J. Fixed Point Theory Appl. (2023) 25:22
https://doi.org/10.1007/s11784-022-01007-x
Published online December 15, 2022

Journal of Fixed Point Theory and Applications

Stability analysis for nonlocal evolution equations involving infinite delays

Nhu Thang Nguyen, Dinh Ke Tran and Van Dac Nguyen

Dedicated to Professor Le Mau Hai on the occasion of his 70th birthday.

Abstract. We deal with the inquiry about stability for nonlocal differential equations involving infinite delays. The dissipativity, stability and weak stability of solutions are addressed by using local estimates, fixed point arguments and a new Halanay-type inequality. Our analysis is based on suitable assumptions on the phase space and nonlinearity function. Our abstract results are illustrated by applying to nonlocal partial differential equations.

Mathematics Subject Classification. 34G20, 34D20, 35B40.

Keywords. Nonlocal differential equation, mild solution, asymptotic stability, weak stability.

1. Introduction

We are interested in the following problem

$$\frac{\mathrm{d}}{\mathrm{d}t}[k * (u - u_0)](t) + Au(t) = f(t, u_t),\ t > 0, \tag{1.1}$$

$$u_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

where the unknown function u takes values in a separable Hilbert space H, the kernel $k \in L^1_{loc}(\mathbb{R}^+)$, the notation '$*$' denotes the Laplace convolution, A is an unbounded linear self-adjoint operator, and $f : \mathbb{R}^+ \times \mathcal{B} \to H$ is a given nonlinear function. The admissible phase space $\mathcal{B}$ satisfies certain conditions that will be defined later. In our model, u_t represents the history of the state function u up to the time t, i.e. $u_t(s) = u(t + s), s \leq 0$.

It is worth pointing out that the system under consideration includes some important classical cases with respect to the kernel function k being of special ones (see, e.g. [14,16]). Namely, if $k(t) = g_{1-\mu}(t) := t^{-\mu}/\Gamma(1 - \mu)$, for $\mu \in (0, 1)$, then equation (1.1) is the fractional differential equations since the convolution represents D^μ_0, the Caputo fractional derivative of order μ.

Regarding the fractional differential systems involving finite delays in Banach spaces, some results on (weak) stability and decay solutions were established in [1,12,13]. Based on the special features (e.g., the analyticity, subordinate principle) associated with the kernel $g_{1-\mu}(t)$, the fractional differential equations can be considered in a more general framework:

$$\begin{cases} D_0^\mu[x(t) - h(t, x_t)] = Bx(t) + f(t, x(t), x_t), \ t > 0 \\ x(\theta) = \varphi(\theta), \theta \leq 0, \end{cases} \tag{1.3}$$

where B is the infinitesimal generator of an analytic semigroup on a Banach space X. Considering abstract neutral functional differential equations like (1.3) involving infinite delays, we refer the reader to [18,21] for the existence of integral solutions, and [2] for existence of integral solutions with a certain decay rate. Noting that the approach in the mentioned works heavily relies on the point-wise decaying of the Mittag-Leffler functions $E_{\mu,\nu}(z)$, which is no longer available for the general nonlocal derivatives. Nevertheless, system (1.1) without delay has received considerable attention over decades. It appears in mathematical models of various processes in materials with memory (see, e.g. [5,7,23]). Particularly, Vergara and Zacher [29] mentioned that equation (1.1) with an appropriate class of kernels can be used to depict the anomalous diffusion phenomena which includes slow/ultraslow diffusions when $H = L^2(\Omega), \Omega \subset \mathbb{R}^N$, and $A = -\Delta$ is the Laplacian associated with the homogeneous Dirichlet/Neumann boundary condition. We also refer to [14,17] and the references therein for recent development on this trend.

It should be mentioned that, in modeling of physical/biological processes, the formulated system is usually subject to the history information, that is, a delay term comes into the model. A class of Caputo fractional integro-differential equations with bounded delays has been investigated recently in [4,19] by Lyapunov-Razumikhin method. The authors in the recent work [16] studied (1.1) in the case of finite delay, i.e. $\varphi \in C([-h, 0]; H)$, where some stability results were obtained. As far as we know, this is the first attempt dealing with stability analysis for nonlocal differential equations involving delays. In this work, we consider the case that φ belongs to the fading memory spaces, which were axiomatically introduced by Hale and Kato in [8]. This situation is entirely different from that in [16] due to the complicated structure of phase spaces. Our aim is to find a class of admissible phase spaces and conditions on the nonlinearity function f under which our problem is solvable, and its solution is stable/weakly stable. To this end, we first set the following fundamental hypotheses.

- (A0) The operator $A : D(A) \subset H \to H$ is self-adjoint on H and its spectral $\sigma(A)$ is bounded from below, that is, there exists $\lambda_1 := \lambda_1(A) \in \mathbb{R}$ such that $\sigma(A) \subset [\lambda_1, +\infty)$.
- (K) The kernel $k \in L^1_{loc}(\mathbb{R}^+)$ is nonnegative and nonincreasing, and there exists a function $l \in L^1_{loc}(\mathbb{R}^+)$ such that $k * l = 1$ on $(0, \infty)$.

Hypothesis (K) enables us to get a representation of solutions for (1.1)–(1.2). This hypothesis has been used in a wide range of works (see [14,16,17,22,24,29,30]).

Let us give a brief on our approach. The well-posedness of linear equation is followed by Prüss' theory of resolvent families and Lemma 2.2 ((2)) below, which extends the recent result [14, Lemma 2.3]. Namely, the existence and qualitative properties of the solution operators are established for a general semibounded self-adjoint operator. The assumption (A0) also covers the case A has a negative spectrum, see Lemma 2.2((1)). The solvability of (1.1)–(1.2) is obtained by a fixed point argument. This will be done by proving the compactness of the Cauchy operator in Proposition 2.3 without regularity assumption on the kernel, see Remark 2.1. The stability of the solution to (1.1)–(1.2) is proved by applying a new Halanay type inequality, which is more flexible, in comparison with the one in [16]. In addition, we utilize of the approach developed in [3,16] to get the weakly asymptotic stability result, which relies on the fixed point principle for condensing maps on a special constructed subset. We find that the sufficient conditions for weak stability result, Theorem 4.4, only depend on the asymptotic behavior of the coefficients of the system for a large time. This phenomenon provides a compatible observation of existence result in finite time, where one has no restriction on the magnitude of Lipschitz constant of the nonlinear function. Our setting is more practical and relaxes some conditions proposed by previous works in the literature.

The paper is organized as follows. In Sect. 2, we collect some necessary results on the theory of resolvent, establish a compactness of the Cauchy operator and propose a new Halanay-type inequality. Section 3 is devoted to studying the existence of mild solutions and the dissipativity via the existence of absorbing sets. In Sect. 4, the stability results and the weakly asymptotic stability of the zero solution are formulated under certain assumptions on the nonlinearity as well as on the phase space. The last section presents an application to a class of nonlocal partial differential equations with infinite delays.

2. Preliminaries

2.1. Phase spaces

We recall in this subsection the axiomatic definition of the phase space $\mathcal{B}$, see [8]. The phase space $\mathcal{B}$ is a linear subspace consisting of functions from $(-\infty, 0]$ into H, which is furnished by a suitable seminorm $|\cdot|_{\mathcal{B}}$ and satisfying the following. If a function $v : (-\infty, T + \sigma] \to H$ is such that $v|_{[\sigma,T+\sigma]} \in C([\sigma, T + \sigma]; H)$ and $v_\sigma \in \mathcal{B}$, then

(B1) $v_t \in \mathcal{B}$ for $t \in [\sigma, T + \sigma]$;

(B2) the function $t \mapsto v_t$ is continuous on $[\sigma, T + \sigma]$;

(B3) $|v_t|_{\mathcal{B}} \le K(t-\sigma) \sup_{\sigma\le s\le t} \|v(s)\| + M(t-\sigma)|v_\sigma|_{\mathcal{B}}$, where $K, M : [0,\infty) \to [0,\infty)$ are independent of v, and K is continuous, M is locally bounded.

In the present work, we put a further assumption on $\mathcal{B}$:

(B4) there exists $\varrho > 0$ such that $\|\varphi(0)\| \le \varrho|\varphi|_{\mathcal{B}}$, for all $\varphi \in \mathcal{B}$.

We recall here some examples of phase spaces $\mathcal{B}$. We refer the readers to the book by Hino, Mukarami and Naito [10] for more details. The first one is given by

$$C_\gamma = \{\varphi \in C((-\infty,0];H) : \lim_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } H\},$$

for a given $\gamma > 0$. It easily sees that C_γ satisfies (B1)–(B3) with

$$K(t) = 1,\ M(t) = e^{-\gamma t},$$

and C_γ is a Banach space with the following norm

$$|\varphi|_{\mathcal{B}} = \sup_{\theta\le 0} e^{\gamma\theta}\|\varphi(\theta)\|.$$

The second example is defined as follows. Assume that $1 \le p < +\infty, 0 \le r < +\infty$ and a function $g : (-\infty,-r] \to \mathbb{R}$ is nonnegative, Borel measurable on $(-\infty,-r)$. Let CL^p_g denote a class of functions $\varphi : (-\infty,0] \to H$ such that φ is continuous on $[-r,0]$ and $g(\theta)\|\varphi(\theta)\|^p \in L^1(-\infty,-r)$. The associated seminorm in CL^p_g is given by

$$|\varphi|_{CL^p_g} = \sup_{-r\le\theta\le 0}\|\varphi(\theta)\| + \Big[\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p\,\mathrm{d}\theta\Big]^{\frac{1}{p}}.$$

Furthermore, suppose that

$$\int_s^{-r} g(\theta)\mathrm{d}\theta < +\infty, \text{ for every } s \in (-\infty,-r) \text{ and} \tag{2.1}$$

$$g(s+\theta) \le G(s)g(\theta) \text{ for } s \le 0 \text{ and } \theta \in (-\infty,-r), \tag{2.2}$$

where $G : (-\infty,0] \to \mathbb{R}^+$ is a locally bounded function. It is shown in [10], CL^p_g satisfies (B1)–(B3) provided that (2.1)–(2.2) hold true. More precisely,

$$K(t) = \begin{cases} 1 & \text{for } 0 \le t \le r, \\ 1+\Big[\int_{-t}^{-r} g(\theta)\,\mathrm{d}\theta\Big]^{\frac{1}{p}} & \text{for } t > r; \end{cases} \tag{2.3}$$

$$M(t) = \begin{cases} \max\Big\{1+\Big[\int_{-t}^{-r} g(\theta)\,\mathrm{d}\theta\Big]^{\frac{1}{p}}, G(-t)^{\frac{1}{p}}\Big\} & \text{for } 0 \le t \le r, \\ \max\Big\{\Big[\int_{-t}^{-r} g(\theta)\,\mathrm{d}\theta\Big]^{\frac{1}{p}}, G(-t)^{\frac{1}{p}}\Big\} & \text{for } t > r. \end{cases} \tag{2.4}$$

2.2. The resolvent families

Consider the following scalar Volterra equations which describe the relaxation functions

$$s(t) + \lambda(l * s)(t) = 1, \quad t \ge 0, \tag{2.5}$$

$$r(t) + \lambda(l * r)(t) = l(t), \quad t > 0. \tag{2.6}$$

The solvability of s and r was mentioned in [20]. The solutions of (2.5) and (2.6) are denoted by $s(\cdot,\lambda)$ and $r(\cdot,\lambda)$, respectively. The kernel l is said to be completely positive if and only if for every $\lambda > 0$, $s(\cdot)$ and $r(\cdot)$ take nonnegative values. An equivalent criterion is that (see [5, Theorem 2.2]), there exist $\alpha \ge 0$ and a nonnegative and nonincreasing kernel $k \in L^1_{loc}(\mathbb{R}^+)$ which satisfy $\alpha l(t)+l*k(t) = 1$ for all $t > 0$. Hence, our assumption (K) yields that l is completely positive and particularly, l takes nonnegative values by [5, Proposition 2.1 (1)]. Consequently, the functions $s(\cdot,\lambda)$ and $r(\cdot,\lambda)$ take nonnegative values (for even $\lambda \le 0$, see also explanation in [30]). We remind some further properties of these relaxation functions.

Proposition 2.1. [14,30] *Let the hypothesis* (K) *hold. Then for every $\lambda \in \mathbb{R}$, $s(\cdot,\lambda), r(\cdot,\lambda) \in L^1_{loc}(\mathbb{R}^+)$. In addition, we have:*

(1) *The function $s(\cdot,\lambda)$ is nonnegative and nonincreasing. Moreover, for $\lambda > 0$,*
$$s(t,\lambda)\left[1+\lambda\int_0^t l(\tau)\mathrm{d}\tau\right] \le 1, \quad \forall t \ge 0. \tag{2.7}$$
Hence if $l \notin L^1(\mathbb{R}^+)$ then $\lim_{t\to\infty} s(t,\lambda) = 0$ for every $\lambda > 0$.

(2) *The function $r(\cdot,\lambda)$ is nonnegative and one has*
$$s(t,\lambda) = 1-\lambda\int_0^t r(\tau,\lambda)\mathrm{d}\tau = k * r(\cdot,\lambda)(t), \quad t \ge 0,$$
so $\int_0^t r(\tau,\lambda)\mathrm{d}\tau \le \lambda^{-1}$, $\forall t > 0$. If $l \notin L^1(\mathbb{R}^+)$ then $\int_0^\infty r(\tau,\lambda)\mathrm{d}\tau = \lambda^{-1}$ for every $\lambda > 0$.

(3) *For each $t > 0$, the functions $\lambda \mapsto s(t,\lambda)$ and $\lambda \mapsto r(t,\lambda)$ are nonincreasing in $\mathbb{R}$.*

(4) *Equation* (2.5) *is equivalent to the problem*
$$\frac{\mathrm{d}}{\mathrm{d}t}[k*(s-1)] + \lambda s = 0,\ s(0) = 1.$$

(5) *Let $v(t) = s(t,\lambda)v_0 + (r(\cdot,\lambda) * g)(t)$, here $g \in L^1_{loc}(\mathbb{R}^+)$. Then v solves the problem*
$$\frac{\mathrm{d}}{\mathrm{d}t}[k*(v-v_0)](t) + \lambda v(t) = g(t),\ v(0) = v_0.$$

Using spectral theorem for self-adjoint operator [26, Theorem 1.7], the hypothesis (A0) implies that there exist a measure space $(\Xi, \mathrm{d}\mu)$, a unitary map $U : L^2(\Xi,\mathrm{d}\mu) \to H$ and a real-valued function a on Ξ such that
$$U^{-1}AUf(\xi) = a(\xi)f(\xi),\ Uf \in D(A). \tag{2.8}$$
Note that for $f \in L^2(\Xi,\mathrm{d}\mu)$, $Uf \in D(A)$ iff $M_a f(\cdot) = a(\cdot)f(\cdot)$ belongs to $L^2(\Xi,\mathrm{d}\mu)$.

Based on this spectral representation of A, the Borelian functional calculus of A is given by
$$\left(U^{-1}g(A)Uv\right)(\xi) = g(a(\xi))v(\xi),\ \text{for almost every } \xi \in \Xi, \tag{2.9}$$
for arbitrary Borel function $g : \mathbb{R} \to \mathbb{C}$. In general, $g(A)$ is unbounded linear self-adjoint in H for an arbitrary Borel real-valued function g. If g is bounded in $\mathbb{R}$ then so is $g(A)$ and $\|g(A)\|_{L(H)} \le \|g\|_{L^\infty(\mathbb{R})}$.

The spectral boundedness from below of A implies that
$$a(\xi) \in [\lambda_1, +\infty),\ \text{for almost every } \xi \in \Xi.$$
Therefore, the functional $g(A)$ only depends on the essential value of g in $[\lambda_1, +\infty)$. In particular, if $g \in L^\infty([\lambda_1,\infty),\mathrm{d}\mu)$ then $g(A)$ is a bounded linear map in H and furthermore
$$\|g(A)\|_{L(H)} \le \operatorname{ess\,sup}_{\xi\in\Xi} |g(a(\xi))|. \tag{2.10}$$

Note that if $\lambda_1 \geq 0$ then for $\gamma \geq 0$, the fractional power of A can be defined as follows

$$D(A^\gamma) = \left\{Uw \in H : w \in L^2(\Xi, \mathrm{d}\mu), (a(\xi))^\gamma\, w(\xi) \in L^2(\Xi, \mathrm{d}\mu)\right\},$$
$$U^{-1}A^\gamma Uw(\xi) = a(\xi)^\gamma w(\xi), Uw \in D(A^\gamma).$$

Let $V_\gamma = D(A^\gamma)$. Then V_γ is a Banach space endowed with the norm

$$\|v\|_{D(A^\gamma)} = \left(\int_\Xi \left(1 + |a(\xi)|^{2\gamma}\right) |U^{-1}v(\xi)|^2 \mathrm{d}\mu\right)^{\frac{1}{2}}.$$

For $\lambda_1 > 0$, this is equivalent to the following norm

$$\|v\|_\gamma = \|A^\gamma v\|_H = \left(\int_\Xi |a(\xi)|^{2\gamma}|U^{-1}v(\xi)|^2 \mathrm{d}\mu\right)^{\frac{1}{2}}. \tag{2.11}$$

Moreover, for $\gamma > 0$, $V_{-\gamma}$ can be identified with the dual space V_γ^* of V_γ.

By formula (2.9) and properties of the functions $s(t,\mu), r(t,\mu)$, we now define the resolvent operators

$$U^{-1}S(t,A)Uv(\xi) = s(t,a(\xi))v(\xi), \xi \in \Xi, t \geq 0, Uv \in H, \tag{2.12}$$
$$U^{-1}R(t,A)Uv(\xi) = r(t,a(\xi))v(\xi), \xi \in \Xi, t > 0, Uv \in H. \tag{2.13}$$

Obviously, $S(t) := S(t,A)$ and $R(t) := R(t,A)$ are linear self-adjoint operators in H and fulfill the following fundamental relation

$$S'(t,A) = -AR(t,A), t > 0, \tag{2.14}$$

due to Proposition 2.1(2) and the relation (2.12)–(2.13). In the following lemma, we prove some properties of $S(t), R(t)$ which extend the recent result [14, Lemma 3.2]. By relation (2.14), the statement in Lemma 2.2 ((2)) below implies that $S(t,A)$ is differentiable in the sense of Prüss [23, Definition 1.4]. A consequence of (2.17) is a smoothing effect of the solution operator. This estimate plays an important role in analyzing semilinear nonlocal evolution equations since the assumption on the nonlinearity can be relaxed considerably, as mentioned in [23, Section 13.5].

Lemma 2.2. *Let $\{S(t,A)\}_{t\geq 0}$ and $\{R(t,A)\}_{t>0}$, be defined by (2.12) and (2.13), respectively and $T > 0$ be given.*

(1) *For each $v \in H$, $S(\cdot,A)v \in C([0,T];H)$ and $AS(\cdot,A)v \in C((0,T];H)$. Moreover,*

$$\|S(t,A)v\| \leq s(t,\lambda_1)\|v\|,\ t \in [0,T];$$
$$\|AS(t,A)v\| \leq \begin{cases} \dfrac{\|v\|}{(1 * l)(t)}, & \text{if } \lambda_1 \geq 0, \\ |\lambda_1| s(t,-|\lambda_1|)\|v\|, & \text{if } \lambda_1 < 0, \end{cases}$$

for $t \in (0,T]$.

(2) *Let $v \in H$. Then $R(\cdot,A)v \in C((0,T];H)$. Furthermore,*

$$\|R(t,A)v\| \leq r(t,\lambda_1)\|v\|,\ t \in (0,T], \tag{2.15}$$
$$\|AR(t,A)v\| \leq r(t,\lambda_1)\|Av\|, v \in D(A), t > 0. \tag{2.16}$$

In particular, for $t > 0$ one has $\|S'(t,A)v\| \le r(t,\lambda_1)\|v\|_{D(A)}$ for all $v \in D(A)$.

(3) *Assume further that $\lambda_1 > 0$. Then the convolution with R possesses a smoothing effect in the sense that if $g \in C([0,T];V_\gamma), \gamma \ge 0$ then $A(R*g) \in C([0,T];V_{\gamma-\frac{1}{2}})$*

$$\|A(R*g)(t)\|_{V_{\gamma-\frac{1}{2}}} \le \left(\int_0^t r(t-\tau,\lambda_1)\|g(\tau)\|^2_{V_\gamma}\mathrm{d}\tau\right)^{\frac{1}{2}},\ t\in[0,T]. \tag{2.17}$$

Proof. The first part in (1) and (2) follow by the same argument as in [14, Lemma 2.3], so we verify only the remain statement in (1). By (2.10), one has

$$\|s(t,A)\|_{L(H)} \le \sup_{\lambda\ge\lambda_1} s(t,\lambda) = s(t,\lambda_1), \tag{2.18}$$

where the last relation follows from the monotonicity of $s(t,\cdot)$ with respect to λ. Analogously, one also gets

$$\|AS(t,A)\|_{L(H)} \le \sup_\xi |a(\xi)|s(t,a(\xi)) \tag{2.19}$$

$$= \sup_{\lambda\ge\lambda_1}|\lambda|s(t,\lambda) \le \sup_{\lambda\ge-|\lambda_1|}|\lambda|s(t,\lambda) \tag{2.20}$$

$$= \max\left\{\sup_{-|\lambda_1|\le\lambda\le 0}|\lambda|s(t,\lambda), \sup_{\lambda>0}\lambda s(t,\lambda)\right\} \tag{2.21}$$

$$\le \max\left\{|\lambda_1|s(t,-|\lambda_1|), \frac{1}{1*l(t)}\right\}, \tag{2.22}$$

here the last inequality follows from Proposition 2.1 (1).

The proof of the first part in (3) goes similarly the one above, hence we show only the last estimate (2.17). Using the representation

$$U^{-1}A(R*U\hat{g})(t,\xi) = \int_0^t a(\xi)r(t-\tau,a(\xi))\hat{g}(\tau,\xi)\mathrm{d}\tau,$$

where $\hat{g}(t,\cdot) = U^{-1}g(t)$, we have

$$\|AR*g(t)\|^2_{V_{\gamma-1/2}} = \int_\Xi a(\xi)^{2\gamma-1}\left(\int_0^t a(\xi)r(t-\tau,a(\xi))\hat{g}(\tau,\xi)\mathrm{d}\tau\right)^2\mathrm{d}\mu,$$

thanks to (2.11). Then utilizing the Hölder inequality, we get

$$\begin{aligned}
&\|AR*g(t)\|^2_{V_{\gamma-1/2}}\\
&\le \int_\Xi a(\xi)^{2\gamma-1}\left(\int_0^t a(\xi)r(t-\tau,a(\xi))\mathrm{d}\tau\right)\left(\int_0^t a(\xi)r(t-\tau,a(\xi))|\hat{g}(\tau,\xi)|^2\mathrm{d}\tau\right)\mathrm{d}\mu\\
&\le \int_\Xi (1-s(t,a(\xi)))\left(\int_0^t a(\xi)^{2\gamma}r(t-\tau,a(\xi))|\hat{g}(\tau,\xi)|^2\mathrm{d}\tau\right)\mathrm{d}\mu\\
&\le \int_0^t\left(\int_\Xi r(t-\tau,a(\xi))a(\xi)^{2\gamma}|\hat{g}(\tau,\xi)|^2\mathrm{d}\mu\right)\mathrm{d}\tau\\
&\le \int_0^t\left(r(t-\tau,\lambda_1)\int_\Xi a(\xi)^{2\gamma}|\hat{g}(\tau,\xi)|^2\mathrm{d}\mu\right)\mathrm{d}\tau = \int_0^t r(t-\tau,\lambda_1)\|g(\tau)\|^2_{V_\gamma}\mathrm{d}\tau,
\end{aligned}$$

which ensures (2.17). In particular, $\|R*g\|_{C([0,T];V_{\gamma+1/2})} \leq \frac{1}{\sqrt{\lambda_1}}\|g\|_{C([0,T],V_\gamma)}$. This is the half smoothing effect of the resolvent operator R. $\square$

Let E and F be Banach spaces. The notations $\mathcal{L}(E,F), \mathcal{K}(E,F)$ stand for spaces of bounded linear operators, linear compact operators from E to F, respectively. Note that $\mathcal{K}(E,F)$ is closed subset in $\mathcal{L}(E,F)$ with respect to the operator norm.

To gain the compactness of the solution operators, we need further assumption as follows:

(A) The operator $A : D(A) \subset H \to H$ is nonnegative, self-adjoint on H with compact resolvent.

This assumption guarantees that H possesses an orthonormal basis $\{e_n\}_{n=1}^{\infty}$, where $e_n, n \geq 1$ are eigenfunctions of the operator A with corresponding eigenvalues $\lambda_n > 0$. The domain

$$D(A) = \left\{ v = \sum_{n=1}^{\infty} v_n e_n : \sum_{n=1}^{\infty} \lambda_n^2 v_n^2 < \infty \right\},$$

and A admits the presentation

$$Av = \sum_{n=1}^{\infty} \lambda_n v_n e_n, \ v_n = (v, e_n), v \in D(A).$$

The assumption (A) implies that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty$ as $n \to \infty$. In this case, Ξ is the set $\mathbb{N}$ of natural numbers and the measure $\mathrm{d}\mu$ is thus the counting measure, the function $a(n) = \lambda_n, \forall n \in \mathbb{N}$ and the unitary map $U : L^2(\mathbb{N}, \mathrm{d}\mu) \to H$ is given by

$$f = (f_1, f_2, \dots, f_n, \dots) \in L^2(\mathbb{N}) \mapsto Uf = \sum_{k=1}^{\infty} f_k e_k \in H.$$

For a real number s we denote $X_s = C([0,T]; D(A^s))$. We need the following result to investigate the existence results.

Proposition 2.3. *Let assumptions* (A) *and* (K) *hold. Then the operator*

$$Q : C([0,T]; D(A^s)) \to C([0,T]; D(A^{s+1/2})), f \mapsto Qf(t) := R * f(t)$$

is compact for any $s \in \mathbb{R}$.

Proof. Based on the approximation argument, the proof is divided into several steps.

Step 1. We first remind that for a given $g \in C[0,T]$, the convolution map $\mathcal{C}_g : C[0,T] \to C[0,T], v \mapsto g*v$ is compact. This is a classical result, but for the convenience of the readers, we provide a proof here. Fix any bounded subset $D \subset C[0,T]$, that is, there exists a positive constant $R > 0$ such that

$$\|v\| \leq R, \quad \forall v \in D.$$

Clearly, $|g*v(t)| \leq \|g\|_{L^1(0,T)} \max_{t\in[0,T]} |v(t)|, v \in D$, which implies the pointwise bounded of $\mathcal{C}_g(D)$.

On the other hand, for any $\epsilon > 0$, by the uniform continuity of g on $[0,T]$, one chooses a positive number $\delta < \epsilon/(2R\|g\|+1)$ such that

$$|g(s_1) - g(s_2)| \le \frac{\epsilon}{2RT} \text{ for any } s_1, s_2 \in [0,T], |s_1 - s_2| \le \delta.$$

For any $t_1, t_2 \in [0,T]$, $0 < t_2 - t_1 < \delta$ and $v \in D$, one has

$$\begin{aligned} |g * v(t_2) - g * v(t_1))| &\le \int_0^{t_1} |g(t_1-\tau) - g(t_2-\tau)||v(\tau)|\mathrm{d}s \\ &\quad + \int_{t_1}^{t_2} |g(t_2-\tau)v(\tau)|\mathrm{d}s \\ &\le \frac{\epsilon}{2RT}\int_0^T |v(\tau)|\mathrm{d}s + \max_{\tau\in[0,T]}|v(\tau)||t_1-t_2| \max_{\tau\in[0,T]}|g(\tau)| \\ &\le \left(\frac{\epsilon}{2RT}T + \|g\|\delta\right)\sup_{v\in D}\|v\| \le \epsilon, \end{aligned}$$

for any $v \in D$. Therefore, the equi-continuity of $\mathcal{C}_g(D)$ is testified. Thus, $\mathcal{C}_g(D)$ is relatively compact in $C[0,T]$ by Arzelà–Ascoli Theorem.

Step 2. Extend the statement above to the singular kernel. For a given $g \in L^1(0,T)$ Then $g* : C[0,T] \to C[0,T]$ is compact. Indeed, by density of smooth functions in $L^1(0,T)$, one can choose a sequence of continuous function g_n such that g_n converges to g in $L^1(0,T)$. Then, we have

$$\begin{aligned} \|(\mathcal{C}_{g_n} - \mathcal{C}_g)v\| &= \sup_{t\in[0,T]} |\int_0^t (g_n(t-\tau) - g(t-\tau))v(\tau)\mathrm{d}\tau| \\ &\le \sup_{t\in[0,T]}\int_0^t |g_n(t-\tau) - g(t-\tau)|.|v(\tau)|\mathrm{d}\tau \\ &\le \|g_n - g\|_{L^1(0,T)}\|v\|. \end{aligned}$$

Thus, $\lim_{n\to\infty}\|\mathcal{C}_{g_n} - \mathcal{C}_g\|_{\mathcal{L}(C[0,T])} = 0$, which implies the compactness of $\mathcal{C}_g$.

Step 3. Let denote

$$Q_n f(t) = \sum_{k=1}^n \left(\int_0^t r(t-\tau,\lambda_k) f_k(\tau)\mathrm{d}\tau\right) e_k$$

for any $f = \sum_{k=1}^\infty f_k(t)e_k \in C([0,T]; D(A^s))$.

By Step 2, Q_n is a compact operator from $C([0,T];D(A^s))$ to $C([0,T]; D(A^{s+1/2}))$.

Step 4. By Step 3, it reduces to show that Q_n converges to Q with respect to the operator norm in $\mathcal{L}(X_s, X_{s+1/2})$. Indeed, we have

$$\begin{aligned} &\|(Q-Q_n)v(t)\|^2_{D(A^{s+1/2})} \\ &\quad = \sum_{k>n} |\lambda_k^{1+2s}\int_0^t r(t-\tau,\lambda_k)v_k(\tau)\mathrm{d}\tau|^2 \\ &\quad \le \sum_{k>n}\int_0^t \lambda_k r(t-\tau,\lambda_k)\mathrm{d}\tau \cdot \int_0^t r(t-\tau,\lambda_k)|\lambda_k^s v_k(\tau)|^2\mathrm{d}\tau \end{aligned}$$

(by Hölder's inequality)

$$
\begin{aligned}
&\leq \int_0^t r(t-\tau,\lambda_n)\sum_{k>n}|\lambda_k^s v_k(\tau)|^2 \mathrm{d}\tau \\
&\qquad \left(\text{since } \int_0^t \lambda_k r(\tau,\lambda_k)\mathrm{d}\tau = 1 - s(t,\lambda_k) \leq 1 \text{ and } r(\cdot,\lambda_k)\right. \\
&\leq r(\cdot,\lambda_n) \text{ for } k>n) \\
&\leq \int_0^t r(t-\tau,\lambda_n)\|v(\tau)\|^2_{D(A^s)}\mathrm{d}\tau \\
&\leq \left(\int_0^t r(t-\tau,\lambda_n)\mathrm{d}\tau\right)\sup_{s\in[0,T]}\|v(\tau)\|^2_{D(A^s)} \\
&= \frac{1-s(t,\lambda_n)}{\lambda_n}\|v\|^2_{X_s}.
\end{aligned}
$$

Hence, we obtain

$$
\|Q_n - Q\|_{X_{s+1/2}} = \sup_{t\in[0,T]}\|(Q-Q_n)v(t)\|^2_{D(A^{s+1/2})} \leq \frac{1}{\lambda_n}\|v\|^2_{X_s}.
$$

In other word, we get

$$
\|Q - Q_n\|_{\mathcal{L}(X_s, X_{s+1/2})} \leq \lambda_n^{-1/2} \to 0 \text{ as } n\to\infty.
$$

This finishes the proof. □

Remark 2.1. The standard argument for checking the compactness of a subset in $C([0,T], D(A^{s+1/2}))$ is applying Arzelà–Ascoli Theorem directly. However, due to the singularity of the kernel l (so $r(\cdot,\lambda)$), it is difficult to testify the equi-continuity of $Q(D)$ directly without further regularity assumption on the function l. So the proof of compactness for Q in this work requires less conditions than those in [15, Lemma 3.5].

2.3. Halanay type inequality

We denote by $BC(\mathbb{R}^+;X)$ the space of continuous and bounded functions defined on $\mathbb{R}^+$ taking values in a Banach space X. It is a Banach space with the norm given by $\|y\|_{BC} = \sup_{t\geq 0}\|y(t)\|$. Let $BC(\mathbb{R}^+) = BC(\mathbb{R}^+;\mathbb{R})$ and $BC_0 := \{v \in BC(\mathbb{R}^+) \mid \lim_{t\to\infty} v(t) = 0\}$. We verify now a Halanay type inequality in the integral form, which plays a crucial role in our approach.

Proposition 2.4. (Halanay type inequality) *Let v be a continuous and non-negative function on $\mathbb{R}^+$. Assume that for any $t>0$, it holds*

$$
v(t) \leq p(t) + \int_0^t r(t-\tau,a)q(\tau)\mathrm{d}\tau + b\int_0^t r(t-\tau,a)\sup_{\xi\in[\tau-\rho(\tau),\tau]} v(\xi)\mathrm{d}\tau, \tag{2.23}
$$

where $p, q \in BC(\mathbb{R}^+)$, $a > b \geq 0$, and ρ is a given function such that $t \geq \rho(t)$ for $t \geq 0$. Then, $v \in BC(\mathbb{R}^+)$ and

$$
v(t) \leq \left(\|p\|_{BC} + \|r(\cdot,a) * q\|_{BC}\right)\frac{a}{a-b},\ \forall t \geq 0. \tag{2.24}
$$

Moreover, if $\lim_{t\to\infty}(t-\rho(t)) = \infty$*, then*

$$\limsup_{t\to\infty} v(t) \le \limsup_{t\to\infty} \Big(p(t) + \big(r(\cdot,a) * q\big)(t)\Big)\frac{a}{a-b}. \tag{2.25}$$

In particular, for any $\epsilon > 0$*, there exists* $T(\epsilon) > 0$ *such that*

$$v(t) \le \frac{ar^*}{a-b} + \epsilon,\ \forall t \ge T(\epsilon),$$

here $r^* = \limsup\limits_{t\to\infty} p(t) + \limsup\limits_{t\to\infty} r(\cdot,a) * q(t)$.

Proof. First, we prove (2.24). By (2.23), for all $t \in [0,T]$, one has

$$\begin{aligned} v(t) &\le \|p + r(\cdot,a) * q\|_{BC} + b \sup_{\xi\in[0,T]} v(\xi) \int_0^t r(t-\tau,a)\mathrm{d}\tau \\ &\le \|p\|_{BC} + \|r(\cdot,a) * q\|_{BC} + b \sup_{\xi\in[0,T]} v(\xi) \frac{1-s(t,a)}{a} \\ &\le \|p\|_{BC} + \|r(\cdot,a) * q\|_{BC} + \frac{b}{a} \sup_{\xi\in[0,T]} v(\xi). \end{aligned}$$

It implies

$$\sup_{\xi\in[0,T]} v(\xi) \le \Big(\|p\|_{BC} + \|r(\cdot,a) * q\|_{BC}\Big)\frac{a}{a-b}.$$

Let $T \to \infty$, we get

$$\sup_{\xi\in[0,\infty)} v(\xi) \le \Big(\|p\|_{BC} + \|r(\cdot,a) * q\|_{BC}\Big)\frac{a}{a-b}.$$

Thus, (2.24) is verified.

We next show that (2.25) holds. Since $t-\rho(t) \to \infty$ as $t \to \infty$, it follows that for any $T > 0$, there exists $T_1 = T_1(T) > 0$ such that

$$t - \rho(t) \ge T, \quad \forall t \ge T_1,$$

and $T_1 \to \infty$ as $T \to \infty$. Using (2.23) with $t \ge T_1$ yields

$$\begin{aligned} v(t) \le\ & p(t) + \big(r(\cdot,a) * q\big)(t) + b\int_0^{T_1} r(t-\tau,a) \sup_{\xi\in[\tau-\rho(\tau),\tau]} v(\xi)\mathrm{d}\tau \\ &+ b\int_{T_1}^t r(t-\tau,a) \sup_{\xi\in[\tau-\rho(\tau),\tau]} v(\xi)\mathrm{d}\tau \\ \le\ & p(t) + \big(r(\cdot,a) * q\big)(t) + b\|v\|_{BC}\int_0^{T_1} r(t-\tau,a)\mathrm{d}\tau \\ &+ b\sup_{\xi\ge T} v(\xi)\int_{T_1}^t r(t-\tau,a)\mathrm{d}\tau \\ \le\ & p(t) + \big(r(\cdot,a) * q\big)(t) + C\int_{t-T_1}^t r(\xi,a)\mathrm{d}\xi + \frac{b}{a}\sup_{\xi\ge T} v(\xi), \end{aligned}$$

here $C = \Big(\|p\|_{BC} + \|r(\cdot,a)*q\|_{BC}\Big)\frac{ab}{a-b}$. Taking the supremum over $[2T_1,\infty)$, we have

$$\sup_{t\geq 2T_1} v(t) \leq \sup_{t\geq 2T_1} \Big(p(t) + \big(r(\cdot,a)*q\big)(t)\Big) + \frac{C}{a}s(T_1,a) + \frac{b}{a}\sup_{\xi\geq T} v(\xi).$$

Let $T\to\infty$, then $T_1\to\infty$. So we obtain

$$\begin{aligned}\lim_{T_1\to\infty}\sup_{t\geq 2T_1} v(t) \leq \lim_{T_1\to\infty}\sup_{t\geq 2T_1} \Big(p(t) + \big(r(\cdot,a)*q\big)(t)\Big)\\ + \lim_{T_1\to\infty}\frac{C}{a}s(T_1,a) + \lim_{T\to\infty}\frac{b}{a}\sup_{\xi\geq T} v(\xi).\end{aligned}$$

Hence, it implies

$$\limsup_{t\to\infty} v(t) \leq \Big(\limsup_{t\to\infty} p(t) + \limsup_{t\to\infty}\big(r(\cdot,a)*q\big)(t)\Big)\frac{a}{a-b}.$$

Consequently, the last statement in Proposition 2.4 holds. The proof is complete. □

Corollary 2.5. *If $p\in BC_0$ and $q(t) = q_0(t) + q_\infty(t)$, $t\geq 0$, $q_0\in BC_0, q_\infty\in BC(\mathbb{R}^+)$ then there exits $T(\epsilon)>0$, for each given $\epsilon>0$, such that*

$$v(t) \leq \frac{\|q_\infty\|_{BC}}{a-b} + \epsilon,\ t\geq T(\epsilon).$$

Halanay-type inequality plays an essential role in the stability analysis of nonlocal evolution equations. Another approach is combining the Lyapunov-Razumikhin method [9] and the nonlocal chain rule [17, Lemma 6.1]. We refer the readers interested in this approach to [28] for asymptotic stability result for a class of nonlinear Volterra integral-differential equations in the finite-dimensional case with the continuous kernel. A version of nonlinear Halanay-type inequality which utilizes the nonlinear structure should be more interesting in applications to nonlinear systems to obtain the optimal results.

2.4. Definition of mild solutions

For $\varphi\in\mathcal{B}$, we define

$$\mathbf{C}_\varphi = \{u\in C([0,T];H) : u(0)=\varphi(0)\}$$

as a closed subset of $C([0,T];H)$ with respect to the supremum norm denoted by $\|\cdot\|_\infty$.

For any $v\in\mathbf{C}_\varphi$, the function $v[\varphi]:\mathbb{R}\to H$ is defined by

$$v[\varphi](t) = \begin{cases}\varphi(t), & -\infty < t\leq 0,\\ v(t), & t>0.\end{cases}$$

Then, obviously

$$v[\varphi]_t(\theta) = \begin{cases}\varphi(t+\theta), & -\infty<\theta<-t,\\ v(t+\theta), & \theta\in[-t,0].\end{cases}$$

Motivated by arguments in [14], the definition of mild solution to the system (1.1)–(1.2) is given as follows.

Definition 2.1. A function $u \in C((-\infty, T]; H)$ is said to be a mild solution to (1.1)–(1.2) on $(-\infty, T]$ iff $u(t) = \varphi(t)$ for $t \in (-\infty, 0]$ and

$$u(t) = S(t)\varphi(0) + \int_0^t R(t-\tau) f(\tau, u[\varphi]_\tau) \mathrm{d}\tau,$$

for $t \in [0, T]$.

3. Existence results and dissipativity of solutions

We use the fixed point method to get our results by considering the operator defined by

$$\mathcal{F} : \mathbf{C}_\varphi \to \mathbf{C}_\varphi$$
$$\mathcal{F}(v)(t) = S(t)\varphi(0) + \int_0^t R(t-\tau) f(\tau, v[\varphi]_\tau) \mathrm{d}\tau, \ t \in [0; T].$$

Obviously, if v is a fixed point of $\mathcal{F}$, then $v[\varphi]$ is a mild solution to (1.1)–(1.2). So $\mathcal{F}$ is referred to as the solution operator.

The first result is obtained in the case that f has a superlinear growth and the initial datum is sufficiently small.

Theorem 3.1. *Let* (A) *and* (K) *hold. Suppose that the function f is continuous and satisfies the following estimate*

$$\|f(t, w)\| \le \beta |w|_{\mathcal{B}} + \Psi(|w|_{\mathcal{B}}), \forall t \ge 0, w \in \mathcal{B}, \tag{3.1}$$

where $\beta > 0$, $\Psi \in C(\mathbb{R}^+; \mathbb{R})$ such that $\lim_{r \to 0} \frac{\Psi(r)}{r} = 0$. If $\beta < \lambda_1 \left(\sup_{s \in [0;T]} K(s)\right)^{-1}$, then there is a positive number δ such that a mild solution to (1.1)–(1.2) *exists globally provided $|\varphi|_{\mathcal{B}} < \delta$. Furthermore, if f is locally Lipschitzian, i.e., for each $\bar{r} > 0$, there is $L(\bar{r}) > 0$ such that*

$$\|f(t, w_1) - f(t, w_2)\| \le L(\bar{r}) |w_1 - w_2|_{\mathcal{B}}, \tag{3.2}$$

for all $t \ge 0$, $|w_i|_{\mathcal{B}} \le \bar{r}, i \in \{1, 2\}$, then the mild solution to (1.1)–(1.2) *is unique.*

Proof. By definition of $\mathcal{F}$, we see that it is a continuous map from $\mathbf{C}_\varphi$ into itself. We employ the Schauder theorem to prove that $\mathcal{F}$ has a fixed point in $\mathbf{C}_\varphi$. Firstly, we find a number $\eta > 0$ such that $\mathcal{F}(B_\eta) \subset B_\eta$, provided that $|\varphi|_{\mathcal{B}}$ is small enough. Here $B_\eta = \{w \in \mathbf{C}_\varphi \mid \sup_{t \in [0,T]} \|w(t)\| \le \eta\}$.

Due to the assumption on f, for $\theta \in \left(0; \frac{\lambda_1}{K_T} - \beta\right)$, where $K_T = \sup_{[0;T]} K(s)$, there exists $\bar{\eta} > 0$ such that

$$\|f(t, w)\| \le (\beta + \theta) |w|_{\mathcal{B}}, \text{ for all } w \in \mathcal{B}, \ |w|_{\mathcal{B}} \le \bar{\eta}.$$

Now we choose $\eta = \frac{\bar{\eta}}{2K_T}$ and let $\|u\|_\infty \le \eta$. If $|\varphi|_{\mathcal{B}} \le \delta_1 := \frac{\bar{\eta}}{2M_T}$, here $M_T = \sup_{s \in [0;T]} M(s)$, then $|u[\varphi]_\tau|_{\mathcal{B}} \le \bar{\eta}$ for $\tau > 0$.

One gets

$$\begin{aligned} \|\mathcal{F}(u)(t)\| &\le s(t, \lambda_1) \|\varphi(0)\| + \int_0^t r(t-\tau, \lambda_1)(\beta + \theta) |u[\varphi]_\tau|_{\mathcal{B}} \mathrm{d}\tau \\ &\le s(t, \lambda_1) \|\varphi(0)\| \end{aligned}$$

$$
\begin{aligned}
&+(\beta+\theta)\int_0^t r(t-\tau,\lambda_1)[K(\tau)\sup_{s\in[0;\tau]}\|u(s)\|+M(\tau)|\varphi|_{\mathcal{B}}]\mathrm{d}\tau\\
&\le s(t,\lambda_1)\varrho|\varphi|_{\mathcal{B}}\\
&+(\beta+\theta)\int_0^t r(t-\tau,\lambda_1)[K_T\eta+M_T|\varphi|_{\mathcal{B}}]\mathrm{d}\tau.
\end{aligned}
$$

Using Proposition 2.1, we have

$$
\begin{aligned}
\|\mathcal{F}(u)(t)\| &\le s(t,\lambda_1)\varrho|\varphi|_{\mathcal{B}}+(\beta+\theta)\lambda_1^{-1}(1-s(t,\lambda_1))[K_T\eta+M_T|\varphi|_{\mathcal{B}}]\\
&\le \varrho|\varphi|_{\mathcal{B}}+\frac{\beta+\theta}{\lambda_1}(K_T\eta+M_T|\varphi|_{\mathcal{B}})\\
&\le \varrho|\varphi|_{\mathcal{B}}+\frac{\beta+\theta}{\lambda_1}(K_T\eta+M_T|\varphi|_{\mathcal{B}})\\
&=\left(\varrho+\frac{M_T(\beta+\theta)}{\lambda_1}\right)|\varphi|_{\mathcal{B}}+\frac{K_T(\beta+\theta)}{\lambda_1}\eta.
\end{aligned}
$$

Putting $\delta_2 := \eta\frac{(\lambda_1-(\beta+\theta)K_T)}{\varrho\lambda_1+(\beta+\theta)M_T}$, we obtain $\|\mathcal{F}(u)(t)\|\le\eta$ if $|\varphi|_{\mathcal{B}}\le\delta_2$.

Thus $\mathcal{F}(B_\eta)\subset B_\eta$ if $|\varphi|_{\mathcal{B}}<\delta:=\min\{\delta_1,\delta_2\}$. Employing Proposition 2.3, we see that $\mathcal{F}$ is compact. Therefore, by the Schauder theorem, the operator $\mathcal{F}: B_\eta\to B_\eta$ possesses a fixed point. We gain the solvability of problem (1.1)–(1.2).

Finally, suppose that the Lipschitz condition (3.2) holds. If $u_i, i\in\{1,2\}$, are solutions of (1.1)–(1.2), then

$$
u_i(t)=S(t)\varphi(0)+\int_0^t R(t-\tau)f(\tau,u_i[\varphi]_\tau)\mathrm{d}\tau.
$$

Set $\bar{r}=\max\{|u_i[\varphi]|_{\mathcal{B}}: i=1,2\}$. Then

$$
\begin{aligned}
\|u_1(t)-u_2(t)\| &\le \int_0^t r(t-\tau,\lambda_1)L(\bar{r})|(u_1-u_2)[\varphi]_\tau|_{\mathcal{B}}\mathrm{d}\tau\\
&\le L(\bar{r})\int_0^t r(t-\tau,\lambda_1)K_T\sup_{[0,\tau]}\|u_1(\xi)-u_2(\xi)\|\mathrm{d}\tau.
\end{aligned}
$$

Since the last term is nondecreasing with respect to t, we get

$$
\sup_{[0,t]}\|u_1(\xi)-u_2(\xi)\|\le L(\bar{r})\int_0^t r(t-\tau,\lambda_1)K_T\sup_{[0,\tau]}\|u_1(\xi)-u_2(\xi)\|\mathrm{d}\tau.
$$

Employing [14, Proposition 2.2], we conclude that $u_1=u_2$. The proof is completed. □

In the next result, we get a global existence to problem (1.1)–(1.2) by relaxing the smallness condition on both the initial datum and coefficients. However, the nonlinearity part must satisfy the sublinear condition.

Theorem 3.2. *Assume the hypotheses* (A) *and* (K). *Let f be continuous and obey the condition given by*

$$
\|f(t,w)\|\le\alpha(t)+\beta|w|_{\mathcal{B}},\forall t\ge 0, w\in\mathcal{B},
$$

where $\alpha \in L^1_{loc}(\mathbb{R}^+;\mathbb{R}^+)$ *and* β *is a nonnegative number. Then the problem* (1.1)–(1.2) *admits at least one global mild solution.*

Proof. Since $\mathcal{F}$ is a compact operator, we just find a closed convex set which is invariant under the solution operator. On the other words, we construct a closed convex set $D \subset \mathbf{C}_\varphi$ such that $\mathcal{F}(D) \subset D$.

Indeed, from the formulation of $\mathcal{F}$, we obtain

$$\begin{aligned}\|\mathcal{F}(u)(t)\| &\le s(t,\lambda_1)\|\varphi(0)\| \\ &\quad + \int_0^t r(t-\tau,\lambda_1)\Big(\alpha(\tau) + \beta|u[\varphi]_\tau|_{\mathcal{B}}\Big)\mathrm{d}\tau \\ &\le s(t,\lambda_1)\|\varphi(0)\| + \int_0^t r(t-\tau,\lambda_1)\alpha(\tau)\mathrm{d}\tau \\ &\quad + \int_0^t r(t-\tau,\lambda_1)\beta\Big(K_T \sup_{\xi\in[0;\tau]}\|u(\xi)\| + M_T|\varphi|_{\mathcal{B}}\Big)\mathrm{d}\tau,\ t\in[0,T].\end{aligned}$$

Then, in view of Proposition 2.1, we have

$$\begin{aligned}\|\mathcal{F}(u)(t)\| &\le s(t,\lambda_1)\|\varphi(0)\| + (r(\cdot,\lambda_1)*\alpha)(t) + \beta M_T\lambda_1^{-1}(1-s(t,\lambda_1))|\varphi|_{\mathcal{B}} \\ &\quad + \beta K_T\int_0^t r(t-\tau,\lambda_1)\sup_{[0,\tau]}\|u(\xi)\|\mathrm{d}\tau \\ &\le (\varrho + \beta M_T\lambda_1^{-1})|\varphi|_{\mathcal{B}} + \sup_{[0,T]}(r(\cdot,\lambda_1)*\alpha)(t) \\ &\quad + \beta K_T\int_0^t r(t-\tau,\lambda_1)\sup_{[0,\tau]}\|u(\xi)\|\mathrm{d}\tau,\ t\in[0,T].\end{aligned}$$

Because $\tau \mapsto \sup_{[0,\tau]}\|u(\xi)\|$ is a nondecreasing function, the last integral is nondecreasing with respect to t. Therefore, one gets

$$\sup_{[0,t]}\|\mathcal{F}(u)(\xi)\| \le M_0 + \beta K_T\int_0^t r(t-\tau,\lambda_1)\sup_{[0,\tau]}\|u(\xi)\|\mathrm{d}\tau,\ t\in[0,T], \quad (3.3)$$

where $M_0 = (\varrho + \beta M_T\lambda_1^{-1})|\varphi|_{\mathcal{B}} + \sup_{[0,T]}(r(\cdot,\lambda_1)*\alpha)(t)$.

Let $v \in C([0,T];\mathbb{R}^+)$ be the unique solution of the integral equation

$$v(t) = M_0 + \beta K_T\int_0^t r(t-\tau,\lambda_1)v(\tau)\mathrm{d}\tau, t\in[0,T].$$

We define the set

$$D = \left\{ w \in \mathbf{C}_\varphi : \sup_{[0,t]}\|w(\xi)\| \le v(t), \forall t\in[0,T]\right\}.$$

Obviously, D is a bounded closed convex set. Then, inequality (3.3) implies that $\mathcal{F}(D)\subset D$. The proof is completed. □

The rest of this section is devoted to showing the dissipativity of the system. Let $\mathbb{S}(\varphi)$ be the solution set corresponding to a given initial datum φ.

The problem (1.1)–(1.2) is said to be uniformly dissipative with an absorbing set B_σ if we can find a constant $\sigma > 0$ such that: For each bounded set $D \subset \mathcal{B}$ there exists $T(D) > 0$ such that $\forall u \in \mathbb{S}(\varphi),\ \varphi \in D$, we have

$$|u_t|_{\mathcal{B}} \le \sigma,\ \forall t > T(D).$$

We now in a position to state a dissipativity result for (1.1)–(1.2).

Theorem 3.3. *Let* (A) *and* (K) *hold. Suppose that f is a continuous function and satisfies the condition*

$$\|f(t,w)\| \le \alpha(t) + \beta |w|_{\mathcal{B}}, \forall\, t \ge 0, w \in \mathcal{B},$$

where $\beta > 0$ such that $\beta K_\infty < \lambda_1$, $K_\infty = \sup_{s\ge 0} K(s)$, $\alpha \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative nondecreasing function such that $r(\cdot,\lambda_1) * \alpha \in BC(\mathbb{R}^+)$. If $l \notin L^1(\mathbb{R}^+)$ and $M \in BC_0$, then the system (1.1)–(1.2) *is dissipative with the absorbing set B_σ for any σ satisfying*

$$\sigma > \frac{\lambda_1 \alpha^* K_\infty}{\lambda_1 - \beta K_\infty},$$

where $\alpha^* = \sup_{\mathbb{R}^+}(r(\cdot,\lambda_1) * \alpha)(t)$.

Proof. Let $D \subset \mathcal{B}$ be a bounded set, $\varphi \in D$ and $u \in \mathbb{S}(\varphi)$. Then

$$u(t) = S(t)\varphi(0) + \int_0^t R(t-\tau) f(\tau, u[\varphi]_\tau)\mathrm{d}\tau,\ t \ge 0.$$

Thus,

$$\begin{aligned}
\|u(t)\| &\le s(t,\lambda_1)\|\varphi(0)\| + \int_0^t r(t-\tau,\lambda_1)\big[\alpha(\tau) + \beta |u[\varphi]_\tau|\big]\mathrm{d}\tau \\
&\le s(t,\lambda_1)\varrho|\varphi|_{\mathcal{B}} + \int_0^t r(t-\tau,\lambda_1)\Big[\alpha(\tau) + \beta M(\tau)|\varphi|_{\mathcal{B}} \\
&\quad + \beta K_\infty \sup_{\xi\in[0;\tau]} \|u(\xi)\|\Big]\mathrm{d}\tau \\
&\le \Big(s(t,\lambda_1)\varrho + \beta \int_0^t r(t-\tau,\lambda_1) M(\tau)\mathrm{d}\tau\Big)|\varphi|_{\mathcal{B}} \\
&\quad + \int_0^t r(t-\tau,\lambda_1)\alpha(\tau)\mathrm{d}\tau + \int_0^t r(t-\tau,\lambda_1)\beta K_\infty \sup_{\xi\in[0;\tau]}\|u(\xi)\|\mathrm{d}\tau \\
&\le \Big(\varrho + \frac{\beta M_\infty}{\lambda_1}\Big)|\varphi|_{\mathcal{B}} + \int_0^t r(t-\tau,\lambda_1)\alpha(\tau)\mathrm{d}\tau + \frac{\beta K_\infty}{\lambda_1}\sup_{\xi\in[0;t]}\|u(\xi)\|.
\end{aligned}$$

It implies

$$\begin{aligned}
\sup_{\xi\in[0;t]}\|u(\xi)\| &\le \frac{\lambda_1}{\lambda_1 - \beta K_\infty}\Big(\varrho + \frac{\beta M_\infty}{\lambda_1}\Big)|\varphi|_{\mathcal{B}} + \frac{\lambda_1}{\lambda_1 - \beta K_\infty}\sup_{\mathbb{R}^+}(r(\cdot,\lambda_1) * \alpha)(t) \\
&\le \frac{\lambda_1}{\lambda_1 - \beta K_\infty}\Big(\varrho + \frac{\beta M_\infty}{\lambda_1}\Big)|D|_{\mathcal{B}} + \frac{\lambda_1 \alpha^*}{\lambda_1 - \beta K_\infty},
\end{aligned}$$

here $|D|_{\mathcal{B}} := \sup\{|w|_{\mathcal{B}} :\ w \in \mathcal{B}\}$ and $\alpha^* = \sup_{\mathbb{R}^+}(r(\cdot,\lambda_1) * \alpha)(t)$.

Combing the last estimate and the formulation of solution, we have

$$\begin{aligned}
\|u(t)\| &\le s(t,\lambda_1)\varrho|\varphi|_{\mathcal{B}} \\
&\quad + \int_0^t r(t-\tau,\lambda_1)\Big[\alpha(\tau)+\beta M(\tau/2)|u_{\frac{\tau}{2}}|_{\mathcal{B}}+\beta K(\tau/2)\sup_{\xi\in[\tau/2;\tau]}\|u(\xi)\|\Big]\mathrm{d}\tau \\
&\le s(t,\lambda_1)\varrho|\varphi|_{\mathcal{B}} + \int_0^t r(t-\tau,\lambda_1)\Big[\alpha(\tau)+\beta M(\tau/2)|u_{\frac{\tau}{2}}|_{\mathcal{B}}\Big]\mathrm{d}\tau \\
&\quad + \beta K_\infty \int_0^t r(t-\tau,\lambda_1)\sup_{\xi\in[\tau/2;\tau]}\|u(\xi)\|\mathrm{d}\tau \\
&\le s(t,\lambda_1)\varrho|D|_{\mathcal{B}} \\
&\quad + \int_0^t r(t-\tau,\lambda_1)\Big[\alpha(\tau)+\beta M(\tau/2)\big(K_\infty K_D + M_\infty|D|_{\mathcal{B}}\big)\Big]\mathrm{d}\tau \\
&\quad + \beta K_\infty \int_0^t r(t-\tau,\lambda_1)\sup_{\xi\in[\tau/2;\tau]}\|u(\xi)\|\mathrm{d}\tau,
\end{aligned}$$

where $K_D = \frac{\lambda_1}{\lambda_1-\beta K_\infty}\Big(\varrho+\frac{\beta M_\infty}{\lambda_1}\Big)|D|_{\mathcal{B}} + \frac{\lambda_1\alpha^*}{\lambda_1-\beta K_\infty}$.

In view of Proposition 2.4 with $v(t)=\|u(t)\|$, $p(t)=s(t,\lambda_1)\varrho|D|_{\mathcal{B}}$, and $q(t)=\alpha(t)+\beta\big(K_\infty K_D+M_\infty|D|_{\mathcal{B}}\big)M(t/2)$, we have $\|u(\cdot)\|\in BC(\mathbb{R}^+)$ and

$$\begin{aligned}
\limsup_{t\to\infty}\|u(t)\| &\le \Big[\limsup_{t\to\infty} s(t,\lambda_1)\varrho|D|_{\mathcal{B}} + \limsup_{t\to\infty}\big(r(\cdot,\lambda_1)*\alpha\big)(t)\Big]\frac{\lambda_1}{\lambda_1-\beta K_\infty} \\
&\quad + \limsup_{t\to\infty}\Big[r(\cdot,\lambda_1)*M\big(\tfrac{\cdot}{2}\big)(t)\Big]\beta\big(K_\infty K_D+M_\infty|D|_{\mathcal{B}}\big)\frac{\lambda_1}{\lambda_1-\beta K_\infty} \\
&\le \frac{\lambda_1\alpha^*}{\lambda_1-\beta K_\infty},
\end{aligned}$$

thanks to the fact that $M\big(\frac{\cdot}{2}\big)\in BC_0$. So for $\epsilon>0$ there exists $T_1(\epsilon)>0$ such that

$$\sup_{\xi\ge t}\|u(\xi)\| \le \frac{\lambda_1\alpha^*}{\lambda_1-\beta K_\infty}+\frac{\epsilon}{2K_\infty},\ \forall t\ge T_1(\epsilon).$$

Thus,

$$|u_t|_{\mathcal{B}} \le K_\infty\Big(\frac{\lambda_1\alpha^*}{\lambda_1-\beta K_\infty}+\frac{\epsilon}{2K_\infty}\Big)+M\Big(\frac{t}{2}\Big)|u_{\frac{t}{2}}|_{\mathcal{B}},\ \forall t\ge 2T_1(\epsilon). \tag{3.4}$$

On the other hand, we get

$$M\Big(\frac{t}{2}\Big)|u_{\frac{t}{2}}|_{\mathcal{B}} \le M\Big(\frac{t}{2}\Big)\big(K_\infty K_D+M_\infty|D|_{\mathcal{B}}\big).$$

By virtue of $M\in BC_0$, there exists $T_2(D,\epsilon)>0$ such that

$$M\Big(\frac{t}{2}\Big)|u_{\frac{t}{2}}|_{\mathcal{B}} \le \frac{\epsilon}{2},\quad \forall t\ge T_2(D,\epsilon). \tag{3.5}$$

From (3.4) and (3.5), we arrive at

$$|u_t|_{\mathcal{B}} \le \frac{\lambda_1\alpha^*K_\infty}{\lambda_1-\beta K_\infty}+\epsilon,\ \forall t\ge T(D,\epsilon),$$

where $T(D,\epsilon)=\max\{T_1(\epsilon),T_2(D,\epsilon)\}$. We choose a sufficiently small number ϵ to get the uniform dissipativity with any $\sigma>\frac{\lambda_1\alpha^*K_\infty}{\lambda_1-\beta K_\infty}$. The proof is completed. □

4. Asymptotic stability and weakly asymptotic stability

In this section, we utilize the well-known Lyapunov stability theory [9] to nonlocal evolution equations involving infinite delays. Note that the Lyapunov stability theory is also a basic tool to analyze other general systems, for instance, fractional with impulsive effects [12, 13], fractal differential systems [27] or Volterra integral-differential systems [28]. We will establish this asymptotic (weak) stability for the system with (without) the uniqueness, respectively.

4.1. Asymptotic stability

In the following theorem, we prove the asymptotic stability of solution to (1.1)–(1.2) when the nonlinearity is globally Lipschitzian.

Theorem 4.1. *Let* (A) *and* (K) *hold. Assume that* f *satisfies the Lipschitz condition*

$$\|f(t, w_1) - f(t, w_2)\| \le \beta |w_1 - w_2|_{\mathcal{B}}, \forall\, t \ge 0, w_1, w_2 \in \mathcal{B},$$

where $\beta > 0$ *such that* $\beta K_\infty < \lambda_1$. *If* $l \notin L^1(\mathbb{R}^+)$ *and* $M \in BC_0$, *then an arbitrary solution of* (1.1)–(1.2) *is asymptotically stable.*

Proof. Let $u(\cdot, \varphi)$ and $v(\cdot, \psi)$ be solutions of (1.1)–(1.2) with initial data φ and ψ, respectively. Then

$$\begin{aligned}
\|u(t) - v(t)\| &\le s(t, \lambda_1)\|\varphi(0) - \psi(0)\| \\
&\quad + \int_0^t r(t-\tau, \lambda_1)\|f(\tau, u[\varphi]_\tau) - f(\tau, v[\psi]_\tau)\| \mathrm{d}\tau \\
&\le s(t, \lambda_1)\|\varphi(0) - \psi(0)\| \\
&\quad + \int_0^t r(t-\tau, \lambda_1)\beta |u[\varphi]_\tau - v[\psi]_\tau|_{\mathcal{B}} \mathrm{d}\tau \\
&\le s(t, \lambda_1)\|\varphi(0) - \psi(0)\| \\
&\quad + \beta \int_0^t r(t-\tau, \lambda_1)\big[K_\infty \sup_{\xi \in [0;\tau]} \|u(\xi) - v(\xi)\| + M_\infty |\varphi - \psi|_{\mathcal{B}}\big] \mathrm{d}\tau \\
&\le \frac{\beta K_\infty}{\lambda_1} \sup_{\xi \in [0;t]} \|u(\xi) - v(\xi)\| + \Big(\varrho + \frac{\beta M_\infty}{\lambda_1}\Big) |\varphi - \psi|_{\mathcal{B}}.
\end{aligned}$$

Therefore

$$\sup_{\xi \in [0;t]} \|u(\xi) - v(\xi)\| \le \frac{\varrho\lambda_1 + \beta M_\infty}{\lambda_1 - \beta K_\infty} |\varphi - \psi|_{\mathcal{B}}. \tag{4.1}$$

Using (B3) and the estimate above, one gets

$$\begin{aligned}
|u_t - v_t|_{\mathcal{B}} &\le K(t) \sup_{\xi \in [0;t]} \|u(\xi) - v(\xi)\| + M(t)|\varphi - \psi|_{\mathcal{B}} \\
&\le \left(K_\infty \frac{\varrho\lambda_1 + \beta M_\infty}{\lambda_1 - \beta K_\infty} + M_\infty\right) |\varphi - \psi|_{\mathcal{B}},\ \forall t \ge 0.
\end{aligned} \tag{4.2}$$

Thus,

$$\|u(t) - v(t)\| \le s(t, \lambda_1)\|\varphi(0) - \psi(0)\| + \beta \int_0^t r(t-\tau, \lambda_1) M(\tau/2) |u_{\frac{\tau}{2}} - v_{\frac{\tau}{2}}|_{\mathcal{B}} \mathrm{d}\tau$$

$$
\begin{aligned}
&+ \beta \int_0^t r(t-\tau,\lambda_1) K(\tau/2) \sup_{\xi\in[\tau/2;\tau]} \|u(\xi)-v(\xi)\| \mathrm{d}\tau \\
\le\; & s(t,\lambda_1)\|\varphi(0)-\psi(0)\| \\
&+ \beta\Big(K_\infty \frac{\varrho\lambda_1+\beta M_\infty}{\lambda_1-\beta K_\infty} + M_\infty\Big)|\varphi-\psi|_{\mathcal{B}} \int_0^t r(t-\tau,\lambda_1) M(\tau/2)\mathrm{d}\tau \\
&+ \beta K_\infty \int_0^t r(t-\tau,\lambda_1) \sup_{\xi\in[\tau/2;\tau]} \|u(\xi)-v(\xi)\| \mathrm{d}\tau
\end{aligned}
$$

By Proposition 2.4, we have

$$
\begin{aligned}
\limsup_{t\to\infty} \|u(t)-v(t)\| \le \frac{\lambda_1}{\lambda_1-\beta K_\infty} & \limsup_{t\to\infty} s(t,\lambda_1)\|\varphi(0)-\psi(0)\| \\
&+ C \limsup_{t\to\infty} \big(r(\cdot,\lambda_1) * M(\cdot/2)\big)(t),
\end{aligned}
$$

here $C = \beta(K_\infty \frac{\varrho\lambda_1+\beta M_\infty}{\lambda_1-\beta K_\infty} + M_\infty)|\varphi-\psi|_{\mathcal{B}}$. Then

$$
\limsup_{t\to\infty} \|u(t)-v(t)\| = 0,
$$

thanks to $M \in BC_0$.

It follows that

$$
\forall \epsilon > 0,\ \exists\, T_1(\epsilon) > 0 : \sup_{\xi\ge t} \|u(\xi)-v(\xi)\| \le \frac{\epsilon}{2K_\infty}, \quad \forall t \ge T_1(\epsilon).
$$

Consequently,

$$
|u_t - v_t|_{\mathcal{B}} \le K_\infty \frac{\epsilon}{2K_\infty} + \frac{\varrho\lambda_1+\beta M_\infty}{\lambda_1-\beta K_\infty}|\varphi-\psi|_{\mathcal{B}} M(t/2),\ \forall t \ge 2T_1(\epsilon),
$$

due to (4.1). Since $M \in BC_0$, there exists $T_2 > 0$ such that

$$
\frac{\varrho\lambda_1+\beta M_\infty}{\lambda_1-\beta K_\infty}|\varphi-\psi|_{\mathcal{B}} M(t/2) \le \frac{\epsilon}{2},\ \forall t \ge T_2.
$$

This gives

$$
|u_t - v_t|_{\mathcal{B}} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\ \forall t \ge T, \tag{4.3}
$$

here $T = \max\{2T_1(\epsilon), T_2\}$.

Combining (4.2) and (4.3) gives us the desired conclusion. □

The next theorem states the asymptotic stability of the zero solution when f satisfies the hypotheses in Theorem 3.1 with K_∞ in place of K_T.

Theorem 4.2. *Let the hypotheses of Theorem 3.1 hold where K_T is replaced by K_∞. If $l \notin L^1(\mathbb{R}^+)$ and $M \in BC_0$, then the zero solution of (1.1) is asymptotically stable.*

Proof. Choose the numbers $\theta \in (0, \frac{\lambda_1}{K_\infty} - \beta)$, η and δ as in the proof of Theorem 3.1. Combining (3.1)–(3.2) and $\|\varphi\|_\infty < \delta$, problem (1.1)–(1.2) possesses a unique solution $u \in B_\eta$. Moreover, this solution verifies the following

$$
\begin{aligned}
\|u(t)\| &\le s(t,\lambda_1)\|\varphi(0)\| + \int_0^t r(t-\tau,\lambda_1)(\beta+\theta)|u[\varphi]_\tau|_{\mathcal{B}}\mathrm{d}\tau \\
&\le s(t,\lambda_1)\|\varphi(0)\|
\end{aligned}
$$

$$
\begin{aligned}
&+ (\beta+\theta)\int_0^t r(t-\tau,\lambda_1)[K_\infty \sup_{s\in[0;\tau]}\|u(s)\| + M(\tau)|\varphi|_{\mathcal{B}}]\mathrm{d}\tau \\
&\le \frac{(\beta+\theta)K_\infty}{\lambda_1}\sup_{s\in[0;t]}\|u(s)\| + \Big(\varrho + (\beta+\theta)\int_0^t r(t-\tau,\lambda_1)M(\tau)\mathrm{d}\tau\Big)|\varphi|_{\mathcal{B}},
\end{aligned}
$$

for all $t \ge 0$. Then

$$
\sup_{s\in[0;t]}\|u(s)\| \le \frac{\lambda_1}{\lambda_1-(\beta+\theta)K_\infty}\Big(\varrho + (\beta+\theta)M^*\Big)|\varphi|_{\mathcal{B}},
$$

for all $t \ge 0$, where $M^* = \sup_{t\ge 0}\int_0^t r(t-\tau,\lambda_1)M(\tau)\mathrm{d}\tau$.

From the last estimate and (B3), we see that

$$
|u_t|_{\mathcal{B}} \le \left(\frac{\lambda_1 K_\infty\big(\varrho + (\beta+\theta)M^*\big)}{\lambda_1-(\beta+\theta)K_\infty} + M_\infty\right)|\varphi|_{\mathcal{B}}. \tag{4.4}
$$

By the same arguments as in Theorem 4.1, one gets

$$
\limsup_{t\to\infty}\|u(t)\| = 0.
$$

Then

$$
\lim_{t\to\infty}|u_t|_{\mathcal{B}} = 0. \tag{4.5}
$$

From (4.4) and (4.5), we obtain the asymptotic stability of the zero solution. □

4.2. Weakly asymptotic stability

In this subsection, we do not assume that f is Lipschitz continuous. Consequently, the solutions of problem (1.1)–(1.2) are not necessarily unique. We aim at the weakly asymptotic stability for the zero solution.

Definition 4.1. [3] Let $\mathbb{S}(\varphi)$ be the solution set of (1.1)–(1.2) with respect to the initial datum φ. Assume that $0 \in \mathbb{S}(0)$, that is (1.1) admits zero solution. The zero solution of (1.1) is said to be weakly asymptotically stable iff

(1) It is stable, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\varphi|_{\mathcal{B}} < \delta$ then $|u_t|_{\mathcal{B}} < \varepsilon$ for all $u \in \mathbb{S}(\varphi)$;
(2) It is weakly attractive, i.e. for each $\varphi \in \mathcal{B}$, there exists $u \in \mathbb{S}(\varphi)$ such that $|u_t|_{\mathcal{B}} \to 0$ as $t \to \infty$.

In the present situation, we employ a version of fixed point theory for condensing maps. We now collect some essential properties of measure of noncompactness (MNC), and fixed point principles.

Definition 4.2. [11] Let E be a Banach space and $\mathcal{B}(E)$ the collection of all nonempty and bounded subsets of E. A function $\omega : \mathcal{B}(E) \to \mathbb{R}^+$ is said to be a measure of noncompactness (MNC) if $\omega(\overline{\mathrm{co}}\, D) = \omega(D)$ for all $D \in \mathcal{B}(E)$. An MNC is called

- nonsingular if $\omega(D \cup \{x\}) = \omega(D)$ for all $D \in \mathcal{B}(E)$, $x \in E$.
- monotone if $\omega(D_1) \le \omega(D_2)$ provided that $D_1 \subset D_2$.

The Hausdorff measure of noncompactness is an MNC which is defined by

$$\chi(D) = \inf\{\varepsilon > 0 : D \text{ admits a finite } \varepsilon - \text{net}\}.$$

Definition 4.3. [11] Let E be a Banach space and $D \in \mathcal{B}(E)$. A continuous map $\mathcal{F} : D \to E$ is said to be condensing with respect to MNC ω (ω-condensing) iff the relation $\omega(B) \leq \omega(\mathcal{F}(B)), B \subset D$, implies that B is relatively compact.

The following theorem states a fixed point principle for condensing maps.

Theorem 4.3. [11] *Let ω be a monotone and nonsingular MNC on E. Assume that $D \subset E$ is a closed convex set and $\mathcal{F} : D \to D$ is ω-condensing. Then $\mathcal{F}$ admits a fixed point.*

In this subsection, we consider the solution operator on $BC_0(\mathbb{R}^+; H)$ where

$$BC_0(\mathbb{R}^+; H) = \{y \in BC(\mathbb{R}^+; H) : \lim_{t\to\infty} \|y(t)\| = 0\}.$$

It is known that $BC_0(\mathbb{R}^+; H)$ is a closed subspace of $BC(\mathbb{R}^+; H)$. Given $\varphi \in \mathcal{B}$, put $\mathcal{BC}_0^\varphi = \{u \in BC_0(\mathbb{R}^+; H) : u(0) = \varphi(0)\}$. Then $\mathcal{BC}_0^\varphi$ is a closed convex set of $BC_0(\mathbb{R}^+; H)$.

Let $\pi_T : \mathcal{BC}_0^\varphi \to C([0, T]; H)$ the restriction operator on $\mathcal{BC}_0^\varphi$, which is defined by $\pi_T(u)(t) = u(t), \forall\, t \in [0, T]$, for all $u \in \mathcal{BC}_0^\varphi$. For a bounded set D in $\mathcal{BC}_0^\varphi$, we set

$$d_\infty(D) = \lim_{T\to\infty} \sup_{u\in D} \sup_{t\geq T} \|u(t)\|,$$

$$\chi_\infty(D) = \sup_{T>0} \chi_T(D),$$

where $\chi_T(\cdot)$ is the Hausdorff MNC in $C([0, T]; H)$. Put

$$\chi^*(D) = d_\infty(D) + \chi_\infty(D).$$

It is shown in [2] that χ^* satisfies all properties stated in Definition 4.2. In addition, if $\chi^*(D) = 0$, then D is relatively compact in $BC_0(\mathbb{R}^+; H)$. Especially, if $u \in C(\mathbb{R}^+; H)$, then $d_\infty(\{u\}) = 0$ if and only if $u \in BC_0(\mathbb{R}^+; H)$.

We are now in a position to present the main result in this section.

Theorem 4.4. *Assume* (A) *and* (K) *hold. Let the nonlinear f be continuous and satisfy the estimate*

$$\|f(t, w)\| \leq \beta(t)|w|_{\mathcal{B}},\ \forall t \geq 0, w \in \mathcal{B}, \tag{4.6}$$

where $\beta \in BC(\mathbb{R}^+)$ is a nonnegative function. Suppose that $l \notin L^1(\mathbb{R}^+)$, $K \in BC(\mathbb{R}^+)$ and $M \in BC_0$. Then the zero solution of (1.1) *is weakly asymptotically stable provided that*

$$\ell = \limsup_{t\to\infty} \int_{t/2}^{t} r(t - \tau, \lambda_1)\beta(\tau)K(\tau/2)\mathrm{d}\tau < 1. \tag{4.7}$$

Let us outline the proof. The idea is employing the fixed point theorem for condensing maps. We first establish the well-defined and condensing property of the solution operator in Lemma 4.5. Then Lemma 4.6 reduces the condition on functions K and M to the auxiliary functions K_1 and M_1. Using this preparation, Lemma 4.7 constructs a closed bounded invariant subset of the solution operator. Combining all these results, the poof is finished by following the standard argument. We are now in a position to show the proof in details.

Lemma 4.5. *Assume the hypotheses of Theorem 4.4. Then*

$$d_\infty(\mathcal{F}(D)) \leq \ell \cdot d_\infty(D)$$

for all bounded set $D \subset \mathcal{BC}_0^\varphi$. In particular, $\mathcal{F}(\mathcal{BC}_0^\varphi) \subset \mathcal{BC}_0^\varphi$.

Proof. We first show that

$$\limsup_{t\to+\infty} \int_0^t r(t-\tau,\lambda_1)M(\tau/2)\beta(\tau)\mathrm{d}\tau = 0. \tag{4.8}$$

Indeed, we see that

$$\begin{aligned}\int_0^t r(t-\tau,\lambda_1)M(\tau/2)\beta(\tau)\mathrm{d}\tau &\leq M_\infty\beta_\infty \int_0^{t/2} r(t-\tau,\lambda_1)\mathrm{d}\tau \\ &\quad + \beta_\infty \int_{t/2}^t r(t-\tau,\lambda_1)M(\tau/2)\mathrm{d}\tau,\end{aligned}$$

where $M_\infty = \sup_{t\geq 0} M(t)$ and $\beta_\infty = \sup_{t\geq 0}\beta(t)$. Moreover,

$$\int_0^{t/2} r(t-\tau,\lambda_1)\mathrm{d}\tau = \int_{t/2}^t r(\tau,\lambda_1)\mathrm{d}\tau \to 0 \text{ as } t\to\infty,$$

according to $r(\cdot,\lambda_1) \in L^1(\mathbb{R}^+)$. In addition,

$$\begin{aligned}\int_{t/2}^t r(t-\tau,\lambda_1)M(\tau/2)\mathrm{d}\tau &\leq \sup_{\tau\geq t/4} M(\tau)\int_0^{t/2} r(\tau,\lambda_1)\mathrm{d}\tau \\ &\leq \lambda_1^{-1}\sup_{\tau\geq t/4} M(\tau) \to 0 \text{ as } t\to\infty,\end{aligned}$$

thanks to the assumption $M \in BC_0$. Thus, (4.8) takes place.

Now let $D \subset \mathcal{BC}_0^\varphi$ be a bounded set and $u \in D$. Put $R_D = \sup_{u\in D}\|u\|_{BC} + |\varphi|_{\mathcal{B}}$. Then

$$\begin{aligned}\|\mathcal{F}(u)(t)\| &\leq s(t,\lambda_1)\|\varphi(0)\| + \int_0^t r(t-\tau,\lambda_1)\beta(\tau)|u[\varphi]_\tau|_{\mathcal{B}}\mathrm{d}\tau \\ &\leq s(t,\lambda_1)\|\varphi(0)\| \\ &\quad + \int_0^t r(t-\tau,\lambda_1)\beta(\tau)\left(M(\tau/2)|u[\varphi]_{\tau/2}|_{\mathcal{B}} + K(\tau/2)\sup_{\xi\in[\tau/2,\tau]}\|u(\xi)\|\right)\mathrm{d}\tau \\ &\leq s(t,\lambda_1)\|\varphi(0)\| \\ &\quad + r * \widetilde{M}(t)[M_\infty|\varphi|_{\mathcal{B}} + K_\infty R_D] + \int_0^t r(t-\tau,\lambda_1)\widetilde{K}(\tau)\sup_{\xi\in[\tau/2,\tau]}\|u(\xi)\|\mathrm{d}\tau,\end{aligned} \tag{4.9}$$

for all $t \geq 0$, where $\widetilde{M}(\tau) = \beta(\tau)M(\tau/2), \widetilde{K}(\tau) = \beta(\tau)K(\tau/2)$.

In order to estimate the last term, fixing a $T > 0$, for $t > 4T$, we get

$$\int_0^t r(t-\tau,\lambda_1)\widetilde{K}(\tau) \sup_{\xi\in[\tau/2,\tau]} \|u(\xi)\|\mathrm{d}\tau \leq R_D\widetilde{K}_\infty \int_0^{t/2} r(t-\tau,\lambda_1)\mathrm{d}\tau + \sup_{\xi\geq T}\|u(\xi)\| \int_{t/2}^t r(t-\tau,\lambda_1)\widetilde{K}(\tau)\mathrm{d}\tau. \tag{4.10}$$

Combining (4.9) and (4.10) yields

$$\begin{aligned}\sup_{u\in D}\sup_{t\geq 4T}\|\mathcal{F}(u)(t)\| \leq{}& \sup_{t\geq 4T} s(t,\lambda_1)\|\varphi(0)\| \\ &+ \sup_{t\geq 4T}\left(r * \widetilde{M}(t)(M_\infty|\varphi|_{\mathcal{B}} + K_\infty R_D) \right.\\ &\left. + R_D\widetilde{K}_\infty \frac{s(t/2,\lambda_1)-s(t,\lambda_1)}{\lambda_1}\right) \\ &+ \left(\sup_{u\in D}\sup_{\xi\geq T}\|u(\xi)\|\right)\sup_{t\geq 4T}\int_{t/2}^t r(t-\tau,\lambda_1)\widetilde{K}(\tau)\mathrm{d}\tau.\end{aligned}$$

Letting $T\to\infty$, we conclude

$$d_\infty(\Phi(D)) \leq \ell \cdot d_\infty(D),$$

thanks to (4.7)–(4.8) and

$$\sup_{t\geq 4T} s(t,\lambda_1) = s(4T,\lambda_1) \to 0 \text{ as } T\to\infty.$$

Consequently, if $D=\{u\}$ then $d_\infty(\{\mathcal{F}(u)\}) \leq \ell\cdot d_\infty(\{u\}) = 0$. This yields $\mathcal{F}(u)\in\mathcal{BC}_0^\varphi$ for all $u\in\mathcal{BC}_0^\varphi$. It follows that $\mathcal{F}(\mathcal{BC}_0^\varphi)\subset\mathcal{BC}_0^\varphi$. □

Lemma 4.6. *Assume that* (4.7) *is satisfied. Let* $K_1(t) = K(t/2)+K(t/2)M(t/2)$. *Then, there exist two positive numbers* $T_1 > 0$ *and* $l_1 < 1$ *such that* $r(\cdot,\lambda_1) * (\beta K_1)(t) \leq l_1$ *for all* $t\geq T_1$.

Proof. We get

$$\begin{aligned} r(\cdot,\lambda_1) * (\beta K_1)(t) ={}& \int_0^t r(t-\tau,\lambda_1)\beta(\tau)K(\tau/2)\mathrm{d}\tau \\ &+ \int_0^t r(t-\tau,\lambda_1)\beta(\tau)K(\tau/2)M(\tau/2)\mathrm{d}\tau \\ \leq{}& \int_0^{t/2} r(t-\tau,\lambda_1)\beta(\tau/2)K(\tau/2)\mathrm{d}\tau \\ &+ \int_{t/2}^t r(t-\tau,\lambda_1)\beta(\tau)K(\tau/2)\mathrm{d}\tau \\ &+ \int_0^t r(t-\tau,\lambda_1)\beta(\tau)K(\tau/2)M(\tau/2)\mathrm{d}\tau \\ \leq{}& K_\infty\beta_\infty \int_{t/2}^t r(\tau,\lambda_1)\mathrm{d}\tau + \int_{t/2}^t r(t-\tau,\lambda_1)\beta(\tau)K(\tau/2)\mathrm{d}\tau \end{aligned}$$

$$+ K_\infty \int_0^t r(t-\tau,\lambda_1)\beta(\tau)M(\tau/2)\mathrm{d}\tau.$$

Obviously, the first term tends to zero when $t \to \infty$ thanks to $r \in L^1(\mathbb{R}^+)$. The last term goes to zero by (4.8). Thus, it follows from (4.7) that

$$\limsup_{t\to\infty} r(\cdot,\lambda_1) * (\beta K_1)(t) = \limsup_{t\to\infty} \int_{t/2}^t r(t-\tau,\lambda_1)\beta(\tau)K(\tau/2) = l < 1.$$

The last inequality implies the desired result. □

Lemma 4.7. *Assume the hypotheses of Theorem 4.4. Then, there exists a bounded closed convex set which is invariant under the solution operator.*

Proof. Take T_1 and l_1 from Lemma 4.6, that is,

$$r(\cdot,\lambda_1) * (\beta K_1)(t) \le l_1, \quad \forall t \ge T_1.$$

The construction of the invariant set consists of two steps. We first use a suitable weight function to obtain the invariant set for finite time. Then we combine the estimate on this finite time interval with Lemma 4.6 to get the estimate for large time, which gives us the invariant set for all time.

Step 1 (Estimate on the interval $[0,T_1]$). We have

$$\begin{aligned}&\int_0^t r(t-\tau,\lambda_1)\beta(\tau)K_1(\tau)e^{-\mu(t-\tau)}\mathrm{d}\tau\\ &\le \|\beta K_1\|_{L^\infty(0,T_1)} \int_0^t r(t-\tau,\lambda_1)e^{-\mu(t-\tau)}\mathrm{d}\tau\\ &\le \|\beta K_1\|_{L^\infty(0,T_1)} \int_0^{T_1} r(\tau,\lambda_1)e^{-\mu\tau}\mathrm{d}\tau, \forall t \in (0,T_1].\end{aligned}$$

Observing that

$$\lim_{\mu\to+\infty} \int_0^{T_1} r(\tau,\lambda_1)e^{-\mu\tau}\mathrm{d}\tau = 0,$$

one can take a positive number μ such that

$$r(\cdot,\lambda_1) * (\beta K_1 m)(t) < \frac{m(t)}{2}, \text{ for all } t \in [0,T_1], \tag{4.11}$$

where $m(t) = e^{-\mu t}$. Let $M_1(t) = M(t/2)^2$, then one sees that

$$\begin{aligned}|u_t|_{\mathcal{B}} &\le K(t/2) \sup_{\xi\in[t/2,t]} \|u(\xi)\| + M(t/2)|u_{t/2}|_{\mathcal{B}}\\ &\le K(t/2) \sup_{\xi\in[t/2,t]} \|u(\xi)\| + M(t/2)\left(K(t/2) \sup_{\xi\in[0,t/2]} \|u(\xi)\| + M(t/2)|\varphi|_{\mathcal{B}}\right)\\ &\le K_1(t) \sup_{\xi\in[0,t]} \|u(\xi)\| + M_1(t)|\varphi|_{\mathcal{B}},\end{aligned} \tag{4.12}$$

according to the axiom (B3) of phase spaces.

Choose

$$R_1 \ge 2 \sup_{t\in[0,T_1]} \frac{(\varrho + \|r(\cdot,\lambda_1) * (\beta M_1)\|_\infty)|\varphi|_{\mathcal{B}}}{m(t)}. \tag{4.13}$$

Then for all $u \in BC_0(\mathbb{R}^+; H)$ such that $\sup_{t\in[0,T_1]} \dfrac{\|u(t)\|}{m(t)} \le R_1$, one has

$$\begin{aligned}
\|\mathcal{F}(u)(t)\| &\le s(t,\lambda_1)\|\varphi(0)\| + \int_0^t r(t-\tau,\lambda_1)\beta(\tau)|u[\varphi]_\tau|_{\mathcal{B}}\mathrm{d}\tau \\
&\le \|\varphi(0)\| + R_1 \int_0^t r(t-\tau,\lambda_1)\beta(\tau)K_1(\tau)m(\tau)\mathrm{d}\tau \\
&\quad + |\varphi|_{\mathcal{B}} \int_0^t r(t-\tau,\lambda_1)\beta(\tau)M_1(\tau)\mathrm{d}\tau \\
&\le \left(\varrho + \|r(\cdot,\lambda_1) * (\beta M_1)\|_\infty\right)|\varphi|_{\mathcal{B}} + R_1\frac{m(t)}{2},
\end{aligned}$$

here we employ (4.11) and (4.12). Hence, combining with (4.13) one gets

$$\sup_{t\in[0,T_1]} \frac{\|\mathcal{F}(u)(t)\|}{m(t)} \le R_1.$$

Step 2 (Estimate on the infinite interval $[T_1,\infty)$). Fixing a number R_2 such that

$$\left(\varrho + \|r(\cdot,\lambda_1) * (\beta M_1)\|_\infty\right)|\varphi|_{\mathcal{B}} + R_1 m(T_1)\|r(\cdot,\lambda_1) * (\beta K_1)\|_{BC} \le (1-l_1)R_2. \tag{4.14}$$

Then for all $u \in BC_0(\mathbb{R}^+; H)$ such that $\sup_{t\ge T_1}\|u(t)\| \le R_2$ and for $t \ge T_1$, we obtain

$$\begin{aligned}
\|\mathcal{F}(u)(t)\| &\le s(t,\lambda_1)\|\varphi(0)\| + \int_0^t r(t-\tau,\lambda_1)\beta(\tau)|u[\varphi]_\tau|_{\mathcal{B}}\mathrm{d}\tau \\
&\le \|\varphi(0)\| + R_1 m(T_1)\int_0^{T_1} r(t-\tau,\lambda_1)\beta(\tau)K_1(\tau)\mathrm{d}\tau \\
&\quad + \int_{T_1}^t r(t-\tau,\lambda_1)\beta(\tau)K_1(\tau)\left[\max_{\xi\in[0,T_1]}\|u(\xi)\| + \max_{T_1\le\xi\le\tau}\|u(\xi)\|\right]\mathrm{d}\tau \\
&\quad + |\varphi|_{\mathcal{B}}\int_0^t r(t-\tau,\lambda_1)\beta(\tau)M_1(\tau)\mathrm{d}\tau \\
&\le \left(\varrho + \|r(\cdot,\lambda_1) * (\beta M_1)\|_\infty\right)|\varphi|_{\mathcal{B}} + R_1 m(T_1)\|r(\cdot,\lambda_1) * (\beta K_1)\|_{BC} + l_1 R_2 \\
&\le R_2,
\end{aligned}$$

thanks to (4.14).

Finally, let consider the set

$$\mathbf{D} = \left\{ u \in \mathcal{BC}_0^\varphi : \sup_{[0,T_1]} \frac{\|u(t)\|}{m(t)} \le R_1; \sup_{t\ge T_1}\|u(t)\| \le R_2 \right\}. \tag{4.15}$$

It is evident that $\mathbf{D}$ is a closed bounded convex subset of $\mathcal{BC}_0^\varphi$ satisfying $\mathcal{F}(\mathbf{D}) \subset \mathbf{D}$. This completes the proof. $\square$

Proof of Theorem 4.4. By Lemma 4.7, we have

$$\mathcal{F}(\mathbf{D}) \subset \mathbf{D},$$

where $\mathbf{D}$ is given by (4.15). Let $D^* = \overline{\mathrm{co}}\mathcal{F}(\mathbf{D})$, then and $\pi_T(D^*)$ is compact for all $T > 0$ thanks to Proposition 2.3, and it is also a convex set. In addition, we get $\mathcal{F}(D^*) \subset D^*$. Considering $\mathcal{F} : D^* \to D^*$, we show that $\mathcal{F}$ is χ^*-condensing. If $D \subset D^*$, then obviously $\chi_T(D) = 0$, which implies $\chi_\infty(D) = 0$. Using Lemma 4.5, we have

$$\chi^*(\mathcal{F}(D)) = \chi_\infty(\mathcal{F}(D)) + d_\infty(\mathcal{F}(D)) = d_\infty(\mathcal{F}(D)) \le \ell \cdot d_\infty(D) \le \ell \cdot \chi^*(D).$$

If $\chi^*(D) \le \chi^*(\mathcal{F}(D))$, then $\chi^*(D) \le \ell \cdot \chi^*(D)$, which yields $\chi^*(D) = 0$, since $\ell < 1$. This implies that D is relatively compact. Therefore, $\mathcal{F}$ is χ^*-condensing. By Theorem 4.3, $\mathcal{F}$ possesses a fixed point.

We now show that for all $u \in \mathbb{S}(\varphi)$, $|u_t|_\mathcal{B} \le C|\varphi|_\mathcal{B}$ for some $C > 0$.

Let $t \in [0, T_1]$. The following estimate holds

$$\begin{aligned}
\|u(t)\| &\le s(t,\lambda_1)\|\varphi(0)\| \\
&\quad + \int_0^t r(t-\tau,\lambda_1)\beta(\tau)\Big(K_1(\tau)\sup_{\xi\in[0,\tau]}\|u(\xi)\| + M_1(\tau)|\varphi|_\mathcal{B}\Big)\mathrm{d}\tau \\
&\le s(t,\lambda_1)\varrho|\varphi|_\mathcal{B} + |\varphi|_\mathcal{B}\int_0^t r(t-\tau,\lambda_1)\beta(\tau)M_1(\tau)\mathrm{d}\tau \\
&\quad + \beta_\infty K_{1\infty}\int_0^t r(t-\tau,\lambda_1)\Big(\sup_{[0,\tau]}\|u(\xi)\|\Big)\mathrm{d}\tau \\
&\le (\varrho + \|r(\cdot,\lambda_1) * (\beta M_1)\|_\infty)|\varphi|_\mathcal{B} + \beta_\infty K_{1\infty}\int_0^t r(t-\tau,\lambda_1)\sup_{[0,\tau]}\|u(\xi)\|\mathrm{d}\tau.
\end{aligned}$$

Since the last integral is nondecreasing with respect to t, one can take the supremum over $[0, t]$ to get

$$\begin{aligned}
\sup_{[0,t]}\|u(\xi)\| &\le (\varrho + \|r(\cdot,\lambda_1) * (\beta M_1)\|_\infty)|\varphi|_\mathcal{B} \\
&\quad + \beta_\infty K_{1\infty}\int_0^t r(t-\tau,\lambda_1)\sup_{[0,\tau]}\|u(\xi)\|\mathrm{d}\tau.
\end{aligned}$$

The Gronwall type inequality [14, Proposition 2.2] gives

$$\sup_{[0,t]}\|u(\xi)\| \le Y(t)C_1(\varphi), \forall t \in [0, T_1],$$

where $C_1(\varphi) = (\varrho + \|r(\cdot,\lambda_1) * (\beta M_1)\|_\infty)|\varphi|_\mathcal{B}$ and $Y(t)$ is the unique solution of Volterra equation

$$Y(t) = 1 + \beta_\infty K_{1\infty}\int_0^t r(t-\tau,\lambda_1)Y(\tau)\mathrm{d}\tau.$$

Particularly,

$$\|u(t)\| \le Y(T_1)C_1(\varphi), \forall t \in [0, T_1]. \tag{4.16}$$

Now estimating for $t \ge T_1$, we have

$$\begin{aligned}
\|u(t)\| &\le s(t,\lambda_1)\|\varphi(0)\| + \int_0^t r(t-\tau,\lambda_1)\beta(\tau)\Big(K_1(\tau)\sup_{\xi\in[0,\tau]}\|u(\xi)\| + M_1(\tau)|\varphi|_\mathcal{B}\Big)\mathrm{d}\tau \\
&\le s(t,\lambda_1)\varrho|\varphi|_\mathcal{B} + |\varphi|_\mathcal{B}\int_0^t r(t-\tau,\lambda_1)\beta(\tau)M_1(\tau)\mathrm{d}\tau \\
&\quad + \int_0^t r(t-\tau,\lambda_1)\beta(\tau)K_1(\tau)\Big(\sup_{[0,T_1]}\|u(\xi)\| + \sup_{[T_1,\tau]}\|u(\xi)\|\Big)\mathrm{d}\tau \\
&\le C_1(\varphi) + \|r(\cdot,\lambda_1) * (\beta K_1)\|_{BC}Y(T_1)C_1(\varphi)
\end{aligned}$$

$$+\sup_{[T_1,t]}\|u(\xi)\|\int_0^t r(t-\tau,\lambda_1)\beta(\tau)K_1(\tau))\mathrm{d}\tau$$
$$\leq C_1(\varphi)[1+Y(T_1)\|r(\cdot,\lambda_1)*(\beta K_1)\|_{BC}]+l_1\sup_{[T_1,t]}\|u(\xi)\|,$$

thanks to Lemma 4.7. Let t vary on $[T_1,T]$ for an arbitrary $T>T_1$, one concludes that

$$\sup_{[T_1,T]}\|u(t)\|\leq C_1(\varphi)\left(1+Y(T_1)\|r(\cdot,\lambda_1)*(\beta K_1)\|_{BC}\right)+l_1\sup_{[T_1,T]}\|u(\xi)\|.$$

Consequently,

$$\sup_{t\geq T_1}\|u(t)\|\leq\frac{1}{1-l_1}C_1(\varphi)\left(1+Y(T_1)\|r(\cdot,\lambda_1)*(\beta K_1)\|_{BC}\right). \tag{4.17}$$

Combing (4.16) with (4.17), we finally obtain

$$\|u(t)\|\leq C_2|\varphi|_{\mathcal{B}},\forall t>0,$$

where

$$C_2=\frac{\varrho+\|r(,\lambda_1)*(\beta M_1)\|_\infty}{1-l_1}\left(1+Y(T_1)\|r(\cdot,\lambda_1)*(\beta K_1)\|_{BC}\right).$$

This implies

$$|u_t|_{\mathcal{B}}\leq M(t)|\varphi|_{\mathcal{B}}+K(t)\sup_{[0,t]}\|u(\xi)\|\leq[M(t)+K(t)C_2]\,|\varphi|_{\mathcal{B}}\leq C|\varphi|_{\mathcal{B}}, \tag{4.18}$$

where $C=M_\infty+K_\infty C_2$.

We now show that $\lim_{t\to\infty}|u_t|_{\mathcal{B}}=0$. By properties of phase space, we have

$$|u_t|_{\mathcal{B}}\leq K(t/2)\sup_{[t/2,t]}\|u(\xi)\|+M(t/2)|u[\varphi]_{t/2}|_{\mathcal{B}} \tag{4.19}$$
$$\leq K_\infty\sup_{[t/2,t]}\|u(\xi)\|+M(t/2)C|\varphi|_{\mathcal{B}}, \tag{4.20}$$

thanks to (4.18).

Because $\lim_{t\to\infty}\|u(t)\|=0$ and $M\in BC_0$, for any $\epsilon>0$, there exists a positive $T(\epsilon)>0$ such that

$$\|u(t)\|<\epsilon,\quad \|M(t)\|<\epsilon,\ \text{for all } t>T(\epsilon).$$

Combining with the inequality (4.20) gives

$$|u_t|_{\mathcal{B}}\leq(K_\infty+C|\varphi|_{\mathcal{B}})\epsilon,\ \forall t>2T(\epsilon).$$

The proof is completed. □

Remark 4.1. The statement in Theorem 4.4 presents a new observation, even for bounded delays. In [16, Theorem 7], the weakly asymptotic stability was proved under a condition on the magnitude of coefficients on the half-line. In contrast to the latter, the conditions in Theorem 4.4 involves only the asymptotic information of $\beta(t)$ near infinity. For example, if $\beta\in BC(\mathbb{R}^+)$ such that

$$\limsup_{t\to\infty}\beta(t)<\lambda_1/K_\infty,$$

then the assumptions in Theorem 4.4 are testified even for β possessing large values in a finite interval.

5. Example

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain which has smooth boundary $\partial\Omega$. We consider the following multi-term fractional-in-time PDE:

$$\sum_{i=1}^{m} \mu_i \partial_t^{\alpha_i} u(t,x) - \Lambda u(t,x) = b(t,x) \int_{-\infty}^{0} \int_{\Omega} \nu(\theta,y)\kappa(y,u(t+\theta,y))\mathrm{d}y\mathrm{d}\theta, \tag{5.1}$$

$$\text{for } t > 0, x \in \Omega,$$

$$u(t,x) = 0, \quad \text{for } t \geq 0,\ x \in \partial\Omega, \tag{5.2}$$

$$u(s,x) = \varphi(s,x), x \in \Omega, s \in (-\infty, 0], \tag{5.3}$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 1$, $\mu_i > 0$, $\partial_t^{\alpha_i}$ denote the Caputo fractional derivatives of order α_i in t, for $i = \overline{1,m}$. The operator Λ is determined by

$$D(\Lambda) = \{u \in H_0^1(\Omega) : \Lambda u \in L^2(\Omega)\},\ \Lambda u = \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x)\partial_{x_j} u),$$

where $a_{ij} \in L^\infty(\Omega), a_{ij} = a_{ji}, 1 \leq i,j \leq N$, and fulfills the uniformly elliptic condition $\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$, for some $\theta > 0$. Applying the Friedrichs theory [25, Prop. 8.5], $-\Lambda$ is a positive self-adjoint operator with compact resolvent.

Let H be the Hilbertian space $L^2(\Omega)$ furnished with the standard inner product $(u,v) = \int_\Omega u(x)v(x)\mathrm{d}x$. Set

$$k(t) = \sum_{i=1}^{m} \mu_i g_{1-\alpha_i}(t),$$
$$A = -\Lambda.$$

Clearly k is completely monotonic, so the associated kernel l exists. Furthermore, the Laplace transform of l is calculated as follows

$$\hat{l}(\lambda) = \lambda^{-1}\hat{k}(\lambda)^{-1} = \frac{1}{\mu_i \sum\limits_{i=1}^{m} \lambda^{\alpha_i}}.$$

Thus

$$\widehat{(1 * l)}(\lambda) = \frac{1}{\mu_i \sum\limits_{i=1}^{m} \lambda^{\alpha_i+1}} \sim \frac{1}{\mu_1 \lambda^{\alpha_1+1}} \text{ as } \lambda \to 0.$$

Hence $l \notin L^1(\mathbb{R}^+)$ which follows from the asymptotic expansion

$$(1 * l)(t) \sim \frac{t^{\alpha_1}}{\mu_1 \Gamma(\alpha_1 + 1)} \to \infty \text{ as } t \to \infty,$$

thanks to the Karamata–Feller Tauberian theorem (see [6]).

We are now in a position to give the description for the nonlinearity:

(A1) $b \in BC(\mathbb{R}^+; L^2(\Omega))$.

(A2) $\nu : (-\infty, 0] \times \Omega \to \mathbb{R}$ is a continuous function and there exist a nonnegative function $\omega \in L^2(\Omega)$ and $\nu_0 \in (0,1)$ such that

$$|\nu(t,x)| \le \omega(x) e^{\nu_0 t}, \text{ for all } t \in (-\infty, 0], x \in \Omega.$$

(A3) $\kappa : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a nonnegative function $p \in L^2(\Omega)$ and $q \in \mathbb{R}^+$ satisfying

$$|\kappa(y,z)| \le p(y) + q|z|.$$

In this example, we choose the phase space $\mathcal{B} = CL^1_g$ with $r = 0$ and $g(s) = e^{\nu_0 s}$. The seminorm in $\mathcal{B}$ is given by

$$|w|_{\mathcal{B}} = \|w(0)\| + \int_{-\infty}^0 e^{\nu_0 \theta} \|w(\theta)\| \mathrm{d}\theta.$$

Then one can see that (2.1)–(2.2) are satisfied with $G(s) = g(s)$. Then $\mathcal{B}$ satisfies (B1)–(B3) with

$$K(t) = 1 + \nu_0^{-1}(1 - e^{-\nu_0 t}),\ M(t) = e^{-\nu_0 t},$$

thanks to the expressions of K and M in (2.3) and (2.4), respectively. Obviously, $M \in BC_0$ and $K_\infty = 1 + \nu_0^{-1}$.

Let $f : \mathbb{R}^+ \times \mathcal{B} \to L^2(\Omega)$ be defined as

$$f(t,\phi)(x) = b(t,x) \int_{-\infty}^0 \int_\Omega \nu(\theta,y)\kappa(y,\phi(\theta,y)) \mathrm{d}y \mathrm{d}\theta.$$

Then the problem (5.1)–(5.3) can be rewritten in the form (1.1)–(1.2).

We now testify the conditions related to f in Theorems 3.3 and 4.4.

For every $\phi \in \mathcal{B}$, we obtain

$$\begin{aligned}
\|f(t,\phi)\|^2 &= \int_\Omega \Big| b(t,x) \int_{-\infty}^0 \int_\Omega \nu(\theta,y)\kappa(y,\phi(\theta,y)) \mathrm{d}y\mathrm{d}\theta \Big|^2 \mathrm{d}x \\
&\le \|b(t,\cdot)\|^2 \Big[\int_{-\infty}^0 \int_\Omega |\nu(\theta,y)| \big(p(y) + q|\phi(\theta,y)|\big) \mathrm{d}y\mathrm{d}\theta) \Big]^2 \\
&\le \|b(t,\cdot)\|^2 \Big[\int_{-\infty}^0 e^{\nu_0\theta} \int_\Omega \big(\omega(y)p(y) + q\omega(y)|\phi(\theta,y)|\big) \mathrm{d}y\mathrm{d}\theta) \Big]^2, \\
&\le \|b(t,\cdot)\|^2 \Big[\int_{-\infty}^0 e^{\nu_0\theta} \|\omega\| \big(\|p\| + q\|\phi(\theta,\cdot)\|\big) \mathrm{d}\theta \Big]^2 \\
&\le \|b(t,\cdot)\|^2 \|\omega\|^2 \Big[\nu_0^{-1}\|p\| + q \int_{-\infty}^0 e^{\nu_0\theta} \|\phi(\theta,\cdot)\| \mathrm{d}\theta \Big]^2,
\end{aligned}$$

thanks to (A2) and (A3) and the Hölder inequality.

Hence

$$\|f(t,\phi)\| \le \|b(t,\cdot)\| \|\omega\| \Big[\nu_0^{-1}\|p\| + q|\phi|_{\mathcal{B}} \Big].$$

By taking

$$\alpha(t) = \|b(t,\cdot)\| \|\omega\| \nu_0^{-1} \|p\|,\ \beta = q\|\omega\| \sup_{t\ge 0} \|b(t,\cdot)\|,$$

we see that $r(\cdot,\lambda_1) * \alpha \in BC(\mathbb{R}^+)$ due to (A1).

Applying Theorem 3.3, if $\beta\big(1+\nu_0^{-1}\big) < \lambda_1$ then our system is dissipative.

On the other hand, let $p = 0$ then (4.6) takes place with

$$\beta(t) = q\|\omega\|\|b(t,\cdot)\|.$$

Let $\tilde{\beta} = \limsup_{t\to\infty} \beta(t)$. Then the condition

$$\tilde{\beta}\big(1+\nu_0^{-1}\big) < \lambda_1 \tag{5.4}$$

implies (4.7). By Theorem 4.4, the zero solution of (1.1) is weakly asymptotically stable if the last inequality (5.4) holds. Note that condition (5.4) holds even for $\sup_{t\ge 0}\beta(t)$ being large.

We now replace (A3) with the following one

(A3b) $\kappa : \Omega\times\mathbb{R} \to \mathbb{R}$ is a continuous function and there exists $q > 0$ such that

$$|\kappa(y,z_1) - \kappa(y,z_2)| \le q|z_1 - z_2|.$$

Then, we have the following estimates

$$\begin{aligned}
&\|f(t,\phi_1) - f(t,\phi_2)\|^2\\
&= \int_\Omega \Big|b(t,x)\int_{-\infty}^0\int_\Omega \nu(\theta,y)\big(\kappa(y,\phi_1(\theta,y)) - \kappa(y,\phi_1(\theta_2,y))\big)\mathrm{d}y\mathrm{d}\theta\Big|^2\mathrm{d}x\\
&\le \|b(t,\cdot)\|^2q^2\Big[\int_{-\infty}^0\int_\Omega |\nu(\theta,y)||\phi_1(\theta,y) - \phi_2(\theta,y)|\mathrm{d}y\mathrm{d}\theta\Big]^2\\
&\le \|b(t,\cdot)\|^2q^2\Big[\int_{-\infty}^0 e^{\nu_0\theta}\int_\Omega \omega(y)|\phi_1(\theta,y) - \phi_2(\theta,y)|\mathrm{d}y\mathrm{d}\theta\Big]^2\\
&\le \|b(t,\cdot)\|^2q^2\Big[\int_{-\infty}^0 e^{\nu_0\theta}\|\omega\|\|\phi_1(\theta,\cdot) - \phi_2(\theta,\cdot)\|\mathrm{d}\theta\Big]^2\\
&\le q^2\|b(t,\cdot)\|^2\|\omega\|^2|\phi_1 - \phi_2|^2_{\mathcal{B}}.
\end{aligned}$$

It leads to

$$\|f(t,\phi_1) - f(t,\phi_2)\| \le q\|\omega\|\|b(t,\cdot)\||\phi_1 - \phi_2|_{\mathcal{B}},$$

that is, f satisfies the Lipschitz condition with Lipschitz constant

$$\beta = q\|\omega\|\sup_{t\ge 0}\|b(t,\cdot)\|.$$

Employing Theorem 4.1, one concludes that the solution to (5.1)–(5.3) is asymptotically stable provided that $\beta(1+\nu_0^{-1}) < \lambda_1$.

6. Conclusion

In this paper, we establish stability results for a class of nonlocal evolution equations in Hilbert spaces involving infinite delays. The techniques are based on local estimates, fixed-point arguments, the resolvent theory of Prüss and the phase space axioms. By relaxing some sufficient conditions, the obtained results have improved and extended the previous works in the literature.

Acknowledgements

The authors are grateful to the anonymous referees for their valuable suggestions and comments leading to improvement in the presentation of this paper. The first and second authors would like to thank Hanoi National University of Education for providing a fruitful working environment. A part of this research was completed while the first author visited the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank VIASM for support and hospitality. This work is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2020.07 and by Vietnam Ministry of Education and Training under grant number B2021-SPH-15.

References

[1] Anh, C.T., Ke, T.D.: On nonlocal problems for retarded fractional differential equations in Banach spaces. Fixed Point Theory **15**, 373–392 (2014)

[2] Anh, N.T., Ke, T.D.: Decay integral solutions for neutral fractional differential equations with infinite delays. Math. Methods Appl. Sci. **38**, 1601–1622 (2015)

[3] Anh, N.T., Ke, T.D., Quan, N.N.: Weak stability for integro-differential inclusions of diffusion-wave type involving infinite delays. Discrete Contin. Dyn. Syst. Ser. B **21**, 3637–3654 (2016)

[4] Bohner, M., Tunç, O., Tunç, C.: Qualitative analysis of caputo fractional integro-differential equations with constant delays. Comput. Appl. Math. **40**, 214 (2021)

[5] Clément, Ph., Nohel, J.A.: Asymptotic behavior of solutions of nonlinear Volterra equations with completely positive kernels. SIAM J. Math. Anal. **12**, 514–535 (1981)

[6] Feller, W.: An Introduction to Probability Theory and Its Applications, vol. II, 2nd edn. Wiley, New York (1971)

[7] Gripenberg, G.: Volterra integro-differential equations with accretive nonlinearity. J. Differ. Equ. **60**, 57–79 (1985)

[8] Hale, J.K., Kato, J.: Phase space for retarded equations with infinite delay. Funkcial. Ekvac. **21**, 11–41 (1978)

[9] Hale, J.K., Verduyn Lunel, S.M.: Introduction to Functional Differential Equations. Springer, New York (1993)

[10] Hino, Y., Murukami, S., Naito, T.: Functional Differential Equations with Unbounded Delay. Lecture Notes in Mathematics, vol. 1473. Springer, Berlin (1991)

[11] Kamenskii, M., Obukhovskii, V., Zecca, P.: Condensing multivalued maps and semilinear differential inclusions in Banach spaces. In: de Gruyter Series in Nonlinear Analysis and Applications, vol. 7. Walter de Gruyter, Berlin (2001)

[12] Ke, T.D., Lan, D.: Fixed point approach for weakly asymptotic stability of fractional differential inclusions involving impulsive effects. J. Fixed Point Theory Appl. **19**, 2185–2208 (2017)

[13] Ke, T.D., Lan, D.: Decay integral solutions for a class of impulsive fractional differential equations in Banach spaces. Fract. Calc. Appl. Anal. **17**, 96–121 (2014)

[14] Ke, T.D., Thang, N.N., Thuy, L.T.P.: Regularity and stability analysis for a class of semilinear nonlocal differential equations in Hilbert spaces. J. Math. Anal. Appl. **483**(2), 123655 (2020)

[15] Ke, T.D., Thuy, L.T.P.: Nonlocal final value problem governed by semilinear anomalous diffusion equations. Evol. Equ. Control Theory **9**(3), 891–914 (2020)

[16] Ke, T.D., Thuy, L.T.P.: Dissipativity and stability for semilinear anomaluos diffusion equations involving delays. Math. Methods Appl. Sci. **43**(15), 8449–8465 (2020)

[17] Kemppainen, J., Siljander, J., Vergara, V., Zacher, R.: Decay estimates for time fractional and other non-local in time subdiffusion equations in $\mathbb{R}^n$. Math. Ann. **366**(3–4), 941–979 (2016)

[18] Li, F.: An existence result for fractional differential equations of neutral type with infinite delay. Electron. J. Qual. Theory Differ. Equ. **52**, 1–15 (2011)

[19] Liu, S., Yang, R., Zhou, X.F., Jiang, W., Li, X., Zhao, X.W.: Stability analysis of fractional delayed equations and its applications on consensus of multi-agent systems. Commun. Nonlinear Sci. Numer. Simul. **73**, 351–362 (2019)

[20] Miller, R.K.: On Volterra integral equations with nonnegative integrable resolvents. J. Math. Anal. Appl. **22**, 319–340 (1968)

[21] Mophou, G.M., NGuérékata, G.M.: Existence of mild solutions for some semilinear neutral fractional functional evolution equations with infinite delay. Appl. Math. Comput. **216**, 61–69 (2010)

[22] Nguyen, N.T.: Notes on ultraslow nonlocal telegraph evolution equations. Proc. Amer. Math. Soc. **151**, 583–593 (2023). https://doi.org/10.1090/proc/15877

[23] Prüss, J.: Evolutionary Integral Equations and Applications. Monographs in Mathematics, vol. 87. Birkhauser, Basel

[24] Samko, S.G., Cardoso, R.P.: Integral equations of the first kind of Sonine type. Int. J. Math. Math. Sci. **57**, 3609–3632 (2003)

[25] Taylor, E.M.: Partial Differential Equations I: Basic Theory. Springer, New York (2013)

[26] Taylor, E.M.: Partial Differential Equations II: Qualitative Studies of Linear Equations. Springer, New York (2013)

[27] Tunç, C., Golmankhaneh, A.K.: On stability of a class of second alpha-order fractal differential equations. AIMS Math. **5**(3), 2126–2142 (2020)

[28] Tunç, C., Tunç, O.: New qualitative criteria for solutions of Volterra integro-differential equations. Arab J. Basic Appl. Sci. **25**(3), 158–165 (2018)

[29] Vergara, V., Zacher, R.: Optimal decay estimates for time-fractional and other nonlocal subdiffusion equations via energy methods. SIAM J. Math. Anal. **47**, 210–239 (2015)

[30] Vergara, V., Zacher, R.: Stability, instability, and blowup for time fractional and other nonlocal in time semilinear subdiffusion equations. J. Evol. Equ. **17**, 599–626 (2017)

Nhu Thang Nguyen and Dinh Ke Tran
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy, Cau Giay
Hanoi
Vietnam
e-mail: thangnn@hnue.edu.vn;
ketd@hnue.edu.vn

Van Dac Nguyen
Faculty of Computer Science and Engineering
Thuyloi University
175 Tay Son, Dong Da
Hanoi
Vietnam
e-mail: nvdac@tlu.edu.vn

Accepted: February 28, 2022.