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Concepts of almost periodicity and ergodic theorems in locally convex spaces

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Abstract. We wish to investigate mean ergodic theorems for generalizations of almost periodic functions on semigroups, as well as for semigroups of operators in the framework of locally convex spaces. Specially, we present functional characterizations of concepts of almost periodicity for vector-valued functions.

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1. Introduction

The theory of almost periodic functions can trace its origins back to the work of Bohr [10,11] involving reduction to the periodic functions on the real line. The definition of an almost periodic function on a semitopological semigroup is due to Bochner [8]. Weakly almost periodic functions were first defined and investigated by Eberlein [15]. Almost periodic and weakly almost periodic functions with values in a Banach space, which are of great interest, were considered by Bochner in [9] and Goldberg and Irwin [16], respectively. Solutions to the abstract Cauchy problem, under suitable conditions, fall into this category (see, e.g., [2,6,31]), and of particular importance is the integration of such functions and convergence in the mean. Ruess and Summers [32,33] showed that if a function f from $\mathbb{R}^+$ to a Banach space E is continuous and weakly almost periodic, then $\frac{1}{t}\int_0^t f(s+h)ds$ converges strongly as $t \to \infty$ to a point z in E, uniformly in $h \in \mathbb{R}^+$. Let us point out that the first nonlinear ergodic theorems for nonexpansive mappings and semigroups were established by Baillon [3,4] and Baillon and Brezis [5].

Nonlinear mean ergodic theorems, together with some generalizations, have been investigated extensively (see [7,12,14,20–22,26–30,35]).

Kada [17,18] extended the mean ergodic result of Ruess and Summers to the case of a strongly asymptotically invariant net of means on a commutative semigroup with identity and applied it to weakly almost periodic representations of nonexpansive mappings. Miyake and Takahashi [24] extended Kada's results to noncommutative semigroups and they also proved mean ergodic theorems to the more general case of asymptotically invariant means, for almost periodic functions with values in a Banach space. These results were extended to the framework of a locally convex space in [25]. However, there are some gaps in the proofs of [25] since those proofs depend upon the Krein–Smulian property and Mazur's theorem, and while they are true for Banach spaces, they may not be true for a general locally convex space.

This paper will discuss the mean ergodic theorems for some generalized almost periodic functions on a semigroup S, the relevant ergodic means of which include the both left and right translates. We use different tools and methods to accomplish our aims, as well as to fix some of the gaps in [25]. It is remarkable that here no Krein–Smulian property is needed. Specially, we present useful characterizations of notions of almost periodicity with values in a locally convex space E, and derive the mean ergodic theorems for such functions, as well as for semigroups of mappings enjoying such properties. The results presented in the paper are new even in Banach spaces.

2. Preliminaries

Throughout the paper, E will denote a Hausdorff locally convex space (l.c.s.) and E' the topological dual of E. The weak topology $\sigma(E, E')$ on E with respect to the dual pair $\langle E, E'\rangle$ is defined as the initial topology with respect to the family $(\langle ., y\rangle : y \in E')$; the weak topology $\sigma(E', E)$ on E' is defined analogously. For $A \subset E$, the polar A° is defined as $A^\circ = \{y \in E' : |\langle x, y\rangle| \leq 1, (x \in A)\}$. Analogously, for $B \subset E'$, B° is defined by $B^\circ := \{x \in E : |\langle x, y\rangle| \leq 1, (y \in B)\}$. The closure of A will be denoted by $\overline{A}$. By the bipolar theorem $A^{\circ\circ} := (A^\circ)^\circ = \overline{acoA}^{\sigma(E,E')}$, where $acoA$ is the absolutely convex hull of A (see, e.g., [36,39]).

In what follows, S denotes a semigroup. Let $\ell^\infty(S)$ denote the C*-algebra of all bounded real valued functions on S. Let $f \in \ell^\infty(S)$ and let $s \in S$. The right and left translation operators $r(s)$ and $l(s)$ are defined by

$$(r(s)f)(t) = f(ts), \ (l(s)f)(t) = f(st). \tag{2.1}$$

A function $f \in \ell^\infty(S)$ is said to be almost periodic (resp. weakly almost periodic) if the set $R_S f$ of right translates of f is norm relatively compact (resp. weakly relatively compact) in $\ell^\infty(S)$, as well as the same is true for $L_S f$ in place of $R_S f$. The set of all almost periodic (resp. weakly almost periodic) functions on S is denoted by $AP(S)$ (resp. $WAP(S)$).

Let X be a linear subspace of $\ell^\infty(S)$. A mean on X is a linear functional μ on X with the property

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s).$$

The set of all means on X is denoted by $M(X)$. Sometimes we write $\mu_s(f(s))$ instead of $\mu(f)$. If X is linear subspace of $\ell^\infty(S)$ containing the constant functions then a linear functional μ on X is a mean if and only if $\mu(1) = \|\mu\| = 1$; see [7]. A subspace X of $\ell^\infty(S)$ is said to be right (resp. left) invariant if $r(s)X \subset X$ (resp. $l(s)X \subset X$) for all $s \in S$. X is translation invariant if it is both right and left invariant. Let X be a left (resp. right) translation-invariant linear subspace of $\ell^\infty(S)$ containing the constant functions. A member μ of X' is said to be left (resp. right) invariant if, for all $f \in X$ and $s \in S$, $\mu(l(s)f) = \mu(f)$ (resp. $\mu(r(s)f) = \mu(f)$). A member μ of X' is said to be invariant if it is both left and right invariant. X is said to be amenable, if there is an invariant mean on X. A net of means μ_α on X is said to be left (resp. right) asymptotically invariant, if for each $s \in S$, $l(s)'\mu_\alpha - \mu_\alpha$ (resp. $r(s)'\mu_\alpha - \mu_\alpha$) converges to 0 in the topology $\sigma(X', X)$. A net of means μ_α on X is said to be left (resp. right) strongly asymptotically invariant if $l(s)'\mu_\alpha - \mu_\alpha$ (resp. $r(s)'\mu_\alpha - \mu_\alpha$) converges to 0 in the norm topology(see [13]).

3. Characterizations of $AP_w(S, E)$ and $W(S, E)$

Here and subsequently, S denotes a semigroup and E is a locally convex space. Let $\ell^\infty(S, E)$ denote the vector space of all vector-valued functions defined on S that take values in E such that, for each $f \in \ell^\infty(S, E)$, $f(S)$ is bounded in E. Let $\mathcal{U}_0$ be a neighborhood base of 0 in E. When V runs through $\mathcal{U}_0$, the family $M(V) = \{f \in \ell^\infty(S, E) : f(S) \subset V\}$ is a 0-neighborhood base in $\ell^\infty(S, E)$ for a unique translation-invariant topology $\mathcal{T}$, called the topology of uniform convergence on S, and $\ell^\infty(S, E)$ is a locally convex space under the $\mathcal{T}$-topology; for more details we refer the reader to [36]. For each $s \in S$, the right and left translation operators $r(s)$ and $l(s)$ on $\ell^\infty(S, E)$ are defined similar to (2.1). Let $AP(S, E)$ be the set of all $f \in \ell^\infty(S, E)$ such that the set $R_S f$ of right translates of f is relatively compact in $(\ell^\infty(S, E), \mathcal{T})$. Denote by $AP_w(S, E)$ the set of all functions $f \in \ell^\infty(S, E)$ such that $x' \circ f \in AP(S)$ for all $x' \in E'$. Similarly, let $WAP(S, E)$ denote the set of all $f \in \ell^\infty(S, E)$ for which $R_S f$ is weakly relatively compact in $(\ell^\infty(S, E), \mathcal{T})$, and, following [16], we denote $W(S, E)$ to be the set of all functions $f \in \ell^\infty(S, E)$ such that $x' \circ f \in WAP(S)$ for all $x' \in E'$.

If E is a Banach space and S is a semigroup with identity, then the following implications about a function $f : S \to E$ with norm relatively compact range hold: $f \in AP(S, E) \Leftrightarrow f \in AP_w(S, E)$ and $f \in WAP(S, E) \Leftrightarrow f \in W(S, E)$; see [1,7,23]. Furthermore, it is known that $AP(S, E) \subset AP_w(S, E)$ and $WAP(S, E) \subset W(S, E)$. On the other hand, there are examples in which the inclusions are strict; for example, $AP(\mathbb{R}, \ell^2) \neq AP_w(\mathbb{R}, \ell^2)$ (see [16]) as well as $WAP(\mathbb{R}, \ell^p) \neq W(\mathbb{R}, \ell^p)$, $1 < p < \infty$ (see [23]).

It is goal of this section to obtain analogous descriptions of $AP_w(S, E)$ and $W(S, E)$ in a locally convex space. It is worth noting that distinctions between closed totally bounded sets and compact sets may appear in locally convex spaces, especially in incomplete spaces. We shall need the following characterization result, which is also a generalization of [1, Theorem X].

Theorem 3.1. *Let f be a bounded function from a semigroup S into a l.c.s. E. Then, $f \in AP_w(S,E)$ if and only if $R_S f$ is totally bounded in the topology of uniform convergence on S with E endowed with the weak topology, and hence $AP(S,E) \subseteq AP_w(S,E)$.*

Proof. Let $R_S f$ be totally bounded in the topology of uniform convergence on S with E equipped with the weak topology, and let $x' \in E'$. We show that $\langle f(\cdot), x'\rangle \in AP(S)$. It suffices to show that $R_S\langle f(\cdot), x'\rangle$ is totally bounded in the complete space $\ell^\infty(S)$. Fix $\epsilon > 0$ and take $V = \{x \in E;\ |\langle x, x'\rangle| < 2\epsilon\}$. By totally boundedness of $R_S f$ in $\ell^\infty(S,E)$, we have

$$R_S f \subset \bigcup_{i=1}^{n} r(t_i)f + M(S,V),$$

for some $t_1,\dots,t_n \in S$, which is equivalent to

$$R_S\langle f(\cdot), x'\rangle \subset \bigcup_{i=1}^{n} B_{\|.\|}(r(t_i)\langle f(\cdot), x'\rangle, \epsilon).$$

That is, $R_S\langle f(\cdot), x'\rangle$ is totally bounded in $\ell^\infty(S)$, for all $x' \in E'$; thus $f \in AP_w(S,E)$. To prove the necessity, assume that $f \in AP_w(S,E)$ and take $W = \{x \in E;\ |\langle x, x'_i\rangle| < \epsilon,\ x'_i \in E', i = 1,\dots,n\}$. Then, the set $\prod_{i=1}^{n} R_S\langle f(\cdot), x'_i\rangle$ is relatively compact, and so is

$$K = \{(r(t)\langle f(\cdot), x'_1\rangle, \dots, r(t)\langle f(\cdot), x'_n\rangle);\ t \in S\}.$$

Then, there exist $t_1 \dots, t_m \in S$ such that

$$K \subset \bigcup_{j=1}^{m} (r(t_j)\langle f(\cdot), x'_1\rangle, \dots, r(t_j)\langle f(\cdot), x'_n\rangle) + \prod_{i=1}^{n} B_{\epsilon/2},$$

where $B_{\epsilon/2}$ is the ball of radius $\epsilon/2$ in $\ell^\infty(S)$. We see immediately that $R_S f \subset \bigcup_{j=1}^{m} r(t_j)f + M(S,W)$. Therefore, $R_S f$ is totally bounded. □

We now present a useful characterizations of $W(S,E)$. For this purpose we introduce a topology on $\ell^\infty(S,E)$ defined as follows. For $x' \in E'$ and $\mu \in \ell^\infty(S)'$, we define $\varphi_{x',\mu} \in \ell^\infty(S,E)'$ by $\varphi_{x',\mu}(g) = \mu\langle g, x'\rangle$. Let $\Gamma_{x'} = \{\varphi_{x',\mu} :\ \mu \in \ell^\infty(S)'\}$ and $\Gamma = \bigcup_{x' \in E'} \Gamma_{x'}$. The $\sigma(\ell^\infty(S,E), \Gamma)$ topology is a locally convex topology on $\ell^\infty(S,E)$ which is weaker than the weak topology on $\ell^\infty(S,E)$ since $\Gamma \subset \ell^\infty(S,E)'$.

Theorem 3.2. *Let f be a bounded function from a semigroup S into a l.c.s. E. Then, for any $x' \in E'$, $R_S f$ is relatively compact in the $\sigma(\ell^\infty(S,E), \Gamma_{x'})$ topology if and only if $x' \circ f \in WAP(S)$. In particular, $WAP(S,E) \subseteq W(S,E)$.*

Proof. Take some $x' \neq 0$ in E' and choose $y \in E$ such that $\langle y, x'\rangle = 1$. Let $R_S f$ be relatively compact in the $\sigma(\ell^\infty(S,E), \Gamma_{x'})$ topology, and suppose that

$\overline{R_S\langle f(\cdot),x'\rangle}^w \subset \bigcup_\alpha V_\alpha$, where the sets V_α are open in the weak topology. We may assume that

$$V_\alpha = \{g \in \ell^\infty(S);\ |\langle g-g_\alpha, \mu_{\alpha_i}\rangle| < 1,\ g_\alpha \in \ell^\infty(S),\ \mu_{\alpha_i} \in \ell^\infty(S)',\ i=1,\ \ldots n_\alpha\}.$$

Set

$$W_\alpha = \{h \in \ell^\infty(S,E);\ |\varphi_{x',\mu_{\alpha_i}}(h-g_\alpha y)| < 1,\ i=1,\ \ldots n_\alpha\},$$

where $\varphi_{x',\mu_{\alpha_i}}(h) = \mu_{\alpha_i}\langle h, x'\rangle$.

Claim 1. For any α and $h \in \ell^\infty(S,E)$, $h \in W_\alpha$ if and only if $x' \circ h \in V_\alpha$, and consequently $\overline{R_S f}^{\Gamma_{x'}} \subseteq \bigcup_\alpha W_\alpha$.

The first part of the claim is trivial. In order to prove the latter statement let $h \in \overline{R_S f}^{\Gamma_{x'}}$ and consider a net $\{r(s_\gamma)f\}$ converging to h in the $\sigma(\ell^\infty(S,E),\Gamma_{x'})$ topology. Then, $\{r(s_\gamma)\langle f(\cdot),x'\rangle\} \subseteq \overline{R_S\langle f(\cdot),x'\rangle}^w$ converges weakly to $x'\circ h$. Since $\overline{R_S\langle f(\cdot),x'\rangle}^w \subset \bigcup_\alpha V_\alpha$, there exists some α for which $x'\circ h \in V_\alpha$. From the first assertion, we have $h \in W_\alpha$. Hence, we have proved that $\overline{R_S f}^{\Gamma_{x'}} \subseteq \bigcup_\alpha W_\alpha$.

Claim 2. $x' \circ W_\alpha = V_\alpha$ and $x' \circ (\overline{R_S f}^{\Gamma_{x'}}) = \overline{R_S\langle f(\cdot),x'\rangle}^w$.

It is easy to see that $x'\circ W_\alpha \subseteq V_\alpha$. The reverse inclusion follows from the fact that, for each $g \in V_\alpha$, we have $gy \in W_\alpha$ and then $g = x'\circ gy \in x' \circ W_\alpha$. To verify the second assertion, let $g \in \overline{R_S\langle f(\cdot),x'\rangle}^w$ and assume that $\{r(s_\gamma)\langle f(\cdot),x'\rangle\}$ converges weakly to g. By definition, $\{r(s_\gamma)f\}$ converges to gy in the $\sigma(\ell^\infty(S,E),\Gamma_{x'})$ topology and thus $g = x'\circ gy \in x'\circ(\overline{R_S f}^{\Gamma_{x'}})$. The reverse inclusion is trivial.

Since $R_S f$ is relatively compact in the $\sigma(\ell^\infty(S,E),\Gamma_{x'})$ topology, we conclude using Claim 1 that there is a finite covering of $R_S f$, say $\overline{R_S f}^{\Gamma_{x'}} \subseteq \bigcup_{j=1}^m W_{\beta_j}$. Hence, $x'\circ(\overline{R_S f}^{\Gamma_{x'}}) \subseteq \bigcup_{j=1}^m x'\circ W_{\beta_j}$, which implies by Claim 2 that $\overline{R_S\langle f(\cdot),x'\rangle}^w \subseteq \bigcup_{j=1}^m V_{\beta_j}$, and therefore, $R_S\langle f(\cdot),x'\rangle$ is relatively compact. The proof of the converse is similar. Indeed, let $\overline{R_S f}^{\Gamma_{x'}} \subseteq \bigcup_\alpha W_\alpha$, where

$$W_\alpha = \{h \in \ell^\infty(S,E);\ |\varphi_{x',\mu_{\alpha_i}}(h-h_\alpha)| < 1,\ h_\alpha \in \ell^\infty(S,E),\ i=1,\ \ldots n_\alpha\},$$

and choose open sets U_α in the weak topology by

$$U_\alpha = \{g \in \ell^\infty(S);\ |\langle g - x'\circ h_\alpha, \mu_{\alpha_i}\rangle| < 1,\ i=1,\ \ldots n_\alpha\}.$$

It is easy to check that similarly the conclusions of Claims 1 and 2 are valid in this case. Then, $\overline{R_S\langle f(\cdot),x'\rangle}^w = x'\circ(\overline{R_S f}^{\Gamma_{x'}}) \subseteq \bigcup_\alpha x'\circ W_\alpha = \bigcup_\alpha U_\alpha$. We conclude using the assumption of weak compactness on $\overline{R_S\langle f(\cdot),x'\rangle}^w$ that there exists a finite covering $\overline{R_S\langle f(\cdot),x'\rangle}^w \subseteq \bigcup_{j=1}^m U_{\beta_j}$. In other words, $x'\circ(\overline{R_S f}^{\Gamma_{x'}}) \subseteq \bigcup_{j=1}^m x'\circ W_{\beta_j}$, and by the conclusions of Claims 1 and 2, we have $(\overline{R_S f}^{\Gamma_{x'}}) \subseteq \bigcup_{j=1}^m W_{\beta_j}$. This shows that $R_S f$ is relatively compact in the $\sigma(\ell^\infty(S,E),\Gamma_{x'})$ topology. □

As a consequence of this, we deduce the following result, extending a previous result of [23, Corollary 5] (where the range $f(S)$ was assumed to be relatively norm-compact in a Banach space).

Theorem 3.3. *Let f be a function on a semigroup S with a relatively weakly compact range in a l.c.s. E. Then, $f \in W(S)$ if and only if $R_S f$ is relatively compact in the $\sigma(\ell^\infty(S,E),\Gamma)$ topology.*

Proof. Let $f \in W(S)$ and choose a net $r(s_\alpha)f$ in $R_S f$. We must show that $r(s_\alpha)f$ contains a $\sigma(\ell^\infty(S,E),\Gamma)$-convergent subnet. Let $\overline{R_S f}^{\Gamma_{x'}}$ denote the closure of $R_S f$ in the $\sigma(\ell^\infty(S,E),\Gamma_{x'})$ topology, and let $\Upsilon = \prod_{x'\in E'} \overline{R_S f}^{\Gamma_{x'}}$. For each α, we define $\mathcal{Z}_\alpha \in \prod_{x'\in E'} \overline{R_S f}^{\Gamma_{x'}}$ by $\mathcal{Z}_\alpha(x') = r(s_\alpha)f$, for all $x' \in E'$. In view of Theorem 3.2, $\overline{R_S f}^{\Gamma_{x'}}$ is compact in the $\sigma(\ell^\infty(S,E),\Gamma_{x'})$ topology, and then, by Tychonoff's compactness theorem, Υ is compact with respect to the product topology. It follows that $(\mathcal{Z}_\alpha)$ has a subnet $(\mathcal{Z}_{\alpha_\beta})$ convergent in Υ to some $\mathcal{Z} = \prod_{x'\in E'} g_{x'} \in \Upsilon$. Thus, by definition of the product topology,

$$\lim_\beta \varphi_{x',\mu}(r(s_{\alpha_\beta})f) = \varphi_{x',\mu}(g_{x'}), \quad \forall x' \in E',\ \forall \mu \in \ell^\infty(S)'. \tag{3.1}$$

for all $x' \in E'$ and $\mu \in \ell^\infty(S)'$. For $t \in S$, let $\delta_t \in \ell^\infty(S)'$ be the evaluation map at t (i.e., $\delta_t(g) = g(t)$, for any $g \in \ell^\infty(S)$). Taking $\mu = \delta_t$ in (3.1), we have

$$\lim_\beta \langle r(s_{\alpha_\beta})f(t), x'\rangle = \langle g_{x'}(t), x'\rangle, \ \ \forall x' \in E'. \tag{3.2}$$

Since $f(S)$ is relatively weakly compact, in view of Tychonoff's compactness theorem for $\prod_{t\in S} \overline{f(tS)}^w$, it is readily seen that there is a subnet $(r(s_{\alpha_{\beta_\gamma}})f)_\gamma$ of $(r(s_{\alpha_\beta})f)_\beta$ such that $(r(s_{\alpha_{\beta_\gamma}})f(t))_\gamma$ converges weakly to some ξ_t in $\overline{f(tS)}^w$, for each $t \in S$. Defining $\xi : S \to E$ by $\xi(t) = \xi_t$, we see that ξ is bounded and $\langle \xi(t), x'\rangle = \langle g_{x'}(t), x'\rangle$, for all $t \in S$ and $x' \in E'$, by (3.2). Consequently, (3.1) shows that $(r(s_{\alpha_{\beta_\gamma}})f)$ is $\sigma(\ell^\infty(S,E),\Gamma)$-convergent to ξ, and the proof of the necessity is complete. The sufficiency follows from Theorem 3.2. □

4. Mean ergodic theorems for almost convergent functions

This section deals with some classes of functions on a semigroup S that are almost convergent. We prove new mean ergodic theorems for almost convergent functions which will be essential to obtain our main results stated in the next section. Throughout this section, we assume that X is a translation-invariant closed linear subspace of $l^\infty(S)$ containing constant functions.

Definition 4.1. For $\mu \in X'$, the left introversion operator determined by μ is the mapping $T_\mu : X \to l^\infty(S)$ defined by

$$(T_\mu f)(s) = \mu(l(s)f) \quad (f \in X,\ s \in S).$$

The right introversion operator determined by μ is the mapping $U_\mu : X \to l^\infty(S)$ defined by

$$(U_\mu f)(s) = \mu(r(s)f) \quad (f \in X,\ s \in S).$$

The properties of right introversion operators are analogous to those of left introversion operators. It is a useful fact that if $f \in X$, then $\{T_\mu f : \mu \in M(X)\}$ is the closure in $l^\infty(S)$ of $co(R_S(f))$ in the topology p of pointwise convergence on S; that is, $\{T_\mu f : \mu \in M(X)\} = \overline{co}^p(R_S(f))$. See, e.g., [7, page 73].

Remark 4.2. A net in $l^\infty(S)$ converges in the weak-star topology of $l^\infty(S)$, as the dual of $l^1(S)$ (see [13]), if it is norm bounded and converges coordinatewise. In other words, in a bounded subset of $l^\infty(S)$, the weak-star limit agrees with the pointwise limit (and then a bounded subset of $l^\infty(S)$ is pointwise closed if and only if it is weak-star closed); in fact, if (f_α) is a net in $\ell^\infty(S)$ converging in the weak-star topology to f, we will have no difficulty verifying that f_α converges pointwise to f. On the other hand, we assume that $(f_\alpha)_{\alpha\in I}$ is a bounded net in $\ell^\infty(S)$ converging in the pointwise topology to f, and prove that it is weak-star convergent. Otherwise, there would exist a weak-star neighborhood V_f of f such that, for each $\alpha \in I$, there exists $\gamma_\alpha \geq \alpha$ such that $f_{\gamma_\alpha} \notin V_f$. Let

$$J = \{\gamma \in I : f_\gamma \notin V_f\}.$$

For any $\gamma_1, \gamma_2 \in J$, choosing some $\alpha \in I$ with $\gamma_1, \gamma_2 \leq \alpha$, we may find $\gamma_\alpha \geq \alpha$ such that $f_{\gamma_\alpha} \notin V_f$. That is, there exists $\gamma_\alpha \in J$ with $\gamma_1, \gamma_2 \leq \gamma_\alpha$. Therefore, $(f_\gamma)_{\gamma\in J}$ is a subnet of $(f_\alpha)_{\alpha\in I}$. Moreover, $(f_\gamma)_{\gamma\in J}$ has no subnet converging weak-star to f. Since, by the Banach–Alaoglu theorem and pointwise convergence assumption on $(f_\alpha)_{\alpha\in I}$, the bounded net $(f_\gamma)_{\gamma\in J}$ has a convergent subnet in the weak-star topology to f, we arrive at a contradiction.

Definition 4.3. [7, page 220] A function $f \in X$ is called left almost convergent if the set $\{\mu(f) : \mu \in LIM(X)\}$ is a singleton. f is called almost convergent if it is both left and right almost convergent.

It is known that if S is a commutative semitopological semigroup, $WAP(S)$ has a unique invariant mean [7, page 162], and then each function in $WAP(S)$ is almost convergent. In the following we are going to present other classes of almost convergent functions.

Lemma 4.4. *If X is amenable and $f \in X$ satisfies $\overline{co}(R_S(f)) = \overline{co}^p(R_S(f))$, then f is almost convergent. In particular, every weakly almost periodic $f \in X$ is almost convergent.*

Proof. Let $\nu \in RIM(X)$. Then, for any $g \in \overline{co}(R_S(f))$ it easily follows $\nu(g) = \nu(f)$. Then, giving $\mu \in LIM$, we have $c := \mu(f) = T_\mu f \in \overline{co}^p(R_S(f)) = \overline{co}(R_S(f))$ and hence $T_\mu f = \nu(c) = \nu(f)$. That is,

$$\{\mu(f) : \mu \in LIM(X)\} = \{\nu(f) : \nu \in RIM(X)\} \tag{4.1}$$

is a singleton. □

Remark 4.5. Under the assumptions of Lemma 4.4, it easily follows by (4.1) that $\overline{co}^p(R_S(f))$ and $\overline{co}^p(L_S(f))$ have the same constant function.

Lemma 4.6. *If X is left amenable and right countable amenable, then each f in X is left almost convergent.*

Proof. Let ϑ be a countable right invariant mean on X and let $f \in X$. Then, $\vartheta(g) = \vartheta(f)$, for all $g \in \mathrm{co}(R_S(f))$. We show that f is left almost convergent. Let μ be a left invariant mean on X. Then, $c = T_\mu f$ is a constant function contained in $\overline{co}^p(R_S(f))$ and hence by Remark 4.2, we may choose a net $\{g_\beta\}$ in $\mathrm{co}(R_S(f))$ converging weak-star to c. On the other hand, by virtue of [13], ϑ may be considered as a member of $l^1(S)$, the predual of $l^\infty(S)$. Hence, $c = \vartheta(c) = \lim \vartheta(g_\beta) = \vartheta(f)$, and therefore f is left almost convergent. □

Let $S = \{s_1, s_2, \dots, s_n\}$ be a finite left (right) cancellative semigroup. Then, $\mu = n^{-1} \sum_{i=1}^{n} \delta(s_i)$ is a countable left (right) invariant mean on $l^\infty(S)$, where δ is the evaluation mapping.

Theorem 4.7. *Let X be a translation-invariant closed subspace of $l^\infty(S)$ containing the constant functions and let $\{\mu_\alpha\}$ be a left (resp. right) asymptotically invariant net of means on X. If $f \in X$ is left (resp. right) almost convergent, then $T_{\mu_\alpha} f$ (resp. $U_{\mu_\alpha} f$) converges weakly-star to the unique constant function in $\overline{co}^p(R_S(f))$ (resp. $\overline{co}^p(L_S(f))$).*

Proof. Since the properties of right introversion operators are analogous to those of left introversion operators, we shall prove the result only for the latter. Let f be left almost convergent. Then, there exists a constant c_f such that $\{c_f\} = \{T_\mu f : \ \mu \in LIM(X)\}$, and this is clear from $\{T_\mu f : \ \mu \in M(X)\} = \overline{co}^p(R_S(f))$ that c_f is the unique constant function in $\overline{co}^p(R_S(f))$. Suppose that a subnet $T_{\mu_{\alpha_\beta}} f$ of $T_{\mu_\alpha} f$ converges pointwise to g in $\overline{co}^p(R_S(f))$. Let μ be a cluster point of $\{\mu_{\alpha_\beta}\}$ in the weak-star topology. Since $\{\mu_\alpha\}$ is left asymptotically invariant, it easily follows that μ is a mean on X which is left invariant. Without loss of generality, we may assume that $\{\mu_{\alpha_\beta}\}$ converges weakly-star to μ. Then, $c_f = \mu(f) = \mu(l(s)f) = \lim_\beta \mu_{\alpha_\beta}(l(s)f) = \lim_\beta T_{\mu_{\alpha_\beta}} f(s) = g(s)$, for each $s \in S$. Thus, $T_{\mu_\alpha} f$ converges weakly-star to the unique constant function c_f in $\overline{co}^p(R_S(f))$. □

By Lemma 4.6 and Theorem 4.7, we have the following corollary.

Corollary 4.8. *Let $\{\mu_\alpha\}$ be a left asymptotically invariant net of means on X. If for some $f \in X$ there is a countable mean on $l^\infty(S)$ that is right invariant on $R_S(f)$, then $T_{\mu_\alpha} f$ converges weakly-star to the unique constant function c_f in $\overline{co}^p(R_S(f))$.*

The following result is immediate in view of Lemma 4.4, Remark 4.5 and Theorem 4.7.

Corollary 4.9. *Let X be an amenable translation-invariant closed subspace of $l^\infty(S)$ containing the constant functions and let $\{\mu_\alpha\}$ be a left (resp. right) asymptotically invariant net of means on X. If $f \in X$ is weakly almost periodic, then $T_{\mu_\alpha} f$ (resp. $U_{\mu_\alpha} f$) converges weakly in X to the unique constant* *function in* $\overline{co}(R_S(f))$*; when* $f \in X$ *is almost periodic, the convergence is strong.*

As has been observed above, the convergence of $T_{\mu_\alpha} f$ relates the left asymptotically invariance of $\{\mu_\alpha\}$; the question arises whether it is true that $T_{\mu_\alpha} f$ converges, when $\{\mu_\alpha\}$ is right asymptotically invariant. In the following we are going to answer it and even more.

Theorem 4.10. *Let* X *be an amenable translation-invariant closed subspace of* $l^\infty(S)$ *containing the constant functions and let* $\{\mu_\alpha\}$ *be a right asymptotically invariant net of means on* X*. If* $f \in X$ *satisfies* $\overline{co}(R_S(f)) = \overline{co}^p(R_S(f))$*, then* $T_{\mu_\alpha} f$ *and* $U_{\mu_\alpha} f$ *converge pointwise to the unique constant function* c_f *in* $\overline{co}(R_S(f))$*; moreover,* $T_{\mu_\alpha} f \to c_f$ *in the sup-norm topology if, in addition,* $\{\mu_\alpha\}$ *is right strongly asymptotically invariant.*

Proof. Since $\{\mu_\alpha\}$ is a right asymptotically invariant, $U_{\mu_\alpha} f$ converges pointwise to the unique constant function c_f in $\overline{co}(R_S(f))$, in view of Lemma 4.4 and Theorem 4.7. We prove that $T_{\mu_\alpha} f$ converges pointwise to c_f. Fix $\varepsilon > 0$. There exists a convex combination $h = \sum_{i=1}^n \lambda_i r(s_i) f$ of right translates of f such that $\|h - c_f\| < \varepsilon/2$, where $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ and $s_1, \ldots, s_n \in S$. Thus, for each α and $t \in S$,

$$|\mu_\alpha(l(t)h - c_f)| < \varepsilon/2. \tag{4.2}$$

On the other hand, for each $t, s \in S$,

$$\begin{aligned} \mu_\alpha(l(t)f) - \mu_\alpha(l(t)r(s)f) &= \mu_\alpha(l(t)f) - \mu_\alpha(r(s)l(t)f) \\ &= (\mu_\alpha - r(s)'\mu_\alpha)(l(t)f) \qquad (4.3) \\ &\leq \|\mu_\alpha - r(s)'\mu_\alpha\| \|f\|. \qquad (4.4) \end{aligned}$$

Using the right asymptotically invariance of $\{\mu_\alpha\}$ in (4.3), it follows that

$$\mu_\alpha(l(t)f) - \mu_\alpha(l(t)r(s)f) \to 0,$$

for each $t, s \in S$. Thus, for any $t \in S$, there exists an α_0 such that, for each $\alpha \geq \alpha_0$ and $i = 1, \ldots, n$,

$$|\mu_\alpha(l(t)f) - \mu_\alpha(l(t)r(s_i)f)| < \varepsilon/2.$$

Then, we have, for each $\alpha \geq \alpha_0$,

$$\begin{aligned} &|\mu_\alpha(l(t)f - c_f)| \\ &\quad \leq |\mu_\alpha(l(t)f) - \mu_\alpha(l(t)h)| + |\mu_\alpha(l(t)h - c_f)| \\ &\quad < \sum_{i=1}^n \lambda_i |\mu_\alpha(l(t)f) - \mu_\alpha(l(t)r(s_i)f)| + \varepsilon/2 < \varepsilon. \end{aligned}$$

This implies that, for each $t \in S$,

$$(T_{\mu_\alpha} f - c_f)(t) = \mu_\alpha(l(t)f - c_f) \to 0,$$

which proves the first assertion. Now let $\{\mu_\alpha\}$ be right strongly asymptotically invariant; then it follows from (4.4) that

$$\sup_{t \in S} |\mu_\alpha(l(t)f) - \mu_\alpha(l(t)r(s)f)| \to 0,$$

for each $s \in S$. Similarly, there exists some α_0 such that, for each $\alpha \geq \alpha_0$ and $i = 1, \ldots, n$,

$$\sup_{t \in S} |\mu_\alpha(l(t)f) - \mu_\alpha(l(t)r(s_i)f)| < \varepsilon/2. \tag{4.5}$$

Thus, in view of (4.2) and (4.5), we have, for each $\alpha \geq \alpha_0$,

$$\begin{aligned} \|T_{\mu_\alpha} f - c_f\| &= \sup_{t \in S} |\mu_\alpha(l(t)f - c_f)| \\ &\leq \sup_{t \in S} |\mu_\alpha(l(t)f) - \mu_\alpha(l(t)h)| + \sup_{t \in S} |\mu_\alpha(l(t)h) - c_f| \\ &\leq \sum_{i=1}^{n} \lambda_i \sup_{t \in S} |\mu_\alpha(l(t)f) - \mu_\alpha(l(t)r(s_i)f)| + \varepsilon/2 < \varepsilon; \end{aligned}$$

that is, $\lim_\alpha \|T_{\mu_\alpha} f - c_f\| = 0$, and the proof of the second assertion is complete. □

Corollary 4.11. *Let X be an amenable translation-invariant closed subspace of $l^\infty(S)$ containing the constant functions and let $\{\mu_\alpha\}$ be a right asymptotically invariant net of means on X. If $f \in X$ is weakly almost periodic, then $T_{\mu_\alpha} f$ and $U_{\mu_\alpha} f$ converge weakly to the unique constant function c_f in $\overline{co}(R_S(f))$; moreover, $T_{\mu_\alpha} f \to c_f$ in the sup-norm topology if, in addition, $\{\mu_\alpha\}$ is right strongly asymptotically invariant.*

Question 1. Does $U_{\mu_\alpha} f$, in the above theorem, converge in the sup-norm topology if $\{\mu_\alpha\}$ is right strongly asymptotically invariant?

5. Mean ergodic theorems in locally convex spaces

We begin this section by stating the following lemma, which is taken from [19] in the case of Banach spaces. For the sake of completeness, we include a proof here.

Lemma 5.1. *Let S be a semigroup, E a l.c.s., f a function of S into a weakly compact convex subset of E and X a subspace of $l^\infty(S)$ containing constants and all functions $s \mapsto \langle f(s), x' \rangle$ with $x' \in E'$. Then, for any $\mu \in X'$, there exists a unique element $\tau(\mu)f$ in E such that $\langle \tau(\mu)f, x' \rangle = \mu_t \langle f(t), x' \rangle$, for all $x' \in E'$. If μ is a mean on X, then $\tau(\mu)f \in \overline{co}\{f(t) : t \in S\}$. $\tau(\mu)f$ will sometimes be denoted by $\int f(s) d\mu(s)$.*

Proof. Set $K = \overline{\mathrm{co}}(\{\|\mu\| f(t) : t \in S\} \cup \{0\})$ and $M = K - K$. Then, M is a balanced weakly compact convex subset of E, since $\mathrm{co}\{f(t) : t \in S\} \subset D$ is relatively weak compact. From this, we may thus consider M as a balanced $\sigma(E'', E')$-compact convex subset of E''. Thus, by the bipolar theorem for the dual system $\langle E'', E' \rangle$ (see, e.g., [36, Theorem 1.5]), we have $M^{\circ\circ} = M$. Now, defining $\tau(\mu)f$ on E' by $\tau(\mu)f : x' \mapsto \mu\langle f(\cdot), x' \rangle$, it easily follows that $\tau(\mu)f$ is a continuous linear functional for which

$$|(\tau(\mu)f)(y')| = |\mu\langle f(\cdot), y' \rangle| \leq \sup_{t \in S} |\langle \|\mu\| f(t), y' \rangle| \leq \sup_{x \in M} \langle x, y' \rangle \leq 1, \ \forall y' \in M^\circ.$$

Therefore, $\tau(\mu)f \in M^{\circ\circ} = M$, which means that $\tau(\mu)f$ can be viewed as an element in E. If μ is a mean on X, there exists a net $\{\lambda_\beta\}_{\beta\in\Lambda}$ of finite means on S such that λ_β converges to μ in the weak-star topology. It is easy to check that $\tau(\lambda_\beta)f \in \text{co}\{f(t) :\ t \in S\}$, $\forall\beta$. Thus,

$$\langle\tau(\mu)f, x'\rangle = \mu\langle f(\cdot), x'\rangle = \lim_\beta \lambda_\beta\langle f(\cdot), x'\rangle = \lim_\beta\langle\tau(\lambda_\beta)f, x'\rangle,$$

for all $x' \in E'$, which yields $\tau(\mu)f \in \overline{\text{co}}\{f(t) :\ t \in S\}$. □

A well-known result that goes back to M. Krein and V. Smulian says the following: the closed convex hull of a weakly compact subset K of a Banach space E is also weakly compact. We point out that this result may not be true for a l.c.s. E even if K is compact: consider $E = \ell^1$ and the topology τ on E induced by the coordinate-axis vectors $\{e_n\} \subset \ell^\infty$. Then, (E, τ) is a locally convex topological space. Set $K = \{ne_n\} \cup \{0\} \subset \ell^1$. Obviously, K is τ-compact, but the τ-closure of $\text{co}(K)$ is not τ-compact.

On the other hand, the Krein–Smulian property plays an important role in generalizations of Baillon's mean ergodic result. The authors of [25] attempted to extend some known mean ergodic results regarding $AP(S, E)$ and $WAP(S, E)$ to locally convex spaces, however, the proofs depend upon the Krein–Smulian property and Mazur's theorem, and while they are true for Banach spaces, they may not be valid for the case of locally convex spaces. In our mean ergodic results, however, we will not use the above property explicitly. With our presented tools, we can obtain results that are new even in the context of Banach spaces.

Theorem 5.2. *Let S be a semigroup, E a l.c.s., f a function of S into a weakly compact convex subset of E and X a translation-invariant amenable subspace of $l^\infty(S)$ containing constants and all functions $s \mapsto \langle f(s), x'\rangle$ with $x' \in E'$. Let $\{\mu_\alpha\}$ be a right asymptotically invariant net of means on X. Assume that $f \in W(S, E)$. Then, $\tau_{\mu_\alpha}(l(\cdot)f)$ and $\tau_{\mu_\alpha}(r(\cdot)f)$ converge in the $\sigma(\ell^\infty(S, E), \Gamma)$ topology to a constant function with a value x in $\cap_{s\in S}\overline{co}\{f(ts) :\ t \in S\}$. If, in addition, $\{\mu_\alpha\}$ is strongly right asymptotically invariant, then $\int f(ts)d\mu_\alpha(s)$ converges weakly to x, uniformly in $t \in S$.*

Proof. Let μ be an arbitrary cluster point of $\{\mu_\alpha\}$ in the weak-star topology, which is right invariant since $\{\mu_\alpha\}$ is right asymptotically invariant. For every $x' \in E'$, let $f_{x'} = x' \circ f$, so that $f_{x'} \in WAP(S)$. It follows from Corollary 4.11 and Remark 4.5 that both of the nets $\langle\tau_{\mu_\alpha}(l(\cdot)f), x'\rangle$ $(= \mu_\alpha(l(\cdot)f_{x'}))$ and $\langle\tau_{\mu_\alpha}(r(\cdot)f), x'\rangle$ $(= \mu_\alpha(r(\cdot)f_{x'}))$ converge weakly to the constant function $\langle\tau_\mu(f), x'\rangle = \mu(f_{x'}) = c_{f_{x'}} \in \overline{co}(L_S(f_{x'})) \cap \overline{co}(R_S(f_{x'}))$, for each $x' \in E'$. Therefore, we find that, for every $x' \in E'$ and $\nu \in \ell^\infty(S)'$, $\varphi_{x',\nu}(\tau_{\mu_\alpha}(l(\cdot)f) - \tau_\mu(f)) \to 0$ and $\varphi_{x',\nu}(\tau_{\mu_\alpha}(r(\cdot)f) - \tau_\mu(f)) \to 0$. We conclude (from the definition of Γ and the right invariance of μ) the first assertion, as well as the second assertion results easily from Theorem 4.10. □

The following result is a direct consequence of Theorem 5.2.

Theorem 5.3. *Let S be a semigroup, let E be a l.c.s. and let f a function of S into a weakly compact convex subset D of E. Let X be a translation-invariant amenable subspace of $l^\infty(S)$ containing constants and all functions*

$s \mapsto \langle f(s), x' \rangle$ with $x' \in E'$, and let $\{\mu_\alpha\}$ be a right asymptotically invariant net of means on X. Assume that $f \in AP_w(S, E)$. Then, $\int f(ts)d\mu_\alpha(s)$ and $\int f(st)d\mu_\alpha(s)$ converge weakly to a point in $\cap_{s\in S}\overline{co}\{f(us) : u \in S\}$, uniformly in $t \in S$.

For the case that X is not assumed to be two-sided amenable, we can obtain an asymptotic version of the ergodic result obtained in Theorem 5.3.

Theorem 5.4. *Let S be a semigroup, let E be a l.c.s., let f be a function of S into a weakly compact convex subset of E, and let X be a translation-invariant subspace of $l^\infty(S)$ containing constants and all functions $s \mapsto \langle f(s), x' \rangle$ with $x' \in E'$. Let $\{\mu_\alpha\}$ be a right asymptotically invariant net of means on X. If $f \in AP_w(S, E)$, then*

$$\int f(st)d\mu_\alpha(s) - \int f(s)d\mu_\alpha(s) \rightharpoonup 0,$$

uniformly in $t \in S$.

Proof. Since $f \in AP_w(S, E)$, it follows by Theorem 3.1 that $R_S f$ is totally bounded in the topology of uniform convergence on S with E endowed with the weak topology. Choose $x' \in E'$ and $\epsilon > 0$, and take $V = \{x \in E;\ |\langle x', x\rangle| < \epsilon\}$; then there exists a finite subset M of S such that

$$R_S f \subset \bigcup_{t\in M} r(t)f + 2^{-1}M(S, V). \tag{5.1}$$

On the other hand, since μ_α is asymptotically invariant, for each $t \in S$, we have

$$|\langle \tau(r(t)'\mu_\alpha)f - \tau(\mu_\alpha)f, x'\rangle| = |(r(t)'\mu_\alpha - \mu_\alpha)\langle f, x'\rangle| \to 0.$$

Thus, there exists α_0, such that, for any $\alpha \ge \alpha_0$ and t in the finite set M,

$$|\langle \tau(r(t)'\mu_\alpha)f - \tau(\mu_\alpha)f, x'\rangle| < 2^{-1}\epsilon. \tag{5.2}$$

Picking up an arbitrary $h \in S$, we can choose some $k \in M$ such that $r(h)f - r(k)f \in 2^{-1}M(S, V)$, by (5.1). Consequently, we have

$$|\langle \tau(r(h)'\mu_\alpha)f - \tau(r(k)'\mu_\alpha)f, x'\rangle| = |\mu_\alpha\langle r(h)f - r(k)f, x'\rangle| \le 2^{-1}\epsilon, \tag{5.3}$$

for each α. Therefore, in view of (5.2) and (5.3), we have

$$\begin{aligned}
&|\langle \tau(r(h)'\mu_\alpha)f - \tau(\mu_\alpha)f, x'\rangle| \\
&\quad \le |\langle \tau(r(h)'\mu_\alpha)f - \tau(r(k)'\mu_\alpha)f, x'\rangle| \\
&\qquad + |\langle \tau(r(k)'\mu_\alpha)f - \tau(\mu_\alpha)f, x'\rangle| < 2^{-1}\epsilon + 2^{-1}\epsilon = \epsilon,
\end{aligned}$$

for all $h \in S$ and $\alpha \ge \alpha_0$; that is, $\tau(r(h)'\mu_\alpha)f - \tau(\mu_\alpha)f \in V$, for all $h \in S$ and $\alpha \ge \alpha_0$. □

Question 2. What can be said in the latter result about the convergence of $\tau(\mu_\alpha)f$?

Let S be a semigroup, let E be a l.c.s., let C be a weakly compact convex subset of E, and let $\mathcal{S} = \{T(t) : t \in S\}$ be a representation of S as mappings of C into itself. Let X be a closed subspace of $l^\infty(S)$ containing constants and all functions $s \mapsto \langle T(s)x, x'\rangle$ with $x \in C$ and $x' \in E'$. In view of Lemma 5.1, for $\nu \in X'$, we can define a mapping $T(\nu) : C \to C$ by $T(\nu)x = \tau_\nu(T(\cdot)x) = \int T(s)xd\nu(s)$, for all $x \in C$; moreover, for a right invariant mean μ on X, we have $T(\mu)x \in \cap_{s\in S}\overline{\text{co}}\{T(ts)x : \ t \in S\}$. If $\mathcal{S}$ is weak to weak equicontinuous at some point x, $T(\nu)$ is immediately weak to weak continuous at x.

Theorem 5.5. *Let S be a semigroup, let E be a l.c.s., let C be a weakly compact convex subset of E, and let $\mathcal{S} = \{T(t) : t \in S\}$ be a representation of S as mappings of C into itself. Let X be a closed, translation invariant and amenable subspace of $l^\infty(S)$ which contains constants and all functions $s \mapsto \langle T(s)x, x'\rangle$ with $x \in C$ and $x' \in E'$. Let $\{\mu_\alpha\}$ be a right asymptotically invariant net of means on X. Fix $x \in C$.*

(i) If, for each $\nu \in X'$, $T(\nu)$ is weak to weak continuous on the weak closure of $\mathcal{S}(x)$, then $T(l(\cdot)'\mu_\alpha)x$ and $T(r(\cdot)'\mu_\alpha)x$ converge in the $\sigma(\ell^\infty(S,E),\Gamma)$ topology to a constant function with a value in $\cap_{s\in S}\overline{co}\{T(ts)x : \ t \in S\}$. If, in addition, $\{\mu_\alpha\}$ is strongly right asymptotically invariant, then $\int T(ts)xd\mu_\alpha(s)$ converges weakly to a point p in $\cap_{s\in S}\overline{co}\{T(ts)x : \ t \in S\}$, uniformly in $t \in S$.

(ii) If $\mathcal{S}$ is weak to weak equicontinuous on the weak closure of $\mathcal{S}(x)$, then $\int T(ts)xd\mu_\alpha(s)$ and $\int T(st)xd\mu_\alpha(s)$ converge weakly to a point in $\cap_{s\in S}\overline{co}\{T(ts)x : \ t \in S\}$, uniformly in $t \in S$.

Proof. To prove (i), suppose that $T(\nu)$ is weak to weak continuous on the weak closure of $\mathcal{S}(x)$, for each $\nu \in X'$. We claim that the function $t \mapsto \langle T(t)x, x'\rangle$ belongs to $WAP(S)$, for each $x' \in E'$. It suffices to prove that any chosen net of the form $r(s_\alpha)\langle T(\cdot)x, x'\rangle$ has a weak convergent subnet in $X \subseteq \ell^\infty(S)$. Since C is weakly compact, we may choose a subnet $\{T(s_{\alpha_\beta})x\}$ in $\{T(s_\alpha)x\}$ converging weakly to some $y \in C$. Then, using the weak to weak continuity of $T(\nu)$ on the weak closure of $\mathcal{S}(x)$, we have $T(\nu)(T(s_{\alpha_\beta})x) \rightharpoonup T(\nu)(y)$, for all $\nu \in X'$. Consequently,

$$\nu\big(r(s_{\alpha_\beta})\langle T(\cdot)x, x'\rangle\big) = \langle T(\nu)(T(s_{\alpha_\beta})x), x'\rangle \to \langle T(\nu)(y), x'\rangle = \nu\big(\langle T(\cdot)y, x'\rangle\big),$$

for all $x' \in E'$ and $\nu \in X'$. Thus, $r(s_{\alpha_\beta})\langle T(\cdot)x, x'\rangle$ is a weak convergent subnet of $r(s_\alpha)\langle T(\cdot)x, x'\rangle$ in X and then $t \mapsto \langle T(t)x, x'\rangle$ belongs to $WAP(S)$, as we claimed. Now, we apply Theorem 5.2.

For (ii), we prove that, for each $x' \in E'$, the function $t \mapsto \langle T(t)x, x'\rangle$ is in $AP(S)$. For any $x' \in E'$, define a function $\psi_{\mathcal{S},x'} : y \longmapsto \langle T(\cdot)y, x'\rangle$ of C into $\ell^\infty(S)$. Since, for each $s \in S$, $r(s)\langle T(\cdot)x, x'\rangle = \langle T(\cdot s)x, x'\rangle = \langle T(\cdot)(T(s)x), x'\rangle = \psi_{\mathcal{S},x'}(T(s)x)$, we have $R_S(\langle T(\cdot)x, x'\rangle) = \psi_{\mathcal{S},x'}(\mathcal{S}(x))$. From the weak to weak equicontinuity assumption of $\mathcal{S}$, $\psi_{\mathcal{S},x'}$ is weak to norm continuous on the weak closure of $\mathcal{S}(x)$ and thus $R_S(\langle T(\cdot)x, x'\rangle)$ is relatively compact. Now, it suffices to apply Theorem 5.3. □

Let us point out that, even in the case of a commutative semigroup acting on a Banach space, $T(\mu)x$ is not necessarily a common fixed point for $\mathcal{S}$ (see, e.g., [37]); although there are appropriate additional conditions under which $T(\mu)x$ is a common fixed point for $\mathcal{S}$ (see, e.g., [5,12,18,19,21, 34,35,38]). It is worth mentioning here that if $\mathcal{S}$ is a representation of S as continuous linear operators on E, and μ is a left invariant mean, then, for any $x \in E$ in which $\overline{\text{co}}\{T(t)x : t \in S\}$ is weakly compact, $T(\mu)x$ is a common fixed point of $\mathcal{S}$; in fact, for each $x' \in E'$ and $s \in S$, we have $x' \circ T(s) \in E'$ and consequently $\langle T(s)T(\mu)x, x'\rangle = \langle T(\mu)x, x' \circ T(s)\rangle = \mu\langle T(\cdot)x, x' \circ T(s)\rangle = \mu\langle T(s\cdot)x, x'\rangle = \mu\langle T(s\cdot)x, x'\rangle = \langle T(\mu)x, x'\rangle$; that is, $T(s)T(\mu)x = T(\mu)x$ for all $s \in S$.

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