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Fixed point theorem for mappings contracting perimeters of triangles

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Abstract. We consider a new type of mappings in metric spaces which can be characterized as mappings contracting perimeters of triangles. It is shown that such mappings are continuous. The fixed point theorem for such mappings is proved and the classical Banach fixed-point theorem is obtained like a simple corollary. Examples of mappings contracting perimeters of triangles which are not contraction mappings are constructed.

Mathematics Subject Classification. Primary 47H10, Secondary 47H09.

Keywords. Fixed point theorems, mappings contracting perimeters of triangles, metric space.

1. Introduction

The Contraction Mapping Principle was established by S. Banach in his dissertation (1920) and published in 1922 [1]. Although the idea of successive approximations in a number of concrete situations (solution of differential and integral equations, approximation theory) had appeared earlier in the works of P. L. Chebyshev, E. Picard, R. Caccioppoli, and others, S. Banach was the first who formulated this result in a correct abstract form suitable for a wide range of applications. After a century, the interest of mathematicians around the world to fixed-point theorems is still very high. This is confirmed by the appearance in recent decades of numerous articles and monographs devoted to the fixed point theory and its applications, see, e.g., the monographs [2–4] for a survey on fixed point results.

The Banach contraction principle has been generalized in many ways over the years. In Ref. [5] authors noted that except Banach's fixed point theorem there are also three classical fixed point theorems against which metric extensions are usually checked. These are, respectively, Nadler's well-known set-valued extension of Banach's theorem [6], the extension of Banach's theorem to nonexpansive mappings [7], and Caristi's theorem [8]. At the same time it is possible to distinguish at least two types of generalizations of such theorems: in the first case the contractive nature of the mapping is weakened, see, e.g. [9–18]; in the second case the topology is weakened, see, e.g. [19–32].

Let X be a metric spaces. In the present paper, we consider a new type of mappings $T\colon X \to X$ which can be characterized as mappings contracting perimeters of triangles and prove the fixed point theorem for such mappings. Although the proof of the main theorem of this work is based on the ideas of the proof of Banach's classical theorem, the essential difference is that the definition of our mappings is based on the mapping of three points of the space instead of two. Moreover, we additionally require a condition which prevents the mapping T from having periodic points of prime period 2. The ordinary contraction mappings form an important subclass of these mappings which immediately allows us to obtain the classical Banach's theorem like a simple corollary. Examples of a mappings contracting perimeters of triangles which are not contraction mappings are constructed for spaces X with $|X| = \aleph_0$, where $|X|$ is the cardinality of the set X.

2. Mappings contracting perimeters of triangles

Definition 2.1. Let (X, d) be a metric space with $|X| \geqslant 3$. We shall say that $T\colon X \to X$ is a *mapping contracting perimeters of triangles* on X if there exists $\alpha \in [0, 1)$ such that the inequality

$$d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \leqslant \alpha(d(x, y) + d(y, z) + d(x, z)) \tag{2.1}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Remark 2.2. Note that the requirement for $x, y, z \in X$ to be pairwise distinct is essential. One can see that otherwise this definition is equivalent to the definition of contraction mapping.

Proposition 2.3. *Mappings contracting perimeters of triangles are continuous.*

Proof. Let (X, d) be a metric space with $|X| \geqslant 3$, $T\colon X \to X$ be a mapping contracting perimeters of triangles on X and let x_0 be an isolated point in X. Then, clearly, T is continuous at x_0. Let now x_0 be an accumulation point. Let us show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(Tx_0, Tx) < \varepsilon$ whenever $d(x_0, x) < \delta$. Since x_0 is an accumulation point, for every $\delta > 0$ there exists $y \in X$ such that $d(x_0, y) < \delta$. By (2.1) we have

$$\begin{aligned} d(Tx_0, Tx) &\leqslant d(Tx_0, Tx) + d(Tx_0, Ty) + d(Tx, Ty) \\ &\leqslant \alpha(d(x_0, x) + d(x_0, y) + d(x, y)). \end{aligned}$$

Using the triangle inequality $d(x, y) \leqslant d(x_0, x) + d(x_0, y)$, we have

$$d(Tx_0, Tx) \leqslant 2\alpha(d(x_0, x) + d(x_0, y)) < 2\alpha(\delta + \delta) = 4\alpha\delta.$$

Setting $\delta = \varepsilon/(4\alpha)$, we obtain the desired inequality. □

Let T be a mapping on the metric space X. A point $x \in X$ is called a *periodic point of period* n if $T^n(x) = x$. The least positive integer n for which $T^n(x) = x$ is called the prime period of x, see, e.g., [33, p. 18].

Theorem 2.4. *Let (X, d), $|X| \geqslant 3$, be a complete metric space and let $T : X \to X$ be a mapping contracting perimeters of triangles on X. Then T has a fixed point if and only if T does not possess periodic points of prime period 2. The number of fixed points is at most two.*

Proof. Let T have no periodic points of prime period 2. Let us show that T has a fixed point. Let $x_0 \in X$, $Tx_0 = x_1$, $Tx_1 = x_2$, $\ldots$, $Tx_n = x_{n+1}$, $\ldots$. Suppose that x_i is not a fixed point of the mapping T for every $i = 0, 1, \ldots$. Let us show that all x_i are different. Since x_i is not fixed, then $x_i \neq x_{i+1} = Tx_i$. Since T have no periodic points of prime period 2 we have $x_{i+2} = T(T(x_i)) \neq x_i$ and by the supposition that x_{i+1} is not fixed we have $x_{i+1} \neq x_{i+2} = Tx_{i+1}$. Hence, x_i, x_{i+1} and x_{i+2} are pairwise distinct. Further, set

$$\begin{aligned} p_0 &= d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_0), \\ p_1 &= d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1), \\ &\cdots \\ p_n &= d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_n), \\ &\cdots. \end{aligned}$$

Since x_i, x_{i+1} and x_{i+2} are pairwise distinct by (2.1) we have $p_1 \leqslant \alpha p_0$, $p_2 \leqslant \alpha p_1$, $\ldots$, $p_n \leqslant \alpha p_{n-1}$ and

$$p_0 > p_1 > \cdots > p_n > \cdots. \tag{2.2}$$

Suppose that $j \geqslant 3$ is a minimal natural number such that $x_j = x_i$ for some i such that $0 \leqslant i < j-2$. Then $x_{j+1} = x_{i+1}$, $x_{j+2} = x_{i+2}$. Hence, $p_i = p_j$ which contradicts to (2.2).

Further, let us show that $\{x_i\}$ is a Cauchy sequence. It is clear that

$$\begin{aligned} d(x_1, x_2) &\leqslant p_0, \\ d(x_2, x_3) &\leqslant p_1 \leqslant \alpha p_0, \\ d(x_3, x_4) &\leqslant p_2 \leqslant \alpha p_1 \leqslant \alpha^2 p_0, \\ &\cdots \\ d(x_n, x_{n+1}) &\leqslant p_{n-1} \leqslant \alpha^{n-1} p_0, \\ d(x_{n+1}, x_{n+2}) &\leqslant p_n \leqslant \alpha^n p_0, \\ &\cdots. \end{aligned}$$

By the triangle inequality,

$$\begin{aligned} d(x_n, x_{n+p}) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leqslant \alpha^{n-1} p_0 + \alpha^n p_0 + \cdots + \alpha^{n+p-2} p_0 = \alpha^{n-1}(1 + \alpha + \cdots + \alpha^{p-1}) p_0 \\ &= \alpha^{n-1} \frac{1-\alpha^p}{1-\alpha} p_0. \end{aligned}$$

Since by the supposition $0 \leqslant \alpha < 1$, then $d(x_n,\, x_{n+p}) < \alpha^{n-1}\frac{1}{1-\alpha}p_0$. Hence, $d(x_n,\, x_{n+p}) \to 0$ as $n \to \infty$ for every $p > 0$. Thus, $\{x_n\}$ is a Cauchy sequence. By completeness of (X, d), this sequence has a limit $x^* \in X$.

Let us prove that $Tx^* = x^*$. By the triangle inequality and by inequality (2.1) we have

$$\begin{aligned} d(x^*, Tx^*) &\leqslant d(x^*, x_n) + d(x_n, Tx^*) = d(x^*, x_n) + d(Tx_{n-1}, Tx^*) \\ &\leqslant d(x^*, x_n) + d(Tx_{n-1}, Tx^*) + d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx^*) \\ &\leqslant d(x^*, x_n) + \alpha(d(x_{n-1}, x^*) + d(x_{n-1}, x_n) + d(x_n, x^*)). \end{aligned}$$

Since all the terms in the previous sum tend to zero as $n \to \infty$, we obtain $d(x^*, Tx^*) = 0$.

Suppose that there exists at least three pairwise distinct fixed points x, y and z. Then $Tx = x$, $Ty = y$ and $Tz = z$, which contradicts to (2.1).

Conversely, let T have a fixed point x^*. Suppose that T has a periodic point x of prime period 2. Set $y = Tx$. Then

$$d(Tx, Ty) + d(Ty, Tx^*) + d(Tx, Tx^*) = d(y, x) + d(x, x^*) + d(y, x^*),$$

which contradicts to (2.1). □

Remark 2.5. Suppose that under the supposition of the theorem the mapping T has a fixed point x^* which is a limit of some iteration sequence $x_0, x_1 = Tx_0, x_2 = Tx_1, \ldots$ such that $x_n \neq x^*$ for all $n = 1, 2, \ldots$. Then x^* is a unique fixed point. Indeed, suppose that T has another fixed point $x^{**} \neq x^*$. It is clear that $x_n \neq x^{**}$ for all $n = 1, 2, \ldots$. Hence, we have that the points x^*, x^{**} and x_n are pairwise distinct for all $n = 1, 2, \ldots$. Consider the ratio

$$\begin{aligned} R_n &= \frac{d(Tx^*, Tx^{**}) + d(Tx^*, Tx_n) + d(Tx^{**}, Tx_n)}{d(x^*, x^{**}) + d(x^*, x_n) + d(x^{**}, x_n)} \\ &= \frac{d(x^*, x^{**}) + d(x^*, x_{n+1}) + d(x^{**}, x_{n+1})}{d(x^*, x^{**}) + d(x^*, x_n) + d(x^{**}, x_n)}. \end{aligned}$$

Taking into consideration that $d(x^*, x_{n+1}) \to 0$, $d(x^*, x_n) \to 0$, $d(x^{**}, x_{n+1}) \to d(x^{**}, x^*)$ and $d(x^{**}, x_n) \to d(x^{**}, x^*)$, we obtain $R_n \to 1$ as $n \to \infty$, which contradicts to condition (2.1).

Example. Let us construct an example of the mapping T contracting perimeters of triangles which has exactly two fixed points. Let $X = \{x, y, z\}$, $d(x, y) = d(y, z) = d(x, z) = 1$, and let $T\colon X \to X$ be such that $Tx = x$, $Ty = y$ and $Tz = x$. One can easily see that (2.1) holds and T does not have periodic points of prime period 2.

Example. Let us construct an example of the mapping T contracting perimeters of triangles which does not have any fixed point. Let $X = \{x, y, z\}$, $d(x, y) = d(y, z) = d(x, z) = 1$, and let $T\colon X \to X$ be such that $Tx = y$, $Ty = x$ and $Tz = x$. In this case the points x and y are periodic points of prime period 2.

Let (X, d) be a metric space. Then a mapping $T\colon X \to X$ is called a *contraction mapping* on X if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \alpha d(x, y) \tag{2.3}$$

for all $x, y \in X$.

Corollary 2.6 (Banach fixed-point theorem). *Let (X, d) be a nonempty complete metric space with a contraction mapping $T: X \to X$. Then T admits a unique fixed point.*

Proof. For $|X| = 1, 2$ the proof is trivial. Let $|X| \geqslant 3$. Suppose that there exists $x \in X$ such that $T(Tx) = x$. Consequently, $d(x, Tx) = d(Tx, x) = d(Tx, T(Tx))$, which contradicts to (2.3). Thus, T does not possess periodic points of prime period 2. Let $x, y, z \in X$ be pairwise distinct. By (2.3) we obtain $d(T(x), T(y)) \leqslant \alpha d(x, y)$, $d(T(y), T(z)) \leqslant \alpha d(y, z)$ and $d(T(x), T(z)) \leqslant \alpha d(x, z)$ which immediately implies that T is a mapping contracting perimeters of triangles on X. By Theorem 2.4 the mapping T has a fixed point.

The uniqueness can be shown in a standard way. □

Recall that for a given metric space X, a point $x \in X$ is said to be an *accumulation point* of X if every open ball centered at x contains infinitely many points of X.

Proposition 2.7. *Let (X, d), $|X| \geqslant 3$, be a metric space and let $T: X \to X$ be a mapping contracting perimeters of triangles. If x is an accumulation point of X, then inequality (2.3) holds for all points $y \in X$.*

Proof. Let $x \in X$ be an accumulation point and let $y \in X$. If $y = x$, then clearly (2.3) holds. Let now $y \neq x$. Since x is an accumulation point, then there exists a sequence $z_n \to x$ such that $z_n \neq x$, $z_n \neq y$ and all z_n are different. Hence, by (2.1) the inequality

$$d(Tx, Ty) + d(Ty, Tz_n) + d(Tx, Tz_n) \leqslant \alpha(d(x, y) + d(y, z_n) + d(x, z_n))$$

holds for every $n \in \mathbb{N}$. Since $d(x, z_n) \to 0$ and every metric is continuous we have $d(y, z_n) \to d(x, y)$. Since T is continuous, we have $d(Tx, Tz_n) \to 0$ and, consequently, $d(Ty, Tz_n) \to d(Tx, Ty)$. Letting $n \to \infty$, we obtain

$$d(Tx, Ty) + d(Tx, Ty) \leqslant \alpha(d(x, y) + d(x, y)),$$

which is equivalent to (2.3). □

Corollary 2.8. *Let (X, d), $|X| \geqslant 3$, be a metric space and let $T: X \to X$ be a mapping contracting perimeters of triangles. If all points of X are accumulation points, then T is a contraction mapping.*

Let (X, d) be a metric space and let $x, y, z \in X$. We shall say that the point y *lies between* x and z in the metric space (X, d) if the extremal version of the triangle inequality

$$d(x, z) = d(x, y) + d(y, z) \tag{2.4}$$

holds.

Example. Let us construct an example of a mapping $T: X \to X$ contracting perimeters of triangles that is not a contraction mapping for a metric space

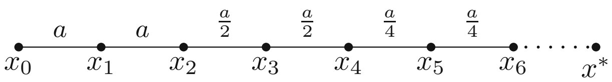

FIGURE 1. The points of the space (X, d) with consecutive distances between them

X with $|X| = \aleph_0$. Let $X = \{x^*, x_0, x_1, \ldots\}$ and let a be positive real number. Define a metric d on X as follows:

$$d(x,y) = \begin{cases} a/2^{\lfloor i/2 \rfloor}, & \text{if } x = x_i, y = x_{i+1}, i = 0,1,2,\ldots, \\ d(x_i, x_{i+1}) + \cdots + d(x_{j-1}, x_j), & \text{if } x = x_i,\ y = x_j,\ i+1 < j, \\ 4a - d(x_0, x_i), & \text{if } x = x_i,\ y = x^*, \\ 0, & \text{if } x = y, \end{cases}$$

where $\lfloor \cdot \rfloor$ is the floor function.

The reader can easily verify that for every three different points from the set X one of them lies between the two others, see Fig. 1. Moreover, the space is complete with the single accumulation point x^*.

Define a mapping $T\colon X \to X$ as $Tx_i = x_{i+1}$ for all $i = 0, 1, \ldots$ and $Tx^* = x^*$. Since $d(x_{2n}, x_{2n+1}) = d(Tx_{2n}, Tx_{2n+1})$, $n = 0, 1, 2 \ldots$, using (2.3) we see that T is not a contraction mapping.

Let us show that inequality (2.1) holds for every three pairwise distinct points from the space X. Consider first triplets of points $x_i, x_j, x^* \in X$ with $0 \leqslant i < j$. According to the definition of the metric d we have

$$d(x_i, x_j) + d(x_j, x^*) + d(x_i, x^*) = 2d(x_i, x^*) = 8a - 2d(x_0, x_i)$$

and

$$d(Tx_i, Tx_j) + d(Tx_j, Tx^*) + d(Tx_i, Tx^*) = 2d(Tx_i, Tx^*) = 8a - 2d(x_0, x_{i+1}).$$

According to the formula for a geometric series that computes the sum of n terms we have

$$d(x_0, x_i) = \begin{cases} 4a(1 - (1/2)^n), & \text{if } i = 2n, \\ 4a(1 - (1/2)^n) - a/2^{n-1}, & \text{if } i = 2n - 1, \end{cases}$$

$n = 1, 2, \ldots$. Note also that $d(x_0, x_{i+1}) = d(x_0, x_i) + a/(2^{\lfloor i/2 \rfloor})$. Consider the ratio

$$\begin{aligned} \frac{d(Tx_i, Tx_j) + d(Tx_j, Tx^*) + d(Tx_i, Tx^*)}{d(x_i, x_j) + d(x_j, x^*) + d(x_i, x^*)} &= \frac{8a - 2d(x_0, x_{i+1})}{8a - 2d(x_0, x_i)} \\ &= \frac{4a - d(x_0, x_i) - a/(2^{\lfloor i/2 \rfloor})}{4a - d(x_0, x_i)} \\ &= \begin{cases} \frac{4a - 4a(1-(1/2)^n) - a/(2^{\lfloor i/2 \rfloor})}{4a - 4a(1-(1/2)^n)}, & \text{if } i = 2n, \\ \frac{4a - 4a(1-(1/2)^n) + a/2^{n-1} - a/(2^{\lfloor i/2 \rfloor})}{4a - 4a(1-(1/2)^n) + a/2^{n-1}}, & \text{if } i = 2n - 1, \end{cases} \\ &= \begin{cases} \frac{3}{4}, & \text{if } i = 2n, \\ \frac{2}{3}, & \text{if } i = 2n - 1. \end{cases} \end{aligned}$$

Let now $x_i, x_j, x_k \in X$ be such that $0 \leqslant i < j < k$. Using Fig. 1, we see that

$$d(x_i, x_j) + d(x_j, x_k) + d(x_i, x_k) - (d(Tx_i, Tx_j) + d(Tx_j, Tx_k) + d(Tx_i, Tx_k)) \\ = 2(a/2^{\lfloor i/2 \rfloor} - a/2^{\lfloor k/2 \rfloor}).$$

Consider the ratio

$$\begin{aligned} R_{i,k} &= \frac{d(Tx_i, Tx_j) + d(Tx_j, Tx_k) + d(Tx_i, Tx_k)}{d(x_i, x_j) + d(x_j, x_k) + d(x_i, x_k)} \\ &= \frac{d(x_i, x_j) + d(x_j, x_k) + d(x_i, x_k) - 2(a/2^{\lfloor i/2 \rfloor} - a/2^{\lfloor k/2 \rfloor})}{d(x_i, x_j) + d(x_j, x_k) + d(x_i, x_k)} \\ &= 1 - 2\frac{(a/2^{\lfloor i/2 \rfloor} - a/2^{\lfloor k/2 \rfloor})}{d(x_i, x_j) + d(x_j, x_k) + d(x_i, x_k)}. \end{aligned}$$

Observe that $i + 1 < k$. Hence,

$$a/2^{\lfloor k/2 \rfloor} \leqslant a/(2 \cdot 2^{\lfloor i/2 \rfloor}). \tag{2.5}$$

Using the structure of (X, d), one can show that $d(x_i, x^*) \leqslant 4d(x_i, x_{i+1})$. Clearly, $d(x_i, x_k) \leqslant d(x_i, x^*)$. Hence, $d(x_i, x_k) \leqslant 4d(x_i, x_{i+1})$. From equality (2.4) and the last inequality it follows that

$$d(x_i, x_j) + d(x_j, x_k) + d(x_i, x_k) = 2d(x_i, x_k) \leqslant 8d(x_i, x_{i+1}) = 8a/(2^{\lfloor i/2 \rfloor}).$$

Using this inequality and inequality (2.5) we obtain

$$R_{i,k} \leqslant 1 - 2\frac{(a/2^{\lfloor i/2 \rfloor} - a/(2 \cdot 2^{\lfloor i/2 \rfloor}))}{8a/2^{\lfloor i/2 \rfloor}} = \frac{7}{8}.$$

Hence, inequality (2.1) holds for every three pairwise distinct points from the space X with the coefficient $\alpha = \frac{7}{8} = \max\{\frac{2}{3}, \frac{3}{4}, \frac{7}{8}\}$.

Example. Note that in the previous example the sequence of iterates of any two points x_i and x_j are overlapping sets. Let us construct an example of a mapping $T\colon X \to X$ contracting perimeters of triangles that is not a contraction mapping having the property that there exists infinitely many points such that the sequences of iterates of these points are disjoint sets. Let $X = \{x_0, x_1, \ldots\} \cup [0, 1] \subseteq \mathbb{R}^1$ where $x_{2k} = -4/2^k$, $x_{2k+1} = -3/2^k$, $k = 0, 1, \ldots$, and let d be the Euclidean metric on X, see Fig. 2.

Define a mapping $T\colon X \to X$ as $Tx_i = x_{i+1}$ for all $i = 0, 1, \ldots$ and $Tx = x/2$ for all $x \in [0, 1]$. It is clear that the above mentioned property holds, e.g., for the sequences of iterates of the points from the interval $[0, 1]$ having the form $p/2^k$, where $p \geqslant 3$ is a prime number and k is the smallest natural number such that $p/2^k \in [0, 1]$.

Note that the metric space from the previous example, if we set $a = 1$, is isometric to the subspace $(\{0, x_0, x_1, \ldots\}, d)$ of the space (X, d). For this

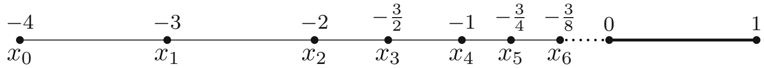

FIGURE 2. The metric space (X, d)

subspace the mapping T is defined analogously. Hence, T is not a contraction mapping.

Let us show that inequality (2.1) holds for every three pairwise distinct points from the space (X, d). For all three pairwise distinct points from the subspace $(\{0, x_0, x_1, \ldots\}, d)$ this property was established in the previous example. Clearly, the metric d is a contraction on the subspace $([0,1], d)$ and every contraction is a mapping contracting perimeters of triangles. It is sufficient to prove inequality (2.1) only for three pairwise distinct points $x, y, z \in X$ such that $x < y < z$, $x \in \{x_0, x_1, \ldots\}$ and $z \in (0, 1]$. Let first $x = x_{2k} = -4/2^k$. Then

$$d(x,y) + d(y,z) + d(x,z) = 2d(x,z) = 2(4/2^k + z).$$

Using that $Tx = Tx_{2k} = x_{2k+1} = -3/2^k$, we have

$$d(Tx,Ty) + d(Ty,Tz) + d(Tx,Tz) = 2d(Tx,Tz) = 2(3/2^k + z/2).$$

To prove (2.1) consider the ratio

$$\frac{d(Tx,Ty) + d(Ty,Tz) + d(Tx,Tz)}{d(x,y) + d(y,z) + d(x,z)} = \frac{2(3/2^k + z/2)}{2(4/2^k + z)} = \frac{6 + z2^k}{8 + 2z2^k} \leqslant \frac{3}{4}.$$

Analogously, let $x = x_{2k+1} = -3/2^k$. Then

$$d(x,y) + d(y,z) + d(x,z) = 2d(x,z) = 2(3/2^k + z).$$

Using that $Tx = Tx_{2k+1} = x_{2(k+1)} = -4/2^{k+1}$, we have

$$d(Tx,Ty) + d(Ty,Tz) + d(Tx,Tz) = 2d(Tx,Tz) = 2(4/2^{k+1} + z/2).$$

To prove (2.1) consider the ratio

$$\frac{d(Tx,Ty) + d(Ty,Tz) + d(Tx,Tz)}{d(x,y) + d(y,z) + d(x,z)} = \frac{2(4/2^{k+1} + z/2)}{2(3/2^k + z)} = \frac{4 + z2^k}{6 + 2z2^k} \leqslant \frac{2}{3}.$$

Hence, using the previous example, we see that inequality (2.1) holds for every three pairwise distinct points from the space X with the coefficient $\alpha = \frac{7}{8} = \max\{\frac{2}{3}, \frac{3}{4}, \frac{7}{8}\}$.

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Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author declares that he has no conflicts of interest to disclose.

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