J. Fixed Point Theory Appl. (2022) 24:82
https://doi.org/10.1007/s11784-022-00999-w
Published online November 22, 2022

Journal of Fixed Point Theory and Applications

Multiplicity and Calabi–Bernstein type asymptotic property of positive solutions for one-dimensional Minkowski-curvature problems

Rui Yang

Abstract. In this paper, we first consider the existence and multiplicity of positive solutions to a Minkowski-curvature problem when nonlinearity is sublinear. We then study the nonexistence and multiplicity of positive solutions for the corresponding one-parameter problem in which its nonlinear function has m zeros in the interval $(0, \frac{1}{2})$ by proving that the problem has no and $m+1$ positive solutions for suitably small and large parameter, respectively. Furthermore, we investigate the Calabi–Bernstein type asymptotic property of each positive solution of the one-parameter problem as the parameter goes to infinity, showing that the ith solution converges to a function whose shape is isosceles trapezoid when $1 \leq i \leq m$ and isosceles triangle when $i = m+1$.

Mathematics Subject Classification. 34B09, 34B16, 34D05.

Keywords. Minkowski-curvature, singular weight, positive solution, multiplicity, Calabi–Bernstein type, truncation techniques.

1. Introduction and main results

We first consider the existence of one or two positive solutions of the following one-dimensional Minkowski-curvature problem

$$\begin{cases} -\Big(\phi(u'(t))\Big)' = r(t)f(u(t)), \quad t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{P}$$

where $\phi(y) = \frac{y}{\sqrt{1-y^2}}$, $y \in (-1,1)$, weight function r satisfies $r \geq 0$, $r \not\equiv 0$ on any compact subinterval of $(0,1)$ and $r \in \mathcal{A}_q$ a class of functions given as

$$\mathcal{A}_q \triangleq \Big\{ r \in L^1_{loc}((0,1),[0,\infty)) : \int_0^1 \tau^q(1-\tau)^q r(\tau)d\tau < \infty, q \geq 1 \Big\}.$$

As an example, consider $r(t) = t^{-\frac{5}{2}}$, then $r \notin L^1(0,1)$ but $r \in \mathcal{A}_2$ ($q = 2$ in $\mathcal{A}_q$). $f : [0,\alpha) \to [0,\infty)$ with $\alpha > \frac{1}{2}$ is a continuous function and $f \not\equiv 0$ on any compact subinterval of $(0, \frac{1}{2})$.

We say u a solution of problem (P) if $u \in C[0,1] \cap C^1(0,1)$, $|u'(t)| < 1$ for $t \in (0,1)$, and $\phi(u')$ is absolutely continuous on any compact subinterval of $(0,1)$, and u satisfies the equation and the boundary conditions in problem (P). Moreover, we say u is a positive solution of problem (P) if solution u satisfies $u(t) \geq 0$ and $u(t) \not\equiv 0$ on $(0,1)$.

This type of problem is related to mean curvature operator in flat Minkowski space endowed with the Lorentzian metric, which has a wide range of applications in physics and geometry. Physically, it naturally appears in Dirichlet p-branes of string theory (see [22]) and nonlinear electrodynamics model of the Born–Infeld theory, see for instance [9–12]. Geometrically, it plays a significant role in determining existence and regularity properties of maximal and constant mean curvature hypersurfaces (see [4,21,35]) and cosmological model, see [5,33] and references therein. In recent decades, many researchers have studied the nonexistence, existence and multiplicity of solutions for boundary value problems of nonlinear Minkowski-curvature equations (see [6–8,15,16,19,20,30,34,40]).

We detail some existence and multiplicity results of solutions for Minkowski-curvature problems with singular nonlinear terms. Specially, Coelho–Corsato–Obersnel–Omari [15] studied the existence and multiplicity of positive solutions for the 0-Dirichlet boundary problem

$$\begin{cases} -\Big(\phi(u'(t))\Big)' = f(t,u(t)), \quad t \in (0,T), \\ u(0) = 0 = u(T), \end{cases} \tag{S}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the L^1-Carathéodory conditions. Under some more assumptions on f, mainly by variational and topological methods, they proved the existence of either one, or two, or three, or infinitely many positive solutions.

In a subsequent paper, Coelho–Corsato–Rivetti [16] studied the existence and multiplicity of positive radial solutions for the null Dirichlet problem of the Minkowski-curvature equation

$$\begin{cases} -\mathrm{div}\Big(\phi_N(\nabla v(x))\Big) = f(|x|, v(x)), & \text{in } B_R, \\ \quad v = 0, & \text{on } \partial B_R, \end{cases} \tag{G}$$

where $\phi_N(y) = \frac{y}{\sqrt{1-|y|^2}}$, $y \in \mathbb{R}^N$, $N \geq 2$, $R > 0$ and B_R is a ball in $\mathbb{R}^N$, $f : [0,R] \times \mathbb{R} \to \mathbb{R}$ satisfies the L^1-Carathéodory conditions. By variational methods, they proved the existence of either one, two or three positive solutions according to the behaviour of $f(r,s)$ near s=0. A model example is $f(r,s) = \lambda a(r)s^p + \mu b(r)s^q$, $0 < p < 1 < q$. Their work extended partially the results for one-dimensional problem (S) obtained in [15] to the radial problem (G).

Recently, Yang–Lee–Sim [38,40] studied the existence of positive or nodal radial solutions for the following problem defined on an exterior domain

$$\begin{cases} -\mathrm{div}\Big(\phi_N(\nabla v(x))\Big) = \lambda K(|x|)f(v(x)), & \text{in } \Omega, \\ v|_{\partial\Omega} = 0, \ \lim\limits_{|x|\to\infty} v(x) = 0, \end{cases} \tag{E_λ}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| > R\}$ and $N \geq 3$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous and odd function satisfying $f(s)s > 0$ for $s \neq 0$, and a function K satisfies

(HK) $K \in L^1([R,\infty),[0,\infty))$ is not identically zero on any subinterval in (R,∞) and satisfies $\int_R^\infty rK(r)dr < \infty$.

After introducing variables $|x| = r$, $v(x) = u(r)$ and $t = (\frac{r}{R})^{-(N-2)}$, problem (E_λ) is transformed into one-dimensional problem of the form

$$\begin{cases} -\Big(\beta(t)\phi\big(\frac{1}{\beta(t)}u'(t)\big)\Big)' = \lambda h(t)f\big(u(t)\big), \ t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{D_λ}$$

where functions β and h are given by

$$\beta(t) = \frac{R}{N-2}t^{-\frac{N-1}{N-2}}, \quad h(t) = \beta^2(t)K\big(Rt^{-\frac{1}{N-2}}\big).$$

We note that weight function h is singular at $t = 0$ and if K in (E_λ) satisfies (HK), then corresponding h in (D_λ) satisfies $h \in L^1_{loc}((0,1),[0,\infty))$ with $\int_0^1 \tau h(\tau)d\tau < \infty$. Under assumptions $0 < f_0^1 < \infty$ and $f_\infty^1 = 0$, where $f_0^1 \triangleq \lim_{s\to 0} \frac{f(s)}{s}$ and $f_\infty^1 \triangleq \lim_{|s|\to\infty} \frac{f(s)}{s}$, they proved that for each $k \in \mathbb{N}$, there exists $\lambda_* \in (0, \frac{\lambda_k(h)}{f_0}]$ such that problem (D_λ) has no $(k-1)$-nodal solution for all $\lambda \in (0,\lambda_*)$ and at least two $(k-1)$-nodal solutions for all $\lambda \in (\frac{\lambda_k(h)}{f_0},\infty)$, where $\lambda_k(h)$ is the kth eigenvalue of corresponding second order linearized problem and for $k = 1$, the 0-nodal solution is a positive solution or a negative solution.

Very recently, Bartolo–Caponio–Pomponio [3] considered the existence of a spacelike solution of the exterior Dirichlet problem

$$\begin{cases} \mathrm{div}\Big(\phi_N(\nabla u(x))\Big) = nH(x,u), & \text{in } \Omega_c, \\ u = \varphi, & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to+\infty} u(x) = 0, \end{cases} \tag{H}$$

where $\Omega_c = \mathbb{R}^N \setminus \bar{\Omega}$ is an exterior domain in $\mathbb{R}^N$, $N \geq 3$, $\varphi : \partial\Omega \to \mathbb{R}$ and $H : \Omega_c \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

(Hh) there exists $h \in L^s(\Omega_c) \cap L^\infty_{loc}(\Omega_c)$, $s \in [1, \frac{2N}{N+2}]$, such that

$$n|H(x,t)| \leq h(x) \text{ for a.e. } x \in \Omega_c \text{ and all } t \in \mathbb{R}.$$

They gave a necessary and sufficient condition for the existence of a spacelike solution of problem (H).

In the study of [38, 40], a novel class of weight functions is generally defined as

$$\mathcal{A}_1 \triangleq \{r \in L^1_{loc}((0,1),[0,\infty)) : \int_0^1 \tau(1-\tau)r(\tau)d\tau < \infty\},$$

admitting the singularity of weight function at $t = 0$ or $t = 1$. In fact, we can easily check that $L^1((0,1),[0,\infty)) \subset \mathcal{A}_1 \subset \mathcal{A}_q$ for $q > 1$. The singularity in nonlinear term may affect the compactness of solution space for such a problem. In paper [41], authors classified the solutions by introducing "non $\frac{\pi}{4}$-tangential solution" defined as $u \in C^1[0,1]$, $|u'(0)| < 1$ and $|u'(1)| < 1$ and "$\frac{\pi}{4}$-tangential solution" defined as $u \in C^1[0,1]$, $|u'(0)| = 1$ or $|u'(1)| = 1$. When weight function in nonlinear term belongs to L^1-class, all solutions of problem (P) are non $\frac{\pi}{4}$-tangential, while solutions of problem (P) may be $\frac{\pi}{4}$-tangential if weight function is of $\mathcal{A}_q$-class and meanwhile, solution operator may loss of compactness. To guarantee compactness of solution operator in this paper, the asymptotic behaviour of $f(s)$ near $s = 0$ is linear or sublinear, which will be specified later. It is also interesting to note that studies of the existence and multiplicity of positive solutions for problem (P), specially with nonlinearity f satisfying $0 \le f_0 < \infty$ and weight function r of $\mathcal{A}_q$-class are rare before this paper as far as the authors know. In the present paper, we will deal with the existence and multiplicity of positive solutions of problem (P) using the Krasnoselskii's theorem of cone expansion and compression, a different approach to the one in [3, 15, 16, 38, 40].

We now give a relation between weight function r and nonlinear function f.

(F) there exist $\delta \in (0, \frac{1}{2})$ and $\rho \in (0, \delta M_\delta)$ such that

$$f(s) \ge \phi\left(\frac{s}{\delta M_\delta}\right) \text{ for } s \in [\delta\rho, \rho],$$

where $M_\delta = \min\left\{\int_\delta^{\frac{1}{2}} \phi^{-1}\left(\int_s^{\frac{1}{2}} r(\tau)d\tau\right) ds, \int_{\frac{1}{2}}^{1-\delta} \phi^{-1}\left(\int_{\frac{1}{2}}^{s} r(\tau)d\tau\right) ds\right\}$.

Remark 1.1. Condition (F) is first introduced in [39] and there is a large number of functions satisfying condition (F), one can refer to [39].

Denote $f_0 \triangleq \lim_{s\to 0^+} \frac{f(s)}{s^q}$, where q is from $\mathcal{A}_q$. The following proposition is an existence result of one positive solution to problem (P).

Proposition 1.1. *Assume $r \in \mathcal{A}_q$, (F) and $0 \le f_0 < \infty$. Then problem (P) has at least one positive solution u satisfying $\rho < \|u\|_\infty < \frac{1}{2}$.*

Our existence result of two positive solutions to problem (P) is the following.

Theorem 1.2. *Assume $r \in \mathcal{A}_q$, (F) and $f_0 = 0$. Then problem (P) has at least two positive solutions u_1 and u_2 such that $0 < \|u_1\|_\infty < \rho < \|u_2\|_\infty < \frac{1}{2}$.*

Next, we consider the corresponding one-parameter problem

$$\begin{cases} -\Big(\phi(u'(t))\Big)' = \lambda r(t)g(u(t)), & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \qquad (P_\lambda)$$

where $\lambda > 0$, weighted function r satisfies the same conditions as the ones in problem (P), $g : [0,\alpha) \to [0,\infty)$ is a continuous function with $\alpha > \frac{1}{2}$ and $g \not\equiv 0$ on $(0,\frac{1}{2})$. A multiplicity result of the one-parameter problem is the following, which shows that the number of solutions of problem (P_λ) is related to the number of interior zeros of function g on $(0,\frac{1}{2})$.

Theorem 1.3. *Assume $r \in \mathcal{A}_q$ and $0 \le g_0 < \infty$. Also assume that there exist numbers $0(= a_0) < a_1 < a_2 < \cdots < a_m < \frac{1}{2}(= a_{m+1})$ such that $(\frac{1}{2} + a_i)a_i > 2a_{i-1}$ for all $i \in \{1, 2, \cdots, m+1\}$, $\sum_{i=0}^{m} g(a_i) = 0$ and $g > 0$ on the set $I \triangleq (0,\frac{1}{2}) \setminus \{a_i\}_{i=1}^{m}$. Then there exist $0 < \lambda_* \le \lambda^* < \infty$ such that problem (P_λ) has no positive solution for all $\lambda < \lambda_*$ and at least $m+1$ distinct positive solutions $\{u_{\lambda,i}\}_{i=1}^{m+1}$ for all $\lambda \ge \lambda^*$, which satisfy*

- (i) $u_{\lambda,i}(t) > 0$ *for* $t \in (0,1)$*;*
- (ii) $0 < b_1 \le \|u_{\lambda,1}\|_\infty \le a_1 < b_2 \le \|u_{\lambda,2}\|_\infty \le a_2 < \cdots \le a_{m-1} < b_m \le \|u_{\lambda,m}\|_\infty \le a_m < b_{m+1} \le \|u_{\lambda,m+1}\|_\infty < \frac{1}{2}$, *where* $b_1, b_2, \cdots, b_{m+1}$ *are constants.*

Remark 1.2. (i) The condition $(\frac{1}{2} + a_i)a_i > 2a_{i-1}$ is crucial to guarantee a suitable interval length between a_{i-1} and a_i. If such condition is $a_i^2 > \frac{1}{4}a_{i-1}$, by a similar fashion, the result is also valid. Previous work on nonlinear problems with nonlinear function which has many zeros has been done in [25,36]. In [25], the author proved $2m-1$ positive solutions result for a nonlinear elliptic eigenvalue problem, in which nonlinear function is of C^1-class and may be negative in some subintervals. Paper [36] is concerned with those quasilinear equations where operators satisfy a homeomorphism condition. Condition $\sum_{i=0}^{m} g(a_i) = 0$ is equivalent to $g(a_0) = g(a_1) = \cdots = g(a_m) = 0$.

(ii) The number $m+1$ of positive solutions is sharp in Theorem 1.3. Specially, situation 1: when $m = 0$, $0 < g_0 < \infty$ and $\frac{g(s)}{s}$ is strictly decreasing on $(0,\frac{1}{2})$, problem (P_λ) has exact one positive solution for suitably large λ. For readers' convenience, we give a brief explanation here. Indeed, by a similar argument of Theorem 2.4 in [37], problem (P_λ) has at most one positive solution for any fixed $\lambda > 0$. And together with the result of Corollary 1.1 in [29], we conclude that there exists $\lambda^* > 0$ such that problem (P_λ) has a unique positive solution for all $\lambda \ge \lambda^*$; situation 2: when $m = 0$ and $g_0 = 0$, similar to Corollary 1.2 in [29], problem (P_λ) has at least two positive solutions for suitably large λ.

The existence, uniqueness and regularity for spacelike hypersurface with zero or constant mean curvature are classical and important problems in general relativity (see [1,2,13,14,18,35] and references therein). There are some well-known results in this perspective which are so called Calabi–Bernstein problem in Minkowski spacetime. In 1968, Calabi [13] studied the maximal spacelike hypersurface equation

$$(1 - |\nabla u|^2)\sum_{i=1}^{N}\frac{\partial^2 u}{\partial x_i^2} + \sum_{i,j=1}^{N}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \tag{1.1}$$

and found a remarkable result that Eq. (1.1) has the Bernstein-type property by proving that (1.1) has only linear entire solutions in dimension $N \leq 4$. In 1976, Cheng–Yau [18] extended the result to all N. In [17,27], authors studied Calabi–Bernstein-type problem for the maximal surface equation in a Robertson–Walker spacetime. Note that the above references [13,17,18,27] considered the equations with zero mean curvature. In a recent paper [29], authors studied bifurcation and Calabi–Bernstein type asymptotic property of one-sign solutions of problem (P_λ) as parameter λ goes to ∞ when nonlinear function g has no interior zero on the interval $(0, \frac{1}{2})$ and proved that one-sign solutions on two bifurcation branches converge to two linear functions. Our next aim is to investigate Calabi–Bernstein type asymptotic property of multiplicity solutions of problem (P_λ) as λ goes to ∞ when nonlinearity g has m zeros on the interval $(0, \frac{1}{2})$.

The Calabi–Bernstein type asymptotic property result of multiplicity solutions for problem (P_λ) is given as follows.

Theorem 1.4. *Under the assumptions in Theorem 1.3, the solutions* $\{u_{\lambda,i}\}_{i=1}^{m+1}$ *of problem* (P_λ)*, obtained from Theorem 1.3, satisfy*

- (i) $u_{\lambda,i}(t)$ *is continuous with respect to* λ $(\geq \lambda^*)$*, that is, there exist a sequence* $\{(\lambda_n, u_{\lambda_n,i})\}$ *and* $(\tilde{\lambda}, u_{\tilde{\lambda},i})$ *such that* $\|u_{\lambda_n,i} - u_{\tilde{\lambda},i}\|_\infty \to 0$ *as* $\lambda \to \tilde{\lambda}$*, here notation* (λ, u_λ) *means a solution pair that* u_λ *is a solution of problem* (P_λ) *at* λ*;*
- (ii) $\lim_{\lambda\to\infty} u'_{\lambda,i}(t) = 1$ *uniformly on* $[0, a_i - \varepsilon]$*, for sufficiently small* $\varepsilon > 0$*,* $i \in \{1, 2, \cdots, m+1\}$*;*
- (iii) $\lim_{\lambda\to\infty} u'_{\lambda,i}(t) = -1$ *uniformly on* $[1 - a_i + \varepsilon, 1]$*, for sufficiently small* $\varepsilon > 0$*,* $i \in \{1, 2, \cdots, m+1\}$*;*
- (iv) $\lim_{\lambda\to\infty} u'_{\lambda,i}(t) = 0$ *uniformly on* $[a_i + \varepsilon, 1 - a_i - \varepsilon]$*, for sufficiently small* $\varepsilon > 0$*,* $i \in \{1, 2, \cdots, m\}$*;*
- (v) $\lim_{\lambda\to\infty} \|u_{\lambda,i}\|_\infty = a_i$*,* $i \in \{1, 2, \cdots, m+1\}$*.*

Note that Theorem 1.4 (i) shows that the solution pairs $(\lambda, u_{\lambda,i})$, $i = 1, 2, \cdots, m+1$, of problem (P_λ) may form at least $m+1$ connected curves, name $\mathcal{C}_i(\lambda, u_{\lambda,i})$, going through $(\lambda, u_{\lambda,i})$, which can extend to infinity in λ-direction. The results $(ii)-(v)$ in Theorem 1.4 manifests that solution $u_{\lambda,i}(t)$ $(i \in \{1, \cdots, m+1\})$ of problem (P_λ) converges to the following function as λ goes to ∞, whose shape is isosceles trapezoid when $1 \leq i \leq m$ and isosceles triangle when $i = m+1$,

$$u_i(t) = \begin{cases} t, & t \in [0, a_i), \\ a_i, & t \in [a_i, 1 - a_i], \\ 1 - t, & t \in (1 - a_i, 1]. \end{cases}$$

It is worth mentioning that in a recent paper [26], Hong-Yuan studied the existence and uniqueness of solution to the exterior Dirichlet problem of the Minkowski spacetime and proved asymptotic properties of the exterior solution at infinity, which shows that the linear growth rate of the exterior solution at infinity is uniformly less that one. Compared with the asymptotic results of Hong-Yuan, the asymptotic property in Theorem 1.4 is concerned with stable state of solution with respect to a parameter near infinity in a bounded domain.

The rest of this paper is organized as follows. In Sect. 2, we prove Proposition 1.1 and Theorem 1.2. In Sect. 3, we prove Theorems 1.3–1.4.

2. Multiplicity result for problem (P)

In this section, we prove Proposition 1.1 and Theorem 1.2 which deal with the existence and multiplicity of positive solutions for problem (P), respectively.

Let $E = C[0,1]$ be the Banach space with supremum norm $\|\cdot\|_\infty$ and denote $K = \{u \in E :\ u(0) = u(1) = 0$ and u is concave on $(0,1)\}$. Then K is a cone in E.

We will mainly use the Krasnoselskii's theorem of cone expansion and compression to prove the existence and multiplicity results in this section.

Lemma 2.1. ([23]) *Let E be a Banach space and let K be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Tx\| \le \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \ge \|x\|$ *for* $x \in K \cap \partial\Omega_2$, *or*
(ii) $\|Tx\| \ge \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \le \|x\|$ *for* $x \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Remark 2.1. Assume $r \in \mathcal{A}_q$ and $0 \le f_0 < \infty$, then $r(\cdot)f(u(\cdot)) \in L^1(0,1)$ and every solution u of problem (P) is of $C^1[0,1]$ and $\|u'\|_\infty < 1$ implying $\|u\|_\infty < \frac{1}{2}$, refer to Theorem 2.1 in [41].

For $u \in K$ and fixed $\sigma \in (0,1)$ in an arbitrary manner, we define an integral operator $T : K \to E$ as

$$(Tu)(t) = \begin{cases} \int_0^t \phi^{-1}\left(\alpha + \int_s^\sigma r(\tau)f(u(\tau))d\tau\right) ds, & t \in (0,\sigma], \\ \int_t^1 \phi^{-1}\left(-\alpha + \int_\sigma^s r(\tau)f(u(\tau))d\tau\right) ds, & t \in [\sigma,1), \end{cases} \tag{2.1}$$

where α satisfies

$$\int_0^\sigma \phi^{-1}\left(\alpha + \int_s^\sigma r(\tau)f(u(\tau))d\tau\right) ds = \int_\sigma^1 \phi^{-1}\left(-\alpha + \int_\sigma^s r(\tau)f(u(\tau))d\tau\right) ds.$$

We easily check by a standard argument that $T(K) \subset K$ and T is completely continuous. Moreover, we can check that u is a solution of problem (P) if and only if $u \in K$ satisfies $u = Tu$. For $u \in K$, Tu is concave and satisfies the Dirichlet boundary condition. Thus we may assume that there exists $t^* \in (0,1)$, a maximal point of Tu, such that $\|Tu\|_\infty = (Tu)(t^*)$ and $(Tu)'(t^*) = 0$. Here t^* need not be unique.

From the fact $(Tu)'(t^*) = 0$, we obtain

$$\alpha = -\int_{t^*}^\sigma r(\tau)f(u(\tau))d\tau.$$

Since $r \in L^1(t^*-\delta, t^*+\delta)$ for any small δ, replacing σ with t^*, we get $\alpha = 0$ and Tu can be written as

$$(Tu)(t) = \begin{cases} \int_0^t \phi^{-1}\left(\int_s^{t^*} r(\tau)f(u(\tau))d\tau\right)ds, & t \in (0, t^*], \\ \int_t^1 \phi^{-1}\left(\int_{t^*}^s r(\tau)f(u(\tau))d\tau\right)ds, & t \in [t^*, 1). \end{cases} \tag{2.2}$$

This operator is first introduced in [39].

Remark 2.2. (i) If there exist $0 < t_1 < t_2 < 1$ such that $u'(t_1) = u'(t_2) = 0$, then the operator T defined in (2.2) is independent of the choice of $t^* \in [t_1, t_2]$. In fact, by the concavity of u, we see that $u'(t) \equiv 0$ for $t \in [t_1, t_2]$ and thus $u(t) \equiv$ constant for $t \in [t_1, t_2]$. We calculate the first equation in problem (P) and get

$$-\frac{u''(t)}{\left(\sqrt{1-|u'(t)|^2}\right)^3} = r(t)f(u(t)),\ t \in [t_1, t_2].$$

Thus $r(t)f(u(t)) \equiv 0$ for $t \in [t_1, t_2]$ implying that $\int_{s_1}^{s_2} r(\tau)f(u(\tau))d\tau = 0$ for any $s_1, s_2 \in [t_1, t_2]$.

(ii) If we find a nontrivial fixed point u of T in K, then $u(t) > 0$ for $t \in (0,1)$ mainly by concavity and double zero property ($u(t) = u'(t) = 0$) of solution u of problem (P) (see Lemma 2.3 in [38]).

We now start to prove Proposition 1.1, an existence result of problem (P).

Proof of Proposition 1.1. Define $\Omega_\rho = \{u \in E : \|u\|_\infty < \rho\}$ and consider $u \in K \cap \partial\Omega_\rho$. Let $\delta \in (0, \frac{1}{2})$ be from condition (F). Recall Tu defined in (2.2). If $t^* \in [\frac{1}{2}, 1)$, then

$$\begin{aligned} \|Tu\|_\infty = (Tu)(t^*) &= \int_0^{t^*} \phi^{-1}\left(\int_s^{t^*} r(\tau)f(u(\tau))d\tau\right)ds \\ &\geq \int_\delta^{\frac{1}{2}} \phi^{-1}\left(\int_s^{\frac{1}{2}} r(\tau)f(u(\tau))d\tau\right)ds. \end{aligned}$$

Since $u \in K \cap \partial\Omega_\rho$, we see

$$\frac{u(t)}{\|u\|_\infty} \geq \frac{t}{t^*} > \delta,\ \text{for } t \in [\delta, t^*],$$

implying $u(t) \in [\delta\rho, \rho]$ for $t \in [\delta, t^*]$. Applying (F), we get

$$\begin{aligned} \int_\delta^{\frac{1}{2}} \phi^{-1}\left(\int_s^{\frac{1}{2}} r(\tau)f(u(\tau))d\tau\right)ds &\geq \int_\delta^{\frac{1}{2}} \phi^{-1}\left(\int_s^{\frac{1}{2}} r(\tau)\phi\left(\frac{u(\tau)}{\delta M_\delta}\right)d\tau\right)ds \\ &\geq \int_\delta^{\frac{1}{2}} \phi^{-1}\left(\int_s^{\frac{1}{2}} r(\tau)\phi\left(\frac{\|u\|_\infty}{M_\delta}\right)d\tau\right)ds \\ &> \int_\delta^{\frac{1}{2}} \phi^{-1}\left(\int_s^{\frac{1}{2}} r(\tau)d\tau\right)ds\frac{\|u\|_\infty}{M_\delta} \geq \|u\|_\infty, \end{aligned}$$

where we used the property $\phi^{-1}(xy) > \phi^{-1}(x)\phi^{-1}(y)$ for $x, y > 0$ in the above inequality. Thus $\|Tu\|_\infty > \|u\|_\infty$ for $u \in K \cap \partial\Omega_\rho$. By a similar argument to the case $t^* \in (0, \frac{1}{2})$, we also get

$$(Tu)(t^*) > \|u\|_\infty, \text{ for } u \in K \cap \partial\Omega_\rho.$$

This implies that $\|Tu\|_\infty > \|u\|_\infty$ for $u \in K \cap \partial\Omega_\rho$. Denote $\Omega_{1/2} = \{u \in E : \|u\|_\infty < \frac{1}{2}\}$. Consider $u \in K \cap \partial\Omega_{1/2}$. From Remark 2.1, we see $\|Tu\|_\infty < \|u\|_\infty$ for any $u \in K \cap \overline{\Omega}_{1/2}$. Therefore, by Lemma 2.1, operator T has a fixed point in $K \cap (\overline{\Omega}_{1/2} \setminus \Omega_\rho)$, which is a positive solution, name $\widetilde{u}$, of problem (P) satisfying $\rho \le \|\widetilde{u}\|_\infty \le \frac{1}{2}$. Also noticing that $Tu \ne u$ for $u \in \partial\Omega_\rho \cup \partial\Omega_{1/2}$, we get $\rho < \|\widetilde{u}\|_\infty < \frac{1}{2}$ and the proof is completed. □

We prove Theorem 1.2, an existence result of two positive solutions for problem (P) below.

Proof of Theorem 1.2. By following the proof of Theorem 3.1 in [39] with obvious modifications and combining with the fact $Tu \ne u$ for $u \in \partial\Omega_\rho$, problem (P) has a positive solution, name u_1, such that $0 < \|u_1\|_\infty < \rho$. Moreover, from Proposition 1.1, we see that problem (P) also has one positive solution u_2 (name $\widetilde{u}$ in Proposition 1.1) such that $\rho < \|u_2\|_\infty < \frac{1}{2}$. Therefore, problem (P) has at least two positive solutions u_1 and u_2 satisfying $0 < \|u_1\|_\infty < \rho < \|u_2\|_\infty < \frac{1}{2}$. The proof is done. □

3. Multiplicity and asymptotic property for (P_λ)

In this section, we show the nonexistence, multiplicity and Calabi–Bernstein type asymptotic property of positive solutions to the one-parameter problem (P_λ) under condition $0 \le g_0 < \infty$. We first show a nonexistence result of positive solutions for problem (P_λ).

Theorem 3.1. *Assume $r \in \mathcal{A}_q$. Also assume that there is a positive constant c_0 such that*

$$\frac{g(s)}{s^q} \le c_0, \tag{3.1}$$

for all $s \in [0, \frac{1}{2}]$. Then problem (P_λ) has no positive solution for all $\lambda \in (0, \lambda_0]$ with $\lambda_0 = \frac{1}{c_0 2^{q-1} \int_0^1 t^q(1-t)^q r(t)dt}$.

Proof. Let $\lambda > 0$ and u be a positive solution of problem (P_λ). Under conditions $r \in \mathcal{A}_q$ and (3.1), by Theorem 2.1 in [41], we know $u \in C^1[0,1]$ and $|u'(t)| < 1$ for $t \in [0,1]$.

Together with the boundary conditions, we obtain

$$|u(t)| = \left|\int_0^t u'(\tau)d\tau\right| \le t\|u'\|_\infty \le 2t(1-t)\|u'\|_\infty, \text{ for } t \in (0, \frac{1}{2}],$$

and

$$|u(t)| = \left|\int_t^1 u'(\tau)d\tau\right| \le (1-t)\|u'\|_\infty \le 2t(1-t)\|u'\|_\infty, \text{ for } t \in [\frac{1}{2}, 1).$$

Thus,

$$|u(t)| \le 2t(1-t)\|u'\|_\infty, \text{ for } t \in [0,1]. \tag{3.2}$$

To eliminate the integration $\int_0^1 |u'(\tau)|^2 d\tau$ later, we now bound $|u(t)|^2$ by $\int_0^1 |u'(\tau)|^2 d\tau$. Applying the Hölder's inequality, we obtain

$$|u(t)| \le \int_0^t |u'(\tau)| d\tau \le t^{\frac{1}{2}} \left(\int_0^t |u'(\tau)|^2 d\tau \right)^{\frac{1}{2}}, \ \forall t \in [0,1].$$

Thus,

$$(1-t)|u(t)|^2 \le t(1-t) \left(\int_0^t |u'(\tau)|^2 d\tau \right), \ \forall t \in [0,1]. \tag{3.3}$$

Similarly, we also obtain

$$t|u(t)|^2 \le t(1-t) \left(\int_t^1 |u'(\tau)|^2 d\tau \right), \ \forall t \in [0,1]. \tag{3.4}$$

Adding (3.3) and (3.4), we get

$$|u(t)|^2 \le t(1-t) \left(\int_0^1 |u'(\tau)|^2 d\tau \right), \ \forall t \in [0,1]. \tag{3.5}$$

Multiplying the first equation in problem (P_λ) by u and then integrating it over $(0,1)$, we obtain

$$\int_0^1 \Big(\phi(u'(t)) \Big) u'(t) dt = \lambda \int_0^1 r(t) g(u(t)) u(t) dt. \tag{3.6}$$

Note that the integration on the right-hand side of (3.6) makes sense, since $r(\cdot)g(u(\cdot)) \in L^1(0,1)$ by (3.1) and (3.2). Combining (3.2), (3.5) and (3.6), we deduce

$$\begin{aligned} \int_0^1 |u'(t)|^2 dt &< \int_0^1 \Big(\phi(u'(t)) \Big) u'(t) dt \\ &= \lambda \int_0^1 r(t) g(u(t)) u(t) dt \\ &\le \lambda c_0 \int_0^1 r(t) |u(t)|^{q+1} dt \\ &= \lambda c_0 \int_0^1 r(t) |u(t)|^{q-1} |u(t)|^2 dt \\ &\le \lambda c_0 2^{q-1} \int_0^1 r(t) t^q (1-t)^q dt \|u'\|_\infty^{q-1} \left(\int_0^1 |u'(\tau)|^2 d\tau \right). \end{aligned}$$

It yields $\lambda > \lambda_0$ and the proof is completed. □

The following lemma shows that a nontrivial solution of problem (P_λ) has no interior zero on $(0,1)$.

Lemma 3.1. *Let u be a nontrivial solution of problem (P_λ), then $u(t) > 0$ for $t \in (0,1)$.*

Proof. Since u is a nontrivial solution of problem (P_λ), by the definition of function g and 0-Dirichlet boundary conditions, u is concave and $u \geq 0$ on $(0,1)$. If u has an interior zero on $(0,1)$, then the interior zero is a double zero of u. Now, we claim that u has no double zero on $(0,1)$. Then $u(t) > 0$ for $t \in (0,1)$ and the proof is done.

Suppose on the contrary that $t_0 \in (0,1)$ is a double zero of u, that is $u(t_0) = u'(t_0) = 0$. Integrating the first equality in problem (P_λ) over (t, t_0) for $t \in [0,1]$, we obtain

$$u'(t) = \phi^{-1}\left(\int_t^{t_0} \lambda r(\tau) g(u(\tau)) d\tau\right).$$

Thus, $u'(t) \geq 0$ for $t \in [0, t_0)$ and $u'(t) \leq 0$ for $t \in (t_0, 1]$. Together with the fact $u(0) = u(t_0) = u(1) = 0$, we deduce $u \equiv 0$ on $[0,1]$, a contradiction. Therefore, solution u has no double zero on $[0,1]$. □

Before proving the multiplicity result of Theorem 1.3, we introduce a new family of truncation functions g^i, $i = 1, 2, \cdots, m+1$, by

$$g^i(s) = \begin{cases} g(s), & 0 \leq s \leq a_i, \\ 0, & s > a_i, \end{cases} \tag{3.7}$$

and consider the following auxiliary family

$$\begin{cases} -\Big(\phi(u'(t))\Big)' = \lambda r(t) g^i(u(t)), & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{P_λ^i}$$

where r is given in problem (P_λ).

Similar to the operator T introduced in (2.2), we define the operator $T_\lambda^i : K \to K$ as

$$(T_\lambda^i u)(t) = \begin{cases} \int_0^t \phi^{-1}\left(\lambda \int_s^{t_i^*} r(\tau) g^i(u(\tau)) d\tau\right) ds, & t \leq t_i^*, \\ \int_t^1 \phi^{-1}\left(\lambda \int_{t_i^*}^s r(\tau) g^i(u(\tau)) d\tau\right) ds, & t \geq t_i^*, \end{cases} \tag{3.8}$$

where $t_i^* \in (0,1)$ is a zero of function $G_\lambda^i : (0,1) \to \mathbb{R}$ with

$$G_\lambda^i(t) = \int_0^t \phi^{-1}\left(\lambda \int_s^t r(\tau) g^i(u(\tau)) d\tau\right) ds - \int_t^1 \phi^{-1}\left(\lambda \int_t^s r(\tau) g^i(u(\tau)) d\tau\right) ds.$$

Then T_λ^i is completely continuous and $u \in K$ is a positive fixed point of T_λ^i if and only if u is a positive solution of problem (P_λ^i).

We consider properties of solutions of problem (P_λ^i) in the following two lemmas.

Lemma 3.2. *Let g satisfy the assumptions in Theorem 1.3. If $u_i \in K$ is a solution of problem (P_λ^i), then $\|u_i\|_\infty \leq a_i$ and u_i is also a solution of problem (P_λ).*

Proof. Let u_i be a solution of problem (P_λ^i). Obviously, $u_i(0) = 0 = u_i(1)$. From the first equation in problem (P_λ^i), we have

$$-\frac{u_i''}{\left(\sqrt{1-|u_i'(t)|^2}\right)^3} = \lambda r(t) g^i(u_i(t)),\ t \in (0,1).$$

Thus u_i is concave on $(0,1)$. By the mean value theorem, there exists $t_i^* \in (0,1)$ such that $u_i'(t_i^*) = 0$, implying that $u_i(t_i^*) = \|u_i\|_\infty$. Suppose on the contrary that $\|u_i\|_\infty > a_i$. Then, combining with the continuity of u_i, there exists a subinterval $(t_i^1, t_i^2) \subset (0,1)$ satisfying $t_i^* \in (t_i^1, t_i^2)$, $u(t) > a_i$ for $t \in (t_i^1, t_i^2)$ and $u_i(t_i^1) = u_i(t_i^2) = a_i$. Recalling the condition $g^i(s) = 0$ for $s \geq a_i$, we have

$$-\Big(\phi(u_i'(t))\Big)' = 0,\ t \in [t_i^1, t_i^2].$$

Thus, $\phi(u_i')$ is a constant on $[t_i^1, t_i^2]$ and so $u_i'(t)$ is a constant on $[t_i^1, t_i^2]$. Together with the fact $u_i'(t_i^*) = 0$, we deduce $u_i'(t) \equiv 0$ for $t \in [t_i^1, t_i^2]$. Hence, $u_i(t) = u_i(t_i^1) = a_i$ for $t \in [t_i^1, t_i^2]$, a contradiction showing $\|u_i\|_\infty \leq a_i$. Due to (3.7), $g^i(u_i) = g(u_i)$ for $0 \leq u_i \leq a_i$. Thus, u_i is also a solution of problem (P_λ). The proof is completed. □

Denote $\Omega_R = \{u \in E : \|u\|_\infty < R\}$.

Lemma 3.3. *Let g satisfy the assumptions in Theorem 1.3 and $b_i > 0$ be such that $\frac{1}{2}(\frac{1}{2}+b_i)b_i \in \big(a_{i-1}, \frac{1}{2}(\frac{1}{2}+a_i)a_i\big)$, for any $i \in \{1,2,\cdots,m+1\}$. Then, there exists $0 < \lambda_i^* < \infty$ such that*

$$\|T_\lambda^i u\|_\infty > \|u\|_\infty,\ \text{for all } \lambda \geq \lambda_i^* \text{ and } u \in K \cap \partial\Omega_{b_i}.$$

Proof. From assumptions on b_i for any $i \in \{1,2,\cdots,m+1\}$, we see $[\frac{1}{2}(\frac{1}{2}+b_i)b_i, b_i] \subset (a_{i-1}, a_i)$. Select

$$\lambda_i^* = \frac{\phi\left(\frac{8b_i}{1+6b_i}\right)}{g_{min}^i \min\left\{\int_{\frac{1}{2}(\frac{1}{2}+b_i)}^{\frac{1}{2}} r(\tau)d\tau, \int_{\frac{1}{2}}^{1-\frac{1}{2}(\frac{1}{2}+b_i)} r(\tau)d\tau\right\}},$$

where

$$g_{min}^i = \min_{s \in [\frac{1}{2}(\frac{1}{2}+b_i)b_i, b_i]} g^i(s).$$

Then, $g_{min}^i > 0$ from the condition $g^i(s) > 0$ for $s \in [\frac{1}{2}(\frac{1}{2}+b_i)b_i, b_i] \subset (a_{i-1}, a_i)$, and thus $0 < \lambda_i^* < \infty$.

Let $u \in K \cap \partial\Omega_{b_i}$ and $t_i^* \in (0,1)$ be such that $u_i'(t_i^*) = 0$. Then from the definition of $T_\lambda^i u$ in (3.8), $(T_\lambda^i u)(t_i^*)$ is the maximum value of $T_\lambda^i u$ on $[0,1]$. Obviously, $t_i^* \in (0, \frac{1}{2})$ or $t_i^* \in [\frac{1}{2}, 1)$. Without loss of generality, we consider the case $t_i^* \in [\frac{1}{2}, 1)$. The argument would be similar for the case $t_i^* \in (0, \frac{1}{2})$ and we omit its details here. Let us fix $\lambda \geq \lambda_i^*$ below.

For $t_i^* \in [\frac{1}{2}, 1)$ and any $\delta \in (0, \frac{1}{2})$, we have

$$\|T_\lambda^i u\|_\infty = (T_\lambda^i u)(t_i^*) = \int_0^{t_i^*} \phi^{-1}\left(\lambda \int_s^{t_i^*} r(\tau) g^i(u(\tau)) d\tau\right) ds$$

$$\geq \int_0^{\frac{1}{2}} \phi^{-1}\left(\lambda_i^* \int_s^{\frac{1}{2}} r(\tau)g^i(u(\tau))d\tau\right) ds$$
$$\geq \int_0^{\delta} \phi^{-1}\left(\lambda_i^* \int_\delta^{\frac{1}{2}} r(\tau)g^i(u(\tau))d\tau\right) ds \triangleq I_0.$$

By the concavity of u, we obtain

$$u(t) \geq \frac{t}{t_i^*}\|u\|_\infty \geq \frac{\delta}{t_i^*}\|u\|_\infty > \delta\|u\|_\infty, \quad \text{for } t \in [\delta, t_i^*].$$

Specially, setting $\delta = \frac{1}{2}(\frac{1}{2}+b_i)$, we get $u(t) \in (\frac{1}{2}(\frac{1}{2}+b_i)b_i, b_i]$ for $t \in [\frac{1}{2}(\frac{1}{2}+b_i), \frac{1}{2}]$. Hence,

$$\begin{aligned} I_0 &> \int_{\frac{1}{4}(\frac{1}{2}-b_i)}^{\frac{1}{2}(\frac{1}{2}+b_i)} \phi^{-1}\left(\lambda_i^* \int_{\frac{1}{2}(\frac{1}{2}+b_i)}^{\frac{1}{2}} r(\tau)g^i(u(\tau))d\tau\right) ds \\ &= \frac{1+6b_i}{8}\phi^{-1}\left(\lambda_i^* \int_{\frac{1}{2}(\frac{1}{2}+b_i)}^{\frac{1}{2}} r(\tau)g^i(u(\tau))d\tau\right) \\ &\geq \frac{1+6b_i}{8}\phi^{-1}\left(\lambda_i^* g^i_{min} \int_{\frac{1}{2}(\frac{1}{2}+b_i)}^{\frac{1}{2}} r(\tau)d\tau\right) \\ &\geq b_i = \|u\|_\infty. \end{aligned}$$

Therefore, $\|T_\lambda^i u\|_\infty > \|u\|_\infty$ for $\lambda \geq \lambda_i^*$ and $u \in K \cap \partial\Omega_{b_i}$. The proof is completed. □

Proof of Theorem 1.3. Let $b_i > 0$ be such that $\frac{1}{2}(\frac{1}{2}+b_i)b_i \in \big(a_{i-1}, \frac{1}{2}(\frac{1}{2}+a_i)a_i\big)$ for any $i \in \{1, 2, \cdots, m+1\}$. Then by Lemma 3.3, there exists $0 < \lambda_i^* < \infty$ such that $\|T_\lambda^i u\|_\infty > \|u\|_\infty$ for $\lambda \geq \lambda_i^*$ and $u \in K \cap \partial\Omega_{b_i}$. On the other hand, by Lemma 3.2, $\|T_\lambda^i u\|_\infty \leq \|u\|_\infty$ for $u \in K \cap \partial\Omega_{a_i}$. Therefore, by Lemma 2.1, for any $\lambda \geq \lambda_i^*$, operator T_λ^i has a fixed point in $K \cap (\overline{\Omega_{a_i}} \setminus \Omega_{b_i})$, which is a positive solution of problem (P_λ^i), denoted by $u_{\lambda,i}$, satisfying

$$a_{i-1} < b_i \leq \|u_{\lambda,i}\|_\infty \leq a_i. \tag{3.9}$$

Now we focus on problem (P_λ). By Theorem 3.1, there exists $\lambda_* > 0$, such that problem (P_λ) has no positive solution for all $\lambda < \lambda_*$. Recall that any solution $u_{\lambda,i}$ of problem (P_λ^i) is also a solution of problem (P_λ) in Lemma 3.2. Combining with Lemma 3.1, we know $u_{\lambda,i}(t) > 0$ for $t \in (0,1)$ and so Theorem 1.3 (i) is derived. We set $\lambda^* = \max_{1\leq i\leq m+1} \lambda_i^*$, then problem (P_λ) has at least $m+1$ distinct positive solutions $\{u_{\lambda,i}\}_{i=1}^{m+1}$ for $\lambda \geq \lambda^*$, which satisfy

$$\begin{aligned} &0 < b_1 \leq \|u_{\lambda,1}\|_\infty \leq a_1 < b_2 \leq \|u_{\lambda,2}\|_\infty \leq a_2 < \cdots \\ &\quad < b_m \leq \|u_{\lambda,m}\|_\infty \leq a_m < b_{m+1} \leq \|u_{\lambda,m+1}\|_\infty < \frac{1}{2}, \end{aligned}$$

and thus Theorem 1.3 (ii) is deduced. The proof is completed. □

We begin to investigate Calabi–Bernstein type asymptotic property of solution $u_{\lambda,i}$, $i \in \{1, 2, \cdots, m+1\}$, of Theorem 1.3. Before showing it, we give some lemmas for later use. The following lemma is essential to show the continuity of solutions of problem (P_λ) with respect to λ.

Lemma 3.4. *Let $\{(\lambda_n, u_{\lambda_n,i})\}$ $(n \in \mathbb{N})$ be a sequence of solution pairs of problem (P_λ) such that $\lambda_n \to \tilde{\lambda}$ as $n \to \infty$. Then there exists a solution $u_{\tilde{\lambda},i}$ of problem (P_λ) such that $\|u_{\lambda_n,i} - u_{\tilde{\lambda},i}\|_\infty \to 0$ as $n \to \infty$.*

Proof. Since $\{(\lambda_n, u_{\lambda_n,i})\}$ is a sequence of solution pairs of problem (P_λ) with $\lambda_n \to \tilde{\lambda}$, combining with the fact $\|u_{\lambda_n,i}\|_\infty < \frac{1}{2}$ and by Arzelà-Ascoli theorem, we obtain a subsequence, say $\{u_{\lambda_n,i}\}$ again and a function $u_{\tilde{\lambda},i} \in C[0,1]$ such that $u_{\lambda_n,i} \to u_{\tilde{\lambda},i}$ in $C[0,1]$. We still need to show that $u_{\tilde{\lambda},i}$ is a solution of problem (P_λ) at $\lambda = \tilde{\lambda}$. Similar to Remark 2.1, we see $r(\cdot)g(u(\cdot)) \in L^1(0,1)$. It suffices to show that $u_{\tilde{\lambda},i}$ is a fixed point of operator T_λ at $\lambda = \tilde{\lambda}$, where $T_\lambda : K \to K$ is obtained after replacing f by λg in T given in (2.1), that is,

$$(T_\lambda u)(t) = \begin{cases} \int_0^t \phi^{-1}\left(\alpha + \int_s^\sigma \lambda r(\tau)g(u(\tau))d\tau\right) ds, & t \in (0,\sigma], \\ \int_t^1 \phi^{-1}\left(-\alpha + \int_\sigma^s \lambda r(\tau)g(u(\tau))d\tau\right) ds, & t \in [\sigma,1), \end{cases}$$

where $\sigma \in (0,1)$ is fixed in any fashion and $\alpha \left(\triangleq \alpha(\lambda r g(u))\right) : L^1(0,1) \to \mathbb{R}$ satisfies

$$\int_0^\sigma \phi^{-1}\left(\alpha + \int_s^\sigma \lambda r(\tau)g(u(\tau))d\tau\right) ds = \int_\sigma^1 \phi^{-1}\left(-\alpha + \int_\sigma^s \lambda r(\tau)g(u(\tau))d\tau\right) ds. \tag{3.10}$$

Indeed, α is continuous with respect to $\lambda r g(u)$ and maps equi-integrable sets of L^1 into bounded sets, similar continuous functions were constructed in [24,28,31,32] and also T_λ is completely continuous.

From the definition of sequence $\{(\lambda_n, u_{\lambda_n,i})\}$ and (3.10), together with the continuity of α, we get a sequence $\{\alpha_n\}$ $\left(\alpha_n \triangleq \alpha_n(\lambda_n r g(u_{\lambda_n,i}))\right)$ satisfying $\alpha_n \to \tilde{\alpha}$ $\left(\triangleq \tilde{\alpha}(\tilde{\lambda} r g(u_{\tilde{\lambda},i}))\right)$. Obviously, $\tilde{\alpha}$ satisfies

$$\int_0^\sigma \phi^{-1}\left(\tilde{\alpha} + \int_s^\sigma \tilde{\lambda} r(\tau)g(u_{\tilde{\lambda},i}(\tau))d\tau\right) ds = \int_\sigma^1 \phi^{-1}\left(-\tilde{\alpha} + \int_\sigma^s \tilde{\lambda} r(\tau)g(u_{\tilde{\lambda},i}(\tau))d\tau\right) ds.$$

By the Lebesgue dominated convergence theorem, it follows from the fact $u_{\lambda_n,i} = T_\lambda u_{\lambda_n,i}$ at $\lambda = \lambda_n$ that

$$\begin{aligned} u_{\tilde{\lambda},i} &= \lim_{n\to\infty} u_{\lambda_n,i} = \lim_{n\to\infty} (T_\lambda u_{\lambda_n,i})(t) \\ &= \begin{cases} \int_0^t \lim_{n\to\infty} \phi^{-1}\left(\alpha_n + \int_s^\sigma \lambda_n r(\tau)g(u_{\lambda_n,i}(\tau))d\tau\right) ds, & t \in (0,\sigma], \\ \int_t^1 \lim_{n\to\infty} \phi^{-1}\left(-\alpha_n + \int_\sigma^s \lambda_n r(\tau)g(u_{\lambda_n,i}(\tau))d\tau\right) ds, & t \in [\sigma,1), \end{cases} \\ &= \begin{cases} \int_0^t \phi^{-1}\left(\lim_{n\to\infty} \alpha_n + \int_s^\sigma \lim_{n\to\infty} \lambda_n r(\tau)g(u_{\lambda_n,i}(\tau))d\tau\right) ds, & t \in (0,\sigma], \\ \int_t^1 \phi^{-1}\left(-\lim_{n\to\infty} \alpha_n + \int_\sigma^s \lim_{n\to\infty} \lambda_n r(\tau)g(u_{\lambda_n,i}(\tau))d\tau\right) ds, & t \in [\sigma,1), \end{cases} \end{aligned}$$

$$= \begin{cases} \int_0^t \phi^{-1}\left(\tilde{\alpha} + \int_s^\sigma \tilde{\lambda} r(\tau) g(u_{\tilde{\lambda},i}(\tau)) d\tau\right) ds, & t \in (0,\sigma], \\ \int_t^1 \phi^{-1}\left(-\tilde{\alpha} + \int_\sigma^s \tilde{\lambda} r(\tau) g(u_{\tilde{\lambda},i}(\tau)) d\tau\right) ds, & t \in [\sigma,1). \end{cases}$$

Therefore, $u_{\tilde{\lambda},i}$ is a fixed point of $T_\lambda u$ at $\lambda = \tilde{\lambda}$ and the proof is done. □

From Theorem 1.3 and Lemma 3.4, we conclude that positive solution pairs $(\lambda, u_{\lambda,i})$ of problem (P_λ) may form at least $m+1$ connected curves for all $\lambda \geq \lambda^*$ in which any element is a positive solution pair of problem (P_λ^i) and of problem (P_λ) as well. Let us denote $\mathcal{C}_i \triangleq \mathcal{C}_i(\lambda, u_{\lambda,i})$, $i \in \{1, 2, \cdots, m+1\}$ as the connected curve going through solution pair $(\lambda, u_{\lambda,i})$ in which $u_{\lambda,i}$ comes from Theorem 1.3. Then $\mathcal{C}_i$ goes to infinity in λ-direction.

Note that any positive solution $u_{\lambda,i}$ of problem (P_λ) is concave on $(0,1)$. Using 0-Dirichlet boundary conditions and the mean value theorem, $u'_{\lambda,i}$ has at least one zero on $(0,1)$. Denote $\underline{t}_{i,0}^\lambda(u'_{\lambda,i}, \lambda) \triangleq \inf\{t \in (0,1) : u'_{\lambda,i}(t) = 0\}$ and $\overline{t}_{i,0}^\lambda(u'_{\lambda,i}, \lambda) \triangleq \sup\{t \in (0,1) : u'_{\lambda,i}(t) = 0\}$. For simplicity, we denote $\underline{t}_{i,0}^\lambda(u'_{\lambda,i}, \lambda)$, $\overline{t}_{i,0}^\lambda(u'_{\lambda,i}, \lambda)$ by $\underline{t}_{i,0}^\lambda$, $\overline{t}_{i,0}^\lambda$, respectively.

In the following lemma, we consider some properties related to solution $u_{\lambda,i}$ of problem (P_λ^i) for fixed λ and to be concise, we denote $u_{\lambda,i}$ by u_i with no confusion.

Lemma 3.5. *Let $0 \leq g_0^i < \infty$ and u_i be a positive solution of problem (P_λ^i), then $0 < \underline{t}_{i,0}^\lambda \leq \overline{t}_{i,0}^\lambda < 1$.*

Proof. The proof is inspired by the one of Theorem 2.1 in [41]. We only prove $\underline{t}_{i,0}^\lambda > 0$. The rest can be obtained after suitable modifications. On the contrary, suppose $\underline{t}_{i,0}^\lambda = 0$ (a similar argument can be applied for the case $\overline{t}_{i,0}^\lambda = 1$). Then there exists a sequence $\{t_n\}$ with $t_n \in (0,1)$ satisfying $u_i'(t_n) = 0$ and $t_n \to 0$ as $n \to \infty$. By the mean value theorem, there exists $\xi \in (0,t)$ for any $t \in (0,1)$ such that

$$u_i(t) = |u_i(t) - u_i(0)| = |u_i'(\xi)|t < t, \text{ for } t \in (0,1),$$

and combining with the boundary conditions, we obtain

$$u_i(t) \leq t, \text{ for } t \in [0,1]. \tag{3.11}$$

Similarly,

$$u_i(t) \leq 1 - t, \text{ for } t \in [0,1]. \tag{3.12}$$

By the condition $0 \leq g_0^i < \infty$, we obtain, for some $c \triangleq c(\|u_i\|_\infty) > 0$,

$$g^i\big(u_i(t)\big) \leq c u_i(t)^q, \text{ for } t \in [0,1]. \tag{3.13}$$

Since $r \in \mathcal{A}_q$, we choose a small constant $\gamma_0 > 0$ satisfying

$$c\int_0^{\gamma_0} \lambda \tau^q r(\tau) d\tau < 1. \tag{3.14}$$

We take $N_0 \in \mathbb{N}$ large enough such that $0 < t_n < \gamma_0$ for all $n \geq N_0$ and consider terms $t_n's$ for all $n \geq N_0$. Integrating the first equation in problem (P_λ^i) on (t, t_n) for $t \in (0, t_n)$, we obtain

$$\phi(u_i'(t)) = \int_t^{t_n} \lambda r(\tau) g^i\big(u_i(\tau)\big) d\tau.$$

It follows that

$$u_i'(t) = \phi^{-1}\left(\int_t^{t_n} \lambda r(\tau) g^i\big(u_i(\tau)\big) d\tau\right). \tag{3.15}$$

Integrating the above equation on $(0, t)$ for $t \in (0, t_n)$, we get

$$u_i(t) = \int_0^t \phi^{-1}\left(\int_s^{t_n} \lambda r(\tau) g^i\big(u_i(\tau)\big) d\tau\right) ds.$$

By (3.11) and (3.13), for $t \in (0, t_n)$, setting $\|u_i\|_{t_n,\infty} \triangleq \max_{0 \leq t \leq t_n} |u_i(t)|$ and applying the Fubini's theorem, we have

$$\begin{aligned}
|u_i(t)| &\leq \int_0^t c \int_s^{t_n} \lambda r(\tau) |u_i(\tau)|^q d\tau ds \\
&\leq c \int_0^{t_n} \int_s^{t_n} \lambda r(\tau) |u_i(\tau)|^q d\tau ds \\
&= c \int_0^{t_n} \lambda \tau r(\tau) |u_i(\tau)|^q d\tau \\
&\leq c \int_0^{t_n} \lambda \tau^q r(\tau) d\tau \|u_i\|_{t_n,\infty}.
\end{aligned}$$

Using (3.14), we obtain $\|u_i\|_{t_n,\infty} = 0$ for all $n \geq N_0$, *i.e.*, u_i is identically zero on $[0, t_n]$ for all $n \geq N_0$. Thus we may assume that $u_i(t_n) = u_i'(t_n) = 0$ for sufficiently large n. For $t \in [t_n, 1]$, by (3.12) and (3.13), using the Fubini's theorem again, we obtain

$$\begin{aligned}
|u_i(t)| &= \left|\int_{t_n}^t \phi^{-1}\left(\int_{t_n}^s \lambda r(\tau) g^i\big(u_i(\tau)\big) d\tau\right) ds\right| \\
&\leq \int_{t_n}^t \int_{t_n}^s \lambda r(\tau) \left|g^i\big(u_i(\tau)\big)\right| d\tau ds \\
&= \int_{t_n}^t \lambda (t - \tau) r(\tau) \left|g^i\big(u_i(\tau)\big)\right| d\tau \\
&\leq c \int_{t_n}^t \lambda (1 - \tau) r(\tau) |u_i(\tau)|^q d\tau \\
&\leq c \int_{t_n}^t \lambda (1 - \tau)^q r(\tau) |u_i(\tau)| d\tau.
\end{aligned}$$

By the Gronwall–Bellman inequality, we obtain $u_i(t) = 0$ for $t \in [t_n, 1]$. Thus, $u_i \equiv 0$ on $[0, 1]$ and this is a contradiction. Hence, $\underline{t}_{i,0}^\lambda > 0$ and the proof is done. □

Lemma 3.6. *Let* $(\lambda, u_{\lambda,i}) \in \mathcal{C}_i$, *then* $0 < \liminf_{\lambda\to\infty} \underline{t}_{i,0}^{\lambda} \leq \limsup_{\lambda\to\infty} \underline{t}_{i,0}^{\lambda} < 1$ *and* $0 < \liminf_{\lambda\to\infty} \overline{t}_{i,0}^{\lambda} \leq \limsup_{\lambda\to\infty} \overline{t}_{i,0}^{\lambda} < 1$.

Proof. Without loss of generality, we take an arbitrary element in the set $\{\liminf_{\lambda\to\infty} \underline{t}_{i,0}^{\lambda}, \limsup_{\lambda\to\infty} \underline{t}_{i,0}^{\lambda}, \liminf_{\lambda\to\infty} \overline{t}_{i,0}^{\lambda}, \limsup_{\lambda\to\infty} \overline{t}_{i,0}^{\lambda}\}$, denoted as β. Claim that $0 < \beta < 1$. Then the results will be obtained.

We now show $\beta > 0$. To prove it by contradiction, we suppose on the contrary that $\beta = 0$. Then there exist sequences $\{(\lambda_n, u_{\lambda_n,i})\} \subset \mathcal{C}_i$ $(n \in \mathbb{N})$ of solution pairs of problem (P_λ^i) and $\{t_{i,0}^n\}$ of zeros of the corresponding $\{u'_{\lambda_n,i}\}$ satisfying

(i) $\lambda_n \to \infty$ as $n \to \infty$;
(ii) $a_{i-1} < b_i \leq \|u_{\lambda_n,i}\|_\infty \leq a_i$;
(iii) $t_{i,0}^n \in (0,1)$ and $t_{i,0}^n \to 0$ as $n \to \infty$.

From definitions of $\mathcal{C}_i$ and β, sequences $\{(\lambda_n, u_{\lambda_n,i})\}$ and $\{t_{i,0}^n\}$ make sense.

Using the fact $\|u'_{\lambda_n,i}\|_\infty < 1$ for any $n \in \mathbb{N}$ and definition of $\{t_{i,0}^n\}$, we get

$$\|u_{\lambda_n,i}\|_\infty = u_{\lambda_n,i}(t_{i,0}^n) = \int_0^{t_{i,0}^n} u'_{\lambda_n,i}(\tau)d\tau < t_{i,0}^n.$$

Since $t_{i,0}^n \to 0$ as $n \to \infty$, we infer $\|u_{\lambda_n,i}\|_\infty \to 0$ as $n \to \infty$, which contradicts the above (ii) and thus $\beta > 0$. Similarly, we can prove $\beta < 1$ and the proof is done. □

Lemma 3.7. *Assume that there exists a sequence* $\{(\lambda_n, u_{\lambda_n,i})\} \subset \mathcal{C}_i$ $(n \in \mathbb{N})$ *satisfying*

(i) $\lambda_n \to \infty$ *as* $n \to \infty$;
(ii) $a_{i-1} < b_i \leq \|u_{\lambda_n,i}\|_\infty \leq a_i$.

Denote $\underline{t}_{i,0}^n \triangleq \inf\{t \in (0,1) : u'_{\lambda_n,i}(t) = 0\}$. *Also assume* $\lim_{n\to\infty} \underline{t}_{i,0}^n = \beta$. *Then* $\lim_{n\to\infty} u'_{\lambda_n,i}(t) = 1$ *uniformly on* $[0, \beta - \varepsilon]$, *for sufficiently small* $\varepsilon > 0$.

Proof. By Lemma 3.6, $0 < \beta < 1$. Directly from the definition of β, for any small $\varepsilon_0 > 0$, there exists $N \in \mathbb{N}$ such that $\beta - \varepsilon_0 < \underline{t}_{i,0}^n < \beta + \varepsilon_0$ for all $n > N$. By (ii), $\|u_{\lambda_n,i}\|_\infty \geq b_i$ for all $n > N$ and together with Lemma 3.1, $u_{\lambda_n,i}(t) > 0$ for $t \in (0,1)$ and all $n > N$. Since $u_{\lambda_n,i}$ is concave and $u'_{\lambda_n,i}(t)$ is nonincreasing on $(0,1)$, we see

$$\|u_{\lambda_n,i}\|_\infty = u_{\lambda_n,i}(\underline{t}_{i,0}^n) = \max_{t\in[0,\underline{t}_{i,0}^n]} u_{\lambda_n,i}(t) = \max_{[\underline{t}_{i,0}^n,1]} u_{\lambda_n,i}(t).$$

Moreover,

$$u_{\lambda_n,i}(t) \geq \frac{t}{\underline{t}_{i,0}^n}\|u_{\lambda_n,i}\|_\infty,\ t \in [0, \underline{t}_{i,0}^n]. \tag{3.16}$$

Specially, setting $t \geq \frac{2a_{i-1}}{a_{i-1}+b_i}\underline{t}_{i,0}^n$ in (3.16), we get

$$u_{\lambda_n,i}(t) \geq \frac{2a_{i-1}b_i}{a_{i-1}+b_i} > a_{i-1},$$

for $t \in [\frac{2a_{i-1}}{a_{i-1}+b_i}\underline{t}^n_{i,0}, \underline{t}^n_{i,0}]$. Let us fix $\varepsilon_0 \in (0, \frac{b_i - a_{i-1}}{4a_{i-1}+2b_i}\beta)$ and $\varepsilon_1 \in \left(\frac{2a_{i-1}}{a_{i-1}+b_i}(\beta+\varepsilon_0), \beta - 2\varepsilon_0\right)$. Then from (3.16), we deduce

$$\min_{t\in[\varepsilon_1, \underline{t}^n_{i,0}]} u_{\lambda_n,i}(t) \geq \frac{\varepsilon_1}{\underline{t}^n_{i,0}}\|u_{\lambda_n,i}\|_\infty \geq \frac{2a_{i-1}(\beta+\varepsilon_0)}{(a_{i-1}+b_i)\underline{t}^n_{i,0}}\|u_{\lambda_n,i}\|_\infty > \frac{2a_{i-1}b_i}{a_{i-1}+b_i} > a_{i-1},$$

for all $n > N$. Let $t \in [\varepsilon_1, \beta - \varepsilon_0]$. Obviously, $t \in [\varepsilon_1, \underline{t}^n_{i,0}]$ for all $n > N$ since $\beta - \varepsilon_0 < \underline{t}^n_{i,0}$. Since $u_{\lambda_n,i}$ is strictly increasing on $(0, \underline{t}^n_{i,0})$, we get

$$a_{i-1} < u_{\lambda_n,i}(t) < a_i, \text{ for all } t \in [\varepsilon_1, \beta - \varepsilon_0].$$

Hence there exists $M_0 (= M_0(\varepsilon_1)) > 0$ such that

$$g^i(u_{\lambda_n,i}(t)) > M_0, \text{ for all } t \in [\varepsilon_1, \beta - \varepsilon_0].$$

Let us first consider $t \in [\varepsilon_1, \beta - 2\varepsilon_0]$, then we get

$$\int_t^{\beta-\varepsilon_0} r(\tau)g^i(u_{\lambda_n,i}(\tau))d\tau > M_0 \int_{\beta-2\varepsilon_0}^{\beta-\varepsilon_0} r(\tau)d\tau > 0,$$

which implies

$$\lim_{n\to\infty} \lambda_n \int_t^{\beta-\varepsilon_0} r(\tau)g^i(u_{\lambda_n,i}(\tau))d\tau = \infty,$$

for $t \in [\varepsilon_1, \beta - 2\varepsilon_0]$. Similar to (3.15),

$$u'_{\lambda_n,i}(t) = \phi^{-1}\left(\lambda_n \int_t^{\underline{t}^n_{i,0}} r(\tau)g^i(u_{\lambda_n,i}(\tau))d\tau\right).$$

Since

$$\phi^{-1}\left(\lambda_n \int_t^{\underline{t}^n_{i,0}} r(\tau)g^i(u_{\lambda_n,i}(\tau))d\tau\right) \geq \phi^{-1}\left(\lambda_n \int_t^{\beta-\varepsilon_0} r(\tau)g^i(u_{\lambda_n,i}(\tau))d\tau\right),$$

and $\phi^{-1}\left(\lambda_n \int_t^{\beta-\varepsilon_0} r(\tau)g^i(u_{\lambda_n,i}(\tau))d\tau\right) \to 1$ as $n \to \infty$, it follows that $\lim_{n\to\infty} u'_{\lambda_n,i}(t) \geq 1$ for all $t \in [\varepsilon_1, \beta - 2\varepsilon_0]$. Since $u'_{\lambda_n,i}(t) < 1$ for $t \in [0, 1]$ and any n, it yields

$$\lim_{n\to\infty} u'_{\lambda_n,i}(t) = 1, \text{ uniformly on } [\varepsilon_1, \beta - 2\varepsilon_0]. \tag{3.17}$$

For $t \in [0, \varepsilon_1]$, from the fact

$$\begin{aligned} u'_{\lambda_n,i}(t) &= \phi^{-1}\left(\lambda_n\left(\int_t^{\varepsilon_1} + \int_{\varepsilon_1}^{\underline{t}^n_{i,0}}\right) r(s)g^i(u_{\lambda_n,i}(s))ds\right) \\ &\geq \phi^{-1}\left(\lambda_n \int_{\varepsilon_1}^{\underline{t}^n_{i,0}} r(s)g^i(u_{\lambda_n,i}(s))ds\right) = u'_{\lambda_n,i}(\varepsilon_1), \end{aligned}$$

we have

$$\lim_{n\to\infty} u'_{\lambda_n,i}(t) = 1 \text{ uniformly on } [0, \varepsilon_1]. \tag{3.18}$$

By combining (3.17) and (3.18), we obtain $\lim_{n\to\infty} u'_{\lambda_n,i}(t) = 1$, uniformly on $[0, \beta - 2\varepsilon_0]$, and we conclude

$$\lim_{n\to\infty} u'_{\lambda_n,i}(t) = 1 \text{ uniformly on } [0, \beta-\varepsilon], \text{ for sufficiently small } \varepsilon > 0.$$

□

Similar to Lemma 3.7, we get the following lemma.

Lemma 3.8. *Assume that there exists a sequence* $\{(\lambda_n, u_{\lambda_n,i})\} \subset \mathcal{C}_i$ $(n \in \mathbb{N})$ *satisfying*

(i) $\lambda_n \to \infty$ *as* $n \to \infty$*;*
(ii) $a_{i-1} < b_i \le \|u_{\lambda_n,i}\|_\infty \le a_i$.

Denote $\bar{t}^n_{i,0} \triangleq \sup\{t \in (0,1) : u'_{\lambda_n,i}(t) = 0\}$. *Also assume* $\lim_{n\to\infty} \bar{t}^n_{i,0} = \beta$. *Then* $\lim_{n\to\infty} u'_{\lambda_n,i}(t) = -1$ *uniformly on* $[\beta+\varepsilon, 1]$*, for sufficiently small* $\varepsilon > 0$.

Lemma 3.9. *Let* $(\lambda, u_{\lambda,i}) \in \mathcal{C}_i$*, then*

(a) $\liminf_{\lambda\to\infty} \underline{t}^\lambda_{i,0} = \limsup_{\lambda\to\infty} \underline{t}^\lambda_{i,0} = a_i$ *and* $\liminf_{\lambda\to\infty} \bar{t}^\lambda_{i,0} = \limsup_{\lambda\to\infty} \bar{t}^\lambda_{i,0} = 1 - a_i$*,* $i \in \{1, 2, \cdots, m+1\}$*;*
(b) $\lim_{\lambda\to\infty} u'_{\lambda,i}(t) = 0$ *uniformly on* $[a_i+\varepsilon, 1-a_i-\varepsilon]$*, for sufficiently small* $\varepsilon > 0$*,* $i \in \{1, 2, \cdots, m\}$*;*
(c) $\lim_{\lambda\to\infty} \|u_{\lambda,i}\|_\infty = a_i$*,* $i \in \{1, 2, \cdots, m+1\}$.

Proof. Let $\{(\lambda_n, u_{\lambda_n,i})\} \subset \mathcal{C}_i$ be a sequence of solution pairs of problem (P^i_λ) such that

(i) $\lim_{n\to\infty} \lambda_n = \infty$;
(ii) $a_{i-1} < b_i \le \|u_{\lambda_n,i}\|_\infty \le a_i$.

From Theorem 1.3, we see that such sequence makes sense. Then, we can get two sequences $\{\underline{t}^n_{i,0}\}$ and $\{\bar{t}^n_{i,0}\}$ where $\underline{t}^n_{i,0} \triangleq \inf\{t \in (0,1) : u'_{\lambda_n,i}(t) = 0\}$ and $\bar{t}^n_{i,0} \triangleq \sup\{t \in (0,1) : u'_{\lambda_n,i}(t) = 0\}$. Due to the fact that $\|u'_{\lambda_n,i}\|_\infty \le 1$ and Arzelà–Ascoli theorem, we obtain a subsequence, say $\{u_{\lambda_n,i}\}$ again and a function $v \in C[0,1]$ such that $u_{\lambda_n,i} \to v$ in $C[0,1]$ as $n \to \infty$. Correspondingly, we obtain two subsequences, named $\{\underline{t}^n_{i,0}\}$ and $\{\bar{t}^n_{i,0}\}$ again. It follows from Lemma 3.6 that

$$0 < \liminf_{n\to\infty} \underline{t}^n_{i,0} \le \limsup_{n\to\infty} \underline{t}^n_{i,0} < 1 \quad \text{and}$$
$$0 < \liminf_{n\to\infty} \bar{t}^n_{i,0} \le \limsup_{n\to\infty} \bar{t}^n_{i,0} < 1.$$

To prove $\liminf_{n\to\infty} \underline{t}^n_{i,0} = \limsup_{n\to\infty} \underline{t}^n_{i,0} = a_i$ and $\liminf_{n\to\infty} \bar{t}^n_{i,0} = \limsup_{n\to\infty} \bar{t}^n_{i,0} = 1 - a_i$, it suffices to show that any two subsequences of $\{\underline{t}^n_{i,0}\}$ and $\{\bar{t}^n_{i,0}\}$ converge to a_i and $1 - a_i$, respectively. We consider subsequences, name $\{\underline{t}^n_{i,0}\}$ and $\{\bar{t}^n_{i,0}\}$ again, if necessary in the following. Denote

$$\lim_{n\to\infty} \underline{t}^n_{i,0} = \underline{t}^\infty_{i,0} \text{ and } \lim_{n\to\infty} \bar{t}^n_{i,0} = \bar{t}^\infty_{i,0}.$$

We divide the rest into four steps:

Step 1. Claim that $0 < \underline{t}^\infty_{i,0} \le a_i$ and $1 - a_i \le \bar{t}^\infty_{i,0} < 1$, for any $i \in \{1, 2, \cdots, m+1\}$.

Without loss of generality, we prove $0 < \underline{t}_{i,0}^{\infty} \leq a_i$ and the other can be proved by a similar argument after suitable modifications. Suppose on the contrary that $\underline{t}_{i,0}^{\infty} > a_i$. Recall the result in Lemma 3.7 that

$$\lim_{n\to\infty} u'_{\lambda_n,i}(t) = 1, \text{ uniformly on } [0, \underline{t}_{i,0}^{\infty} - \varepsilon_1], \text{ for sufficiently small } \varepsilon_1 > 0. \tag{3.19}$$

For any small $\varepsilon_0 > 0$, there exists $N(\varepsilon_0) \in \mathbb{N}$ such that

$$\underline{t}_{i,0}^{n} > \underline{t}_{i,0}^{\infty} - \varepsilon_0, \text{ for all } n \geq N(\varepsilon_0).$$

Specially, we set $0 < \varepsilon_1 < \frac{\underline{t}_{i,0}^{\infty} - a_i}{2}$ and $\varepsilon_0 = \varepsilon_1$ in the following proof. Noting $u'_{\lambda_n,i}(t) \geq 0$ for $t \in [0, \underline{t}_{i,0}^{n}]$, we get

$$\begin{aligned} u_{\lambda_n,i}(\underline{t}_{i,0}^{n}) &= \int_0^{\underline{t}_{i,0}^{n}} u'_{\lambda_n,i}(\tau)d\tau \geq \int_0^{\underline{t}_{i,0}^{n} - \varepsilon_1} u'_{\lambda_n,i}(\tau)d\tau \\ &\geq \int_0^{\underline{t}_{i,0}^{\infty} - 2\varepsilon_1} u'_{\lambda_n,i}(\tau)d\tau, \text{ for all } n \geq N(\varepsilon_1). \end{aligned} \tag{3.20}$$

Taking $\varepsilon_2 > 0$ sufficiently small satisfying

$$0 < \varepsilon_2 < \frac{\underline{t}_{i,0}^{\infty} - 2\varepsilon_1 - a_i}{\underline{t}_{i,0}^{\infty} - 2\varepsilon_1},$$

by applying (3.19), we may choose $N(\varepsilon_2) > 0$ such that

$$1 - \varepsilon_2 < u'_{\lambda_n,i}(t) < 1 \quad \text{for all } n > N(\varepsilon_2) \quad \text{and } t \in [0, \underline{t}_{i,0}^{\infty} - \varepsilon_1].$$

Since ε_2 satisfies $(1 - \varepsilon_2)(\underline{t}_{i,0}^{\infty} - 2\varepsilon_1) > a_i$, from (3.20), we have

$$u_{\lambda_n,i}(\underline{t}_{i,0}^{n}) \geq \int_0^{\underline{t}_{i,0}^{\infty} - 2\varepsilon_1} u'_{\lambda_n,i}(\tau)d\tau > (1 - \varepsilon_2)(\underline{t}_{i,0}^{\infty} - 2\varepsilon_1) > a_i,$$

for all $n \geq \max\{N(\varepsilon_1), N(\varepsilon_2)\}$. Thus we get

$$a_i \geq v(\underline{t}_{i,0}^{\infty}) = \lim_{n\to\infty} u_{\lambda_n,i}(\underline{t}_{i,0}^{n}) > a_i.$$

This contradiction proves $\underline{t}_{i,0}^{\infty} \leq a_i$. Similarly, we can show $1 - a_i \leq \overline{t}_{i,0}^{\infty} < 1$.

Step 2. Claim that $\underline{t}_{i,0}^{\infty} = a_i$ and $\overline{t}_{i,0}^{\infty} = 1 - a_i$, for any $i \in \{1, 2, \cdots, m+1\}$.

Suppose that $0 < \underline{t}_{i,0}^{\infty} < a_i$. Then, for $0 < \varepsilon_3 < \frac{a_i - \underline{t}_{i,0}^{\infty}}{4}$, there exists $N(\varepsilon_3) \in \mathbb{N}$ such that

$$\underline{t}_{i,0}^{\infty} - \varepsilon_3 < \underline{t}_{i,0}^{n} < \underline{t}_{i,0}^{\infty} + \varepsilon_3, \text{ for all } n > N(\varepsilon_3).$$

Similarly, for $0 < \varepsilon_4 < \frac{1 - a_i - \underline{t}_{i,0}^{\infty}}{4}$, there exists $N(\varepsilon_4) \in \mathbb{N}$ such that

$$\overline{t}_{i,0}^{\infty} - \varepsilon_4 < \overline{t}_{i,0}^{n} < \overline{t}_{i,0}^{\infty} + \varepsilon_4, \text{ for all } n > N(\varepsilon_4).$$

Recall that $1 - a_i \leq \overline{t}_{i,0}^{\infty} < 1$ in Step 1 and denote $N_3 \triangleq \max\{N(\varepsilon_3), N(\varepsilon_4)\}$. Then, for $n > N_3$, we have

$$\underline{t}_{i,0}^{n} < \frac{a_i + 3\underline{t}_{i,0}^{\infty}}{4} < \frac{3(1 - a_i) + \underline{t}_{i,0}^{\infty}}{4} < \overline{t}_{i,0}^{n}.$$

Since

$$u_{\lambda_n,i}(\underline{t}^n_{i,0}) = \int_0^{\underline{t}^n_{i,0}} u'_{\lambda_n,i}(\tau)d\tau < \underline{t}^n_{i,0} < \frac{a_i + 3\underline{t}^\infty_{i,0}}{4} < a_i, \text{ for all } n \geq N_3,$$

we get $b_i \leq \|u_{\lambda_n,i}\|_\infty < a_i$ for all $n \geq N_3$. Combining with the concavity of $u_{\lambda_n,i}$ on $[0,1]$, we deduce

$$u'_{\lambda_n,i}(t) = 0, \ u_{\lambda_n,i}(t) = \|u_{\lambda_n,i}\|_\infty \in [b_i, a_i) \ \text{ on } t \in [\underline{t}^n_{i,0}, \overline{t}^n_{i,0}], \text{ for all } n,$$

and so $g^i(u_{\lambda_n,i}) > 0$ on $t \in [\underline{t}^n_{i,0}, \overline{t}^n_{i,0}]$, for all $n \geq N_3$. However, from the calculation

$$0 = u'_{\lambda_n,i}(\overline{t}^n_{i,0}) = -\phi^{-1}\left(\lambda_n \int_{\underline{t}^n_{i,0}}^{\overline{t}^n_{i,0}} r(s)g^i(u_{\lambda_n,i}(s))ds\right) < 0, \text{ for all } n \geq N_3,$$

we get a contradiction. Hence $\underline{t}^\infty_{i,0} = a_i$. By a similar fashion, we also conclude $\overline{t}^\infty_{i,0} = 1 - a_i$. Consequently, result (a) is obtained.

Step 3. Prove that $\lim_{n\to\infty} u'_{\lambda_n,i}(t) = 0$ *uniformly on* $[a_i+\varepsilon, 1-a_i-\varepsilon]$, for sufficiently small $\varepsilon > 0$, $i \in \{1, 2, \cdots, m\}$.

By the concavity of $u_{\lambda_n,i}$ on $[0,1]$, we know that $u'_{\lambda_n,i} = 0$ on $[\underline{t}^n_{i,0}, \overline{t}^n_{i,0}]$, for all n. Since $\underline{t}^\infty_{i,0} = a_i$, for small $0 < \varepsilon < \frac{1-2a_i}{2}$, there exists $N_1(\varepsilon) \in \mathbb{N}$ such that

$$a_i - \varepsilon < \underline{t}^n_{i,0} < a_i + \varepsilon, \text{ for all } n > N_1(\varepsilon).$$

Similarly, thanks to $\overline{t}^\infty_{i,0} = 1 - a_i$, there exists $N_2(\varepsilon) \in \mathbb{N}$ such that

$$1 - a_i - \varepsilon < \overline{t}^n_{i,0} < 1 - a_i + \varepsilon, \text{ for all } n > N_2(\varepsilon).$$

Thus, $[a_i + \varepsilon, 1 - a_i - \varepsilon] \subset [\underline{t}^n_{i,0}, \overline{t}^n_{i,0}]$ for $n > N_4 \triangleq \max\{N_1(\varepsilon), N_2(\varepsilon)\}$ and we have

$$u'_{\lambda_n,i} = 0, \text{ on } [a_i + \varepsilon, 1 - a_i - \varepsilon], \text{ for all } n > N_4.$$

Further, we obtain

$$\lim_{n\to\infty} u'_{\lambda_n,i}(t) = 0 \text{ uniformly on } [a_i + \varepsilon, 1 - a_i - \varepsilon],$$

for sufficiently small $\varepsilon > 0$, $i \in \{1, 2, \cdots, m\}$, which is result (b).

Step 4. Prove that $\lim_{n\to\infty} \|u_{\lambda_n,i}\|_\infty = a_i$ for any $i \in \{1, 2, \cdots, m+1\}$.

From (3.20), for any $n \geq N(\varepsilon_1)$, we obtain

$$\|u_{\lambda_n,i}\|_\infty = u_{\lambda_n,i}(\underline{t}^n_{i,0}) \geq \int_0^{\underline{t}^\infty_{i,0}-2\varepsilon_1} u'_{\lambda_n,i}(\tau)d\tau.$$

Since $\underline{t}^\infty_{i,0} = a_i$ as proved in previous steps, by using Lemma 3.7 and the bounded convergence theorem, we get

$$\begin{aligned}\lim_{n\to\infty} \|u_{\lambda_n,i}\|_\infty &\geq \lim_{n\to\infty} \int_0^{\underline{t}^\infty_{i,0}-2\varepsilon_1} u'_{\lambda_n,i}(\tau)d\tau \\ &= \int_0^{a_i-2\varepsilon_1} \lim_{n\to\infty} u'_{\lambda_n,i}(\tau)d\tau = a_i - 2\varepsilon_1.\end{aligned}$$

Thus by the arbitrariness of ε_1, we get $\lim_{n\to\infty} \|u_{\lambda_n,i}\|_\infty \geq a_i$. On the other hand, we know from Lemma 3.2 that $\|u_{\lambda_n,i}\|_\infty \leq a_i$ for all n. Therefore, we conclude $\lim_{n\to\infty} \|u_{\lambda_n,i}\|_\infty = a_i$. Specially, $\lim_{n\to\infty} \|u_{\lambda_n,m+1}\|_\infty = \frac{1}{2}$. Result (c) is deduced and the proof is completed. □

Proof of Theorem 1.4. By Lemma 3.4, we deduce (i). By Lemmas 3.5–3.9, assertions $(ii) - (iv)$ are verified. The proof is completed. □

Finally, we give an example to illustrate the applicability of our multiplicity and asymptotic property results.

Example 3.1. We consider the problem

$$\begin{cases} -\Big(\phi(u'(t))\Big)' = \lambda t^{-2} u^2(t) \left|(u(t) - \frac{1}{64})(u(t) - \frac{1}{16})(u(t) - \frac{1}{4})\right|, & t \in (0,1), \\ u(0) = 0 = u(1). \end{cases} \tag{3.21}$$

Note that $r(t) = t^{-2}$ and $g(s) = s^2|(s - \frac{1}{64})(s - \frac{1}{16})(s - \frac{1}{4})|$ as in problem (P_λ). It is obvious that $r \in \mathcal{A}_2$ and $g_0 = \lim_{s\to 0} \left|(s - \frac{1}{64})(s - \frac{1}{16})(s - \frac{1}{4})\right| = \frac{1}{4096}$. Let $a_0 = 0$, $a_1 = \frac{1}{64}$, $a_2 = \frac{1}{16}$, $a_3 = \frac{1}{4}$ and $a_4 = \frac{1}{2}$. Then we can check $\sum_{i=0}^{3} g(a_i) = 0$ and $g(s) > 0$ on $(0, \frac{1}{2}) \setminus \{a_i\}_{i=1}^{3}$. Thus, assumptions in Theorem 1.3 are satisfied and by applying Theorem 1.3, there exist $0 < \lambda_* \leq \lambda^* < \infty$ such that problem (3.21) has no positive solution for all $\lambda \in (0, \lambda_*)$ and at least four positive solutions $u_{\lambda,1}, u_{\lambda,2}, u_{\lambda,3}$ and $u_{\lambda,4}$ for all $\lambda \in (\lambda^*, \infty)$. Combining with Theorem 1.4, $u_{\lambda,i}$ $(i \in \{1,2,3,4\})$ convergences to function u_i given below as λ goes to ∞,

$$u_i(t) = \begin{cases} t, & t \in [0, a_i), \\ a_i, & t \in [a_i, 1 - a_i], \\ 1 - t, & t \in (1 - a_i, 1]. \end{cases}$$

The shape of u_i is isosceles trapezoid when $1 \leq i \leq 3$ and isosceles triangle when $i = 4$.

Acknowledgements

The author is grateful to the anonymous reviewers and editors for their valuable comments and professional suggestions that improve the original manuscript. The paper is dedicated to professor Yong-Hoon Lee on the occasion of his 65th birthday. This work was partially supported by the National Natural Science Foundation of China (No. 12201648) and the Natural Science Foundation of Hunan Province, China (No. 2022JJ40568).

Data Availability Statement Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declarations

Conflict of interest The authors declare no competing interests.

References

[1] Bartnik, R.: Existence of maximal surfaces in asymptotically flat spacetimes. Commun. Math. Phys. **95**, 155–175 (1984)

[2] Bartnik, R.: Regularity of variational maximal surfaces. Acta Math. **161**, 145–181 (1988)

[3] Bartolo, R., Caponio, E., Pomponio, A.: Spacelike graphs with prescribed mean curvature on exterior domains in the Minkowski spacetime. Proc. Am. Math. Soc. **149**, 5139–5151 (2021)

[4] Bartnik, R., Simon, L.: Spacelike hypersurfaces with prescribed boundary values and mean curvature. Commun. Math. Phys. **87**, 131–152 (1982)

[5] Bereanu, C., Fuente, D., Romero, A., Torres, P.J.: Existence and multiplicity of entire radial space like graphs with prescribed mean curvature function in certain Friedmann-Lemaitre-Robertson-Walker space times. Commun. Contemp. Math. **19**, 1–18 (2017)

[6] Bereanu, C., Jebelean, P., Mawhin, J.: Radial solutions for some nonlinear problems involving mean curvature operators in Euclidean and Minkowski spaces. Proc. Am. Math. Soc. **137**, 161–169 (2009)

[7] Bereanu, C., Jebelean, P., Torres, P.J.: Positive radial solutions for Dirichlet problems with mean curvature operators in Minkowski space. J. Funct. Anal. **264**, 270–287 (2013)

[8] Bereanu, C., Jebelean, P., Torres, P.J.: Multiple positive radial solutions for a Dirichlet problem involving the mean curvature operator in Minkowski space. J. Funct. Anal. **265**, 644–659 (2013)

[9] Born, M.: Modified field equations with a finite radius of the electron. Nature **132**, 282–282 (1933)

[10] Born, M.: On the quantum theory of the electromagnetic field. Proc. R. Soc. Lond. Ser. A **143**, 410–437 (1934)

[11] Born, M., Infeld, L.: Foundations of the new field theory. Nature **132**, 1004–1004 (1933)

[12] Born, M., Infeld, L.: Foundations of the new field theory. Proc. R. Soc. Lond. Ser. A **144**, 425–451 (1934)

[13] Calabi, E.: Examples of Bernstein problems for some nonlinear equations. Proc. Sympos. Pure Math. **15**, 223–230 (1970)

[14] Choquet-Bruhat, Y.: Maximal submanifolds and submanifolds with constant mean extrinsic curvature of a Lorentzian manifold. Ann. Scuola Norm. Sup. Pisa **3**, 361–376 (1976)

[15] Coelho, I., Corsato, C., Obersnel, F., Omari, P.: Positive solutions of the Dirichlet problem for one-dimensional Minkowski-curvature equation. Adv. Nonlinear Stud. **12**, 621–638 (2012)

[16] Coelho, I., Corsato, C., Rivetti, S.: Positive radial solutions of the Dirichlet problem for the Minkowski-curvature equation in a ball. Topol. Methods Nonlinear Anal. **44**, 23–39 (2014)

[17] Caballero, M., Romero, A., Rubio, R.M.: Calabi-Bernstein-type problems for some nonlinear equations arising in Lorentzian geometry. J. Math. Sci. **207**, 544–550 (2015)

[18] Cheng, S., Yau, S.: Maximal spacelike hypersurfaces in the Lorentz–Minkowski spaces. Ann. Math. **104**, 407–419 (1976)

[19] Dai, G.: Bifurcation and positive solutions for problem with mean curvature operator in Minkowski space. Calc. Var. Partial Differ. Equ. **55**, 1–17 (2016)

[20] Dai, G., Wang, J.: Nodal solutions to problem with mean curvature operator in Minkowski space. Differ. Integral Equ. **30**, 463–480 (2017)

[21] Gerhardt, C.: H-surfaces in Lorentzian manifolds. Commun. Math. Phys. **89**, 523–553 (1983)

[22] Gibbons, G.W.: Born-Infeld particles and Dirichlet p-branes. Nucl. Phys. B **514**, 603–639 (1998)

[23] Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cones. Academic Press, Boston (1988)

[24] García-Huidobro, M., Manásevich, R., Ward, J.R.: A homotopy along p for systems with a vector p-Laplace operator. Adv. Differ. Equ. **8**, 337–356 (2003)

[25] Hess, P.: On multiple positive solutions of nonlinear elliptic eigenvalue problems. Comm. Partial Differ. Equ. **6**, 951–961 (1981)

[26] Hong, G.H., Yuan, Y.: Maximal hypersurfaces over exterior domains. Comm. Pure Appl. Math. **74**, 589–614 (2021)

[27] Latorre, J.M., Romero, A.: New examples of Calabi–Bernstein problems for some nonlinear equations. Differ. Geom. Appl. **15**, 153–163 (2001)

[28] Lee, Y.H., Sim, I.: Global bifurcation phenomena for singular one-dimensional p-Laplacian. J. Differ. Equ. **229**, 229–256 (2006)

[29] Lee, Y.H., Sim, I., Yang, R.: Bifurcation and Calabi-Bernstein type asymptotic property of solutions for the one-dimensional Minkowski-curvature equation. J. Math. Anal. Appl. **507**, 125725–16 (2022)

[30] Ma, R., Gao, H., Lu, Y.: Global structure of radial positive solutions for a prescribed mean curvature problem in a ball. J. Funct. Anal. **270**, 2430–2455 (2016)

[31] Manásevich, R., Mawhin, J.: Periodic solutions for nonlinear systems with p-Laplacian-Like operators. J. Differ. Equ. **145**, 367–393 (1998)

[32] Manásevich, R., Mawhin, J.: Boundary value problems for nonlinear perturbations of vector p-Laplacian-Like operator. J. Korean Math. Soc. **47**, 47–11 (2000)

[33] Mawhin, J., Torres, P.J.: Prescribed mean curvature graphs with Neumann boundary conditions in some FLRW spacetimes. J. Differ. Equ. **261**, 7145–7156 (2016)

[34] Pei, M., Wang, L., Lv, X.: Existence and multiplicity of positive solutions of a one-dimensional mean curvature equation in Minkowski space, Bound. Value Probl. **2018** (2018). https://doi.org/10.1186/s13661-018-0963-5

[35] Treibergs, A.E.: Entire spacelike hypersurfaces of constant mean curvature in Minkowski space. Invent. Math. **66**, 39–56 (1982)

[36] Wang, H.: Multiple positive radial solutions for quasilinear equations in annular domains, Proceedings of the Conference on Differential and Difference Equations and Applications, 1121-1130. Hindawi Publishing Corporation, New York, (2006)

[37] Wong, F.: Uniqueness of positive solutions for Sturm-Liouville boundary value problems. Proc. Am. Math. Soc. **126**, 365–374 (1998)

[38] Yang, R., Lee, Y.H.: Bifurcation of positive radial solutions for a prescribed mean curvature problem on an exterior domain. Adv. Differ. Equ. **25**, 161–190 (2020)

[39] Yang, R., Lee, J.K., Lee, Y.H.: A constructive approach about the existence of positive solutions for Minkowski curvature problems. Bull. Malays. Math. Sci. Soc. **45**, 1–16 (2022)

[40] Yang, R., Lee, Y.H., Sim, I.: Bifurcation of nodal radial solutions for a prescribed mean curvature problem on an exterior domain. J. Differ. Equ. **268**, 4464–4490 (2020)

[41] Yang, R., Sim, I., Lee, Y.H.: $\frac{\pi}{4}$-tangential solution for Minkowski-curvature problems. Adv. Nonlinear Anal. **9**, 1463–1479 (2020)

Rui Yang
School of Mathematics and Statistics
HNP-LAMA, Central South University
Changsha Hunan 410083
China
e-mail: ruiyang@csu.edu.cn

Accepted: November 11, 2022.