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A variational principle, fixed points and coupled fixed points on $\mathbb{P}$ sets

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Abstract. We prove a generalization of Ekeland's variational principal using the notion of $\mathbb{P}$ sets. Using this result, we give proofs for fixed point theorems on partially ordered sets. Furthermore, one can obtain theorems for coupled fixed points using this technique. We demonstrate the procedure for proving such theorems.

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Keywords. Fixed point, variational principle, partially ordered set, $\mathbb{P}$ set, coupled fixed points.

1. Introduction

Ekeland formulated a variational principle in Ref. [11]. In a series of articles [11–13] he enriches the results. Later, he presented a more concise proof [14], the techniques from which we will use. In the same article [14], various applications of the variational principle in different fields of mathematics are presented: Gateaux-differentiability, optimization problems, minimal hypersurfaces, and partial differential equations. Ekeland's variational principle has many generalizations and applications [1,3,4,8,15,18,21,28,29].

There is a close relationship between fixed point theorems and variational principles [9]. Actually, Ekeland's variational principle is equivalent to Caristi's fixed point theorem in the sense that Caristi's fixed point theorem can be proven with Ekeland's variational principle and vice versa [16,26]. The same equivalence between Ekeland's variational principle and Takahashi's fixed point theorem is obtained in Ref. [26]. Some new connections between Ekeland's variational principle and Caristi's fixed point theorem are obtained in Ref. [19].

There are a great number of generalizations of Ekeland's variational principle by changing the underlying complete metric space. A variant of Ekeland's variational principle is presented in weighted graphs [2] and in weighted digraphs [3]. A generalization in b-metric spaces is obtained in Ref.

[8]. A version in fuzzy quasi-normed spaces is proved in Ref. [31]. A type of Ekeland's variational principle in quasimetric spaces is presented in Ref. [28]. Many other recent generalizations can be found in Refs. [15, 18, 21, 29]

A generalization of the Banach fixed point theorem [5] for coupled fixed points has been presented in Ref. [17] with applications in solving of systems of differential equations. Unfortunately, this result remained unseen for a long time, until the publication of [6], where, instead of a Banach space partially ordered by a cone, one considers a partially ordered complete metric space. The idea to consider coupled fixed points for a map $F : X \times X \to X$, i.e., to search for a solution to the system of equations $x = F(x, y)$ and $y = F(y, x)$ has one drawback. If there is a solution (x, y), then usually it holds $x = y$. This drawback has been overcome in Ref. [33] by considering an ordered pair of maps (F, G), such that $F, G : X \times X \to X$ and introducing the notion of a coupled fixed point (x, y) for the ordered pair of maps (F, G) to satisfy $x = F(x, y)$ and $y = G(x, y)$. If $G(x, y) = F(y, x)$ then the notion of coupled fixed points from [6, 17] is obtained.

Let $(x, \preccurlyeq)$ be a partially ordered set. Two kinds of maps $F : X \times X \to X$ are usually investigated: maps with the mixed monotone property [6], i.e., F is increasing on its first variable and decreasing on its second variable ($F(u, y) \preccurlyeq F(v, y)$ for $u \preccurlyeq v$ and $F(x, w) \succcurlyeq F(x, z)$ for $w \preccurlyeq z$) or without the mixed monotone property [10], i.e., the values of F are comparable, whenever the variables are comparable ($F(u, y) \asymp F(v, y)$ for $u \asymp v$ and $F(x, w) \asymp F(x, z)$ $w \asymp z$).

A different approach Proposition has been used in Refs. [20, 22–24, 27] by considering F-invariant and P-closed sets instead of mixed monotone maps or maps without the mixed monotone property.

A deep result in Ref. [23] presents the connection between fixed and coupled fixed points. The results from [23] present a possibility to investigate coupled fixed points with the help of results about fixed points.

We will try to combine all the above-mentioned results to get a generalization of Ekeland's variational principle on a domain, which is a F-invariant and P-closed set, to get a fixed point result on such domains and to apply it in the investigation of coupled fixed points on such sets by using the ideas from Proposition [23].

2. Preliminaries

Let X be a set, $\mathbb{R}$ be the real numbers, $\mathbb{N}$ be the naturals, and $d : X \to [0, +\infty)$ be a metric space. Following [6, 17] an ordered pair $(X, \preccurlyeq)$, where X is a set and $\preccurlyeq$ is a partial order on X, is called a partially ordered set. We call two elements $x, y \in X$ comparable if either $x \preccurlyeq y$ or $y \preccurlyeq x$. We denote by $x \succcurlyeq y$ if $y \preccurlyeq x$. We say that $x \prec y$ if $\preccurlyeq y$ but $x \neq y$. Let (X, d) be a metric space with a partial order $\preccurlyeq$, then the ordered triple $(X, d, \preccurlyeq)$ is called a partially ordered metric space. It is fair to note that fixed point theory in partially ordered metric spaces, in a more sophisticated context, was initiated by Turinici [30]. However, only after Ran and Reurings published [25] was there a surge in research on this subject matter. Following [7] let (X, d) be a metric space. The function $T : X \to (-\infty, \infty]$ is called lower semicontinuous (upper semicontinuous) [for short, l.s.c. (u.s.c.)] if at any point $x_0 \in X$ there holds $\liminf_{x \to x_0} T(x) \geq T(x_0)(\limsup_{x \to x_0} T(x) \leq T(x_0))$. Additionally, if $T \not\equiv +\infty$, it is called a proper function.

Inspired by [24], we give the following definition:

Definition 2.1. Let (X, d) be a metric space, $F : X \to X$ and $\mathbb{P} \subset X \times X$. The set $\mathbb{P}$ is called F-regular provided that $(x, F(x)) \in \mathbb{P}$ for every sequence $(x_n, F(x_n)), n \in \mathbb{N}$ in $\mathbb{P}$ such that $\lim_{n \to \infty} x_n = x$.

3. Main result

Definition 3.1. Let (X, d) be a metric space. $F : X \to X$ be a map and $\mathbb{P} \subset X \times X$ be F-regular. Let $V = \{x \in X : (x, F(x)) \in \mathbb{P}\}$ we say that the function $T : X \to \mathbb{R} \cup \{+\infty\}$ is lower l.s.c. on V (u.s.c on V) if at any $x_0, x_n \in V$, such that $\lim_{n\to\infty} x_n = x_0$, there holds $\liminf_{n\to\infty} T(x_n) \geq T(x_0)(\limsup_{n\to\infty} T(x_n) \leq T(x_0))$. Additionally, if $T(x) \not\equiv +\infty$ for $x \in V$, it is called a proper function on V.

Theorem 3.2. *Let (X, d) be a complete metric space, $F : X \to X$ be a map, $\mathbb{P} \subset X \times X$ and let $\mathbb{P}$ be F-regular. Let $V = \{x \in X : (x, F(x)) \in \mathbb{P}\}$ and $T : X \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c. bounded from below function on V.*

Let $\varepsilon > 0$ be arbitrary but fixed. Let $u_0 \in V$ be such that

$$T(u_0) \leq \inf_{v \in V} T(v) + \varepsilon. \tag{3.1}$$

Then there exists $x \in V$ such that

(i) $T(x) \leq \inf_{v \in V} T(v) + \varepsilon$

(ii) $d(x, u_0) \leq 1$

(iii) *for all $w \in V, w \neq x$ there holds $T(w) > T(x) - \varepsilon d(w, x)$.*

Proof. Let us define inductively a sequence $\{u_n\}_{n=0}^{\infty} \subset V$, starting with $u_0 \in V$, that satisfies (3.1).

Suppose that we have already chosen $u_n \in V$. There holds either of the following:

1) for every $w \in V \setminus \{u_n\}$ there holds the inequality

$$T(w) > T(u_n) - \varepsilon d(w, u_n)$$

2) there exists $w \in V \setminus \{u_n\}$, such that the following inequality holds:

$$T(w) \leq T(u_n) - \varepsilon d(w, u_n). \tag{3.2}$$

If case 1) holds, we choose $u_{n+1} = u_n$. In case 2), let us denote by $S_n \subset V \setminus \{u_n\}$ the set of all $w \in V \setminus \{u_n\}$, which satisfy (3.2). We choose $u_{n+1} \in S_n$ so that

$$T(u_{n+1}) \leq \frac{T(u_n)}{2} + \frac{\inf_{v \in S_n} T(v)}{2}. \tag{3.3}$$

We claim that in both cases $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Indeed, if case 1) ever occurs, the sequence is stationary, starting from some index n. If case 1) does not occur for any index $n \in \mathbb{N}$, then it should be case 2) for all indexes $n \in \mathbb{N}$. Therefore, by $T(w) \leq T(u_n) - \varepsilon d(w, u_n)$, we have the inequalities

$$\varepsilon d(u_k, u_{k+1}) \leq T(u_k) - T(u_{k+1})$$

for $k = 0, 1, 2, \ldots$. Summing up the above inequalities for k from n to $p-1$, we get

$$\begin{aligned} \varepsilon d(u_n, u_p) &\leq \sum_{k=n}^{p-1} \varepsilon d(u_k, u_{k+1}) \\ &\leq \sum_{k=n}^{p-1} (T(u_k) - T(u_{k+1})) = T(u_n) - T(u_p). \end{aligned} \tag{3.4}$$

From the inequality

$$T(u_{n+1}) \leq T(u_n) - \varepsilon d(u_n, u_{n+1}) < T(u_n),$$

it follows that the sequence $\{T(u_n)\}_{n=0}^{\infty}$ is a decreasing one and bounded from below. Hence, it is convergent. Therefore, the right-hand side in (3.4) goes to zero, when n and p go to infinity simultaneously. Consequently, $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since (X, d) is a complete metric space, it follows that the sequence $\{u_n\}_{n=0}^{\infty}$ converges to some $x \in X$.

We claim that the limit point $x \in V$ and satisfies (i), (ii) and (iii).

Indeed, from $\mathbb{P}$ being F-regular and $(u_n, F(u_n) \in \mathbb{P}$, we get that the inclusion $(x, F(x) \in \mathbb{P}$ holds. Therefore, $x \in V$.

(i) By construction, the sequence $\{T(u_n)\}_{n=0}^{\infty}$ is monotonously decreasing, and consequently using the l.s.c. of T on the set V we get that the inequalities $T(x) \leq \lim_{n\to\infty} T(u_n) \leq T(u_0) \leq \inf_{v \in V} T(v) + \varepsilon$ hold true, and consequently ((i)) holds.

(ii) Let us put $n = 0$ in (3.4), i.e.

$$\varepsilon d(u_0, u_p) \leq T(u_0) - T(u_p) \leq T(u_0) - \inf_{v \in V} T(v) \leq \varepsilon.$$

Letting p to infinity in the last inequality we get

$$\varepsilon d(u_0, x) = \lim_{p\to\infty} \varepsilon d(u_0, u_p) \leq \varepsilon,$$

i.e., $d(x, u) \leq 1$.

(iii) Let us suppose that (iii) were not true for all $w \in V$. Therefore, we can choose $w \neq x$, $w \in V$, so that

$$T(w) \leq T(x) - \varepsilon d(w, x) < T(x). \tag{3.5}$$

Letting $p \to \infty$ in (3.4), we obtain

$$\varepsilon d(u_n, x) \leq T(u_n) - T(x). \tag{3.6}$$

From (3.5) and (3.6), we get the chain of inequalities

$$\begin{aligned} T(w) &\leq T(x) - \varepsilon d(w, x) \leq T(u_n) - \varepsilon d(u_n, x) - \varepsilon d(x, w) \\ &= T(u_n) - \varepsilon(d(u_n, x) + d(x, w)) \leq T(u_n) - \varepsilon d(u_n, w) \end{aligned}$$

and thus $w \in S_n$ for all $n \in \mathbb{N}$. From (3.3), we have

$$2T(u_{n+1}) - T(u_n) \le \inf_{S_n} T \le T(w), \tag{3.7}$$

because $w \in \cap_{n=0}^{\infty} S_n$. From the existence of $\lim_{n\to\infty} T(u_n) = l$ and (3.7) it follows that

$$\lim_{n\to\infty} (2T(u_{n+1}) - T(u_n)) = \lim_{n\to\infty} T(u_n) = l \le T(w). \tag{3.8}$$

Since T is l.s.c., we have inequality

$$T(x) \le \lim_{n\to\infty} T(u_n) = l \tag{3.9}$$

and thus (3.8) and (3.9) imply that $T(x) \le T(w)$, a contradiction with (3.5). □

4. Applications

4.1. Finding minima of l.s.c. functions

Example 1. Let us consider $(\mathbb{R}, |\cdot - \cdot|, \le)$ with the metric $|\cdot - \cdot|$ induced by the modulus $|\cdot|$, the total order $\le$, the function $T : \mathbb{R} \to [0, +\infty) \cup \{+\infty\}$, defined by

$$T(x) = \begin{cases} \left|\frac{1}{x}\sin\left(\frac{1}{x}\right)\right|, & x \ne 0, \\ +\infty, & x = 0. \end{cases}$$

and the sets $\mathbb{P} = \{(x, y) \in \mathbb{R}\times\mathbb{R} : \ x \ge y\}$, $V = \{x \in \mathbb{R} : (x, Tx) \in \mathbb{P}\}$.

It is easy to observe that T is a proper l.s.c function that is bounded from below, and thus T is a proper l.s.c function that is bounded from below on V, $\inf_{v\in V} T(v) = 0$ and that if $x_n \in V$ for all $n \in \mathbb{N}$ is a convergent sequence, then $\lim_{n\to\infty} x_n = x \in V$. Let us consider the sequence $x_n = \frac{1}{\pi} + \frac{1}{89+n}$. The first term $x_1 \approx 0.329 > T(x_1) \approx 0.321$ is in V and satisfies the inequality $T(x_1) < \inf_{v\in V} T(v) + \varepsilon$ for $\varepsilon > T(x_1)$. Then there exists an $x \in V$ such that ((i)), ((ii)) and ((iii)) of Theorem 3.2 hold. In fact, $x = \lim_{n\to\infty} x_n = \frac{1}{\pi}$.

Example 2. Let us consider the function $T : \mathbb{R} \to \mathbb{R}$, defined by

$$T(x) = \begin{cases} e^{\frac{1}{x}}, & x < 0, \\ e^{-\frac{1}{x}} + x^{\frac{1}{x}}, & x > 0, \\ 0, & x = 0. \end{cases}$$

Let us denote by $F_{p,t}(x) : \mathbb{R} \to \mathbb{R}$ a function of two parameters, defined by $F_{p,t}(x) = H\left(\cos\left(\frac{2\pi x}{p}\right) - \cos\left(\frac{\pi t}{p}\right)\right), t, p > 0$, where $H(x) = 1$ for $x \ge 0$ and $H(x) = 0$ for $x < 0$ is the Heaviside step function, p is the period of F and t is the length of the interval on which $F(x) = 1$. Let us define $\mathbb{P} = \{(x, 1) : x \in \mathbb{R}\}$ and the set $V = \{x \in \mathbb{R} : (x, F(x)) \in \mathbb{P}\}$, i.e. $V = \{x \in \mathbb{R} : F(x) = 1\}$.

The map T is proper l.s.c., bounded from below on V and $\inf_{v\in V} T(v) = 0$. By the definition of $F_{p,t}$ if $u_n \in V$ for all $n \in \mathbb{N}$ and $u_n \to u$, then $u \in V$. Therefore, by Theorem 3.2, there exists an $x \in V$ so that ((i)),((ii)) and ((iii)) hold. In fact, for all $x \in [-\frac{t}{2}, \frac{t}{2}]$ we have $F(x) = 1$ and so it is enough to select a sequence that converges to $x = 0$ to get $T(x) = 0 \leq \inf_{v \in V} T(v) + \varepsilon = \varepsilon$ for all $\varepsilon > 0$.

Definition 4.1. [24] Let (X, d) be a metric space, $\mathbb{P} \subset X \times X$ and $F : X \to X$ be a mapping. $\mathbb{P}$ is called F-closed if

$$(x, y) \in \mathbb{P} \Rightarrow (F(x), F(y)) \in \mathbb{P}.$$

The next examples are well known [22].

Example 3. Let $(X, d, \preccurlyeq)$ be a partially ordered metric space. Let $F : X \to X$ be an increasing function, i.e., $F(x) \preccurlyeq F(y)$, provided that $x \preccurlyeq y$. Then the set $\mathbb{P} = \{(x, y) \in X \times X : x \preccurlyeq y\}$ is F-closed.

Example 4. Let $(X, d, \preccurlyeq)$ be a partially ordered metric space. For $F : X \to X$ let $F(x)$ be comparable with $F(y)$, i.e. $F(x) \asymp F(y)$. Then the set $\mathbb{P} = \{(x, y) \in X \times X : x \asymp y\}$ is F-closed.

We will first prove a corollary of Theorem 3.2.

Definition 4.2. Let $(X, d, \preccurlyeq)$ be a partially ordered metric space. We say that a map $f : X \to X$ is a l.s.c. (u.s.c.) with respect to $\preccurlyeq$ if any sequence x_n, such that $x_n \asymp x_0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x_0$ there holds $\liminf_{n\to\infty} f(x_n) \succcurlyeq f(x_0) (\limsup_{n\to\infty} f(x_n) \preccurlyeq f(x_0))$.

Corollary 4.3. *Let $(X, \rho, \preccurlyeq)$ be a partially ordered complete metric space, $\mathbb{P} = \{(x, y) \in X \times X : x \succcurlyeq y\}$, $F : X \to X$ be a l.s.c. map with respect to $\preccurlyeq$ and $V = \{x \in X : (x, F(x)) \in \mathbb{P}\} \neq \emptyset$. Let $T : X \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c., bounded from below function on the set V. Let $\varepsilon > 0$ be arbitrarily chosen and fixed and let $u_0 \in V$ be such that the inequality*

$$T(u_0) \leq \inf_V T(v) + \varepsilon$$

holds. Then there exists $x \in V$, such that

(I) $T(x) \leq \inf_{v \in V} T(v) + \varepsilon$
(II) $d(x, u_0) \leq 1$
(III) for all $w \in V$, $w \neq x$ there holds $T(w) > T(x) - \varepsilon d(w, x)$.

Proof. The only requirement of Theorem 3.2 that is not apparent is the F-regularity of $\mathbb{P}$. Indeed, if we have $\{u_n\}$ from Theorem 3.2, then for all $n \in \mathbb{N}$ it holds that $(u_n, F(u_n)) \in \mathbb{P}$. Then we have

$$u = \lim_{n\to\infty} u_n \succcurlyeq \lim_{n\to\infty} F(u_n) \succcurlyeq \liminf_{n\to\infty} F(u_n) \succcurlyeq F(u).$$

Therefore, $(u, F(u)) \in \mathbb{P}$, i.e., $u \in V$. □

Remark 4.4. A similar theorem holds if we replace F with a u.s.c. function and $\mathbb{P} = \{(x, y) \in X : x \preccurlyeq y\}$. Furthermore, if instead $\mathbb{P} = \{(x, y) \in X : x \asymp y\}$ and we require that F is continuous, we get another analogous result.

Example 5. Let us consider $\mathbb{R}$ with the canonical metric $|\cdot - \cdot|$ and the usual ordering $\leq$, applied only to the rationals. Let $F(x) = x$ be the identity map, and let $\mathbb{P} = \{(x, y) \in \mathbb{R} : x \asymp y\}$, that is, $\mathbb{P} = \mathbb{Q}^2$. Clearly, $V = \{x \in \mathbb{R} : (x, F(y)) \in \mathbb{P}\} = \mathbb{Q}$. Let T be defined as

$$T(x) = \begin{cases} \arctan(x), & x \in \mathbb{Q} \\ -2\pi, & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

Seeing as

$$\lim_{x_n \to x} T(x_n) \geq T(x)$$

for every $x_n, x \in V$ such that $\lim_{n\to\infty} x_n = x$, we get that T is a proper l.s.c. function on V. F is continuous and $\mathbb{P}$ is F-regular. Let us fix $\varepsilon > 0$ and pick $u_0 = q$,where q is a rational number, satisfying the inequality $q \leq \tan\left(-\frac{\pi}{2} + \varepsilon\right)$. Then u_0 satisfies

$$T(u_0) \leq \inf_{v\in V} T(v) + \varepsilon.$$

Thus, we can find $x \in V$ such that all three conclusions of Theorem 4.3 hold. Let us note that $\inf_{v\in V} T(v) = -\frac{\pi}{2}$, while $\inf_{x\in\mathbb{R}} T(x) = -2\pi$.

4.2. Fixed point theorems

Theorem 4.5. *Let $(X, d, \preccurlyeq)$ be a partially ordered metric space, $\mathbb{P} = \{(x, y) \in X \times X : x \succcurlyeq y\}, F : X \to X$ be a mapping l.s.c. with respect to $\preccurlyeq$ and $V = \{x \in X : (x, F(x)) \in \mathbb{P}\}$. Suppose that $\mathbb{P}$ is F-closed, $V \neq \emptyset$ and the function $x \mapsto d(x, F(x))$ is l.s.c. on V.*

If there exists $\alpha \in [0, 1)$ such that

$$d(F(x), F(y)) \leq \alpha d(x, y)$$

for all $(x, y) \in \mathbb{P}$, then F has a fixed point in X.

Proposition If, additionally, for every pair x, y of fixed points there exists $z \in X$ such that one of the inclusions $(x, z), (z, y) \in \mathbb{P}, (x, z), (y, z) \in \mathbb{P}$ or $(z, x), (z, y) \in \mathbb{P}$ holds, then the fixed point is unique.

Proof. Let us consider the mapping $T : X \to \mathbb{R}$ defined by $T(u) = d(u, F(u))$. Then T, is a proper l.s.c. function, bounded from below by 0, and $V \neq \emptyset$. Therefore, by Corollary 4.3 there exists $x \in V$ such that $T(x) \leq T(w) + \varepsilon d(x, w)$ for every $w \in V$. Let us choose $\varepsilon \in (0, k - \alpha]$, where $k \in (\alpha, 1)$.

Let us set $w = F(x)$. Because $\mathbb{P}$ is F-closed and $x \in V$, that is, $(x, F(x)) \in \mathbb{P}$, it follows that $(F(x), F^2(x)) \in \mathbb{P}$ and $w \in V$. Then we observe

$$\begin{aligned} d(x, F(x)) = T(x) &\leq T(w) + \varepsilon d(x, w) = d(F(x), F(F(x))) + \varepsilon d(x, F(x)) \\ &\leq \alpha d(x, F(x)) + \varepsilon d(x, F(x)) = (\alpha + \varepsilon) d(x, F(x)) \leq k d(x, F(x)) \end{aligned}$$

and, by $k < 1$, we conclude that $d(x, F(x)) = 0$. Therefore, $x = F(x)$.

Let there be two fixed points $x, y \in X$. Without loss of generality, let there exist $z \in X$ such that $(x, z), (z, y) \in \mathbb{P}$. Then, by $\mathbb{P}$ being F-closed, we get that $(F(x), F(z)) = (x, F(z)) \in \mathbb{P}$ and $(F(z), F(y)) = (F(z), y) \in \mathbb{P}$. It follows that $(x, F^n(z)), (F^n(z), y) \in \mathbb{P}, n \in \mathbb{N}$. Then

$$d(x, y) \leq d(x, F^n(z)) + d(F^n(z), y) \leq \alpha^n (d(x, z) + d(z, y)) \to 0.$$

Thus, $x = y$. □

Theorem 4.6. *Let $(X, d, \preccurlyeq)$ be a partially ordered metric space, $\mathbb{P} = \{(x, y) \in X \times X : x \succcurlyeq y\}$, $F : X \to X$ be a mapping l.s.c. with respect to $\preccurlyeq$ and $V = \{x \in X : (x, F(x)) \in \mathbb{P}\}$. Suppose that $\mathbb{P}$ is F-closed, $V \neq \emptyset$ and the function $x \mapsto d(x, F(x))$ is l.s.c. on V.*

If there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that

$$d(F(x), F(y)) \leq \alpha d(x, F(x)) + \alpha d(y, F(y))$$

for all $(x, y) \in \mathbb{P}$, then F has a fixed point in X.

If, additionally, for every pair x, y of fixed points there exists $z \in X$ such that one of the inclusions $(x, z), (z, y) \in \mathbb{P}$, $(x, z), (y, z) \in \mathbb{P}$ or $(z, x), (z, y) \in \mathbb{P}$ holds, then the fixed point is unique.

Proof. It is easy to show that $\frac{\alpha}{1-\alpha} \in [0, 1)$ when $\alpha \in \left[0, \frac{1}{2}\right)$. Let us consider the mapping $T : X \to \mathbb{R}$ defined by $T(u) = d(u, F(u))$. Then T is a proper l.s.c. function, bounded from below by 0, and $V \neq \emptyset$. Therefore, by Theorem 4.3 there exists $x \in V$ such that $T(x) \leq T(w) + \varepsilon d(x, w)$ for every $w \in V$. Let us choose $\varepsilon \in \left(0, k - \frac{\alpha}{1-\alpha}\right]$, where $k \in \left(\frac{\alpha}{1-\alpha}, 1\right)$.

Let us set $w = F(x)$.Because $\mathbb{P}$ is F-closed and $x \in V$, that is, $(x, F(x)) \in \mathbb{P}$, it follows that $(F(x), F^2(x)) \in \mathbb{P}$ and $w \in V$. We also note that

$$\begin{aligned} d(F(x), F^2(x)) &\leq \alpha d(x, F(x)) + \alpha d(F(x), F^2(x)) \\ d(F(x), F^2(x)) &\leq \tfrac{\alpha}{1-\alpha} d(x, F(x)). \end{aligned}$$

From here, we get

$$\begin{aligned} d(x, F(x)) = T(x) &\leq T(w) + \varepsilon d(x, w) = d(F(x), F(F(x))) + \varepsilon d(x, F(x)) \\ &\leq \tfrac{\alpha}{1-\alpha} d(x, F(x)) + \varepsilon d(x, F(x)) = \left(\tfrac{\alpha}{1-\alpha} + \varepsilon) d(x, F(x)\right) \\ &\leq k d(x, F(x)) \end{aligned}$$

and, by $k < 1$, we conclude that $d(x, F(x)) = 0$. Therefore, $x = F(x)$.

The uniqueness can be proven in the same fashion as in Theorem 4.5.

Remark 4.7. Due to Remark 4.4, we can get similar results to those from Theorems 4.5 and 4.6, given that we

1. either replace F with a u.s.c function and $\mathbb{P} = \{(x, y) \in X : x \preccurlyeq y\}$,
2. or we require that F is continuous and $\mathbb{P} = \{(x, y) \in X : x \asymp y\}$.

We have seen that the lower semicontinuity of the function $T(x) = d(x, F(x))$ is crucial. The next proposition presents a sufficient condition for it to hold.

Proposition 4.8. *Let $(X, d, \preccurlyeq)$ be a partially ordered metric space and $F : X \to X$ be a mapping l.s.c. with respect to $\preccurlyeq$ such that for every $x \in X$ there exists a non-constant sequence $\{x_n\}_{n=0}^{\infty}$ converging to x such that $F(x_n) \to F(x)$. Then the mapping $T : X \to [0, +\infty)$ given by $T(x) = d(x, F(x))$ is l.s.c.*

Proof. Let $F(x)$ be a l.s.c. function and $\{x_n\}_{n=0}^{\infty}$ be a non-constant sequence such that when $x_n \to x$, we have $F(x_n) \to F(x)$. Then, the following holds:

$$\liminf_{x \to x_0} d(F(x), F(x_0)) = 0.$$

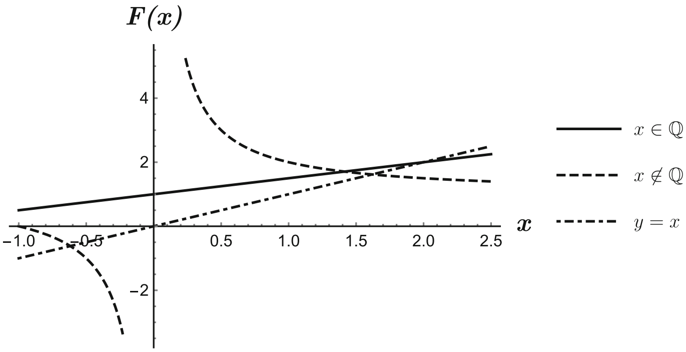

FIGURE 1. Fixed points of $F(x)$

By the inequality

$$T(x_0) = d(x_0, F(x_0)) \leq d(x_0, x) + d(x, F(x)) + d(F(x), F(x_0))$$

we get

$$\begin{aligned} \liminf_{x\to x_0} T(x_0) &= T(x_0) \\ &\leq \liminf_{x\to x_0} d(x_0, x) + d(x, F(x)) + d(F(x), F(x_0)) \\ &= \liminf_{x\to x_0} d(x, F(x)) = \liminf_{x\to x_0} T(x) \end{aligned}$$

Thus, $T(x)$ is a l.s.c. function. □

Remark 4.9. By Proposition 4.8, we can see that if F is continuous or if it is l.s.c. with only jumps, that is, without isolated values, the results of Theorems 4.5 and 4.6 hold.

Example 6. Let us consider $F : \mathbb{R} \to \mathbb{R}$ defined as

$$F(x) = \begin{cases} \frac{x}{2} + 1, & x \in \mathbb{Q} \\ \frac{1}{x} + 1, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

and let us use the usual ordering on the reals, applied only to the rationals. Let us define $\mathbb{P} = \{(x, y) \in \mathbb{R}^2 : x \asymp y\}$, that is, $x, y \in \mathbb{Q}$. Then, F is a proper l.s.c. function with respect to $\leq$ and $d(x, F(x))$ is l.s.c. on the set $V = \{x \in X : (x, F(x)) \in \mathbb{P}\}$. Also, F maps rationals to rationals. Therefore, $\mathbb{P}$ is F-closed and V is not empty. Finally, for all $(x, y) \in \mathbb{P}$ the following inequality holds:

$$d(F(x), F(y)) \leq \frac{1}{2} d(x, y).$$

Then, F has a single fixed point in $\mathbb{Q}$ - $x = 2$. However, since we cannot compare rationals and irrationals, we cannot guarantee the uniqueness of the fixed point. Indeed, there are two more – $x = \varphi, x = -\frac{1}{\varphi}$, where φ is the golden ratio. This can be seen in Fig. 1.

5. Coupled fixed points

In this section, we provide a method for obtaining proofs for coupled fixed point theorems in partially ordered metric spaces. The technique consists of considering the Cartesian product of the underlying metric space with itself and defining an appropriate partial order on the product. Then, one can prove a corollary of Theorem 3.2 for the partial order and the type of mapping being analyzed. Finally, the coupled fixed point theorems are reduced to a corollary of Theorems 4.5 and 4.6.

Definition 5.1. Let $(X, \preccurlyeq)$ be a partially ordered set and let $F : X \times X \to X$. The function F is said to have the mixed monotone property if

for any $x_1, x_2, y \in X$ such that $x_1 \preccurlyeq x_2$ there holds $F(x_1, y) \preccurlyeq F(x_2, y)$
for any $x, y_1, y_2 \in X$ such that $y_1 \preccurlyeq y_2$ there holds $F(x, y_1) \succcurlyeq F(x, y_2)$.

Let us use the following notation—for any $u = (u^{(1)}, u^{(2)}) \in X \times X$, we will denote $\bar{u} = (u^{(2)}, u^{(1)})$. In what follows if $(X, \rho, \preccurlyeq)$ then the space $(X \times X, d, \preceq)$ will be endowed with $d((x, y), (u, v)) = \rho(x, u) + \rho(y, v)$ and $(x, y) \preceq (u, v)$ if $x \preccurlyeq u$ and $y \succcurlyeq v$.

We have mentioned that a deep observation in Ref. [23] suggests that coupled fixed points can be obtained with the help of similar fixed point results. We will show in the next corollary that results connected with maps with the mixed monotone property can be obtained with the help of similar results on $\mathbb{P}$ sets.

We will now prove an earlier result [32] using Theorem 4.3.

Corollary 5.2. [32] *Let $(X, \rho, \preccurlyeq)$ be a partially ordered complete metric space, $(X \times X, d, \preceq)$, where $(u, v) \preceq (x, y)$ if $u \preccurlyeq x$ and $v \succcurlyeq y$, and $F : X \times X \to X$ be a continuous map with the mixed monotone property. Let*

$$V \times V = \{x = (x^{(1)}, x^{(2)}) \in X \times X : x^{(1)} \preccurlyeq F(x) \text{ and } x^{(2)} \succcurlyeq F(\bar{x})\} \neq \emptyset.$$

Let $T : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c., bounded from below function. Let $\varepsilon > 0$ be arbitrarily chosen and fixed, and let $u_0 \in V \times V$ be an ordered pair such that the inequality

$$T(u_0) \leq \inf_{V \times V} T(v) + \varepsilon$$

holds. Then, there exists an ordered pair $x \in V \times V$, such that

(a) $T(x) \leq \inf_{v \in V \times V} T(v) + \varepsilon$
(b) $d(x, u_0) \leq 1$
(c) for all $w \in V, w \neq x$ there holds $T(w) > T(x) - \varepsilon d(w, x)$.

We would like to point out that in the statement in Ref. [32] in (a) $''+\varepsilon$" is missing, but in the article, it is actually proven that $T(x) \leq \inf_{v \in V \times V} T(v) + \varepsilon$.

Proof. Let us consider the set $\mathscr{X} = X \times X$ endowed with the metric d and the partial order $\preceq$ in $X \times X$, $\mathscr{F}(x) = (F(x), F(\bar{x}))$, $\mathbb{P} = \{(x, y) \in \mathscr{X} \times \mathscr{X}, x \preceq y\}$ and $\mathscr{V} = \{x \in \mathscr{X} : (x, \mathscr{F}(x)) \in \mathbb{P}\}$. Let us note that $\mathscr{X}$ is complete,

$\mathscr{V} = V \times V$, $\mathscr{F}$ is continuous (therefore, l.s.c.) and the l.s.c. function T remains unchanged, only changing the notation to $T : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$.

We will prove that $\mathbb{P}$ is $\mathscr{F}$ regular.

Indeed, if we have $x_n = \left(x_n^{(1)}, x_n^{(2)}\right) \to x = \left(x^{(1)}, x^{(2)}\right)$ and $(x_n, \mathscr{F}(x_n)) \in \mathbb{P}$, where $(x_n, x \in \mathscr{X})$, then, due to $\mathscr{F}$ being continuous,

$$x^{(1)} = \lim_{n\to\infty} x_n^{(1)} \preccurlyeq \lim_{n\to\infty} F(x_n) = F(x)$$
$$x^{(2)} = \lim_{n\to\infty} x_n^{(2)} \succcurlyeq \lim_{n\to\infty} F(\bar{x_n}) = F(\bar{x}),$$

i.e. $x = \left(x^{(1)}, x^{(2)}\right) \preceq \mathscr{F}(x) = (F(x), F(\bar{x}))$. Therefore, $(x, \mathscr{F}(x)) \in \mathbb{P}$, that is, $\mathbb{P}$ is $\mathscr{F}$-regular.

We also have $u_0 \in \mathscr{V}$, such that

$$T(u_0) \leq \inf_{\mathscr{V}} T(v) + \varepsilon.$$

All of the conditions of Theorem 4.3 are fulfilled. □

The next corollary is proven in Ref. [32] with the help of Corollary 5.2. We will show that it can be proven with Theorem 4.5, which will justify the observation in Ref. [23] that fixed points and coupled fixed points are closely related.

Corollary 5.3. [32] *Let $(X, \rho, \preccurlyeq)$ be a partially ordered complete metric space, $(X \times X, d, \preceq)$ and $F : X \times X \to X$ be a continuous map with the mixed monotone property. Let there exist $\alpha \in [0, 1)$, so that the inequality*

$$\rho(F(x, y), F(u, v)) + \rho(F(y, x), F(v, u)) \leq \alpha\rho(x, u) + \alpha\rho(y, v)$$

holds for all $x \succcurlyeq u$ and $y \preccurlyeq v$. If there exists at least one ordered pair (x, y) such that $x \preccurlyeq F(x, y)$ and $y \succcurlyeq F(y, x)$, then there exists a coupled fixed point (x, y) of F.

If, in addition, every pair of elements in $X \times X$ has a lower or an upper bound, then the coupled fixed point is unique.

Proof. Using the same notation as in the proof of Corollary 5.3, we will prove it with Theorem 4.5. Due to Remark 4.7, we can use the same definition for $\mathbb{P}$. The fact that $\mathbb{P}$ is $\mathscr{F}$ closed is proven in Ref. [32](Proposition 3.1). It is clear that that

$$d(\mathscr{F}(x), \mathscr{F}(y)) \leq \alpha d(x, y)$$

holds for all $(x, y) \in \mathbb{P}$.

Due to Remark 4.9, we have that $d(x, y)$ is a proper l.s.c. function (in fact, a continuous one) and $\mathscr{V} \neq \emptyset$. Therefore, by Theorem 4.5, there exists a fixed point of $\mathscr{F}$, i.e. a coupled fixed point of F.

The requirement that every pair of elements in $X \times X$ has an upper or a lower bound guarantees that for every two fixed points $x, y \in \mathscr{X}$ of $\mathscr{F}$, there exists $z \in \mathscr{X}$ such that $(x, z), (y, z) \in \mathbb{P}$ or $(z, x), (z, y) \in \mathbb{P}$. Therefore, the fixed point of $\mathscr{F}$ is unique, i.e., the coupled fixed point of F is unique. □

Even though this approach gives us an easy way to prove fixed point theorems, we will show some of its limitations. For this purpose, we will examine maps without the mixed monotone property [10].

Definition 5.4. [10] Let $(X, d, \preccurlyeq)$ be a partially ordered metric space and $F : X \times X \to X$ be a map. We will say that F is a map without the mixed monotone property when for all $\xi, \eta, s \in X$, if $\xi \asymp F(\xi,\eta)$ then $F(\xi,\eta) \asymp F(F(\xi,\eta), s)$.

Corollary 5.5. *Let $(X, \rho, \preccurlyeq)$ be a partially ordered complete metric space, $(X \times X, d, \preceq)$, where $\preceq$ is induced by $\preccurlyeq$ in such a way as to follow that $(x, y) \asymp (u, v)$, provided that $x \asymp u$ and $y \asymp v$. Let $F : X \times X \to X$ be a continuous map without the mixed monotone property. Let*

$$V \times V = \{x = (x^{(1)}, x^{(2)}) \in X \times X : x^{(1)} \asymp F(x) \text{ and } x^{(2)} \asymp F(\bar{x})\} \neq \emptyset.$$

Let $T : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c., bounded from below function. Let $\varepsilon > 0$ be arbitrarily chosen and fixed, and let $u_0 \in V \times V$ be an ordered pair such that the inequality

$$T(u_0) \leq \inf_{V \times V} T(v) + \varepsilon$$

holds. Then there exists an ordered pair $x \in V \times V$, such that

(A) $T(x) \leq \inf_{v \in V \times V} T(v) + \varepsilon$

(B) $d(x, u_0) \leq 1$

(C) for all $w \in V, w \neq x$ there holds $T(w) > T(x) - \varepsilon d(w, x)$.

Proof. Much like as in the proof of Corollary 5.2 we define set $\mathscr{X} = X \times X$, the partially ordered metric space $(\mathscr{X}, d, \preceq)$, $\mathscr{F}(x) = (F(x), F(\bar{x}))$, $\mathbb{P} = \{(x, y) \in \mathscr{X} \times \mathscr{X} : x \asymp y\}$ and $\mathscr{V} = \{x \in \mathscr{X} : (x, \mathscr{F}(x)) \in \mathbb{P}\}$. Let us note that $\mathscr{X}$ is complete, $\mathscr{V} = V \times V$, $\mathscr{F}$ is continuous (therefore, l.s.c.) and the l.s.c. function T remains unchanged, only changing the notation to $T : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$.

We will prove that $\mathbb{P}$ is $\mathscr{F}$ regular.

Indeed, if we have $x_n = \left(x_n^{(1)}, x_n^{(2)}\right) \to x = \left(x^{(1)}, x^{(2)}\right)$ and $(x_n, \mathscr{F}(x_n)) \in \mathbb{P}$, where $(x_n, x \in \mathscr{X})$, then, due to $\mathscr{F}$ being continuous,

$$x^{(1)} = \lim_{n\to\infty} x_n^{(1)} \asymp \lim_{n\to\infty} F(x_n) = F(x)$$
$$x^{(2)} = \lim_{n\to\infty} x_n^{(2)} \asymp \lim_{n\to\infty} F(\bar{x_n}) = F(\bar{x}),$$

i.e. $x = \left(x^{(1)}, x^{(2)}\right) \asymp \mathscr{F}(x) = (F(x), F(\bar{x}))$. Therefore, $(x, \mathscr{F}(x)) \in \mathbb{P}$, that is, $\mathbb{P}$ is $\mathscr{F}$-regular.

We also have $u_0 \in \mathscr{V}$, such that

$$T(u_0) \leq \inf_{\mathscr{V}} T(v) + \varepsilon.$$

All of the conditions of Theorem 4.3 are fulfilled. □

We can achieve a variational principle with such maps. However, the transitivity of the relation used in $\mathbb{P}$ is paramount for the application of the result. That is why we need to add an additional assumption to the result in Ref. [10] in order to use the variational principal.

Corollary 5.6. *Let $(X, d, \preccurlyeq)$ be a complete partial ordered metric space and $F : X \times X \to X$ be a mapping. Suppose that the following conditions hold:*

(a) $x \asymp y$ and $y \asymp z$ implies that $x \asymp z$ for all $x, y, z \in X$,
(b) F is a map without the mixed monotone property,
(c) there exist $\xi_0, \eta_0 \in X$ such that $\xi_0 \asymp F(\xi_0, \eta_0)$ and $\eta_0 \asymp F(\eta_0, \xi_0)$,
(d) there exists $k \in [0, 1)$ such that

$$d(F(\xi, \eta), F(r, s)) \leq k \max\{d(\xi, r), d(\eta, s)\}$$

for all $\xi, \eta, r, s \in X$ satisfying $\xi \asymp r$ and $\eta \asymp s$,
(e) F is continuous or if $\xi_n \to \xi$ when $n \to \infty$ in X, then $\xi_n \asymp \xi$ for n sufficiently large.

Then, F has a coupled fixed point.

Proof. We use same notation as in the proof of Corollary 5.5. We redefine $\mathbb{P} = \{(x, y) \in \mathscr{X} \times \mathscr{X} : x \asymp y, x \asymp \mathscr{F}(x), y \asymp \mathscr{F}(y)\}$. We will prove this corollary using Theorem 4.5. Let us prove that $\mathbb{P}$ is $\mathscr{F}$ closed. If $(x, y) \in \mathbb{P}$, by (a) and the definition of $\mathbb{P}$ we get

$$\mathscr{F}(x)^{(1)} \asymp x^{(1)} \asymp y^{(1)} \asymp \mathscr{F}(y)^{(1)}$$
$$\mathscr{F}(x)^{(2)} \asymp x^{(2)} \asymp y^{(2)} \asymp \mathscr{F}(y)^{(2)}$$

Therefore, $\mathscr{F}(x) \asymp \mathscr{F}(y)$. By (b) we get that $\mathscr{F}(x) \asymp \mathscr{F}(\mathscr{F}(x))$ and $\mathscr{F}(y) \asymp \mathscr{F}(\mathscr{F}y)$. Therefore, we conclude that $(\mathscr{F}(x), \mathscr{F}(y)) \in \mathbb{P}$.

For $x, y \in \mathscr{X}$, let us define $d^*(x, y)$ as

$$d^*(x, y) = \max\{d(x^{(1)}, y^{(1)}), d(x^{(2)}, y^{(2)})\}.$$

It is clear that this is also a metric and due to the fact that

$$\begin{aligned} d(F(x), F(y)) &\leq k \max\{d(x^{(1)}, y^{(1)}), d(x^{(2)}, y^{(2)})\} \\ d(F(\bar{x}), F(\bar{y})) &\leq k \max\{d(x^{(2)}, y^{(2)}), d(x^{(1)}, y^{(1)})\} \\ &= k \max\{d(x^{(1)}, y^{(1)}), d(x^{(2)}, y^{(2)})\} \end{aligned}$$

we get that

$$d^*(\mathscr{F}(x), \mathscr{F}(y)) \leq k d^*(x, y)$$

for all $(x, y) \in \mathbb{P}$.

If F is continuous, then due to Remark 4.9, we have that $d^*(x, y)$ is a proper l.s.c. function (in fact, a continuous one) and $\mathscr{V} \neq \emptyset$ by (c). Therefore, by Theorem 4.5, there exists a fixed point of $\mathscr{F}$, i.e., a coupled fixed point of F.

If F is not continuous, the variational principle cannot be used. The classic proof can be found in Ref. [10]. □

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Declarations

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