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# An application of Edelstein's contraction principle: the attractor of a graph-directed generalized iterated function system

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**Abstract.** This paper gathers two generalizations of iterated function systems, namely the one introduced by the first two authors under the name of generalized iterated function systems and the one introduced by Mauldin and Williams and Boore and Falconer under the label of graph-directed iterated function systems. By combining them we introduce the concept of a graph-directed generalized iterated function system. We prove that, under suitable contractivity assumptions on the constitutive functions of such a system and structural assumptions on the underlying metric space, it generates, via Edelstein's contraction principle, a unique attractor. The result is illustrated by two examples.

**Mathematics Subject Classification.** Primary 28A80; Secondary 54H20.

**Keywords.** Edelstein's contraction principle, graph-directed generalized iterated function system, attractor.

## 1. Introduction

Since the appearance of the seminal Hutchinson's paper [21] introducing iterated function systems, many authors proposed different types of generalizations of this concept which represents a convenient way to describe and classify fractal sets. Having in mind the aim of this paper, we are going to emphasize two types of such generalizations, namely generalized iterated function systems introduced by Miculescu and Mihail (see [32] and [34]) and graph-directed iterated function systems introduced Mauldin and Williams (see [27]) and Boore and Falconer (see [5]).

On the one hand, given $m \in \mathbb{N}$ and a metric space $(X, d)$, one such generalization is obtained by considering (rather than selfmaps of $X$) functions defined on the finite Cartesian product $X^m$ with values in $X$ . In this way one gets the so called generalized iterated function systems (for short GIFSs) of order $m$. Miculescu and Mihail (see [32] and [34]) proved the existence and uniqueness of the attractor of a GIFS and pointed out some of its properties. A big step forward was made by F. Strobin (see [40]) who proved that GIFSs are real generalizations of iterated function systems. To be more precise, he proved that, for any $m \geq 2$, there exists a Cantor subset of the plane which is an attractor of some GIFS of order $m$ , but is not an attractor of a GIFS of order $m-1$. Algorithms generating images of attractors of GIFSs could be found in [6,19,22] and [29]. Various aspects concerning Hutschinson's measure in the framework of GIFS were studied in [7,28,33] and [38]. Finally let us mention that the papers [14,15,25,30,31,35,36,39,41–43] are dealing with certain extensions of GIFSs.

On the other hand, another generalization of the concept of iterated function system is based on the replacement of a single fixed point equation that is satisfied by the attractor with a system of equations. In this way one yields an attractor of a more complicated type of system. More precisely, a family of compact sets (also called invariant list) is considered such that each of them is a union of contracted copies of some (but not necessarily all) family's components. It was Mauldin and Williams' idea of using graphs in the theory of iterated function systems which leads to the concept of graph-directed iterated function system (also referred to as recurrent iterated function system). A classical iterated function system can be presented as a 1-vertex graph-directed iterated function system. Moreover, Boore and Falconer (see [5]) provided a class of 2-vertex graph-directed iterated function systems having attractors which cannot be the attractors of standard iterated function systems. Therefore, Mauldin and Williams' concept is a genuine generalization of Hutchinson' one. For some other works related with this topic see [1–4,8–13,18,20,23] and [44]. One can also consult [17, section 4.3] and [24, section 2.6.3].

The motivation of our study derives from the previously mentioned generalizations of the concept of iterated function system. These generalizations are part of the ongoing research program whose purpose is to obtain a larger class of fractals by extending Hutchinson's framework. By combining these two lines of research, we introduce the concept of graph-directed generalized iterated function system and prove the existence and uniqueness for the attractor of such a system. The main tool that we use is Edelstein's contraction principle (see [16]) concerning $(\varepsilon, \lambda)$-uniformly locally contractive functions on complete $\varepsilon$-chainable metric spaces. Note that iterated function systems, on a $\varepsilon$-chainable metric space, consisting on $(\varepsilon, \lambda)$ -contractions were studied by L. Máté (see [26]) and Gwóźd ź-Łukawska and Jachymski (see [20]).

## 2. Preliminaries

### Notation and terminology

For a function $f : X \to X$ and $n \in \mathbb{N}$, by $f^{[n]}$ we designate the composition of $f$ by itself $n$ times.

For metric spaces $(X, d)$ and $(Y, \rho)$ and a function $f : X \to Y$, we shall use the following notation:

$$\{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ is compact}\} \overset{not}{=} P_{cp}(X)$$

and

$$\sup\left\{\frac{\rho(f(x), f(y))}{d(x,y)} \mid x, y \in X,\, x \neq y\right\} \overset{not}{=} lip(f).$$

Moreover, the function $h : P_{cp}(X) \times P_{cp}(X) \to [0, \infty)$, described by

$$h(K_1, K_2) = \max\{\sup_{x\in K_1} d(x, K_2), \sup_{x\in K_2} d(x, K_1)\},$$

for every $K_1, K_2 \in P_{cp}(X)$ (which turns out to be a metric), is called the Hausdorff-Pompeiu metric on $X$.

Note that:

- $(P_{cp}(X), h)$ is a complete metric space, provided that $(X, d)$ is complete;
- $$h(K_1 \cup K_2, C_1 \cup C_2) \leq \max\{h(K_1, C_1), h(K_2, C_2)\},$$

  for every $K_1, K_2, C_1, C_2 \in P_{cp}(X)$.

**Definition 2.1.** Given $\varepsilon > 0$, a metric space $(X, d)$ is called $\varepsilon$-chainable if for every $x, y \in X$ there exist $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_{n-1}, x_n \in X$ such that $x_0 = x$, $x_n = y$ and $d(x_k, x_{k+1}) < \varepsilon$ for each $k \in \{0, 1, \ldots, n-1\}$.

**Proposition 2.2.** *If $(X, d)$ is an $\varepsilon$ -chainable metric space for some $\varepsilon > 0$, then $(P_{cp}(X), h)$ is $2\varepsilon$-chainable.*

*Proof.* Let us consider $A, B \in P_{cp}(X)$ fixed, but arbitrarily chosen.

Then there exist $m \in \mathbb{N}$, $x_1, \ldots, x_m \in A$ and $y_1, \ldots, y_m \in B$ such that

$$A \subseteq \bigcup_{l=1}^{m} B(x_l, \varepsilon) \text{ and } B \subseteq \bigcup_{l=1}^{m} B(y_l, \varepsilon). \tag{1}$$

One should note that we used the fact that, via a possible repetition of the points $x_l$ and $y_l$, we can suppose that the set of the points $x_l$ has the same number of elements as the set of the points $y_l$.

Since $x_l$ and $y_l$ are elements of the $\varepsilon$-chainable metric space $(X, d)$, there exist $n \in \mathbb{N}$ and $z_{l,0}, \ldots, z_{l,n} \in X$ such that

$$z_{l,0} = x_l,\ z_{l,n} = y_l$$

and

$$d(z_{l,k}, z_{l,k+1}) < \varepsilon, \tag{2}$$

for all $k \in \{0, 1, \ldots, n-1\}$ and $l \in \{1, ...., m\}$. One should note that we used the fact that, via a possible repetition of the elements $z_{l,k}$, the number $n$ (which basically depends on $x_l$ and $y_l$) can be chosen to be the same for all $l \in \{1, ...., m\}$.

Let us consider the finite (so compact) sets

$$C_k = \{z_{1,k}, \ldots, z_{m,k}\} \subseteq X,$$

where $k \in \{1, \ldots, n-1\}$.

Then,

$$d(z_{l,k}, C_{k+1}) \overset{z_{l,k+1}\in C_{k+1}}{\leq} d(z_{l,k}, z_{l,k+1}) \overset{(2)}{<} \varepsilon,$$

for every $k \in \{1, \dots, n-2\}$ and $l \in \{1, \dots, m\}$, so

$$\sup_{x\in C_k} d(x, C_{k+1}) < \varepsilon, \tag{3}$$

for every $k \in \{1, \dots, n-2\}$.

In a similar manner, for every $k \in \{1, \dots, n-2\}$, we obtain $\sup\limits_{x\in C_{k+1}} d(x, C_k) < \varepsilon$. Using this and (3), we get $h(C_k, C_{k+1}) < \varepsilon$ for every $k \in \{1, \dots, n-2\}$.

We also have

$$d(z_{l,1}, A) \overset{x_l\in A}{\leq} d(z_{l,1}, x_l) = d(z_{l,1}, z_{l,0}) \overset{(2)}{<} \varepsilon,$$

for each $l \in \{1, \dots, m\}$, so

$$\sup_{x\in C_1} d(x, A) < \varepsilon.$$

Choose any $x \in A$. Then taking into account (1), there exists $l \in \{1, \dots, m\}$ such that

$$d(x, z_{l,0}) = d(x, x_l) < \varepsilon. \tag{4}$$

Consequently

$$d(x, C_1) \overset{z_{l,1}\in C_1}{\leq} d(x, z_{l,1}) \leq d(x, z_{l,0}) + d(z_{l,0}, z_{l,1}) \overset{(2)\&(4)}{<} \varepsilon + \varepsilon = 2\varepsilon,$$

which gives $\sup\limits_{x\in A} d(x, C_1) < 2\varepsilon$, and hence $h(A, C_1) < 2\varepsilon$. In a similar manner one can prove that $h(C_{n-1}, B) < 2\varepsilon$. The proof is complete. □

**Definition 2.3.** Let $(X, d)$ be a metric space, $\varepsilon > 0$ and $C \in [0, 1)$. A function $f : X \to X$ is called $(\varepsilon, C)$-uniformly locally contractive if $d(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$ such that $d(x, y) < \varepsilon$.

**Edelstein's contraction principle 2.4.** (see [16]) *Let $(X, d)$ be a complete $\varepsilon$-chainable metric space and $f : X \to X$ a $(\varepsilon, C)$ -uniformly locally contractive function. Then $f$ has a unique fixed point $x^*$ and $\lim\limits_{n\to\infty} f^{[n]}(x) = x^*$ for every $x \in X$, i.e., $f$ is a Picard operator.*

In connection with the above mentioned result, see also [37].

**Definition 2.5.** Let $(X, d)$ be a metric space, $m \in \mathbb{N}$ and $f : X^m \to X$. An element $x$ of $X$ is called fixed point of $f$ if $f(\underbrace{x, \dots, x}_{m \text{ times}}) = x$.

*Remark 2.6.* The fixed points of the above mentioned function $f$ coincide with the fixed points of $g : X \to X$ given by $g(x) = f(\underbrace{x, \dots, x}_{m \text{ times}})$ for every $x \in X$.

In this paper, for a metric space $(X, d)$ and $m \in \mathbb{N}$, on $X^m$ we consider the maximum metric $d_{\max}$, given by

$$d_{\max}((x_1, \dots, x_m), (y_1, \dots, y_m)) = \max\{d(x_1, y_1), \dots, d(x_m, y_m)\},$$

for every $(x_1, \dots, x_m), (y_1, \dots, y_m) \in X^m$.

**Definition 2.7.** Given $m \in \mathbb{N}$, a pair $(((X_j, d_j))_{j\in\{1,...,p\}}, (f_i)_{i\in\{1,...,q\}})$ is called a graph-directed generalized iterated function system (abbreviated g-dGIFS) of order $m$ if:

i) for each $j \in \{1, \ldots, p\}$ there exists $\varepsilon_j > 0$ such that $(X_j, d_j)$ is a complete $\varepsilon_j$-chainable metric space;

ii) there exist $D, C : \{1, \ldots, q\} \to \{1, \ldots, p\}$, $C$ onto, such that $f_i : X^m_{D(i)} \to X_{C(i)}$ for each $i \in \{1, \ldots, q\}$;

iii) $f_i$ is a Banach contraction for every $i \in \{1, \ldots, q\}$ i.e., there exists $C_i \in [0, 1)$ such that

$$d_{C(i)}(f_i(x_1, \ldots, x_m), f_i(y_1, \ldots, y_m)) \le C_i \max\{d_{D(i)}(x_1, y_1), \ldots, d_{D(i)}(x_m, y_m)\},$$

for every $(x_1, \ldots, x_m), (y_1, \ldots, y_m) \in X^m_{D(i)}$.

We denote such a system by $\mathcal{S}$.

Note that the $\varepsilon$-chainability assumption for some $\varepsilon > 0$ is quite weak as, for example, each bounded metric space has this property.

Concerning the comparison of graph-directed generalized iterated function systems with graph-directed iterated function systems considered in literature, we mention that, for $m = 1$, Definition 2.7 yields a particular case of the concept of a directed graph IFS considered in [5] (denoted by $(V, E^*, i, t, r, ((C_v, d_v))_{v\in V}, (S_e)_{e\in E^1}))$. Namely we have $V = \{1, \ldots, p\}$, $E^* = E^1 = \{e_k \mid k \in \{1, \ldots, q\}\}$, where the functions $i, t : E^* \to V$ are given by $i(e_k) = D(k)$ and $t(e_k) = C(k)$ for every $k \in \{1, \ldots, q\}$, $((C_v, d_v))_{v\in V} = ((X_j, d_j))_{j\in\{1,...,p\}}$, $(S_e)_{e\in E^1} = (f_k)_{k\in\{1,...,q\}}$ and the function $r : E^* \to (0, 1)$ is given by $r(e_k) = lip(f_k)$ for every $k \in \{1, \ldots, q\}$.

**Additional notation**

For a given g-dGIFS $\mathcal{S} = ((X_j, d_j)_{j\in\{1,...,p\}}, (f_i)_{i\in\{1,...,q\}})$, let us consider the fixed elements $\overline{x_j} \in X_j$, where $j \in \{1, \ldots, p\}$ and $L \in [2\max\{\varepsilon_1, \ldots, \varepsilon_p\}, \infty)$. In the sequel, we shall use the following notations:

–
$$\{(x, j) \mid x \in X_j\} \overset{not}{=} \mathcal{X}^j,$$
for every $j \in \{1, \ldots, p\}$

–
$$\bigcup_{j=1}^{p} \mathcal{X}^j \overset{not}{=} X$$

–
$$K \cap \mathcal{X}^j \overset{not}{=} \mathcal{K}^j,$$
for every $K \subseteq X$ and every $j \in \{1, \ldots, p\}$;

–
$$\{K \in P_{cp}(X) \mid \mathcal{K}^j \neq \emptyset \text{ for every } j \in \{1, \ldots, p\}\} \overset{not}{=} P^{\mathcal{S}}_{cp}(X),$$
where (see Lemma 3.1, a)) $X$ is endowed with the metric $d : X \times X \to [0, \infty)$ described by

$$d((x, k), (y, l)) = \begin{cases} d_k(x, y), & \text{if } l = k \\ d_k(x, \overline{x_k}) + d_l(y, \overline{x_l}) + L, & \text{if } l \neq k \end{cases},$$

for every $(x,k),(y,l) \in X$.

*Remark 2.8.* $X$ is the disjoint union of the sets $X_j$ . It is also denoted by $\bigsqcup_{j=1}^{p} X_j$.

*Remark 2.9.* For every $K \subseteq X$, we have

$$K = \bigcup_{j=1}^{p} \mathcal{K}^j.$$

If $K \in P_{cp}^{\mathcal{S}}(X)$, then $\mathcal{K}^j \in P_{cp}(\mathcal{X}^j)$ for every $j \in \{1,\ldots,p\}$ .

*Remark 2.10.* For every $K_j \in P_{cp}(X_j)$, where $j \in \{1,\ldots,p\}$, we have

$$\bigcup_{j=1}^{p} \mathcal{K}_j^j \in P_{cp}^{\mathcal{S}}(X).$$

**Definition 2.11.** Given a g-dGIFS $\mathcal{S}=(((X_j,d_j))_{j\in\{1,\ldots,p\}},(f_i)_{i\in\{1,\ldots,q\}})$ of order $m$, one can consider the fractal operator $F_{\mathcal{S}} : (P_{cp}^{\mathcal{S}}(X))^m \to P_{cp}^{\mathcal{S}}(X)$ given by

$$F_{\mathcal{S}}(K_1,\ldots,K_m) = \bigcup_{i=1}^{q} \overline{f_i}(\mathcal{K}_1^{D(i)} \times ... \times \mathcal{K}_m^{D(i)}),$$

for every $K_1,\ldots,K_m \in P_{cp}^{\mathcal{S}}(X)$, where $\overline{f_i} : (\mathcal{X}^{D(i)})^m \to \mathcal{X}^{C(i)}$ is described by

$$\overline{f_i}((x_1,D(i)),\ldots,(x_m,D(i))) = (f_i(x_1,\ldots,x_m),C(i)),$$

for every $x_1,\ldots,x_m \in X_{D(i)}$.

*Remark 2.12.* For every $i \in \{1,\ldots,q\}$, $lip(\overline{f_i}) = lip(f_i)$.

*Remark 2.13.* $F_{\mathcal{S}}$ is well defined since:

a) Taking into account Remark 2.9 and Tikhonov's theorem, $\mathcal{K}_1^{D(i)} \times ... \times \mathcal{K}_m^{D(i)}$ is compact. Since $\overline{f_i}$ is continuous (see Remark 2.12), we infer that $\overline{f_i}(\mathcal{K}_1^{D(i)} \times ... \times \mathcal{K}_m^{D(i)})$ is a compact subset of $\mathcal{X}^{C(i)}$, so it is a compact subset of $X$. Therefore

$$\bigcup_{i=1}^{q} \overline{f_i}(\mathcal{K}_1^{D(i)} \times ... \times \mathcal{K}_m^{D(i)}) \in P_{cp}(X).$$

b) In addition,

$$\bigcup_{i=1}^{q} \overline{f_i}(\mathcal{K}_1^{D(i)} \times ... \times \mathcal{K}_m^{D(i)}) \in P_{cp}^{\mathcal{S}}(X)$$

since, as $C$ is onto, we have $(\bigcup_{i=1}^{q} \overline{f_i}(\mathcal{K}_1^{D(i)} \times ... \times \mathcal{K}_m^{D(i)})) \cap \mathcal{X}^j \neq \emptyset$ for every $j \in \{1,\ldots,p\}$.

## 3. The main result

**Lemma 3.1.** *Let* $\mathcal{S} = ((X_j, d_j)_{j\in\{1,\ldots,p\}}, (f_i)_{i\in\{1,\ldots,q\}})$ *be a g-dGIFS. Let us also consider the fixed elements* $\overline{x_j} \in X_j$*, where* $j \in \{1,\ldots,p\}$ *and* $L \in [2\max\{\varepsilon_1,\ldots,\varepsilon_p\},\infty)$*. Then:*

a) $d : X \times X \to [0,\infty)$*, given by*

$$d((x,k),(y,l)) = \begin{cases} d_k(x,y), & \text{if } l = k \\ d_k(x,\overline{x_k}) + d_l(y,\overline{x_l}) + L, & \text{if } l \neq k \end{cases},$$

*for every* $(x,k),(y,l) \in X$*, is a metric on* $X$*.*

b) $(X,d)$ *is a complete metric space.*
*Moreover, if* $h$ *is the Hausdorff–Pompeiu metric associated with* $d$*, we have:*

c)

$$h(\mathcal{K}_1^j, \mathcal{K}_2^j) \leq h(K_1, K_2),$$

*for every* $j \in \{1,\ldots,p\}$ *and every* $K_1, K_2 \in P_{cp}^{\mathcal{S}}(X)$ *such that* $h(K_1,K_2) < L$*.*

d) $(P_{cp}^{\mathcal{S}}(X), h)$ *is a complete metric space.*

e) $(P_{cp}^{\mathcal{S}}(X), h)$ *is a L-chainable metric space.*

*Proof.* a) It is clear that, for every $(x,k),(y,l) \in X$, we have

$$d((x,k),(y,l)) \geq 0;$$
$$d((x,k),(y,l)) = 0 \text{ if and only if } (x,k) = (y,l);$$
$$d((x,k),(y,l)) = d((y,l),(x,k)).$$

To conclude that $d$ is a metric, we have to check that

$$d((x,k),(y,l)) \leq d((x,k),(z,m)) + d((z,m),(y,l)),$$

for every $(x,k),(y,l),(z,m) \in X$.

The justification of the above inequality is divided into three situations, according to the cardinality of the set $\{k,l,m\}$.

If $card\{k,l,m\} = 1$, the inequality is obvious as it takes the form

$$d_k(x,y) \leq d_k(x,z) + d_k(z,y),$$

for every $x,y,z \in X_k$.

If $card\{k,l,m\} = 2$, we have the following cases:

j) $k = l \neq m$
jj) $l = m \neq k$
jjj) $k = m \neq l$.

In the first case, we have

$$\begin{aligned} d((x,k),(y,l)) &= d_l(x,y) \leq d_l(x,\overline{x_l}) + d_l(\overline{x_l},y) \\ &\leq d_k(x,\overline{x_k}) + d_m(z,\overline{x_m}) + L + d_l(y,\overline{x_l}) + d_m(z,\overline{x_m}) + L \\ &= d((x,k),(z,m)) + d((z,m),(y,l)). \end{aligned}$$

In the second case, we have

$$d((x,k),(y,l)) = d_k(x,\overline{x_k}) + d_l(y,\overline{x_l}) + L$$

$$\leq d_k(x,\overline{x_k}) + d_l(y,z) + d_m(z,\overline{x_m}) + L$$
$$= d((x,k),(z,m)) + d((z,m),(y,l)).$$

The third case is similar to the second one.

If $card\{k,l,m\} = 3$, then we have

$$\begin{aligned} d((x,k),(y,l)) &= d_k(x,\overline{x_k}) + d_l(y,\overline{x_l}) + L \\ &\leq d_k(x,\overline{x_k}) + d_m(z,\overline{x_m}) + L + d_l(y,\overline{x_l}) + d_m(z,\overline{x_m}) + L \\ &= d((x,k),(z,m)) + d((z,m),(y,l)). \end{aligned}$$

Hence, the proof of a) is complete.

b) Let us consider a Cauchy sequence $((x_n,k_n))_{n\in\mathbb{N}}$ of elements from $X$.

Then for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d((x_n,k_n),(x_m,k_m)) < \varepsilon, \tag{1}$$

for every $m,n \in \mathbb{N}$, $m,n \geq n_\varepsilon$.

In particular, there exists $n_0 \in \mathbb{N}$ such that

$$d((x_n,k_n),(x_m,k_m)) < L,$$

for every $m,n \in \mathbb{N}$, $m,n \geq n_0$ which means that one can find $j \in \{1,\ldots,p\}$ having the property that $k_n = k_m = j$ for every $m,n \in \mathbb{N}$, $m,n \geq n_0$.

Then, (1) takes the form

$$d_j(x_n,x_m) < \varepsilon,$$

for every $m,n \in \mathbb{N}$, $m,n \geq \max\{n_0,n_\varepsilon\}$, so $(x_n)_{n\in\mathbb{N}}$ turns out to be a Cauchy sequence of elements from the complete metric space $(X_j,d_j)$.

Consequently there exists $x^* \in X_j$ such that $\lim\limits_{n\to\infty} d_j(x_n,x^*) = 0$ which implies that $\lim\limits_{n\to\infty} d((x_n,k_n),(x^*,j)) = 0$, i.e., $((x_n,k_n))_{n\in\mathbb{N}}$ is convergent.

c) Let us consider $j \in \{1,\ldots,p\}$ and $K_1,K_2 \in P_{cp}^{\mathcal{S}}(X)$ such that $h(K_1,K_2) < L$.

For $x \in \mathcal{K}_1^j$ arbitrarily chosen, the compactness of $K_2$ ensures the existence of $y_x \in K_2$ such that

$$d(x,y_x) = d(x,K_2) \leq \sup_{u\in\mathcal{K}_1^j} d(u,K_2) \overset{\text{Remark 2.9}}{\leq} \sup_{u\in K_1} d(u,K_2) \leq h(K_1,K_2). \tag{2}$$

Then,

$$y_x \in \mathcal{K}_2^j, \tag{3}$$

because otherwise we get the contradiction $L \leq d(x,y_x) \overset{(2)}{\leq} h(K_1,K_2) < L$.

Consequently, we get $d(x,\mathcal{K}_2^j) \overset{(3)}{\leq} d(x,y_x) \overset{(2)}{\leq} h(K_1,K_2)$ and hence

$$\sup_{x\in\mathcal{K}_1^j} d(x,\mathcal{K}_2^j) \leq h(K_1,K_2). \tag{4}$$

In a similar manner, we get

$$\sup_{x\in\mathcal{K}_2^j} d(x,\mathcal{K}_1^j) \leq h(K_1,K_2). \tag{5}$$

Therefore, via (4) and (5), we infer that

$$h(\mathcal{K}_1^j,\mathcal{K}_2^j) = \max\{\sup_{x\in\mathcal{K}_1^j} d(x,\mathcal{K}_2^j), \sup_{x\in\mathcal{K}_2^j} d(x,\mathcal{K}_1^j)\} \leq h(K_1,K_2).$$

d) Let us consider a Cauchy sequence $(K_n)_{n\in\mathbb{N}}$ of elements from $P_{cp}^{\mathcal{S}}(X)$.

Then, $(K_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of elements from the complete metric space $(P_{cp}(X),h)$, so there exists $K\in P_{cp}(X)$ such that

$$\lim_{n\to\infty} h(K_n,K) = 0. \tag{6}$$

As $K_n \in P_{cp}^{\mathcal{S}}(X)$ for every $n\in\mathbb{N}$, we have $\emptyset \neq \mathcal{K}_n^j$ for every $j\in\{1,\ldots,p\}$ and every $n\in\mathbb{N}$.

Using c), the sequence $(\mathcal{K}_n^j)_{n\in\mathbb{N}}$ is Cauchy for every $j\in\{1,\ldots,p\}$.

As $(\mathcal{X}^j,d)$ is complete (since it is isometric with the complete metric space $(X_j,d_j)$), we infer that $(P_{cp}(\mathcal{X}^j),h)$ (which is a subspace of $(P_{cp}(X),h)$) is complete, so there exists $C_j\in P_{cp}(\mathcal{X}^j)\subseteq P_{cp}(X)$ such that

$$\lim_{n\to\infty} h(\mathcal{K}_n^j,C_j) = 0, \tag{7}$$

for every $j\in\{1,\ldots,p\}$.

We have

$$h(K_n,\bigcup_{j=1}^{p} C_j) \overset{\text{Remark 2.9}}{=} h(\bigcup_{j=1}^{p}\mathcal{K}_n^j,\bigcup_{j=1}^{p} C_j) \leq \max_{j=1}^{p} h(\mathcal{K}_n^j,C_j),$$

for every $n\in\mathbb{N}$, so, via (7), we get

$$\lim_{n\to\infty} h(K_n,\bigcup_{j=1}^{p} C_j) = 0. \tag{8}$$

In view of (6) and (8), we get $\bigcup_{j=1}^{p} C_j = K$ and therefore

$$\mathcal{K}^j = C_j \neq \emptyset,$$

for every $j\in\{1,\ldots,p\}$, so $K\in P_{cp}^{\mathcal{S}}(X)$. In view of (6), this ends the proof.

e) Let $A,B\in P_{cp}^{\mathcal{S}}(X)$ fixed, but arbitrarily chosen.

Then, $\mathcal{A}^j\neq\emptyset$ and $\mathcal{B}^j\neq\emptyset$ for every $j\in\{1,\ldots,p\}$.

As $(\mathcal{X}^j,d)$ is $\varepsilon_j$-chainable (since it is isometric with the $\varepsilon_j$-chainable metric space $(X_j,d_j)$ ), taking into account Proposition 2.2, there exist $n\in\mathbb{N}$ and the finite subsets $C_1^j,\ldots,C_n^j$ of $\mathcal{X}^j$ such that for every $k\in\{1,\ldots,n-1\}$ and every $j\in\{1,\ldots,p\}$,

$$h(\mathcal{A}^j,C_1^j) < 2\varepsilon_j \leq L,\ h(C_n^j,\mathcal{B}^j) < 2\varepsilon_j \leq L \tag{9}$$

and

$$h(C_k^j,C_{k+1}^j) < 2\varepsilon_j \leq L. \tag{10}$$

One should note that we used the fact that, via a possible repetition of the elements $C_k^j$, the number $n$ (which basically depends on $\mathcal{A}^j$ and $\mathcal{B}^j$) can be chosen to be the same for all $j\in\{1,\ldots,p\}$.

Now, for every $k \in \{1, \ldots, n\}$, let us consider the sets

$$C_k = \overset{p}{\underset{j=1}{\cup}} C_k^j.$$

Since each $C_k$ is finite and $\emptyset \neq C_k^j = \mathcal{C}_k^j$ for every $j \in \{1, \ldots, p\}$, we note that $C_k \in P_{cp}^{\mathcal{S}}(X)$.

Moreover we have

$$h(A, C_1) \overset{\text{Remark 2.9}}{=} h\left(\overset{p}{\underset{j=1}{\cup}} \mathcal{A}^j, \overset{p}{\underset{j=1}{\cup}} C_1^j\right) \leq \overset{p}{\underset{j=1}{\max}}\, h(\mathcal{A}^j, C_1^j) \overset{(9)}{<} L,$$

$$h(C_n, B) \overset{\text{Remark 2.9}}{=} h\left(\overset{p}{\underset{j=1}{\cup}} C_n^j, \overset{p}{\underset{j=1}{\cup}} \mathcal{B}^j\right) \leq \overset{p}{\underset{j=1}{\max}}\, h(C_n^j, \mathcal{B}^j) \overset{(9)}{<} L$$

and for every $k \in \{1, \ldots, n-1\}$,

$$h(C_k, C_{k+1}) \overset{\text{Remark 2.9}}{=} h\left(\overset{p}{\underset{j=1}{\cup}} C_k^j, \overset{p}{\underset{j=1}{\cup}} C_{k+1}^j\right) \leq \overset{p}{\underset{j=1}{\max}}\, h(C_k^j, C_{k+1}^j) \overset{(10)}{<} L.$$

We conclude that $(P_{cp}^{\mathcal{S}}(X), h)$ is a $L$-chainable metric space. □

**Proposition 3.2.** *For every g-dGIFS* $\mathcal{S} = (((X_j, d_j))_{j \in \{1,\ldots,p\}}, (f_i)_{i \in \{1,\ldots,q\}})$ *we have*

$$h(F_{\mathcal{S}}(M_1, \ldots, M_m), F_{\mathcal{S}}(N_1, \ldots, N_m)) \leq C \max\{h(M_1, N_1), \ldots, h(M_m, N_m)\},$$

*for every* $M_1, \ldots, M_m, N_1, \ldots, N_m \in P_{cp}^{\mathcal{S}}(X)$ *such that*

$$\max\{h(M_1, N_1), \ldots, h(M_m, N_m)\} < L,$$

*where* $C = \overset{q}{\underset{i=1}{\max}}\, lip(f_i)$.

*Proof.* We have

$$\begin{aligned}
&h(F_{\mathcal{S}}(M_1, \ldots, M_m), F_{\mathcal{S}}(N_1, \ldots, N_m)) \\
&\quad = h\left(\overset{q}{\underset{i=1}{\cup}} \overline{f_i}(\mathcal{M}_1^{D(i)} \times \ldots \times \mathcal{M}_m^{D(i)}), \overset{q}{\underset{i=1}{\cup}} \overline{f_i}(\mathcal{N}_1^{D(i)} \times \ldots \times \mathcal{N}_m^{D(i)})\right. \\
&\quad \leq \overset{q}{\underset{i=1}{\max}}\, h(\overline{f_i}(\mathcal{M}_1^{D(i)} \times \ldots \times \mathcal{M}_m^{D(i)}), \overline{f_i}(\mathcal{N}_1^{D(i)} \times \ldots \times \mathcal{N}_m^{D(i)})) \overset{(*)}{\leq} \\
&\quad \leq \overset{q}{\underset{i=1}{\max}}\, lip(\overline{f_i}) \max\{h(\mathcal{M}_1^{D(i)}, \mathcal{N}_1^{D(i)}), \ldots, h(\mathcal{M}_m^{D(i)}, \mathcal{N}_m^{D(i)})\} \overset{\text{Remark 2.12.}}{\leq} \\
&\quad \leq C \overset{q}{\underset{i=1}{\max}} \max\{h(\mathcal{M}_1^{D(i)}, \mathcal{N}_1^{D(i)}), \ldots, h(\mathcal{M}_m^{D(i)}, \mathcal{N}_m^{D(i)})\} \\
&\quad = C \max\{\overset{q}{\underset{i=1}{\max}}\, h(\mathcal{M}_1^{D(i)}, \mathcal{N}_1^{D(i)}), \ldots, \overset{q}{\underset{i=1}{\max}}\, h(\mathcal{M}_m^{D(i)}, \mathcal{N}_m^{D(i)})\} \overset{\text{Lemma 3.1, c)}}{\leq} \\
&\quad \leq C \max\{h(M_1, N_1), \ldots, h(M_m, N_m)\},
\end{aligned}$$

for every $M_1, \ldots, M_m, N_1, \ldots, N_m \in P_{cp}^{\mathcal{S}}(X)$ such that

$$\max\{h(M_1, N_1), \ldots, h(M_m, N_m)\} < L.$$

Note that the details concerning the validity of $(*)$ are given in the proof of Theorem 3.5 from [15]. □

**Theorem 3.3.** *The fractal operator* $F_{\mathcal{S}}$ *associated with every g-dGIFS* $\mathcal{S} = ((X_j, d_j)_{j\in\{1,\ldots,p\}}, (f_i)_{i\in\{1,\ldots,q\}})$ *has a unique fixed point. Moreover,*

$$\lim_{n\to\infty} G_{\mathcal{S}}^{[n]}(K) = A_{\mathcal{S}},$$

*for every* $K \in P_{cp}^{\mathcal{S}}(X)$*, where the function* $G_{\mathcal{S}} : P_{cp}^{\mathcal{S}}(X) \to P_{cp}^{\mathcal{S}}(X)$ *is given by*

$$G_{\mathcal{S}}(K) = F_{\mathcal{S}}(K, \ldots, K),$$

*for every* $K \in P_{cp}^{\mathcal{S}}(X)$.

*Proof.* By Proposition 3.2, the operator $G_{\mathcal{S}}$ is $(L, C)$-uniformly locally contractive for $C = \max_{i=1}^{q} lip(f_i)$.

Therefore, taking into account Lemma 3.1, d) and e) and Edelstein's contraction principle, we infer that $G_{\mathcal{S}}$ has a unique fixed point, so, via Remark 2.6, $F_{\mathcal{S}}$ has a unique fixed point and

$$\lim_{n\to\infty} G_{\mathcal{S}}^{[n]}(K) = A_{\mathcal{S}},$$

for every $K \in P_{cp}^{\mathcal{S}}(X)$. □

**Definition 3.4.** The unique fixed point of $F_{\mathcal{S}}$ is called the attractor of the g-dGIFS $\mathcal{S}$ and we denote it by $A_{\mathcal{S}}$.

*Remark 3.5.* In the framework of Theorem 3.3, if $p = 1$ , then $C$ is onto and, in this case, we obtain (a particular case of) the result from [34].

**Proposition 3.6.** *Let* $\mathcal{S} =((X_j, d_j)_{j\in\{1,\ldots,p\}}, (f_i)_{i\in\{1,\ldots,q\}})$ *be a g-dGIFS and* $K_1, \ldots, K_m \in P_{cp}^{\mathcal{S}}(X)$*. Then*

$$\lim_{n\to\infty} K_n = A_{\mathcal{S}},$$

*where the sequence* $(K_n)_{n\in\mathbb{N}}$ *is defined inductively by the relation*

$$K_{n+m} = F_{\mathcal{S}}(K_{n+m-1}, \ldots, K_n),$$

*for every* $n \in \mathbb{N}$.

*Proof.* We start with the following:

**Claim**. Let $M_1, \ldots, M_m, N_1, \ldots, N_m \in P_{cp}^{\mathcal{S}}(X)$ such that

$$h(M_k, N_k) < L,$$

for every $k \in \{1, \ldots, m\}$.

Then,

$$\lim_{n\to\infty} h(M_n, N_n) = 0,$$

where the sequences $(M_n)_{n\in\mathbb{N}}$ and $(N_n)_{n\in\mathbb{N}}$ are defined inductively by the relation from the assertion.

*Justification of the claim.* We are going to use the following notation:

–

$$h(M_n, N_n) \overset{not}{=} d_n$$

–

$$\max\{d_n,\ldots,d_{n+m-1}\} \overset{not}{=} e_n$$

–

$$\max_{i=1}^{q} lip(f_i) \overset{not}{=} C < 1\ .$$

Since

$$\begin{aligned} d_{n+m} &= h(M_{n+m},N_{n+m}) \\ &= h(F_{\mathcal{S}}(M_{n+m-1},\ldots,M_n),F_{\mathcal{S}}(N_{n+m-1},\ldots,N_n)) \overset{\text{Proposition 3.2}}{\leq} \\ &\leq C\max\{h(M_{n+m-1},N_{n+m-1}),\ldots,h(M_n,N_n)\} = C\max\{d_{n+m-1},\ldots,d_n\}, \end{aligned}$$

provided that $d_{n+m-1},\ldots,d_n < L$, we obtain, by mathematical induction, that

$$d_n < L,$$

for all $n \in \mathbb{N}$.

Hence, for every $n \in \mathbb{N}$,

$$d_{n+m} \leq Ce_n < e_n, \tag{1}$$

As

$$d_{n+k} \leq e_n, \tag{2}$$

for every $n \in \mathbb{N}$ and $k \in \{1,\ldots,m-1\}$, via (1) and (2), we get

$$e_{n+1} = \max\{d_{n+1},\ldots,d_{n+m}\} \leq e_n, \tag{3}$$

for every $n \in \mathbb{N}$. Consequently, for every $n \in \mathbb{N}$,

$$e_{n+m} = \max\{d_{n+m},\ldots,d_{n+2m-1}\} \overset{(1)}{\leq} C\max\{e_n,\ldots,e_{n+m-1}\} \overset{(3)}{=} Ce_n. \tag{4}$$

Taking into account (4), we conclude that

$$\lim_{n\to\infty} e_n = 0. \tag{5}$$

As $0 \leq d_n \leq e_n$ for every $n \in \mathbb{N}$, relation (5) ensures $\lim_{n\to\infty} d_n = 0$, so the justification of the Claim is complete.

Since $(P_{cp}^{\mathcal{S}}(X),h)$ is $L$-chainable (see Lemma 3.1, e)), there exists $u \in \mathbb{N}$ and $C_v^l \in P_{cp}^{\mathcal{S}}(X)$, $l \in \{0,\ldots,u\}$, $v \in \{1,\ldots,m\}$, such that

$$C_v^0 = K_v,\ C_v^p = A_{\mathcal{S}} \text{ and } h(C_v^l,C_v^{l+1}) < L, \tag{6}$$

for every $l \in \{0,\ldots,u-1\}$ and $v \in \{1,\ldots,m\}$.

For every $l \in \{0,\ldots,u\}$, we consider the sequence $(K_n^l)_{n\in\mathbb{N}}$ defined inductively by the relation

$$K_{n+m}^l = F_{\mathcal{S}}(K_{n+m-1}^l,\ldots,K_n^l),$$

for every $n \in \mathbb{N}$ and $K_1^l = C_1^l,\ldots,K_m^l = C_m^l$.

Note that, in view of (6) and the Claim, we have

$$\lim_{n\to\infty} h(K_n^l,K_n^{l+1}) = 0, \tag{7}$$

for every $l \in \{0,\ldots,u-1\}$.

Taking into account that $(K_n^0)_{n\in\mathbb{N}} = (K_n)_{n\in\mathbb{N}}$ and $(K_n^u)_{n\in\mathbb{N}} = (A_{\mathcal{S}})_{n\in\mathbb{N}}$, we get

$$0 \leq h(K_n, A_{\mathcal{S}}) = h(K_n^0, K_n^u) \leq \sum_{l=0}^{u-1} h(K_n^l, K_n^{l+1}), \tag{8}$$

for every $n \in \mathbb{N}$.

From (7) and (8), we conclude that

$$\lim_{n\to\infty} h(K_n, A_{\mathcal{S}}) = 0.$$

□

## 4. Examples

**A**. Let us consider the g-dGIFS $\mathcal{S} = (((X_j, d_j))_{j\in\{1,2,3\}}, (f_i)_{i\in\{1,\ldots,5\}})$, where:

- $(X_j, d_j) = (\mathbb{C}, |.|)$ for each $j \in \{1, 2, 3\}$
- $D, C : \{1, 2, 3, 4, 5\} \to \{1, 2, 3\}$ are given by

$$D(1) = D(2) = D(3) = 1,\ D(4) = 2,\ D(5) = 3$$

and

$$C(1) = 1,\ C(2) = 2,\ C(3) = 3,\ C(4) = 1,\ C(5) = 1$$

- $f_1 : X_1^2 \to X_1$, $f_2 : X_1^2 \to X_2$, $f_3 : X_1^2 \to X_3$, $f_4 : X_2^2 \to X_1$ and $f_5 : X_3^2 \to X_1$ are given by

$$\begin{aligned} f_1(z_1, z_2) &= 0,2x_1 + 0,2y_2 + i(0,2x_2 + 0,1y_2),\\ f_2(z_1, z_2) &= \frac{3}{2}[0,15x_1 + 0,07x_2 + 0,4 + i(0,15y_1 + 0,07y_2)],\\ f_3(z_1, z_2) &= \frac{3}{2}[0,15y_1 + 0,07x_2 + i(0,15x_1 + 0,07y_2 + 0,4)],\\ f_4(z_1, z_2) &= \frac{2}{3}z_1 \end{aligned}$$

and

$$f_5(z_1, z_2) = \frac{2}{3}z_2,$$

for every $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}$.

Using an algorithm similar to the one described in [29], we get the graphic representations of $\mathcal{A}^1$, $\mathcal{A}^2$ and $\mathcal{A}^3$ indicated in Figs. 1, 2 and respectively 3, where by $A$ we designate the attractor of the g-dGIFS $\mathcal{S}$.

**B**. Let us consider the g-dGIFS $\mathcal{S} = (((X_j, d_j))_{j\in\{1,2\}}, (f_i)_{i\in\{1,2,3\}})$, where:

- $(X_1, d_1) = (\mathbb{R}, |.|)$ and $(X_2, d_2) = (\mathbb{R}^2, \|.\|_1)$
- $D, C : \{1, 2, 3\} \to \{1, 2\}$ are given by

$$D(1) = D(2) = 1,\ D(3) = 2,\ C(1) = C(3) = 1,\ C(2) = 2$$

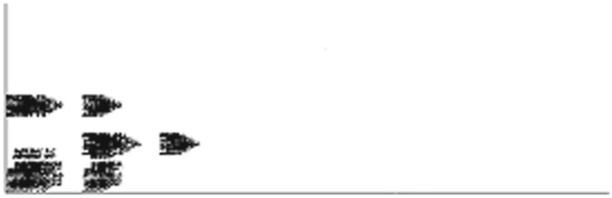

FIGURE 1. The graphic representation of $\mathcal{A}^1$

FIGURE 2. The graphic representation of $\mathcal{A}^2$

FIGURE 3. The graphic representation of $\mathcal{A}^3$

– $f_1 : X_1^2 \to X_1$, $f_2 : X_1^2 \to X_2$ and $f_3 : X_2^2 \to X_1$ are given by

$$f_1(x, y) = \frac{5}{18}x + \frac{5}{18}y + \frac{4}{9},$$

$$f_2(x, y) = \left(\frac{x + y}{3}, 0\right),$$

and

$$f_3((x_1,x_2),(y_1,y_2)) = \frac{x_1+y_1}{3} + \frac{x_2+y_2}{10},$$

for every $x, y, x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Then, one can easily check that

$$A_{\mathcal{S}} = ([0,1]\times\{1\}) \cup \left(\left[0,\frac{2}{3}\right]\times\{0\}\times\{2\}\right).$$

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