Proposition 2.4 (i), idempotents of the form $apa^{-1} \equiv a\mathrm{G}_A(p)$ with $a \in \mathrm{G}_A$, can be alternatively represented as pairs $(a[p]a^{-1}, (a^*)^{-1}[p]a^*)$ so that the "orbit" $\mathrm{G}_A/\mathrm{G}_A(p)$ can be identified with a subset of the Cartesian product $\mathrm{G}_A([p]) \times \mathrm{G}_A([p])$. In this perspective, the preceding projection and diffeomorphism are given, respectively, by

$$\left(a[p]a^{-1}, (a^*)^{-1}[p]a^*\right) \mapsto a[p]a^{-1} \equiv \left(a[p]a^{-1}, a[p]a^{-1}\right), \ \mathrm{G}_A/\mathrm{G}_A(p) \to \mathrm{G}_A/\mathrm{G}_A([p])$$

$$\Big(\text{so } \left(u[p]u^{-1}, u[p]u^{-1}\right) \mapsto upu^{-1} \equiv (u[p]u^{-1}, u[p]u^{-1}), \text{ when } u \in \mathrm{U}_A\Big) \quad \text{and}$$

$$\left(a[p]a^{-1}, (a^*)^{-1}[p]a^*\right) \mapsto \left((a^*)^{-1}[p]a^*, a[p]a^{-1}\right),$$

for every $a \in \mathrm{G}_A$.

*Remark* 3.4. Theorem 3.2 relates to the setting of [5]. Namely, assume that $B$ is a $C^*$-subalgebra of $A$, with $\mathbf{1} \in B \subseteq A$, for which there exist a conditional expectation $E\colon A \to B$ and a state $\varphi\colon A \to \mathbb{C}$ such that $\varphi \circ E = \varphi$. For $X \in \{A, B\}$, we denote by $\varphi_X$ the state $\varphi$ restricted to $X$. Let $\mathcal{H}_X$ be the Hilbert space, and let $\pi_X\colon X \to \mathcal{B}(\mathcal{H}_X)$ be the corresponding cyclic representation obtained by the GNS construction applied to the state $\varphi_X\colon X \to \mathbb{C}$. Thus, $\mathcal{H}_X$ is the completion of $X/N_X$ with respect to the norm $\|y + N_X\|_\varphi := \varphi(y^*y)$, where $N_X := \{y \in X \mid \varphi(y^*y) = 0\}$. The representation $\pi_X$ is then defined as the extension to $\mathcal{H}_X$ of the left multiplication of $X$ on $X/N_X$. Let $P$ denote the orthogonal projection $P\colon \mathcal{H}_A \to \mathcal{H}_B$.

An equivalence relation can be defined in $\mathrm{G}_A \times \mathcal{H}_B$ by $(g_1, h_1) \sim (g_2, h_2)$ (with $g_1, g_2 \in \mathrm{G}_A$, $h_1, h_2 \in \mathcal{H}_B$) if and only if there exists $w \in \mathrm{G}_B$ such that $g_2 = g_1 w$ and $h_2 = \pi_B(w^{-1})h_1$. The corresponding quotient space will be denoted by $\mathrm{G}_A \times_{\mathrm{G}_B} \mathcal{H}_B$, and the equivalence class in $\mathrm{G}_A \times_{\mathrm{G}_B} \mathcal{H}_B$ of any $(g, h) \in \mathrm{G}_A \times \mathcal{H}_B$ will be denoted by $[(g, h)]$. Define $\mathrm{U}_A \times_{\mathrm{U}_B} \mathcal{H}_B$ in an analogous fashion. Then the mappings

$$\Pi_G\colon [(g,h)] \mapsto g\mathrm{G}_B\,, \quad \mathrm{G}_A \times_{\mathrm{G}_B} \mathcal{H}_B \to \mathrm{G}_A/\mathrm{G}_B$$

and

$$\Pi_U\colon [(u,h)] \mapsto u\mathrm{U}_B\,, \quad \mathrm{U}_A \times_{\mathrm{U}_B} \mathcal{H}_B \to \mathrm{U}_A/\mathrm{U}_B$$

are vector bundles, $\Pi_U$ being Hermitian, in fact. Moreover, $\Pi_U$ admits a reproducing kernel $K$ with the associated Hilbert space $\mathcal{H}^K$, formed by continuous sections of $\Pi_U$, such that the restriction of the GNS representation $\pi_A$ to $\mathrm{U}_A$ can be realized on $\mathcal{H}^K$, see [5].

Let us apply the above theory to the case when, for a given unital $C^*$-algebra $A$, we take $B := \{p\}'$ in $A$, where $p = p^* \in \mathcal{P}(A)$. Then $E_p\colon a \mapsto pap + \hat{p}a\hat{p}$, $A \to B$ is a conditional expectation from $A$ onto $B$. Let $\mathcal{H}$ be a Hilbert space such that $A \hookrightarrow \mathcal{B}(\mathcal{H})$. Pick $x_0 \in p\mathcal{H}$ such that $\|x_0\| = 1$. Then $\varphi_0\colon A \to \mathbb{C}$, given by $\varphi_0(a) := (ax_0 \mid x_0)_{\mathcal{H}}$ for all $a \in A$, is a state of $A$ such that $\varphi_0 \circ E_p = \varphi_0$. The GNS representation of $A$ associated with $\varphi_0$ is as follows. Set $(a_1 \mid a_2)_0 := \varphi_0(a_2^* a_1) = (a_2^* a_1 x_0 \mid x_0)_{\mathcal{H}} = (a_1 x_0 \mid a_2 x_0)_{\mathcal{H}}$ for every $a_1, a_2 \in A$. So $\varphi_0(a^*a) = \|a(x_0)\|^2$ for all $a \in A$, whence the null space of $(\cdot \mid \cdot)_0$ is $N_0 := \{a \in A : (a \mid a)_0 = 0\} = \{a \in A : a(x_0) = 0\}$. The norm $\|\cdot\|_0$ induced by $(\cdot \mid \cdot)_0$ on $A/N_0$ is given by $\|h\|_0 \equiv \|a + N_0\|_0 := \varphi_0(a^*a)^{1/2} = \|a(x_0)\|_{\mathcal{H}} = \|h\|_{\mathcal{H}}$ for every $h \in A(x_0) \subset \mathcal{H}$, where $a(x_0) = h \leftrightarrow a + N_0$. Hence $\mathcal{H}_A$ is a closed subspace of $\mathcal{H}$ such that $a\mathcal{H}_A \subset \mathcal{H}_A$ for every $a \in A$. Note that $\mathcal{H}_A$ coincides with $\mathcal{H}$ provided that we can choose $x_0$ in $\mathcal{H}$ such that $A(x_0)$ is dense in $\mathcal{H}$. This will be of interest in Remark 4.7 below.

Analogously, we can consider the restriction of $(\cdot \mid \cdot)_0$ to $B$ and proceed in the same way as above. Thus we obtain that the corresponding null space is $B \cap N_0$, that the norm in $B/(B \cap N_0)$ is that one of $p\mathcal{H}$ (so that one of $\mathcal{H}$), and that $\mathcal{H}_B$ is a closed subspace of $p\mathcal{H}$ such that $b\mathcal{H}_B \subset \mathcal{H}_B$ for every $b \in B$. Also, $\mathcal{H}_B = p\mathcal{H}$ if $x_0$ can be chosen in $p\mathcal{H}$ and such that $B(x_0)$ is dense in $p\mathcal{H}$.

The representation $\pi_A\colon a \mapsto \pi(a)$, $A \to \mathcal{B}(\mathcal{H}_A)$ is the extension to $\mathcal{H}_A$ of the left multiplication $\pi_A(a)\colon a' + N_0 \mapsto (aa') + N_0$, $A/N_0 \to A/N_0$. Thus it satisfies $\pi_A(a' + N_0) = (aa') + N_0 \equiv a(a'x_0) = a(h)$, if $(a' + N_0) \leftrightarrow a(x_0) = h$. In other words, $\pi_A$ is the inclusion operator (by restriction) from $A$ into $\mathcal{B}(\mathcal{H}_A)$. Also, $\pi_B$ is in turn the inclusion operator from $B$ into $\mathcal{B}(\mathcal{H}_B)$.

Since $E_p(N_0) \subseteq N_0$, the conditional expectation $E_p$ induces a well-defined projection $P\colon A/N_0 \to B/(N_0 \cap B)$. On the other hand, $E_p(a^*a) - E_p(a)^*E_p(a) = pa^*\hat{p}ap + \hat{p}a^*pap \geq 0$ since $p, \hat{p} \geq 0$. Hence $P$ extends once again as a bounded projection $P\colon \mathcal{H}_A \to \mathcal{H}_B$. Indeed, if $h = a(x_0)$ with $a \in A$, we have

$$P(h) \equiv P(a + N_0) = E(a) + (B \cap N_0) = E(a)(x_0) = (pa)(x_0) = p(h)\,,$$

that is, $P = p_{|\mathcal{H}_A}$.

In the above setting, note that $\mathrm{U}_B = \mathrm{U}_A(p)$. Let $\Gamma(\mathrm{U}_A/\mathrm{U}_A(p), \mathrm{U}_A \times_{\mathrm{U}_A(p)} \mathcal{H}_B)$ be the section space of the bundle $\Pi_U$. The reproducing kernel associated with $\Pi_U$ is given by $K_p(u_1\mathrm{U}_A(p), u_2\mathrm{U}_A(p))[(u_2, f_2)] := [(u_1, pu_1^{-1}u_2f_2)]$ for $u_1, u_2 \in \mathrm{U}_A$ and $f_2 \in \mathcal{H}_B$. The kernel $K_p$ generates a Hilbert subspace $\mathcal{H}^{K_p}$ of sections in $\Gamma(\mathrm{U}_A/\mathrm{U}_A(p), \mathrm{U}_A \times_{\mathrm{U}_A(p)} \mathcal{H}_B)$. Let $\gamma_p\colon \mathcal{H}_A \to \Gamma(\mathrm{U}_A/\mathrm{U}_A(p), \mathrm{U}_A \times_{\mathrm{U}_A(p)} \mathcal{H}_B)$ be the mapping defined by $\gamma_p(h)(u\mathrm{U}_A(p)) := [(u, pu^{-1}h)]$ for every $h \in \mathcal{H}_A$ and $u \in \mathrm{U}_A$. Then $\gamma_p$ is injective and it intertwines the representation $\pi_A$ of $\mathrm{U}_A$ on $\mathcal{H}_A$ and the natural action of $\mathrm{U}_A$ on $\mathcal{H}^{K_p}$; that is, the diagram

$$\begin{array}{ccc} \mathcal{H}_A & \xrightarrow{\;u\;} & \mathcal{H}_A \\ \gamma_p\Big\downarrow & & \Big\downarrow\gamma_p \\ \mathcal{H}^{K_p} & \xrightarrow{\;\mu(u)\;} & \mathcal{H}^{K_p}, \end{array} \tag{3.1}$$

is commutative for all $u \in \mathrm{U}_A$, where $\mu(u)F := uF(u^{-1}\,\cdot\,)$ for every cross-section $F \in \Gamma(\mathrm{U}_A/\mathrm{U}_A(p), \mathrm{U}_A \times_{\mathrm{U}_A(p)} \mathcal{H}_B)$. In fact

$$\gamma(uh)\big(v\mathrm{U}_A(p)\big) := [(v, pv^{-1}uh)] = u[(u^{-1}v, pv^{-1}uh)] =: u\Big\{\gamma(h)\big(u^{-1}v\mathrm{U}_A(p)\big)\Big\}$$

for all $u, v \in \mathrm{U}_A$. See Theorem 5.4 of [5] for details in the general case. We next show that $\mathcal{H}^{K_p}$ in fact consists of holomorphic sections.

**Proposition 3.5.** *Let $A$ be a unital $C^*$-algebra, $p = p^* \in \mathcal{P}(A)$, and $B := \{p\}'$. Then the homogeneous Hermitian vector bundle $\Pi_U\colon \mathrm{U}_A \times_{\mathrm{U}_A(p)} \mathcal{H}_B \to \mathrm{U}_A/\mathrm{U}_A(p)$ is holomorphic, and the image of $\gamma_p$ consists of holomorphic sections. Thus $\mathcal{H}^{K_p}$ is a Hilbert space of holomorphic sections of $\Pi_U$.*

*Proof.* Let $u_0 \in \mathrm{U}_A$. Then $\Omega_G := \{u_0 g \mid g \in \mathrm{G}_A,\ \|\mathbf{1} - g^{-1}\| < 1\}$ is open in $\mathrm{G}_A$ and contains $u_0$, and similarly with $\Omega_U := \Omega_G \cap \mathrm{U}_A$ in $\mathrm{U}_A$.

It is readily seen that the mapping

$$\psi_0\colon [(u, f)] \mapsto \big(u\mathrm{U}_A(p), E_p(u^{-1}u_0^{-1})^{-1}f\big)\,, \quad \Pi_U^{-1}(\Omega_U) \to \Omega_U \times \mathcal{H}_B$$

is a diffeomorphism, with inverse map $(u\mathrm{U}_A(p), h) \mapsto [(uE_p(u^{-1}), h)]$ (this shows the local triviality of $\Pi_U$). Thus every point in the manifold $\mathrm{U}_A \times_{\mathrm{U}_A(p)} \mathcal{H}_B$ has an open neighborhood which is diffeomorphic to the manifold product $W \times \mathcal{H}_B$, where $W$ is an open subset of $\mathrm{U}_A/\mathrm{U}_A(p)$. By Theorem 3.2, $\mathrm{U}_A/\mathrm{U}_A(p)$ is a complex homogeneous manifold and therefore the manifold $\mathrm{U}_A \times_{\mathrm{U}_A(p)} \mathcal{H}_B$ is locally complex, i.e., holomorphic. Also the bundle map $\Pi_U$ is holomorphic.

On the other hand, for fixed $h \in \mathcal{H}_A$, the mapping

$$\sigma_0 \colon g\mathrm{G}_A([p]) \mapsto E_p(g^{-1}u_0^{-1})^{-1}pg^{-1}h\,, \quad \Omega_G \to \mathcal{H}_B$$

is holomorphic on $\Omega_G$, so it defines a holomorphic function $\tilde{\sigma}_0 \colon \Omega_G\mathrm{G}_A([p]) \to \mathcal{H}_B$. By Theorem 3.2 the injection $j \colon \mathrm{U}_A/\mathrm{U}_A(p) \hookrightarrow \mathrm{G}_A/\mathrm{G}_A([p])$ is holomorphic, and so the restriction map $r := \tilde{\sigma}_0 \circ j$ is holomorphic around $u_0\mathrm{U}_A(p)$. Since $\gamma(h) = \psi_0^{-1} \circ (I_{\Omega_U} \times r)$ around $u_0\mathrm{U}_A(p)$, it follows that $\gamma(h)$ is (locally) holomorphic.

Finally, by applying Theorem 4.2 in [5] we obtain that $K_p$ is holomorphic. $\square$

The starting point for the holomorphic picture given in Proposition 3.5 has been the fact that $\mathrm{U}_A/\mathrm{U}_A(p)$ enjoys a holomorphic structure induced by the one of $\mathrm{G}_A/\mathrm{G}([p])$, see Theorem 3.2. Such a picture can be made even more explicit if we have a global diffeomorphism $\mathrm{U}_A/\mathrm{U}_A(p) \simeq \mathrm{G}_A/\mathrm{G}_A([p])$. The prototypical example is to be found when $A$ is the algebra of bounded operators on a complex Hilbert space. We examine this case more closely in the next section.

## 4. Tautological universal vector bundles

Let us recall the specific definition and some properties of the Grassmannian manifold associated with a complex Hilbert space.

*Notation* 4.1. We shall use the standard notation $\mathcal{B}(\mathcal{H})$ for the $C^*$-algebra of bounded linear operators on the complex Hilbert space $\mathcal{H}$ with the involution $T \mapsto T^*$. Let $\mathrm{GL}(\mathcal{H})$ be the Banach–Lie group of all invertible elements of $\mathcal{B}(\mathcal{H})$, and $\mathrm{U}(\mathcal{H})$ its Banach–Lie subgroup of all unitary operators on $\mathcal{H}$. Also,

- $\mathrm{Gr}(\mathcal{H}) := \{\mathcal{S} \mid \mathcal{S} \text{ closed linear subspace of } \mathcal{H}\}$;
- $\mathcal{T}(\mathcal{H}) := \{(\mathcal{S}, x) \in \mathrm{Gr}(\mathcal{H}) \times \mathcal{H} \mid x \in \mathcal{S}\} \subseteq \mathrm{Gr}(\mathcal{H}) \times \mathcal{H}$;
- $\Pi_{\mathcal{H}} \colon \ (\mathcal{S}, x) \mapsto \mathcal{S}$, $\mathcal{T}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H})$;
- for every $\mathcal{S} \in \mathrm{Gr}(\mathcal{H})$ we denote by $p_{\mathcal{S}} \colon \mathcal{H} \to \mathcal{S}$ the corresponding orthogonal projection.

*Remark* 4.2. The objects introduced in Notation 4.1 have the following well known properties:

(a) Both $\mathrm{Gr}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ have structures of complex Banach manifolds, and $\mathrm{Gr}(\mathcal{H})$ carries a natural (non-transitive) action of $\mathrm{U}(\mathcal{H})$. (See Examples 3.11 and 6.20 in [30], or Chapter 2 in [15].)

(b) For every $\mathcal{S}_0 \in \mathrm{Gr}(\mathcal{H})$ the corresponding connected component of $\mathrm{Gr}(\mathcal{H})$ is the $\mathrm{GL}(\mathcal{H})$-orbit and is also the $\mathrm{U}(\mathcal{H})$-orbit of $\mathcal{S}_0$, that is,

$$\begin{aligned}\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}) &= \bigl\{g\mathcal{S}_0 \mid g \in \mathrm{GL}(\mathcal{H})\bigr\} = \bigl\{u\mathcal{S}_0 \mid u \in \mathrm{U}(\mathcal{H})\bigr\} \\ &= \bigl\{\mathcal{S} \in \mathrm{Gr}(\mathcal{H}) \mid \dim \mathcal{S} = \dim \mathcal{S}_0 \text{ and } \dim \mathcal{S}^\perp = \dim \mathcal{S}_0^\perp\bigr\} \\ &\simeq \mathrm{U}(\mathcal{H})/\bigl(\mathrm{U}(\mathcal{S}_0) \times \mathrm{U}(\mathcal{S}_0^\perp)\bigr)\,.\end{aligned}$$

(See Proposition 23.1 in [30] or Lemma 4.3 below, alternatively.)

(c) The mapping $\Pi_{\mathcal{H}}\colon \mathcal{T}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H})$ is a holomorphic Hermitian vector bundle, and we call it the *universal* (*tautological*) *vector bundle* associated with the Hilbert space $\mathcal{H}$. Set $\mathcal{T}_{\mathcal{S}_0}(\mathcal{H}) := \{(\mathcal{S}, x) \in \mathcal{T}(\mathcal{H}) \mid \mathcal{S} \in \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})\}$. The vector bundle $\mathcal{T}_{\mathcal{S}_0}(\mathcal{H}) \to \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ obtained by restriction of $\Pi_{\mathcal{H}}$ to $\mathcal{T}_{\mathcal{S}_0}(\mathcal{H})$ will be called here the universal vector bundle *at* $\mathcal{S}_0$. It is also Hermitian and holomorphic.

Property (b) in Remark 4.2 means that $U_A/U_A(p_{\mathcal{S}_0}) \simeq G_A/G_A([p_{\mathcal{S}_0}])$ for $A = \mathcal{B}(\mathcal{H})$. For the sake of clarification we now connect Notation 2.1 and Notation 4.1 in more detail. For $A = \mathcal{B}(\mathcal{H})$ we have $\mathrm{Gr}(A) = \mathrm{Gr}(\mathcal{H})$, and with this identification the action $\beta$ of Lemma 2.2 corresponds to the natural action (so-called *collineation* action) of the group of invertible operators on $\mathcal{H}$ upon the set of all closed linear subspaces of $\mathcal{H}$. The following lemma gives us the collineation orbits of $\mathrm{Gr}(\mathcal{H})$ in terms of orbits of projections, and serves in particular to explain the property stated in Remark 4.2(b).

For short, denote $\mathcal{G} = \mathrm{GL}(\mathcal{H})$ and $\mathcal{U} = \mathrm{U}(\mathcal{H})$.

**Lemma 4.3.** *Let $\mathcal{S}_0 \in \mathrm{Gr}(\mathcal{H})$. Then the following assertions hold.*

(i) $\mathcal{G}([p_{\mathcal{S}_0}]) = \{g \in \mathcal{G} \mid g\mathcal{S}_0 = \mathcal{S}_0\}$ *and* $\mathcal{U}([p_{\mathcal{S}_0}]) = \mathcal{U}(p_{\mathcal{S}_0}) = \{u \in \mathcal{U} \mid u\mathcal{S}_0 = \mathcal{S}_0\}$.

(ii) *For every $g \in \mathcal{G}$ and $\mathcal{S} = g\mathcal{S}_0$ we have $\mathcal{S}^\perp = (g^*)^{-1}(\mathcal{S}_0^\perp)$.*

(iii) *We have*

$$\begin{aligned}\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}) &= \{g\mathcal{S}_0 \mid g \in \mathcal{G}\} \simeq \big\{[gp_{\mathcal{S}_0}g^{-1}] \mid g \in \mathcal{G}\big\}\\ &= \{u\mathcal{S}_0 \mid u \in \mathcal{U}\} \simeq \{up_{\mathcal{S}_0}u^{-1} \mid u \in \mathcal{U}\}\,.\end{aligned}$$

(iv) *We have*

$$\mathcal{U}/\mathcal{U}(p_{\mathcal{S}_0}) \simeq \mathcal{G}/\mathcal{G}([p_{\mathcal{S}_0}]) \simeq \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})\,,$$

*where the symbol "$\simeq$" means diffeomorphism between the respective differentiable structures, and that the differentiable structure of the quotient spaces is the one associated with the corresponding quotient topologies.*

(v) *We have $\mathcal{G}/\mathcal{G}(p_{\mathcal{S}_0}) \simeq \{(a\mathcal{S}_0, (a^*)^{-1}\mathcal{S}_0) \mid a \in \mathcal{G}\}$, and the map*

$$\big(a\mathcal{S}_0, (a^*)^{-1}\mathcal{S}_0\big) \mapsto \big((a^*)^{-1}\mathcal{S}_0, a\mathcal{S}_0\big)$$

*is an involutive diffeomorphism on $\mathcal{G}/\mathcal{G}(p_{\mathcal{S}_0})$ whose fixed-point set is*

$$\big\{(u\mathcal{S}_0, u\mathcal{S}_0) \mid u \in \mathcal{U}\big\} \equiv \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})\,.$$

*Proof.* (i) As shown in Proposition 2.4, an element $g$ of $\mathcal{G}$ belongs to $\mathcal{G}([p_{\mathcal{S}_0}])$ if and only if $p_{\mathcal{S}_0}g^{-1}p_{\mathcal{S}_0} = g^{-1}p_{\mathcal{S}_0}$ and $p_{\mathcal{S}_0}g\,p_{\mathcal{S}_0} = g\,p_{\mathcal{S}_0}$. From this, it follows easily that $g(\mathcal{S}_0) \subset \mathcal{S}_0$ and $g^{-1}(\mathcal{S}_0) \subset \mathcal{S}_0$, that is, $g(\mathcal{S}_0) = \mathcal{S}_0$. Conversely, if $g(\mathcal{S}_0) \subset \mathcal{S}_0$ then $(g\,p_{\mathcal{S}_0})(\mathcal{H}) \subset p_{\mathcal{S}_0}(\mathcal{H})$ whence $p_{\mathcal{S}_0}g\,p_{\mathcal{S}_0} = g\,p_{\mathcal{S}_0}$; similarly, $g^{-1}(\mathcal{S}_0) \subset \mathcal{S}_0$ implies that $p_{\mathcal{S}_0}g^{-1}p_{\mathcal{S}_0} = g^{-1}p_{\mathcal{S}_0}$. In conclusion, $\mathcal{G}([p_{\mathcal{S}_0}]) = \{g \in \mathcal{G} \mid g\mathcal{S}_0 = \mathcal{S}_0\}$.

Now, the above equality and Remark 2.5 imply that $\mathcal{U}([p_{\mathcal{S}_0}]) = \mathcal{U}(p_{\mathcal{S}_0}) = \{u \in \mathcal{U} \mid u\mathcal{S}_0 = \mathcal{S}_0\}$.

(ii) Let $x \in \mathcal{S}_0^\perp$, $y \in \mathcal{S}$. Then

$$\big((g^*)^{-1}(x) \mid y\big) = \big((g^{-1})^*(x) \mid y\big) = \big(x \mid g^{-1}(y)\big) = 0\,,$$

so $(g^*)^{-1}(\mathcal{S}_0^\perp) \subset \mathcal{S}^\perp$. Take now $y \in \mathcal{S}^\perp, x = g^*(y)$ and $z \in \mathcal{S}_0$. Then $(x \mid z) = (g^*(y) \mid z) = (y \mid g(z)) = 0$, whence $x \in \mathcal{S}_0^\perp$ and therefore $y = (g^*)^{-1}(g^*y) = (g^*)^{-1}(x) \in (g^*)^{-1}(\mathcal{S}_0^\perp)$. In conclusion, $\mathcal{S}^\perp = (g^*)^{-1}(\mathcal{S}_0^\perp)$.

(iii) By (ii), we have $u(\mathcal{S}_0^\perp) = u(\mathcal{S}_0)^\perp$ for $u \in \mathcal{U}$. Thus $\mathcal{S} = u(\mathcal{S}_0)$ if and only if $\dim \mathcal{S} = \dim \mathcal{S}_0$ and $\dim \mathcal{S}^\perp = \dim \mathcal{S}_0^\perp$. Also, if $\mathcal{S} = u(\mathcal{S}_0)$ and $\mathcal{S}^\perp = u(\mathcal{S}_0^\perp)$, then $up_{\mathcal{S}_0} = p_{\mathcal{S}}u$, that is, $p_{\mathcal{S}} = up_{\mathcal{S}_0}u^{-1}$. Hence

$$\begin{aligned}\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}) &= \{u\mathcal{S}_0 \mid u \in \mathcal{U}\}\\ &= \left\{\mathcal{S} \in \mathrm{Gr}(\mathcal{H}) \mid \dim \mathcal{S} = \dim \mathcal{S}_0 \text{ and } \dim \mathcal{S}^\perp = \dim \mathcal{S}_0^\perp\right\}\\ &\simeq \{up_{\mathcal{S}_0}u^{-1} \mid u \in \mathcal{U}\}\,.\end{aligned}$$

Suppose now that $\mathcal{S} = g\mathcal{S}_0$ with $g \in \mathcal{G}$. Then $\dim \mathcal{S} = \dim \mathcal{S}_0$. By (ii) again, $\mathcal{S}^\perp = (g^*)^{-1}(\mathcal{S}_0^\perp)$ and so $\dim \mathcal{S}^\perp = \dim \mathcal{S}_0^\perp$. Hence $\mathcal{S} \in \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$. Finally, the bijective correspondence between $g\mathcal{S}_0$ and $g[p_{\mathcal{S}_0}]g^{-1}$ is straightforward.

(iv) This is clearly a consequence of parts (iii) and (i) from above, and Theorem 2.2 in [17].

(v) For every $a \in \mathcal{G}$, the pairs $(a\mathcal{S}_0, (a^*)^{-1}\mathcal{S}_0)$ and $(a[p]a^{-1}, (a^*)^{-1}[p]a^*)$ are in a one-to-one correspondence, by part (iii) from above. Hence, this part (v) is a consequence of Remark 3.3. □

Parts (iv) and (v) of Lemma 4.3 tell us that the Grassmannian orbit $\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ is a complex manifold which in turn admits a complexification, namely the orbit $\mathcal{G}/\mathcal{G}(p_{\mathcal{S}_0})$.

*Remark* 4.4. As said in Remark 4.2 (b), every $\mathrm{GL}(\mathcal{H})$-orbit (so every $\mathrm{U}(\mathcal{H})$-orbit) is a connected component of $\mathrm{Gr}(\mathcal{H})$. Let us briefly discuss the connected components of $\mathrm{Gr}(A)$ when $A$ is an arbitrary unital $C^*$-algebra. Every element $g \in \mathrm{G}_A$ has a unique polar decomposition $g = ua$ with $u \in \mathrm{U}_A$ and $0 \leq a \in \mathrm{G}_A$, hence there exists a continuous path $t \mapsto u \cdot ((1-t)\mathbf{1} + ta)$ in $\mathrm{G}_A$ that connects $u = u \cdot \mathbf{1}$ to $g = u \cdot a$. Thus every connected component of the $\mathrm{G}_A$-orbit of $[p] \in \mathrm{Gr}(A)$ contains at least one connected component of the $\mathrm{U}_A$-orbit of $[p] \in \mathrm{Gr}(A)$ for any idempotent $p \in \mathcal{P}(A)$. (Loosely speaking, the $\mathrm{U}_A$-orbit of $[p]$ has more connected components than the $\mathrm{G}_A$-orbit of $[p]$.) Example 7.13 in [25] shows that the $C^*$-algebra $A$ of the continuous functions $S^3 \to M_2(\mathbb{C})$ has the property that there indeed exist $\mathrm{G}_A$-orbits of elements $[p] \in \mathcal{P}(A)$ which are nonconnected.

If the unitary group $\mathrm{U}_A$ is connected (so that the invertible group $\mathrm{G}_A$ is connected), then all the $\mathrm{U}_A$-orbits and the $\mathrm{G}_A$-orbits in $\mathrm{Gr}(A)$ are connected since continuous images of connected sets are always connected. On the other hand, as said formerly, the Grassmannian $\mathrm{Gr}(A)$ is the discrete union of these $\mathrm{G}_A$-orbits. Thus if the unitary group $\mathrm{U}_A$ is connected, then the connected components of $\mathrm{Gr}(A)$ are precisely the $\mathrm{G}_A$-orbits in $\mathrm{Gr}(A)$. One important case of connected unitary group $\mathrm{U}_A$ is when $A$ is a $W^*$-algebra (since every $u \in \mathrm{U}_A$ can be written as $u = \exp(\mathrm{i}a)$ for some $a = a^* \in A$ by the Borel functional calculus, hence the continuous path $t \mapsto \exp(\mathrm{i}ta)$ connects $\mathbf{1}$ to $u$ in $\mathrm{U}_A$). For $W^*$-algebras such that $\mathrm{Gr}(A)$ is the discrete union of $\mathrm{U}_A$-orbits, it is then clear that the $\mathrm{G}_A$-orbits and the $\mathrm{U}_A$-orbits coincide. This is the case if $A$ is the algebra of bounded operators on a complex Hilbert space, as we have seen before.

The universal bundle $\mathcal{T}_{\mathcal{S}_0}(\mathcal{H}) \to \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ can be expressed as a vector bundle obtained from the so-called (principal) *Stiefel bundle* associated to $p_{\mathcal{S}_0} \leftrightarrow \mathcal{S}_0$, see [17]. A similar result holds, by replacing the Stiefel bundle with certain, suitable, of its sub-bundles. To see this, let us now introduce several mappings.

Put $p := p_{\mathcal{S}_0}$. We consider $\mathcal{G} \times_{\mathcal{G}([p])} \mathcal{S}_0$ and $\mathcal{U} \times_{\mathcal{U}(p)} \mathcal{S}_0$ as in Remark 3.4. Note that $g_1\mathcal{S}_0 = g_2\mathcal{S}_0$ and $g_1(h_1) = g_2(h_2)$ $(g_1, g_2 \in \mathcal{G},\ h_1, h_2 \in \mathcal{S}_0)$ if and only if $(g_1, h_1) \sim (g_2, h_2)$, via $w = g_1^{-1} g_2 \in \mathcal{G}([p])$, in $\mathcal{G} \times \mathcal{S}_0$. Hence, the mapping $\upsilon_{\mathcal{G}} \colon \mathcal{G} \times \mathcal{S}_0 \to \mathcal{T}_{\mathcal{S}_0}(\mathcal{H})$ defined by $\upsilon_{\mathcal{G}}((g,h)) = (g\mathcal{S}_0, g(h))$ for $(g,h) \in \mathcal{G} \times \mathcal{S}_0$, induces the usual quotient map $\tilde{\upsilon}_{\mathcal{G}} \colon \mathcal{G} \times_{\mathcal{G}([p])} \mathcal{S}_0 \to \mathcal{T}_{\mathcal{S}_0}(\mathcal{H})$. We denote by $\upsilon_{\mathcal{U}}$ the restriction of $\upsilon_{\mathcal{G}}$ on $\mathcal{G} \times \mathcal{S}_0$. As above, the quotient mapping $\tilde{\upsilon}_{\mathcal{U}} \colon \mathcal{U} \times_{\mathcal{U}(p)} \mathcal{S}_0 \to \mathcal{T}_{\mathcal{S}_0}(\mathcal{H})$ is well defined.

Since $\mathcal{U}(p) = \mathcal{U} \cap \mathcal{G}([p])$, the inclusion mapping $j \colon \mathcal{U} \times_{\mathcal{U}(p)} \mathcal{S}_0 \to \mathcal{G} \times_{\mathcal{G}([p])} \mathcal{S}_0$ is well defined. Note that $j = (\tilde{\upsilon}_{\mathcal{G}})^{-1} \circ \tilde{\upsilon}_{\mathcal{U}}$.

Finally, let $P_{\mathcal{G}} \colon \mathcal{G} \times_{\mathcal{G}([p])} \mathcal{S}_0 \to \mathcal{G}/\mathcal{G}([p])$ and $P_{\mathcal{U}} \colon \mathcal{U} \times_{\mathcal{U}(p)} \mathcal{S}_0 \to \mathcal{U}/\mathcal{U}(p)$ denote the vector bundles built in the standard way from the Stiefel sub-bundles

$$g \mapsto g\mathcal{G}([p]) \simeq g(\mathcal{S}_0)\,, \quad \mathcal{G} \to \mathcal{G}/\mathcal{G}([p]) \simeq \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$$

and

$$u \mapsto u\mathcal{U}(p) \simeq u(\mathcal{S}_0)\,, \quad \mathcal{U} \to \mathcal{U}/\mathcal{U}(p) \simeq \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$$

respectively.

**Proposition 4.5.** *The following diagram is commutative in both sides, and the horizontal arrows are biholomorphic diffeomorphisms between the corresponding holomorphic structures*

$$\begin{array}{ccccc}
\mathcal{T}_{\mathcal{S}_0}(\mathcal{H}) & \xrightarrow{(\tilde{\upsilon}_{\mathcal{U}})^{-1}} & \mathcal{U} \times_{\mathcal{U}(p)} \mathcal{S}_0 & \xrightarrow{\ j\ } & \mathcal{G} \times_{\mathcal{G}([p])} \mathcal{S}_0 \\
\Pi_{\mathcal{H}} \downarrow & & \downarrow P_{\mathcal{U}} & & \downarrow P_{\mathcal{G}} \\
\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}) & \xrightarrow{\ \simeq\ } & \mathcal{U}/\mathcal{U}(p) & \xrightarrow{\ \simeq\ } & \mathcal{G}/\mathcal{G}([p])
\end{array}$$

*Proof.* By construction, the mapping $\tilde{\upsilon}_{\mathcal{U}}$ is clearly one-to-one. Now we show that it is onto. Let $(\mathcal{S}, h) \in \mathcal{T}_{\mathcal{S}_0}(\mathcal{H})$. This means that $h \in \mathcal{S}$ and that $\mathcal{S} = u\mathcal{S}_0$ for some $u \in \mathcal{U}$. Then $f := u^{-1}(h) \in \mathcal{S}_0$ and $h = u(f)$, whence $\tilde{\upsilon}_{\mathcal{U}}([(u,f)]) = (\mathcal{S}, h)$, where $[(u,f)]$ is the equivalence class of $(u,f)$ in $\mathcal{U} \times_{\mathcal{U}(p)} \mathcal{S}_0$. Hence $\tilde{\upsilon}_{\mathcal{U}}$ is a bijective map.

Analogously, we have that $\tilde{\upsilon}_{\mathcal{G}}$ is bijective from $\mathcal{G} \times_{\mathcal{G}([p])} \mathcal{S}_0$ onto $\mathcal{T}_{\mathcal{S}_0}(\mathcal{H})$ as well. As a consequence, $j = (\tilde{\upsilon}_{\mathcal{G}})^{-1} \circ \tilde{\upsilon}_{\mathcal{U}}$ is also bijective. It is straightforward to check that all the maps involved in the diagram above are smooth. □

*Example.* By Proposition 4.5 one can show that the universal, tautological bundle $\Pi_{\mathcal{H}} \colon \mathcal{T}_{\mathcal{S}_0}(\mathcal{H}) \to \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ enters, as a canonical example, the framework outlined in Theorem 5.4 and Theorem 5.8 of [5]. To see this in terms of the bundle $\Pi_{\mathcal{H}}$ itself, first note that the commutant algebra $\{p_{\mathcal{S}_0}\}'$ of $p_{\mathcal{S}_0}$ coincides with the Banach subalgebra $B$ of $A$ formed by the operators $T$ such that $T(\mathcal{S}_0) \subset \mathcal{S}_0$, $T(\mathcal{S}_0^{\perp}) \subset \mathcal{S}_0^{\perp}$.

(It is straightforward to check directly on $B$ that it is stable under the adjoint operation, so that $B$ is a $C^*$-subalgebra of $A$, as it had to be.) Put $p = p_{\mathcal{S}_0}$. From Lemma 4.3, $u \in \mathcal{U}([p])$ if and only if $u\mathcal{S}_0 = \mathcal{S}_0$. Hence $u \in \mathcal{U}(p) = \mathcal{U}([p]) \cap \mathcal{U}([1-p])$ if and only if $u\mathcal{S}_0 = \mathcal{S}_0$ and $u\mathcal{S}_0^\perp = \mathcal{S}_0^\perp$, that is, $\mathcal{U}(p) = \mathcal{U}_A \cap B = \mathcal{U}_B$.

Similarly to what has been done in Remark 3.4, let $E_p\colon A \to B$ denote the canonical expectation associated to the tautological bundle at $\mathcal{S}_0$; that is, $E_p(T) := pTp + \hat{p}T\hat{p}$ for every $T \in A$. Also, for a fixed $x_0 \in \mathcal{S}_0$ such that $\|x_0\| = 1$, let $\varphi\colon A \to \mathbb{C}$ be the state of $A$ given by $\varphi_0(T) := (Tx_0 \mid x_0)_{\mathcal{H}}$. Then $\varphi_0 \circ E_p = \varphi_0$. Since the mappings $T \mapsto T(x_0)$, $\mathcal{B}(\mathcal{H}) \to \mathcal{H}$ and $T \mapsto T(x_0)$, $B \to \mathcal{S}_0$ are surjective, we obtain that $\mathcal{H}_A = \mathcal{H}$ and $\mathcal{H}_B = \mathcal{S}_0$ in the GNS construction associated with $A = \mathcal{B}(\mathcal{H})$, $B$ and $\varphi_0$. Moreover, in this case, $\pi_A$ coincides with the identity operator and the extension $P\colon \mathcal{H}_A \to \mathcal{H}_B$ of $E_p$ is $P = p$. Denote by $p_1, p_2\colon \mathrm{Gr}(\mathcal{H}) \times \mathrm{Gr}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H})$ the natural projections and define

$$Q_{\mathcal{H}}\colon \mathrm{Gr}(\mathcal{H}) \times \mathrm{Gr}(\mathcal{H}) \to \mathrm{Hom}\left(p_2^*(\Pi_{\mathcal{H}}), p_1^*(\Pi_{\mathcal{H}})\right)$$

by

$$Q_{\mathcal{H}}(\mathcal{S}_1, \mathcal{S}_2) = (p_{\mathcal{S}_1})|_{\mathcal{S}_2}\colon \mathcal{S}_2 \to \mathcal{S}_1$$

whenever $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Gr}(\mathcal{H})$. This mapping $\mathcal{Q}_{\mathcal{H}}$ is called the *universal reproducing kernel* associated with the Hilbert space $\mathcal{H}$. In fact, for $\mathcal{S}_1, \dots, \mathcal{S}_n \in \mathrm{Gr}(\mathcal{H})$ and $x_j \in \mathcal{S}_j$ $(j = 1, \dots, n)$,

$$\sum_{j,l=1}^{n} \left(Q_{\mathcal{H}}(\mathcal{S}_l, \mathcal{S}_j)x_j \mid x_l\right)_{\mathcal{H}} = \sum_{j,l=1}^{n} (p_{\mathcal{S}_l} x_j \mid x_l)_{\mathcal{H}} = \sum_{j,l=1}^{n} (x_j \mid x_l)_{\mathcal{H}}$$
$$= \left(\sum_{j=1}^{n} x_j \mid \sum_{l=1}^{n} x_l\right)_{\mathcal{H}} \geq 0\,,$$

so $\mathcal{Q}_{\mathcal{H}}$ is certainly a reproducing kernel in the sense of [5].

Using Proposition 4.5 and Example 4 we get the following special case of Theorem 5.8 in [5].

**Corollary 4.6.** *For a complex Hilbert space $\mathcal{H}$, the action of $\mathcal{U}$ on $\mathcal{H}$ can be realized as the natural action of $\mathcal{U}$ on a Hilbert space of holomorphic sections from $\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ into $\mathcal{H}$, such a realization being implemented by $\gamma(uh) = u\ \gamma(h)u^{-1}$, for every $h \in \mathcal{H}$, $u \in \mathcal{U}$.*

*Proof.* If $\mathcal{S} \in \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$, there exists $u \in \mathcal{U}$ such that $u\mathcal{S}_0 = \mathcal{S}$ and then $p_{\mathcal{S}} = up_{\mathcal{S}_0}u^{-1}$. Thus for all $u_1, u_2 \in \mathcal{U}$ and $x_1, x_2 \in \mathcal{S}_0$ we have $Q_{\mathcal{H}}(u_1\mathcal{S}_0, u_2\mathcal{S}_0)(u_2x_2) = p_{u_1\mathcal{S}_0}(u_2x_2) = u_1p_{\mathcal{S}_0}(u_1^{-1}u_2x_2)$. This formula shows that for every connected component $\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ the restriction of $Q_{\mathcal{H}}$ to $\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}) \times \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ is indeed a special case of the reproducing kernels considered in Remark 3.4. For every $h \in \mathcal{H}$, the mapping $\gamma_{p_{\mathcal{S}_0}}(h)\colon \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}) \to \mathcal{T}_{\mathcal{S}_0}(\mathcal{H})$ which corresponds to $Q_{\mathcal{H}}$ can be identified to the holomorphic map $u\mathcal{S}_0 \mapsto upu^{-1}h$, $\mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H}) \to \mathcal{H}$. Then the conclusion follows by using the diffeomorphism $\mathcal{U}/\mathcal{U}(p) \simeq \mathcal{G}/\mathcal{G}([p]) \simeq \mathrm{Gr}_{\mathcal{S}_0}(\mathcal{H})$ of Lemma 4.3, together with Proposition 4.5. $\square$

*Remark* 4.7. Assume again the situation where $A$ and $B$ are arbitrary $C^*$-algebras, $B$ is a $C^*$-subalgebra of $A$, with unit $\mathbf{1} \in B \subseteq A, E\colon A \to B$ is a conditional expectation, and $\varphi\colon A \to \mathbb{C}$ is a state such that $\varphi \circ E = \varphi$. With the same notations as in Remark 3.4, take $x_0 := \mathbf{1} + N_B \in B/N_B \subset A/N_A$. It is well known that $x_0$ is a cyclic vector of $\pi_X$, for $X \in \{A; B\}$: let $h \in \mathcal{H}_X$ such that $0 = (\pi(c)x_0 \mid h)_{\mathcal{H}_X} \equiv (c + N_X \mid h)_{\mathcal{H}_X}$ for all $c \in X$; since $X/N_X$ is dense in $\mathcal{H}_X$ we get $0 = (h \mid h)_{\mathcal{H}_X} = \|h\|^2$, that is, $h = 0$. Thus $\pi_X(X)x_0$ is dense in $\mathcal{H}_X$.

Inspired by [3], we now consider the $C^*$-subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H}_A)$ generated by $\pi_A(A)$ and $p$, where $p$ is the orthogonal projection from $\mathcal{H}_A$ onto $\mathcal{H}_B$. Set $\mathfrak{B} := \mathfrak{A} \cap \{p\}'$. Clearly, the GNS procedure is applicable to $\mathfrak{B} \subset \mathfrak{A} \subset \mathcal{B}(\mathcal{H}_A)$, for the expectation $E_p\colon \mathfrak{A} \to \mathfrak{B}$ and state $\varphi_0$ defined by $x_0$, as we have done in Remark 3.4. Then $\pi_A(A)(x_0) \subset \mathfrak{A}(x_0) \subset \mathcal{H}_A$ and $\pi_A(B)(x_0) \subset \mathfrak{B}(x_0) \subset \mathcal{H}_B$, whence, by the choice of $x_0$, we obtain that $\overline{\mathfrak{A}(x_0)} = \mathcal{H}_A$ and $\overline{\mathfrak{B}(x_0)} = \mathcal{H}_B$. Thus we have that $\mathcal{H}_{\mathfrak{A}} = \mathcal{H}_A$ and $\mathcal{H}_{\mathfrak{B}} = \mathcal{H}_B$.

According to former discussions there are two (composed) commutative diagrams, namely

$$\begin{array}{ccccccc}
\mathrm{G}_A \times_{\mathrm{G}_B} \mathcal{H}_B & \xrightarrow{\pi_A \tilde{\times} I} & \mathrm{G}_{\mathfrak{A}} \times_{\mathrm{G}_{\mathfrak{A}}(p)} \mathcal{H}_B & \longrightarrow & \mathrm{G}_{\mathfrak{A}} \times_{\mathrm{G}_{\mathfrak{A}}([p])} \mathcal{H}_B & \xrightarrow{j \tilde{\times} I} & \mathcal{G} \times_{\mathcal{G}([p])} \mathcal{H}_B \\
\Pi_G \Big\downarrow & & \Big\downarrow \Pi_{G_{\mathfrak{A}}} & & \Big\downarrow & & \Big\downarrow \Pi_{\mathcal{H}_B} \\
\mathrm{G}_A/\mathrm{G}_B & \xrightarrow{\tilde{\pi}_A} & \mathrm{G}_{\mathfrak{A}}/\mathrm{G}_{\mathfrak{A}}(p) & \longrightarrow & \mathrm{G}_{\mathfrak{A}}/\mathrm{G}_{\mathfrak{A}}([p]) & \xrightarrow{\tilde{j}} & \mathcal{G}/\mathcal{G}([p])
\end{array} \tag{4.1}$$

and

$$\begin{array}{ccccccc}
\mathrm{U}_A \times_{\mathrm{U}_B} \mathcal{H}_B & \xrightarrow{\pi_A \tilde{\times} I} & \mathrm{U}_{\mathfrak{A}} \times_{\mathrm{U}_{\mathfrak{A}}(p)} \mathcal{H}_B & \xrightarrow{j \tilde{\times} I} & \mathcal{U} \times_{\mathcal{U}(p)} \mathcal{H}_B & \xrightarrow{\simeq} & \mathcal{T}_{\mathcal{H}_B}(\mathcal{H}_A) \\
\Pi_U \Big\downarrow & & \Big\downarrow \Pi_{U_{\mathfrak{A}}} & & \Big\downarrow \Pi_{\mathcal{U}} & & \Big\downarrow \Pi_{\mathcal{H}_B} \\
\mathrm{U}_A/\mathrm{U}_B & \xrightarrow{\tilde{\pi}_A} & \mathrm{U}_{\mathfrak{A}}/\mathrm{U}_{\mathfrak{A}}(p) & \xrightarrow{\tilde{j}} & \mathcal{U}/\mathcal{U}(p) & \xrightarrow{\simeq} & \mathrm{Gr}_{\mathcal{H}_B}(\mathcal{H}_A)
\end{array} \tag{4.2}$$

(where the meaning of the arrows is clear). We suggest to call

$$\Pi_G\colon \mathrm{G}_A \times_{\mathrm{G}_B} \mathcal{H}_B \to \mathrm{G}_A/\mathrm{G}_B \quad \text{and} \quad \Pi_U\colon \mathrm{U}_A \times_{\mathrm{U}_B} \mathcal{H}_B \to \mathrm{U}_A/\mathrm{U}_B$$

the *GNS vector bundle* and the *unitary GNS vector bundle*, respectively, for data $E\colon A \to B$ and $\varphi\colon A \to \mathbb{C}$. Following the terminology used in [2, 3] for the maps $\mathrm{G}_A/\mathrm{G}_B \to \mathrm{G}_{\mathfrak{A}}/\mathrm{G}_{\mathfrak{A}}(p)$, $\mathrm{U}_A/\mathrm{U}_B \to \mathrm{U}_{\mathfrak{A}}/\mathrm{U}_{\mathfrak{A}}(p)$, we could refer to the left sub-diagrams of (4.1) and (4.2) as the *basic* vector bundle representations of $\Pi_G$ and $\Pi_U$, respectively. Since $\mathcal{H}_{\mathfrak{A}} = \mathcal{H}_A$ and $\mathcal{H}_{\mathfrak{B}} = \mathcal{H}_B$, the process to construct such "basic" objects, of Grassmannian type, is stationary. Also, since there is another way to associate Grassmannians to the GNS and unitary GNS bundles, which is that one of considering the tautological bundle of $\mathcal{H}_A$ (see the right diagrams in (4.1), (4.2)), we might call $\mathrm{G}_{\mathfrak{A}} \times_{\mathrm{G}_{\mathfrak{A}}(p)} \mathcal{H}_B \to \mathrm{G}_{\mathfrak{A}}/\mathrm{G}_{\mathfrak{A}}(p)$ the *minimal* Grassmannian vector bundle, and call $\mathcal{T}_{\mathcal{H}_B}(\mathcal{H}_A) \to \mathrm{Gr}_{\mathcal{H}_B}(\mathcal{H}_A)$ the *universal* Grassmannian vector bundle, associated with data $E\colon A \to B$ and $\varphi\colon A \to \mathbb{C}$. In the unitary case, we should add the adjective "unitary" to both bundles.

Note that the vector bundles

$$\mathcal{G} \times_{\mathcal{G}([p])} \mathcal{H}_B \to \mathcal{G}/\mathcal{G}([p]) \quad \text{and} \quad \mathcal{T}_{\mathcal{H}_B}(\mathcal{H}_A) \to \mathrm{Gr}_{\mathcal{H}_B}(\mathcal{H}_A)$$

are isomorphic. In this sense, both diagrams (4.1) and (4.2) "converge" towards the tautological bundle for $\mathcal{H}_A$. Let us remark that (4.1) is holomorphic, and everything in (4.2) is holomorphic with the only possible exception of the bundle $\Pi_U$. On the other hand, we have that $\mathrm{G}_{\mathfrak{A}}/\mathrm{G}_{\mathfrak{A}}(p)$ and $\mathcal{G}/\mathcal{G}(p)$ are complexifications of $\mathrm{U}_{\mathfrak{A}}/\mathrm{U}_{\mathfrak{A}}(p)$ and $\mathcal{U}/\mathcal{U}(p)$ respectively, on account of Remark 3.3 and Lemma 4.3. Note in passing that the fact that $\mathrm{G}_A/\mathrm{G}_B$ is such a complexification implies interesting properties of metric nature in the differential geometry of $\mathrm{U}_A/\mathrm{U}_B$, see [2].

The above considerations strongly suggest to investigate the relationships between (4.1) and (4.2) in terms of holomorphy and geometric realizations. In this respect, note that the commutativity of (4.2) corresponds, on the level of reproducing kernels, with the equality

$$(\pi_A \tilde{\times} I) \circ K(u_1\mathrm{U}_B, u_2\mathrm{U}_B) = Q_{\mathcal{H}_B}\big(\pi_A(u_1)\mathcal{U}(p), \pi_A(u_2)\mathcal{U}(p)\big) \circ (\pi_A \tilde{\times} I)$$

for all $u_1, u_2 \in \mathrm{U}_A$ (where the holomorphy supplied by $Q_{\mathcal{H}_B}$ appears explicitly). From this, a first candidate to reproducing kernel on $\mathrm{G}_A/\mathrm{G}_B$, in order to obtain a geometric realization of $\pi_A$ on $G_A$, would be defined by

$$K(g_1G_B, g_2G_B)[(g_2, f)] := \Big[\Big(g_1, p\big(\pi_A(g_1^{-1})\pi_A(g_2)f\big)\Big)\Big]$$

for every $g_1, g_2 \in G_A$ and $f \in \mathcal{H}_B$. Nevertheless, since the elements $g_1, g_2$ are not necessarily unitary, it is readily seen that the kernel $K$ so defined need not be definite-positive in general. There is also the problem of the existence of a suitable structure of Hermitian type in $\Pi_G$.

It would be interesting to have a theory of bundles $\mathrm{G}_A \times_{\mathrm{G}_B} \mathcal{H}_B \to \mathrm{G}_A/\mathrm{G}_B$ and kernels $K$ taking into account natural involutive diffeomorphisms in $\mathrm{G}_A/\mathrm{G}_B$, which would allow to incorporate those bundles to a framework containing as a special case the one established in [5]. This will be the subject of a forthcoming paper by the authors.

## 5. Almost hypercomplex structures associated with Grassmann manifolds

The following definition provides the infinite-dimensional version of the terminology of quaternionic structures on finite-dimensonal manifolds; see for instance Subsection 2.5 in [1].

**Definition 5.1.** Let $Y$ be a Banach manifold. An *almost hypercomplex structure* on $Y$ is a pair of almost complex structures $J_1, J_2 \colon TY \to TY$ satisfying $J_1J_2 = -J_2J_1$.

*Remark* 5.2. Let $\mathbb{H} = \mathbb{R}+\mathbb{R}i_1+\mathbb{R}i_2+\mathbb{R}i_3$ be the quaternion field with the imaginary units $i_1, i_2, i_3 \in \mathbb{H}$ satisfying $(i_1)^2 = (i_2)^2 = (i_3)^2 = -1$, $i_1i_2 = -i_2i_1 = i_3, i_2i_3 = -i_3i_2 = i_1$, and $i_3i_1 = -i_1i_3 = i_2$. In the setting of Definition 5.1, it is easy to see that the hypercomplex structure of $Y$ gives rise to a fiberwise linear action of $\mathbb{H}$ on $TY$ by $i_1 \cdot v = J_1 v, i_2 \cdot v = J_2 v$ and $i_3 \cdot v = J_1 J_2 v$ for every $v \in TY$. Thus for every $y \in Y$ the tangent space $T_y Y$ has a natural structure of quaternionic vector space.

In the following statement we need the notion of spray on Banach manifolds in the sense of [21].

**Theorem 5.3.** *Assume that $X$ is an almost complex Banach manifold. Then the following assertions hold:*

(i) *There exists a natural correspondence from the sprays on $X$ to the almost hypercomplex structures on $TX$.*

(ii) *If there exist a unital $C^*$-algebra $A$ and a projection $p = p^2 = p^* \in A$ such that $X = \mathrm{U}_A/\mathrm{U}_A(p)$, then the almost hypercomplex structure associated with the natural spray on $X$ induces an almost hypercomplex structure on the complexification $\mathrm{G}_A/\mathrm{G}_A(p)$.*

*Proof.* (i) Denote $Y = TX$ and $\pi\colon TX \to X$ the natural projection, and consider the commutative diagram

$$\begin{array}{ccc} \pi^*(TX) & \longrightarrow & TX \\ {\scriptstyle \pi^*(\pi)}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\ \pi\ } & X \end{array}$$

where the left-hand vertical arrow is the pull-back of the right-hand vertical arrow by $\pi\colon Y \to X$. Assume that we have got the covariant derivative associated with some spray on $X$. It then follows by the tensorial splitting theorem (Theorem 4.3 in Chapter X of [21]) that there exists an isomorphism

$$TY \simeq \pi^*(TX) \oplus_{TX} \pi^*(TX) \tag{5.1}$$

of vector bundles over $TX$. Note that the fiber of $\pi^*(TX)$ over any $y \in Y$ is

$$\bigl(\pi^*(TX)\bigr)_y = \bigl\{(y,z) \in Y \times TX \mid \pi(y) = \pi(z)\bigr\} \simeq \pi^{-1}\bigl(\pi(y)\bigr)$$

hence the fiber of the Whitney sum $\pi^*(TX) \oplus_{TX} \pi^*(TX)$ over $y \in Y = TX$ is

$$\bigl(\pi^*(TX) \oplus_{TX} \pi^*(TX)\bigr)_y \bigl\{(y_1,y_2) \in TX \times TX \mid \pi(y_1) = \pi(y_2) = \pi(y)\bigr\}\,,$$

which is isomorphic to $T_{\pi(y)}X \times T_{\pi(y)}X$. By taking into account the isomorphism (5.1) we can now define two almost complex structures on $Y$ by

$$(y_1,y_2) \mapsto (-y_2,y_1)\,, \quad TY \xrightarrow{J_1} TY\,, \tag{5.2}$$

and

$$(y_1,y_2) \mapsto (\mathrm{i}y_1,-\mathrm{i}y_2)\,, \quad TY \xrightarrow{J_2} TY\,. \tag{5.3}$$

For every pair $(y_1,y_2) \in \pi^*(TX) \oplus_{TX} \pi^*(TX) \simeq TY$ we have $J_1J_2(y_1,y_2) = J_1(\mathrm{i}y_1,-\mathrm{i}y_2) = (\mathrm{i}y_2,\mathrm{i}y_1)$ and $J_2J_1(y_1,y_2) = J_2(-y_2,y_1) = (-\mathrm{i}y_2,-\mathrm{i}y_1)$. Hence $J_1J_2 = -J_2J_1$, and thus the pair of almost complex structures $J_1, J_2$ defines an almost hypercomplex structure on $Y = TX$.

(ii) Now assume that $X = \mathrm{U}_A/\mathrm{U}_A(p)$ as in the statement. This is a complex homogeneous space by Theorem 3.2. The natural connection on this Grassmann manifold is the connection associated with the conditional expectation

$$E\colon A \to B\,, \quad E(a) = pap + (\mathbf{1} - p)a(\mathbf{1} - p)\,,$$

where $B = \{a \in A \mid ap = pa\}$. Recall that this conditional expectation induces a connection in the principal bundle $\mathrm{U}_A \to \mathrm{U}_A/\mathrm{U}_A(p)$ (see [2] and [18]). On the other hand, if we denote $\mathfrak{p} = \{a \in \operatorname{Ker} E \mid a^* = -a\}$, then $\mathrm{U}_A(p)$ acts upon $\mathfrak{p}$ by means of the adjoint action and it is well known that there exists an isomorphism of vector bundles $\mathrm{U}_A \times_{\mathrm{U}_A(p)} \mathfrak{p} \simeq TX$ over $\mathrm{U}_A/\mathrm{U}_A(p) = X$. In particular the tangent bundle of $X$ is a vector bundle associated with the principal bundle $\mathrm{U}_A \to \mathrm{U}_A/\mathrm{U}_A(p)$. Thus we get a linear connection on the vector bundle $TX \to X$ which is associated with a connection map (or connector) $TTX \to TX$, and the latter map gives rise to a spray on $X$ by means of the Christoffel symbols. (See [21] and Subsections 37.24–27 in [20] for more details.) Now assertion (i) shows that there exists an almost hypercomplex structure on $TX$ associated with the spray we got.

To complete the proof we have to show that there exists a projection

$$\pi\colon \mathrm{G}_A/\mathrm{G}_A(p) \to X$$

making $\mathrm{G}_A/\mathrm{G}_A(p)$ into a vector bundle which is isomorphic to the tangent bundle $TX \to X$. Recall from the above reasonings that $TX \simeq \mathrm{U}_A \times_{\mathrm{U}_A(p)} \mathfrak{p}$ as vector bundles over $\mathrm{U}_A/\mathrm{U}_A(p)$. Now define the mapping $(u, a) \mapsto u \exp(\mathrm{i}a)\mathrm{G}_A(p), \mathrm{U}_A \times \mathfrak{p} \to \mathrm{G}_A/\mathrm{G}_A(p)$. It is straightforward to check that this induces an injective mapping $\mathrm{U}_A \times_{\mathrm{U}_A(p)} \mathfrak{p} \to \mathrm{G}_A/\mathrm{G}_A(p)$, which is actually a diffeomorphism as a consequence of Theorem 8 in [26]. This makes $\mathrm{G}_A/\mathrm{G}_A(p)$ into a vector bundle isomorphic to $TX$ over $X$, and the proof ends. □

*Remark* 5.4. The corespondence between affine connections on finite dimensional manifolds and almost (hyper)complex structures goes back to [19] and [14]. See [9, 10], and [7] for more recent advances.

*Remark* 5.5. Let us note another general way to construct almost hypercomplex structures associated with the infinite-dimensional complex Grassmann manifolds. Quite generally, assume that $X$ is an almost complex Banach manifold. If we denote by $\overline{X}$ the complex-conjugate manifold of $X$, then the direct product $X \times \overline{X}$ is a complexification of $X$ and has a natural almost hypercomplex structure.

This fact was noted in the paper [13] in the case of finite-dimensional manifolds and can be proved in the general case as follows. Let $I\colon TX \to TX$ be the almost complex structure of $M$. Then $\overline{X}$ is just the underlying real analytic manifold of $X$ thought of as an almost complex manifold with respect to the almost complex structure $-I\colon TX \to TX$. Let us denote by $\theta\colon X \to \overline{X}$ the identity mapping, which is an anti-holomorphic mapping. Also denote $Z = X \times \overline{X}$. Now consider the direct product almost complex structure of $Z$,

$$J_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} : TZ \to TZ$$

and define

$$J_2 = \begin{pmatrix} 0 & T\theta \\ -(T\theta)^{-1} & 0 \end{pmatrix} : TZ \to TZ\,.$$

It is straightforward to check that $(J_1)^2 = (J_2)^2 = -\mathrm{id}_{TZ}$ and $J_1 J_2 = -J_2 J_1$, where the latter equality follows by the fact that $\theta$ is antiholomorphic.

## Acknowledgements

We wish to thank Professor Karl-Hermann Neeb and Professor Radu Pantilie for pointing out to us some pertinent references. The present research was begun during the first-named author's visit at the Department of Mathematics of the University of Zaragoza with financial support from Proyecto MTM2004-03036, DGI-FEDER, of the MCYT, Spain. Our research was further developed during the visit of the second-named author's visit at the Institute of Mathematics "Simion Stoilow" of the Romanian Academy with support from the aforementioned grant and from the Softwin Group. Partial financial support from Proyecto E-64 of the DG Aragón, Spain, and from Grant 2-CEx06-11-22/25.07.06 of the Romanian Government is also acknowledged.

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Daniel Beltiţă
Institute of Mathematics "Simion Stoilow" of the Romanian Academy
P.O.Box 1-764
Bucharest
Romania
e-mail: `Daniel.Beltita@imar.ro`

José E. Galé
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Zaragoza
E-50009 Zaragoza
Spain
e-mail: `gale@unizar.es`

Communicated by G. Misra.
Submitted: January 4, 2008.
Accepted: May 5, 2008.