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Nodal solutions for Kirchhoff equations with Choquard nonlinearity

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Abstract. In this paper, we consider the following Kirchhoff equation with Choquard nonlinearity:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=\left(\int_{\mathbb{R}^3}\frac{|u(y)|^p}{|x-y|^{3-\alpha}}\,\mathrm{d}y\right)|u|^{p-2}u+|u|^{q-2}u,\quad \text{in } \mathbb{R}^3, \tag{0.1}$$

where $V(x)$ is a smooth function, a and b are positive constants, $\alpha \in (1,3)$, $q \in (4,6)$. By employing the results from the matrix theory, gluing approach and Brouwer degree theory, we prove that for any integer k, the above equation with $p \in (4, 3+\alpha)$ has a sign changing radial solution, which changes sign k times.

Mathematics Subject Classification. 35J15, 35J20.

Keywords. Kirchhoff type equations, Choquard nonlinearity, nodal solutions, gluing method, Brouwer degree theory.

1. Introduction and statement of main result

Consider the Kirchhoff type equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=f(x,u),\quad u\in H^1(\mathbb{R}^3), \tag{1.1}$$

where $a, b > 0$, $V(x)$ is a smooth function. This problem is related to the stationary analogue of the equation

$$\frac{\partial^2 u}{\partial t^2}-\left(a+b\int_0^L\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2}=0,$$

which was proposed by Kirchhoff [12] as an extension of the classical D'Alembert's wave equations for free vibration of elastic string. After Lions [15] introduced an abstract function analysis framework to the problem, many researchers have paid attention to it, see [1,4] and the references therein.

There are many interesting results on the existence and multiplicity of solutions for problem (1.1), such as [6,8–10,13,19,21,23] and references therein. In particular, He and Zou [10] studied the existence of positive solutions for (1.1) by the variational method, and obtained the multiplicity by means of Category theory. Moreover, they also studied the concentrated behavior of positive solutions. When $f(x,u) = |u|^{p-1}u$, Li and Ye [13] proved (1.1) has a positive ground state solution by using a monotonicity trick and global compactness lemma. This result can be seen as a partial extension of the results of He and Zou in [10]. Wu [19] considered the existence of nontrivial solutions and infinitely many high energy solutions for (1.1) using a symmetric Mountain Pass Theorem. Guo [6] considered the positive ground state solution of (1.1) without classical Ambrosetti–Rabinowitz condition by using variational methods. The existence of sign-changing solution of (1.1) has also been studied in [16,22].

Moreover, there are many results about the existence of nodal solutions for elliptic problems. Bartsch and Willem [2], Cao and Zhu [3] proved that for any positive integer k, there exists a pair of solutions ${u_k}^{\pm}$ having exact k nodes for the following equation:

$$-\Delta u + V(|x|)u = f(|x|,u), \quad u \in H^1(\mathbb{R}^3).$$

By gluing method, Deng, Peng and Shuai [5] considered the existence and asymptotic behavior of nodal solutions for the following Kirchhoff equation:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(|x|)u = f(|x|,u), x \in \mathbb{R}^3. \tag{1.2}$$

Due to the existence of nonlocal items, gluing method cannot be used to solve this problem directly. To solve this difficulty, they regard this problem as a system of $k+1$ equations with $k+1$ unknown functions u_i, each u_i is supported on only one annulus and vanishing on the complement. Huang, Yang and Yu [11] showed the existence of nodal solutions of Choquard equation by the same method as in [5]. Wang and Guo [20] proved the existence and nonexistence of nodal solutions for Choquard type equations with perturbation by employing the variational method, gluing approach and the Brouwer degree. Recently, Guo and Wu [7] showed the existence of nodal solutions for the Schrödinger–Poisson equations with convolution terms.

Motivated by the above results, we intend to establish infinitely many nodal solutions to the following equation:

$$\begin{aligned} &-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u \\ &\quad = \left(\int_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|^{3-\alpha}}\mathrm{d}y\right)|u|^{p-2}u + |u|^{q-2}u, \quad u \in H^1(\mathbb{R}^3), \end{aligned} \tag{1.3}$$

where $V(x)$ is a smooth function, a, b are positive constants, $\alpha \in (1,3)$ and $q \in (4,6)$.

Our main result can be stated as follows.

Theorem 1.1. *Suppose that V(x) satisfies*

(V) $V(x) = V(|x|) \in C([0,\infty),\mathbb{R})$ *is bounded from below by a positive constant* V_0.

Moreover, if $\alpha \in (1,3)$*,* $p \in (4, 3+\alpha)$ *and* $q \in (4,6)$*. Then for any positive integer* k*,* Eq. (1.3) *has a radial solution* u_k *changing sign exactly* k *times.*

The paper is organized as follows. In Sect. 2, we give some notations and preliminary results. In Sect. 3, we are devoted to the proof of our main result which mainly show the construction of least energy nodal solutions changing sign exactly k times.

2. Variational framework and some results in the matrix theory

In this section, we give some notations and preliminary results. First, we present the variational framework. The space $H_V(\mathbb{R}^3)$ is defined by

$$H_V(\mathbb{R}^3) := \left\{ u \in H^1_r(\mathbb{R}^3) \ : \int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(|x|)u^2\right) \mathrm{d}x < \infty \right\},$$

with the norm

$$\|u\|_V = \left(\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + V(|x|)u^2\right) \mathrm{d}x \right)^{1/2}.$$

The energy functional associated with problem (1.3) is given by

$$I_b(u) = \frac{1}{2}\|u\|_V^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\Big)^2 - \frac{1}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q \mathrm{d}x.$$

By Hardy–Littlewood–Sobolev inequality ([14, Theorem 4.3]), I_b is well defined on $H^1(\mathbb{R}^3)$ when $p \in (4, 3+\alpha)$.

For any integer k, we define

$$\Gamma_k = \left\{ \boldsymbol{\rho}_k = (\rho_1, \dots, \rho_k) \in \mathbb{R}^k | \quad 0 := \rho_0 < \rho_1 < \cdots < \rho_{k+1} := \infty \right\},$$

and for each $\boldsymbol{\rho}_k \in \Gamma_k$, set

$$\begin{aligned} B_1^{\boldsymbol{\rho}_k} &= \left\{ x \in \mathbb{R}^3 : 0 \le |x| < \rho_1 \right\}, \\ B_i^{\boldsymbol{\rho}_k} &= \left\{ x \in \mathbb{R}^3 : \rho_{i-1} < |x| < \rho_i \right\}, \quad \text{for } i = 1, 2, \dots, k; \\ B_{k+1}^{\boldsymbol{\rho}_k} &= \left\{ x \in \mathbb{R}^3 : |x| \ge \rho_k \right\}. \end{aligned}$$

Fix $\boldsymbol{\rho}_k=(\rho_1,\dots,\rho_k) \in \Gamma_k$ and thereby a family of $\left\{B_i^{\boldsymbol{\rho}_k}\right\}_{i=1}^{k+1}$, we denote

$$H_i^{\boldsymbol{\rho}_k} = \left\{ u \in H_0^1(B_i^{\boldsymbol{\rho}_k}) |\ u(x) = u(|x|), u(x) = 0 \quad if\ x \notin B_i^{\boldsymbol{\rho}_k} \right\},$$

for $i = 1, \dots, k+1$. Therefore, $H_i^{\boldsymbol{\rho}_k}$ is a Hilbert space with the norm

$$\|u\|_i = \left(\int_{B_i^{\boldsymbol{\rho}_k}} (a|\nabla u|^2 + V(|x|)u^2)\mathrm{d}x \right)^{1/2}.$$

Let us set

$$(u_i,u_j)_* = \int_{B_i^{\rho_k}} |\nabla u_i|^2 \mathrm{d}x \int_{B_j^{\rho_k}} |\nabla u_j|^2 \mathrm{d}x, \tag{2.1}$$

and

$$(u_i,u_j)_\alpha = \int_{B_i^{\rho_k}} \int_{B_j^{\rho_k}} \frac{|u_i(x)|^p |u_j(y)|^p}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y. \tag{2.2}$$

We now define the function J_b: $\mathcal{H}_k^{\rho_k} \to \mathbb{R}$ by

$$\begin{aligned}
J_b(u_1,\ldots,u_{k+1}) &:= I_b\left(\sum_{i=1}^{k+1} u_i\right) \\
&= \sum_{i=1}^{k+1} \left(\frac{1}{2}\|u_i\|_i^2 + \frac{b}{4}\left(\int_{B_i^{\rho_k}} |\nabla u_i|^2 \mathrm{d}x\right)^2 \right. \\
&\quad \left. - \frac{1}{2p}\int_{B_i^{\rho_k}} \int_{B_i^{\rho_k}} \frac{|u_i(x)|^p |u_i(y)|^p}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y - \frac{1}{q}\int_{B_i^{\rho_k}} |u_i|^q \mathrm{d}x \right) \\
&\quad + \sum_{\substack{i,j=1 \\ j\neq i}}^{k+1} \left(\frac{b}{4}\int_{B_i^{\rho_k}} |\nabla u_i|^2 \mathrm{d}x \int_{B_j^{\rho_k}} |\nabla u_j|^2 \mathrm{d}x - \right. \\
&\qquad \left. \frac{1}{2p}\int_{B_i^{\rho_k}} \int_{B_j^{\rho_k}} \frac{|u_i(x)|^p |u_j(y)|^p}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y \right) \\
&= \sum_{i=1}^{k+1} \left(\frac{1}{2}\|u_i\|_i^2 + \frac{b}{4}(u_i,u_i)_* - \frac{1}{2p}(u_i,u_i)_\alpha - \frac{1}{q}\int_{B_i^{\rho_k}} |u_i|^q \mathrm{d}x \right) \\
&\quad + \sum_{\substack{i,j=1 \\ j\neq i}}^{k+1} \left(\frac{b}{4}(u_i,u_j)_* - \frac{1}{2p}(u_i,u_j)_\alpha \right),
\end{aligned}$$

where $\mathcal{H}_k^{\rho_k} = H_1^{\rho_k} \times \cdots \times H_{k+1}^{\rho_k}$ and $u_i \in H_i^{\rho_k}$ for $i = 1,\ldots,k+1$. Obviously, for each $i = 1,\ldots,k+1$

$$\begin{aligned}
\partial_{u_i} J_b(u_1,\ldots,u_{k+1})u_i &= \|u_i\|_i^2 + b(u_i,u_i)_* - (u_i,u_i)_\alpha \\
&\quad - \int_{B_i^{\rho_k}} |u_i|^q \mathrm{d}x + \sum_{\substack{j=1 \\ j\neq i}}^{k+1} \left(b(u_i,u_j)_* - (u_i,u_j)_\alpha\right).
\end{aligned}$$

If $(u_1,\ldots,u_{k+1}) \in \mathcal{H}_k^{\rho_k}$ is a critical point of J_b, then every component u_i satisfies

$$\begin{cases} -\left(a + b\sum_{j=1}^{k+1} \int_{B_j^{\rho_k}} |\nabla u_j|^2 \mathrm{d}x\right)\Delta u_i + V(|x|)u_i = \\ \qquad \left(\int_{\mathbb{R}^3} \frac{|\sum_{j=1}^{k+1} u_j(y)|^p}{|x-y|^{3-\alpha}} \mathrm{d}y\right)|u_i|^{p-2}u_i + |u_i|^{q-2}u_i & x \in B_i^{\rho_k}, \\ u_i = 0 & x \notin B_i^{\rho_k}. \end{cases} \tag{2.3}$$

We regard $u_i \in H_i^{\boldsymbol{\rho}_k}$ as an element in $H^1(\mathbb{R}^3)$ by setting $u \equiv 0$ in $\mathbb{R}^3 \setminus B_i^{\boldsymbol{\rho}_k}$. To find least energy radial solution which changes signs exactly k times of (1.3), let *Nehari* manifold

$$\mathcal{N}_k = \big\{u \in H_V : \text{there exists } \boldsymbol{\rho}_k \ \text{ such that } (-1)^{i+1}u_i > 0 \ in \ B_i^{\boldsymbol{\rho}_k}, \\ u_i = 0 \ on \ \partial B_i^{\boldsymbol{\rho}_k}, \ \text{ and } I_b'(u)u_i = 0, \quad \forall\, 1 \le i \le k+1\big\},$$

and $c_k = \inf\limits_{\mathcal{N}_k} J_b$. Obviously, $u = \sum_{i=1}^{k+1} u_i$ and $\mathcal{N}_k$ consists of nodal functions with precisely k nodes.

Least energy radial solution of (1.3) which changes signs exactly k times will constructed by gluing the solutions of the system (2.3). To this end, we look for critical points of J_b with nonzero component by considering the following *Nehari* type set

$$\mathcal{M}_k^{\boldsymbol{\rho}_k} = \big\{(u_1, \ldots, u_{k+1}) \in \mathcal{H}_k^{\boldsymbol{\rho}_k} : u_i \neq 0, \ \langle \partial_{u_i} J_b(u_1, u_2, \ldots, u_{k+1}), u_i\rangle \\ = 0, i = 1, \ldots, k+1\big\}.$$

Next, we will present some results of the matrix theory in order to prove that $\mathcal{M}_k^{\boldsymbol{\rho}_k}$ is nonempty.

Lemma 2.1. [11] *For any* $(u_1, \ldots, u_{k+1}) \in \mathcal{H}_k^{\boldsymbol{\rho}_k}$ *with* $u_i \neq 0$,$i = 1, \ldots, k+1$, *define the matrix* $A := (a_{ij})_{(k+1)\times(k+1)}$ *by* $a_{ij} = (u_i, u_j)_\alpha$. *Then the matrix* A *is positive definite.*

Lemma 2.2. (Gersgorin Disc Theorem [18, Theorem 1.1]) *For any matrix* $B = (b_{ij}) \in \mathbb{C}^{n\times n}$ *and any eigenvalue* $\lambda \in \sigma(B) := \{\mu \in \mathbb{C} : det(\mu E - B) = 0\}$, *there is a positive integer* $m \in \{1, \ldots, n\}$ *such that*

$$|\lambda - b_{m,m}| \le \sum_{\substack{j=1\\ j\neq m}}^{k+1} |b_{m,j}|.$$

By this lemma, we have the following result.

Lemma 2.3. [7] *For any* $b_{ij} = b_{ji} > 0$ *with* $i \neq j \in \{1, \ldots, n\}$ *and* $s_i > 0$ *with* $i = 1, \ldots, n$, *define the matrix* $C := (c_{ij})_{n\times n}$ *by*

$$c_{ij} = \begin{cases} -\sum\limits_{\substack{m=1\\ m\neq i}}^{k+1} s_m b_{im}/s_i & \text{for} \quad i = j, \\ b_{ij} > 0 & \text{for} \quad i \neq j. \end{cases}$$

Then the real symmetric matrix $(c_{ij})_{n\times n}$ *is non-positive definite.*

3. The proof of Theorem 1.1

In this section, we are devoted to the proof of Theorem 1.1. First, we give the following lemma.

Lemma 3.1. *Assume that* $\boldsymbol{\rho}_k \in \Gamma_k$ *is fixed. Then for each* $(v_1, \ldots, v_{k+1}) \in \mathcal{H}_k^{\boldsymbol{\rho}_k}$ *with* $v_i \neq 0$ *for* $i = 1, \ldots, k+1$, *there exists a unique* $(k+1)$ *tuple* $(t_1, \ldots, t_{k+1}) \in (\mathbb{R} > 0)^{k+1}$ *such that* $(t_1 v_1, \ldots, t_{k+1} v_{k+1}) \in \mathcal{M}_k^{\boldsymbol{\rho}_k}$.

Proof. When $q \geq p$, for each $(v_1, \ldots, v_{k+1}) \in \mathcal{H}_k^{\rho_k}$ with $v_i \neq 0$ for $i = 1, \ldots, k+1$, we define $G : (\mathbb{R} \geq 0)^{k+1} \to \mathbb{R}$ by

$$G(c_1, \ldots, c_{k+1}) = J_b(c_1^{\frac{1}{p}} v_1, \ldots, c_{k+1}^{\frac{1}{p}} v_{k+1}).$$

Then

$$\begin{aligned} G(c_1, \ldots, c_{k+1}) =& \sum_{i=1}^{k+1} \left[\frac{1}{2} c_i^{\frac{2}{p}} \|v_i\|_i^2 + \frac{b}{4} c_i^{\frac{4}{p}} (v_i, v_i)_* - \frac{1}{2p} c_i^2 (v_i, v_i)_\alpha - \frac{1}{q} c_i^{\frac{q}{p}} \int_{B_i^{\rho_k}} |v_i|^q \mathrm{d}x \right] \\ &+ \sum_{\substack{i,j=1 \\ j \neq i}}^{k+1} \left(\frac{b}{4} c_i^{\frac{2}{p}} c_j^{\frac{2}{p}} (v_i, v_j)_* - \frac{1}{2p} c_i c_j (v_i, v_j)_\alpha \right). \end{aligned}$$

It is clearly that G is continuous and $G(c_1, \ldots, c_{k+1}) \to 0$ as $|(c_1, \ldots, c_{k+1})| \to 0$ and $G(c_1, \ldots, c_{k+1}) \to -\infty$ as $|(c_1, \ldots, c_{k+1})| \to \infty$, due to $p \in (4, \alpha + 3)$ and $q \in (4, 6)$. Thus, G possesses a global maximum point $(\bar{c}_1, \ldots, \bar{c}_{k+1}) \in (\mathbb{R} \geq 0)^{k+1}$.

We claim that $\bar{c}_i > 0$ for all $i = 1, \ldots, k+1$. Otherwise, there exists $i_0 \in \{1, \ldots, k+1\}$ such that $\bar{c}_i = 0$. Without loss of generality, we assume $\bar{c}_1 = 0$. Then since

$$\begin{aligned} &G(\tau, \bar{c}_2, \ldots, \bar{c}_{k+1}) \\ &\quad = \frac{\tau^{\frac{2}{p}}}{2} \|v_1\|_1^2 + \frac{b}{4} \tau^{\frac{4}{p}} (v_1, v_1)_* - \frac{\tau^2}{2p} (v_1, v_1)_\alpha - \frac{\tau^{\frac{q}{p}}}{q} \int_{B_1^{\rho_k}} |v_1|^q \mathrm{d}x \\ &\qquad + \sum_{j=2}^{k+1} \left[\frac{b}{2} \tau^{\frac{2}{p}} \bar{c}_j^{\frac{2}{p}} (v_1, v_j)_* - \frac{1}{p} \tau \bar{c}_j (v_1, v_j)_\alpha \right] \\ &\qquad + \sum_{i=2}^{k+1} \left[\frac{1}{2} \bar{c}_i^{\frac{2}{p}} \|v_i\|_i^2 + \frac{b}{4} \bar{c}_i^{\frac{4}{p}} (v_i, v_i)_* - \frac{\bar{c}_i^2}{2p} (v_i, v_i)_\alpha - \frac{\bar{c}_i^{\frac{q}{p}}}{q} \int_{B_i^{\rho_k}} |v_i|^q \mathrm{d}x \right] \\ &\qquad + \sum_{\substack{i,j=2 \\ j \neq i}}^{k+1} \left[\frac{b}{4} \bar{c}_i^{\frac{2}{p}} \bar{c}_j^{\frac{2}{p}} (v_i, v_j)_* - \frac{1}{2p} \bar{c}_i \bar{c}_j (v_i, v_j)_\alpha \right] \end{aligned}$$

is increasing with respect to $\tau > 0$ when τ is small enough. Thus, $(0, \bar{c}_2, \ldots, \bar{c}_{k+1})$ is not a maximum point of G. This contradicts the assumption above. Therefore, the claim follows.

Next, we prove that this global maximum point is unique in $(\mathbb{R} > 0)^{k+1}$. In fact, by direct computation, we have

$$\begin{aligned} \frac{\partial G}{\partial c_i} =& \frac{1}{p} c_i^{\frac{2}{p}-1} \|v_i\|_i^2 + \frac{b}{p} c_i^{\frac{4}{p}-1} (v_i, v_i)_* - \frac{1}{p} c_i (v_i, v_i)_\alpha - \frac{1}{p} c_i^{\frac{q}{p}-1} \int_{B_i^{\rho_k}} |v_i|^q \mathrm{d}x \\ &+ \frac{b}{p} \sum_{\substack{j=1 \\ j \neq i}}^{k+1} c_i^{\frac{2}{p}-1} c_j^{\frac{2}{p}} (v_i, v_j)_* - \frac{1}{p} \sum_{\substack{j=1 \\ j \neq i}}^{k+1} c_j (v_i, v_j)_\alpha, \\ \frac{\partial^2 G}{\partial c_i^2} =& \frac{2-p}{p^2} c_i^{\frac{2}{p}-2} \|v_i\|_i^2 + \frac{b(4-p)}{p^2} c_i^{\frac{4}{p}-2} (v_i, v_i)_* - \frac{1}{p} (v_i, v_i)_\alpha \end{aligned}$$

$$- \frac{q-p}{p^2} c_i^{\frac{q}{p}-2} \int_{B_i^{\rho_k}} |v_i|^q \mathrm{d}x + \sum_{\substack{j=1\\ j\neq i}}^{k+1} \frac{b(2-p)}{p^2} c_i^{\frac{2}{p}-2} c_j^{\frac{2}{p}} (v_i, v_j)_*,$$

$$\frac{\partial^2 G}{\partial c_i \partial c_j} = \frac{2b}{p^2} c_i^{\frac{2}{p}-1} c_j^{\frac{2}{p}-1} (v_i, v_j)_* - \frac{1}{p}(v_i, v_j)_\alpha \quad for \quad i \neq j.$$

Let the matrix

$$\left(\frac{\partial^2 G}{\partial c_i \partial c_j}\right)_{(k+1)\times(k+1)} = \frac{1}{p^2}(a_{ij})_{(k+1)\times(k+1)} + \frac{2b}{p^2}(b_{ij})_{(k+1)\times(k+1)} + \frac{1}{p}(c_{ij})_{(k+1)\times(k+1)},$$

where

$$a_{ij} = \begin{cases} (2-p) c_i^{\frac{2}{p}-2} \|v_i\|_i^2 + (4-p) b c_i^{\frac{4}{p}-2} (v_i, v_i)_* \\ +(4-p) \sum_{\substack{m=1\\ m\neq i}}^{k+1} b c_i^{\frac{2}{p}-2} c_m^{\frac{2}{p}} (v_i, v_m)_* - \\ (q-p) c_i^{\frac{2}{p}-2} \int_{B_i^{\rho_k}} |v_i|^q \mathrm{d}x, & i = j, \\ 0 & i \neq j \end{cases}$$

$$b_{ij} = \begin{cases} -\sum_{\substack{m=1\\ m\neq i}}^{k+1} c_i^{\frac{2}{p}-2} c_m^{\frac{2}{p}} (v_i, v_m)_* & i = j, \\ c_i^{\frac{2}{p}-1} c_j^{\frac{2}{p}-1} (v_i, v_j)_* & i \neq j, \end{cases} \qquad c_{ij} = \begin{cases} -(v_i, v_i)_\alpha & i = j, \\ -(v_i, v_j)_\alpha & i \neq j. \end{cases}$$

Note the fact that $p > 4$ and $q \geq p$, thus (a_{ij}) is negative definite. By Lemma 2.3, (b_{ij}) is non-positive definite. By Lemma 2.1, (c_{ij}) is negative definite. Thus, $\left(\frac{\partial^2 G}{\partial_{c_i} \partial_{c_j}}\right)$ is negative definite and G is strictly concave in $(\mathbb{R} > 0)^{k+1}$. Therefore, G has a unique maximum point in $(\mathbb{R} > 0)^{k+1}$.

When $q < p$, for each $(v_1, \ldots, v_{k+1}) \in \mathcal{H}_k^{\rho_k}$ with $v_i \neq 0$ for $i = 1, \ldots, k+1$, we define $G : (\mathbb{R} \geq 0)^{k+1} \to \mathbb{R}$ by

$$G(c_1, \ldots, c_{k+1}) = J_b(c_1^{\frac{1}{q}} v_1, \ldots, c_{k+1}^{\frac{1}{q}} v_{k+1}).$$

Then

$$\begin{aligned} G(c_1, \ldots, c_{k+1}) &= \sum_{i=1}^{k+1} \left[\frac{1}{2} c_i^{\frac{2}{q}} \|v_i\|_i^2 + \frac{b}{4} c_i^{\frac{4}{q}} (v_i, v_i)_* - \frac{1}{2p} c_i^{\frac{2p}{q}} (v_i, v_i)_\alpha - \frac{1}{q} c_i \int_{B_i^{\rho_k}} |v_i|^q \mathrm{d}x\right] \\ &+ \sum_{\substack{i,j=1\\ j\neq i}}^{k+1} \left(\frac{b}{4} c_i^{\frac{2}{q}} c_j^{\frac{2}{q}} (v_i, v_j)_* - \frac{1}{2p} c_i^{\frac{p}{q}} c_j^{\frac{p}{q}} (v_i, v_j)_\alpha\right). \end{aligned}$$

By the same arguments as above, the conclusion follows. □

We define $\psi_b : (\mathbb{R} > 0)^{k+1} \to \mathbb{R}$ by

$$\psi_b(t_1, \ldots, t_{k+1}) = J_b(t_1 v_1, \ldots, t_{k+1} v_{k+1}),$$

where $(v_1,\ldots,v_{k+1})\in\mathcal{H}_k^{\boldsymbol{\rho}_k}$. Then we have the following corollary.

Corollary 3.2. *Let $\boldsymbol{\rho}_k\in\Gamma_k$. Then for any $(v_1,\ldots,v_{k+1})\in\mathcal{H}_k^{\boldsymbol{\rho}_k}$ with $v_i\neq 0$ for $i=1,\ldots,k+1$, there exists a unique global maximal point $(\bar{t}_1,\ldots,\bar{t}_{k+1})\in\mathbb{R}_+^{k+1}$ of ψ_b such that*

$$\psi_b(\bar{t}_1,\ldots,\bar{t}_{k+1})=\sup_{\mathbb{R}_+^{k+1}}\psi_b(t_1,\ldots,t_{k+1}),$$

and $(\bar{t}_1v_1,\ldots,\bar{t}_{k+1}v_{k+1})\in\mathcal{M}_k^{\boldsymbol{\rho}_K}$.

Lemma 3.3. *For $p\in(4,3+\alpha)$, $q\in(4,6)$, $\mathcal{M}_k^{\boldsymbol{\rho}_k}$ is a differentiable manifold in $\mathcal{H}_k^{\boldsymbol{\rho}_k}$. Moreover, all critical points of $J_b|_{\mathcal{M}_k^{\boldsymbol{\rho}_k}}$ are critical points of J_b in $\mathcal{H}_k^{\boldsymbol{\rho}_k}$ with no zero component.*

Proof. Note that

$$\mathcal{M}_k^{\boldsymbol{\rho}_k}=\left\{(u_1,\ldots,u_{k+1})\in\mathcal{H}_k^{\boldsymbol{\rho}_k}: u_i\neq 0, F(u_1,\ldots,u_{k+1})=0, i=1,\ldots,k+1\right\},$$

where $F=(F_1,\ldots,F_{k+1}):\mathcal{H}_k^{\boldsymbol{\rho}_k}\to\mathbb{R}$ is given by

$$F_i=\partial_{u_i}J_b(u_1,\ldots,u_{k+1})u_i.$$

Then

$$F_i=\|u_i\|_i^2+b(u_i,u_i)_*-(u_i,u_i)_\alpha-\int_{B_i^{\rho_k}}|u_i|^q\mathrm{d}x+\sum_{\substack{j=1\\ j\neq i}}^{k+1}b(u_i,u_j)_*-\sum_{\substack{j=1\\ j\neq i}}^{k+1}(u_i,u_j)_\alpha.$$

When $q\geq p$, by direct computation, we have that at each point $(u_1,\ldots,u_{k+1})\in\mathcal{M}_k^{\boldsymbol{\rho}_k}$, there holds that

$$\begin{aligned}M_{ii}&:=\langle\partial_{u_i}F_i(u_1,\ldots,u_{k+1}),u_i\rangle\\&=2\|u_i\|_i^2+4b(u_i,u_i)_*-2p(u_i,u_i)_\alpha-q\int_{B_i^{\rho_k}}|u_i|^q\mathrm{d}x\\&\quad+2b\sum_{\substack{j=1\\ j\neq i}}^{k+1}(u_i,u_j)_*-p\sum_{\substack{j=1\\ j\neq i}}^{k+1}(u_i,u_j)_\alpha\\&=(2-p)\|u_i\|_i^2+(4b-bp)(u_i,u_i)_*-p(u_i,u_i)_\alpha\\&\quad-(q-p)\int_{B_i^{\rho_k}}|u_i|^q\mathrm{d}x+(2b-bp)\sum_{\substack{j=1\\ j\neq i}}^{k+1}(u_i,u_j)_*\end{aligned}$$

for $i=1,\ldots,k+1$, and

$$\begin{aligned}M_{ij}&:=\langle\partial_{u_i}F_j(u_1,\ldots,u_{k+1}),u_i\rangle\\&=2b(u_i,u_j)_*-p(u_i,u_j)_\alpha,\quad\text{for } i\neq j\quad\text{and}\quad i,j=1,\ldots,k+1.\end{aligned}$$

By the same arguments as Lemma 3.1, when $q\geq p$, the matrix

$$(M_{ij})_{(k+1)\times(k+1)}=(a_{ij})_{(k+1)\times(k+1)}+(b_{ij})_{(k+1)\times(k+1)}+2b(c_{ij})_{(k+1)\times(k+1)}$$

is negative definite and, therefore, $\det(M_{ij}) \neq 0$, where

$$a_{ij} = \begin{cases} (2-p)\|u_i\|^2 + (4-p)b(u_i,u_i)_* - (q-p)\int_{B_i^{\rho_k}} |u_i|^q \mathrm{d}x \\ +(4-p)b\sum\limits_{\substack{m=1\\ m\neq i}}^{k+1}(u_i,u_m)_*, & \text{if } i=j, \\ 0, & \text{if } i \neq j, \end{cases}$$

$$b_{ij} = \begin{cases} -p(u_i,u_i)_\alpha, & \text{if } i=j, \\ -p(u_i,u_j)_\alpha, & \text{if } i\neq j, \end{cases} \qquad c_{ij} = \begin{cases} -\sum\limits_{\substack{m=1\\ m\neq i}}^{k+1}(u_i,u_m)_*, & \text{if } i=j, \\ (u_i,u_j)_*, & \text{if } i\neq j. \end{cases}$$

When $q<p$, by direct computation, we have that at each point $(u_1,\ldots,u_{k+1}) \in \mathcal{M}_k^{\rho_k}$, there holds that

$$\begin{aligned} M_{ii} :=& \langle \partial_{u_i} F_i(u_1,\ldots,u_{k+1}), u_i\rangle \\ =& 2\|u_i\|_i^2 + 4b(u_i,u_i)_* - 2p(u_i,u_i)_\alpha - q\int_{B_i^{\rho_k}} |u_i|^q \mathrm{d}x \\ &+ 2b\sum_{\substack{j=1\\ j\neq i}}^{k+1}(u_i,u_j)_* - p\sum_{\substack{j=1\\ j\neq i}}^{k+1}(u_i,u_j)_\alpha \\ =& (2-q)\|u_i\|_i^2 + b(4-q)(u_i,u_i)_* - (2p-q)(u_i,u_i)_\alpha \\ &+ b(2-q)\sum_{\substack{j=1\\ j\neq i}}^{k+1}(u_i,u_j)_* - (p-q)\sum_{\substack{j=1\\ j\neq i}}^{k+1}(u_i,u_j)_\alpha, \end{aligned}$$

for $i=1,\ldots,k+1$, and

$$M_{ij} = 2b(u_i,u_j)_* - p(u_i,u_j)_\alpha, \quad \text{for } i\neq j \quad \text{and} \quad i,j=1,\ldots,k+1.$$

By the same arguments as above, when $q<p$, the matrix

$$(M_{ij})_{(k+1)\times(k+1)} = (a_{ij})_{(k+1)\times(k+1)} + (b_{ij})_{(k+1)\times(k+1)} + 2b(c_{ij})_{(k+1)\times(k+1)},$$

is also negative definite and, therefore, $\det(M_{ij}) \neq 0$, where

$$a_{ij} = \begin{cases} (2-q)\|u_i\|^2 + (4-q)b(u_i,u_i)_* - (p-q)(u_i,u_i)_\alpha \\ +b(4-q)\sum\limits_{\substack{j=1\\ m\neq i}}^{k+1}(u_i,u_m)_*, & \text{if } i=j, \\ 0, & \text{if } i \neq j, \end{cases}$$

$$b_{ij} = \begin{cases} -p(u_i,u_i)_\alpha, & \text{if } i=j, \\ -p(u_i,u_j)_\alpha, & \text{if } i\neq j, \end{cases} \qquad c_{ij} = \begin{cases} -\sum\limits_{\substack{m=1\\ m\neq i}}^{k+1}(u_i,u_m)_*, & \text{if } i=j, \\ (u_i,u_j)_*, & \text{if } i\neq j. \end{cases}$$

Thus, (M_{ij}) is nonsingular at each point $(u_1,\ldots,u_{k+1}) \in \mathcal{M}_k^{\rho_k}$. So $\mathcal{M}_k^{\rho_k}$ is differentiable in $\mathcal{H}_k^{\rho_k}$.

If $(u_1,\ldots,u_{k+1})$ is a critical point of $J_b|_{\mathcal{M}_k^{\boldsymbol{\rho}_k}}$, by the Lagrange multiplier principle, there exist $\eta_1,\ldots,\eta_{k+1}$ such that

$$J_b'(u_1,\ldots,u_{k+1})=\eta_1F_1'(u_1,\ldots,u_{k+1})+\cdots+\eta_{k+1}F_{k+1}'(u_1,\ldots,u_{k+1}).$$

Applying $(u_1,0,\ldots,0)$,$(0,u_2,\ldots,0)$,$(0,\ldots,0,u_{k+1})$ into the identity above, we get

$$(M_{ij})\begin{bmatrix}\eta_1\\ \vdots\\ \eta_{k+1}\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}.$$

Since $det\,(M_{ij})\neq 0$, we see that $\eta_i=0$ for all $i=1,\ldots,k+1$. Thus $(u_1,\ldots,u_{k+1})$ is a critical point of J_b. □

Consider the infimum level

$$d(\boldsymbol{\rho}_k):=\inf_{(u_1,\ldots,u_{k+1})\in\mathcal{M}_k^{\boldsymbol{\rho}_k}}J_b(u_1,\ldots,u_{k+1}).$$

Then we have the following result.

Lemma 3.4. *For any $p\in(4,3+\alpha)$, $q\in(4,6)$ and $\boldsymbol{\rho}_k\in\Gamma_k$, there is a minimizer $(\xi_1^{\boldsymbol{\rho}_k},\ldots,\xi_{k+1}^{\boldsymbol{\rho}_k})\in\mathcal{M}_k^{\boldsymbol{\rho}_k}$ of $J_b|_{\mathcal{M}_k^{\boldsymbol{\rho}_k}}$ with $(-1)^{i+1}\xi_i^{\boldsymbol{\rho}_k}>0$ in $B_i^{\boldsymbol{\rho}_k}$, for $i=1,\ldots,k+1$ such that*

$$J_b(\xi_1^{\boldsymbol{\rho}_k},\ldots,\xi_{k+1}^{\boldsymbol{\rho}_k})=d(\boldsymbol{\rho}_k).\tag{3.1}$$

Moreover, $(\xi_1^{\boldsymbol{\rho}_k},\ldots,\xi_{k+1}^{\boldsymbol{\rho}_k})$ satisfies (2.3).

Proof. For $(u_1,\ldots,u_{k+1})\in\mathcal{M}_k^{\boldsymbol{\rho}_k}$, denote by $u=\sum_{i=1}^{k+1}u_i$, then

$$\begin{aligned}0=&\sum_{i=1}^{k+1}\partial_{u_i}J_b(u_1,\ldots,u_{k+1})u_i=I_b'\left(\sum_{i=1}^{k+1}u_i\right)\left(\sum_{i=1}^{k+1}u_i\right)\\ =&\|u\|_V^2+b\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^2-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{3-\alpha}}\mathrm{d}y\mathrm{d}x-\int_{\mathbb{R}^3}|u|^q\mathrm{d}x.\end{aligned}$$

By Hardy–Littlewood–Sobolev inequality and Sobolev embedding theorem, we can see

$$\|u\|_V^2\le\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{3-\alpha}}\mathrm{d}y\mathrm{d}x+\int_{\mathbb{R}^3}|u|^q\mathrm{d}x\le c\|u\|_V^{2p}+c\|u\|_V^q.$$

Since $p\in(4,3+\alpha)$, $q\in(4,6)$, we have $\|u\|_V\ge c_1>0$ for some $c_1>0$.

When $q \geq 2p$, we have

$$\begin{aligned}
J_b(u_1,\ldots,u_{k+1}) &= I_b\left(\sum_{i=1}^{k+1} u_i\right) = I_b(u) - \frac{1}{2p}I_b'(u)u \\
&= \left(\frac{1}{2}-\frac{1}{2p}\right)\|u\|_V^2 + b\left(\frac{1}{4}-\frac{1}{2p}\right)\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^2 \\
&\quad + \left(\frac{1}{2p}-\frac{1}{q}\right)\int_{\mathbb{R}^3}|u|^q\mathrm{d}x \\
&\geq \left(\frac{1}{2}-\frac{1}{2p}\right)\|u\|_V^2 \geq c_2 > 0. \qquad (3.2)
\end{aligned}$$

When $q < 2p$, we have

$$\begin{aligned}
J_b(u_1,\ldots,u_{k+1}) &= I_b\left(\sum_{i=1}^{k+1} u_i\right) = I_b(u) - \frac{1}{q}I_b'(u)u \\
&= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|_V^2 + b\left(\frac{1}{4}-\frac{1}{q}\right)\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^2 \\
&\quad + \left(\frac{1}{q}-\frac{1}{2p}\right)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{3-\alpha}}\mathrm{d}y\mathrm{d}x \\
&\geq \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|_V^2 \geq c_2 > 0. \qquad (3.3)
\end{aligned}$$

Thus, $d(\boldsymbol{\rho}_k) \geq c_2 > 0$. We can choose a minimizing sequence $\{(u_1^n,\ldots,u_{k+1}^n)\}_{n=1}^\infty \subset \mathcal{M}_k^{\boldsymbol{\rho}_k}$ of $J_b|_{\mathcal{M}_k^{\boldsymbol{\rho}_k}}$. From (3.2), (3.3), we know that $\{u_i^n\}_{n=1}^\infty$ is bounded in $\mathcal{H}_k^{\boldsymbol{\rho}_k}$. Up to a subsequence, $(u_1^n,\ldots,u_{k+1}^n)$ converges to an element $(u_1^0,\ldots,u_{k+1}^0)$ weakly in $\mathcal{H}_k^{\boldsymbol{\rho}_k}$.

We claim that for all $i=1,\ldots,k+1$, $u_i^0 \neq 0$. If $u_i^n \to u_i^0$ strongly in $H_i^{\boldsymbol{\rho}_k}$, for any $i=1,\ldots,k+1$,

$$\begin{aligned}
\|u_i^n\|_i^2 &\leq \int_{\mathbb{R}^3}\int_{B_i^{\rho_k}}\frac{\left|\sum_{j=1}^{k+1}u_j^n(y)\right|^p|u_i^n(x)|^p}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y + \int_{B_i^{\rho_k}}|u_i^n|^q\mathrm{d}x \\
&\leq c\left(\left\|\sum_{i=1}^{k+1}u_i^n\right\|^p\|u_i^n\|_i^p + \|u_i^n\|_i^q\right) \leq c\left(\|u_i^n\|_i^p + \|u_i^n\|_i^q\right).
\end{aligned}$$

Hence,

$$\liminf_{n\to\infty}\|u_i^n\|_i > 0. \qquad (3.4)$$

This implies that $u_i^0 \neq 0$ for all $i=1,\ldots,k+1$. Thus, the claim follows.

If $u_i^n \rightharpoonup u_i^0$ weakly but not strongly in $H_i^{\boldsymbol{\rho}_k}$, then there exists $i \in \{1, \ldots, k+1\}$ such that $\|u_i^0\|_i < \liminf_{n\to\infty} \|u_i^n\|_i$, we have

$$\|u_i^0\|_i^2 + b\int_{\mathbb{R}^3}\left|\nabla\left(\sum_{i=1}^{k+1} u_i^0(x)\right)\right|^2 \mathrm{d}x \int_{B_i^{\rho_k}} |\nabla u_i^0|^2 \mathrm{d}x$$

$$< \int_{\mathbb{R}^3}\int_{B_i^{\rho_k}} \frac{\left|\sum_{j=1}^{k+1} u_j^0(y)\right|^p |u_i^0(x)|^p}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y + \int_{B_i^{\rho_k}} |u_i^0(x)|^q \mathrm{d}x.$$

By Hardy–Littlewood–Sobolev inequality and Sobolev embedding theorem, the claim also follows.

We further claim that $(u_1^n, \ldots, u_{k+1}^n) \to (u_1^0, \ldots, u_{k+1}^0)$ in $\mathcal{H}_k^{\boldsymbol{\rho}_k}$. Suppose by contradiction that the claim does not hold. There exists $i \in \{1, \ldots, k+1\}$ such that $\|u_i^0\|_i < \liminf_{n\to\infty} \|u_i^n\|_i$. By Lemma 3.1, there is $(t_1^0, \ldots, t_{k+1}^0) \neq (1, \ldots, 1)$ satisfying $(t_1^0 u_1^0, \ldots, t_{k+1}^0 u_{k+1}^0) \in \mathcal{M}_k^{\boldsymbol{\rho}_k}$, then

$$\begin{aligned} d(\boldsymbol{\rho}_k) &\leq J_b(t_1^0 u_1^0, \ldots, t_{k+1}^0 u_{k+1}^0) \\ &< \liminf_{n\to\infty} \left\{ \frac{1}{2}\sum_{i=1}^{k+1} (t_i^0)^2 \|u_i^n\|_i^2 + \frac{b}{4}\left(\sum_{i=1}^{k+1} (t_i^0)^2 \int_{\mathbb{R}^3} |\nabla u_i^n(x)|^2 \mathrm{d}x\right)^2 \right. \\ &\quad - \frac{1}{2p}\sum_{i,j=1}^{k+1} (t_i^0)^p (t_j^0)^p \int_{B_i^{\rho_k}}\int_{B_j^{\rho_k}} \frac{(|u_i^n(y)|^p)(|u_j^n(x)|^p)}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y \\ &\quad \left. - \sum_{i=1}^{k+1} (t_i^0)^q \int_{B_i^{\rho_k}} |u_i^n(x)|^q \mathrm{d}x \right\} \\ &\leq \liminf_{n\to\infty} J_b(u_1^n, \ldots, u_{k+1}^n) = d(\boldsymbol{\rho}_k), \end{aligned}$$

which is a contradiction. Thus, the claim follows and $(u_1^0, \ldots, u_{k+1}^0) \in \mathcal{M}_k^{\boldsymbol{\rho}_k}$ is a minimizer of $J_b|_{\mathcal{M}_k^{\rho_k}}$.

It is easy to check that

$$(\xi_1^{\boldsymbol{\rho}_k}, \ldots, \xi_{k+1}^{\boldsymbol{\rho}_k}) := (|u_1^0|, -|u_2^0|, \ldots, (-1)^k |u_{k+1}^0|)$$

belongs to $\mathcal{M}_k^{\boldsymbol{\rho}_k}$ and is a minimizer of $J_b|_{\mathcal{M}_k^{\rho_k}}$ satisfying (3.1). From Lemma 3.3, it is a critical point of J_b in $\mathcal{H}_k^{\boldsymbol{\rho}_k}$ and satisfying (2.3). Using the strong maximum principle, each component $(-1)^{i+1}\xi_i^{\boldsymbol{\rho}_k} > 0$ in $B_i^{\boldsymbol{\rho}_k}$, for $i = 1, \ldots, k+1$. The proof is complete. □

Lemma 3.5. *For any $p \in (4, 3+\alpha)$, $q \in (4, 6)$ and $\boldsymbol{\rho}_k = (\rho_1, \ldots, \rho_k) \in \Gamma_k$*

- (i) *For uniformly bounded $\boldsymbol{\rho}_k$, if $\rho_i - \rho_{i-1} \to 0$ for some $i \in \{1, \ldots, k\}$, then $d(\boldsymbol{\rho}_k) \to +\infty$.*
- (ii) *If $\rho_k \to \infty$, then $d(\boldsymbol{\rho}_k) \to +\infty$.*
- (iii) *d is continuous in Γ_k. Therefore, there exists a $\bar{\boldsymbol{\rho}}_k \in \Gamma_k$ such that*

$$d(\bar{\boldsymbol{\rho}}_k) = \inf_{\boldsymbol{\rho}_k \in \Gamma_k} d(\boldsymbol{\rho}_k).$$

Proof. (i) By lemma 3.4, it is easy to see that for each $\boldsymbol{\rho}_k \in \Gamma_k$, there exists a solution $\boldsymbol{\xi}^{\boldsymbol{\rho}_k} = (\xi_1^{\boldsymbol{\rho}_k}, \ldots, \xi_{k+1}^{\boldsymbol{\rho}_k}) \in \mathcal{M}_k^{\boldsymbol{\rho}_k}$ such that $d(\boldsymbol{\rho}_k) = J_b(\xi_1^{\boldsymbol{\rho}_k}, \ldots, \xi_{k+1}^{\boldsymbol{\rho}_k})$. By Hardy–Littlewood–Sobolev inequality, Hölder inequality and embedding inequality, we have

$$\begin{aligned}
\|\xi_i^{\boldsymbol{\rho}_k}\|_i^2 &= \int_{\mathbb{R}^3}\int_{B_i^{\rho_k}} \frac{|\sum_{j=1}^{k+1} \xi_j^{\boldsymbol{\rho}_k}(y)|^p |\xi_i^{\boldsymbol{\rho}_k}(x)|^p}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y + \int_{B_i^{\rho_k}} |\xi_i^{\boldsymbol{\rho}_k}|^q \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} \sum_{j=1}^{k+1} |\nabla \xi_j^{\boldsymbol{\rho}_k}(x)|^2 \mathrm{d}x \int_{B_i^{\rho_k}} |\nabla \xi_i^{\boldsymbol{\rho}_k}|^2 \mathrm{d}x \\
&\le c \left(|\sum_{j=1}^{k+1} \xi_j^{\boldsymbol{\rho}_k}|^p_{L^{\frac{6p}{3+\alpha}}} |\xi_i^{\boldsymbol{\rho}_k}|^p_{L^{\frac{6p}{3+\alpha}}} + |\xi_i^{\boldsymbol{\rho}_k}|^q_{L^q} \right) \\
&\le c \left(|\sum_{j=1}^{k+1} \xi_j^{\boldsymbol{\rho}_k}|^p_{L^{\frac{6p}{3+\alpha}}} |\xi_i^{\boldsymbol{\rho}_k}|^p_{L^6} |B_i^{\boldsymbol{\rho}_k}|^{\frac{3+\alpha-p}{6}} + |\xi_i^{\boldsymbol{\rho}_k}|^q_{L^6} |B_i^{\boldsymbol{\rho}_k}|^{\frac{6-q}{6}} \right) \\
&\le c \left[\left(\sum_{j=1}^{k+1} \|\xi_j^{\boldsymbol{\rho}_k}\|_j^p \right) \|\xi_i^{\boldsymbol{\rho}_k}\|_i^p |B_i^{\boldsymbol{\rho}_k}|^{\frac{3+\alpha-p}{6}} + \|\xi_i^{\boldsymbol{\rho}_k}\|_i^q |B_i^{\boldsymbol{\rho}_k}|^{\frac{6-q}{6}} \right].
\end{aligned}$$

Since $\boldsymbol{\rho}_k$ is uniformly bounded, then if $\rho_i - \rho_{i-1} \to 0$ for some $i \in \{1, \ldots, k\}$, we have $|B_i^{\boldsymbol{\rho}_k}| \to 0$. From $p \in (4, 3+\alpha)$, $q \in (4,6)$, we have $\sum_{j=1}^{k+1} \|\xi_j^{\boldsymbol{\rho}_k}\|_j^p \to \infty$. By the same arguments as in (3.2) or (3.3), we have

$$d(\boldsymbol{\rho}_k) \ge \left(\frac{1}{2} - \frac{1}{2p} \right) \sum_{j=1}^{k+1} \|\xi_j^{\boldsymbol{\rho}_k}\|_j^2 \to \infty,$$

or

$$d(\boldsymbol{\rho}_k) \ge \left(\frac{1}{2} - \frac{1}{q} \right) \sum_{j=1}^{k+1} \|\xi_j^{\boldsymbol{\rho}_k}\|_j^2 \to \infty.$$

Then, (i) holds.

(ii) By the Strauss inequality [2], for every radial function $u \in H_V$, we can find $a_0 > 0$ such that $u(x) \le \frac{a_0\|u\|_V}{|x|}$ for a.e. $|x| > 1$.This combined with Hardy–Littlewood–Sobolev inequality and embedding theorem, yields

$$\begin{aligned}
\|\xi_{k+1}^{\boldsymbol{\rho}_k}\|_{k+1}^2 &= \int_{\mathbb{R}^3}\int_{B_{k+1}^{\rho_k}} \frac{|\sum_{j=1}^{k+1} \xi_j^{\boldsymbol{\rho}_k}(y)|^p |\xi_{k+1}^{\boldsymbol{\rho}_k}(x)|^p}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y + \int_{B_{k+1}^{\rho_k}} |\xi_{k+1}^{\boldsymbol{\rho}_k}|^q \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} \sum_{j=1}^{k+1} |\nabla \xi_j^{\boldsymbol{\rho}_k}(x)|^2 \mathrm{d}x \int_{B_{k+1}^{\rho_k}} |\nabla \xi_{k+1}^{\boldsymbol{\rho}_k}(x)|^2 \mathrm{d}x \\
&\le \int_{\mathbb{R}^3}\int_{B_{k+1}^{\rho_k}} \frac{|\sum_{j=1}^{k+1} \xi_j^{\boldsymbol{\rho}_k}(y)|^p |\xi_{k+1}^{\boldsymbol{\rho}_k}(x)|^p}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y + \int_{B_{k+1}^{\rho_k}} |\xi_{k+1}^{\boldsymbol{\rho}_k}|^q \mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
&\le c\left(|\sum_{j=1}^{k+1}\xi_j^{\boldsymbol{\rho}_k}|^p_{L^{\frac{6p}{3+\alpha}}}\right)\left(\int_{B_{k+1}^{\boldsymbol{\rho}_k}}|\xi_{k+1}^{\boldsymbol{\rho}_k}|^{\frac{6p}{3+\alpha}}\mathrm{d}x\right)^{\frac{3+\alpha}{6}}+c\int_{B_{k+1}^{\boldsymbol{\rho}_k}}|\xi_{k+1}^{\boldsymbol{\rho}_k}|^q\mathrm{d}x\\
&\le c\left(|\sum_{j=1}^{k+1}\xi_j^{\boldsymbol{\rho}_k}|^p_{L^{\frac{6p}{3+\alpha}}}\right)\left(\int_{B_{k+1}^{\boldsymbol{\rho}_k}}|\xi_{k+1}^{\boldsymbol{\rho}_k}|^{\frac{6p}{3+\alpha}-2}|\xi_{k+1}^{\boldsymbol{\rho}_k}|^2\mathrm{d}x\right)^{\frac{3+\alpha}{6}}\\
&\quad+c\int_{B_{k+1}^{\boldsymbol{\rho}_k}}|\xi_{k+1}^{\boldsymbol{\rho}_k}|^{q-2}|\xi_{k+1}^{\boldsymbol{\rho}_k}|^2\mathrm{d}x\\
&\le c\left(\sum_{j=1}^{k+1}\|\xi_j^{\boldsymbol{\rho}_k}\|_j^p\right)\|\xi_{k+1}^{\boldsymbol{\rho}_k}\|_{k+1}^{\left(\frac{6p}{3+\alpha}-2\right)\frac{3+\alpha}{6}}\left(\int_{B_{k+1}^{\boldsymbol{\rho}_k}}|x|^{-\left(\frac{6p}{3+\alpha}-2\right)}|\xi_{k+1}^{\boldsymbol{\rho}_k}(x)|^2\mathrm{d}x\right)^{\frac{3+\alpha}{6}}\\
&\quad+c\|\xi_{k+1}^{\boldsymbol{\rho}_k}\|_{k+1}^{q-2}\left(\int_{B_{k+1}^{\boldsymbol{\rho}_k}}|x|^{-(q-2)}|\xi_{k+1}^{\boldsymbol{\rho}_k}(x)|^2\mathrm{d}x\right)\\
&\le c\left(\sum_{j=1}^{k+1}\|\xi_j^{\boldsymbol{\rho}_k}\|_j^p\right)|\rho_k|^{-\left(\frac{6p}{3+\alpha}-2\right)\frac{3+\alpha}{6}}\|\xi_{k+1}^{\boldsymbol{\rho}_k}\|_{k+1}^{\left(\frac{6p}{3+\alpha}-2\right)\frac{3+\alpha}{6}}\\
&\quad\left(\int_{B_{k+1}^{\boldsymbol{\rho}_k}}|\xi_{k+1}^{\boldsymbol{\rho}_k}(x)|^2\mathrm{d}x\right)^{\frac{3+\alpha}{6}}\\
&\quad+c|\rho_k|^{-(q-2)}\|\xi_{k+1}^{\boldsymbol{\rho}_k}\|_{k+1}^{q-2}\left(\int_{B_{k+1}^{\boldsymbol{\rho}_k}}|\xi_{k+1}^{\boldsymbol{\rho}_k}(x)|^2\mathrm{d}x\right)\\
&\le c\left(\sum_{j=1}^{k+1}\|\xi_j^{\boldsymbol{\rho}_k}\|_j^p\right)|\rho_k|^{-\left(\frac{6p}{3+\alpha}-2\right)\frac{3+\alpha}{6}}\|\xi_{k+1}^{\boldsymbol{\rho}_k}\|_{k+1}^p+c|\rho_k|^{-(q-2)}\|\xi_{k+1}^{\boldsymbol{\rho}_k}\|_{k+1}^q\\
&= c\left(\sum_{j=1}^{k+1}\|\xi_j^{\boldsymbol{\rho}_k}\|_j^p\right)|\rho_k|^{-\left(p-\frac{3+\alpha}{3}\right)}\|\xi_{k+1}^{\boldsymbol{\rho}_k}\|_{k+1}^p+c|\rho_k|^{-(q-2)}\|\xi_{k+1}^{\boldsymbol{\rho}_k}\|_{k+1}^q,
\end{aligned}$$

which yields that $\sum_{j=1}^{k+1}\|\xi_j^{\boldsymbol{\rho}_k}\|_j^p\to\infty$ as $\rho_k\to\infty$, due to $p\in(4,3+\alpha), q\in(4,6), \alpha\in(1,3)$. So $d(\boldsymbol{\rho}_k)\to\infty$ and (ii) follows.

(iii) Take a sequence $\{\boldsymbol{\rho}_k^n\}_{n=1}^{\infty}$ satisfying $\boldsymbol{\rho}_k^n\to\bar{\boldsymbol{\rho}}_k\in\Gamma_k$. We will prove the conclusion by showing $d(\bar{\boldsymbol{\rho}}_k)\ge\limsup_{n\to\infty}d(\boldsymbol{\rho}_k^n)$, $d(\bar{\boldsymbol{\rho}}_k)\le\liminf_{n\to\infty}d(\boldsymbol{\rho}_k^n)$.

First, we prove that $d(\bar{\boldsymbol{\rho}}_k)\ge\limsup_{n\to\infty}d(\boldsymbol{\rho}_k^n)$. In order to emphasize that $v_i^{\boldsymbol{\rho}_k^n}$ is radial in $B_i^{\boldsymbol{\rho}_k^n}$, we will rewrite $v_i^{\boldsymbol{\rho}_k^n}(|x|)=v_i^{\boldsymbol{\rho}_k^n}(\rho)$. Define $v_i^{\boldsymbol{\rho}_k^n}:[\rho_{i-1}^n,\rho_i^n]\to\mathbb{R}$ by

$$v_i^{\boldsymbol{\rho}_k^n}(\rho)=\begin{cases} t_i^n\xi_i^{\bar{\boldsymbol{\rho}}_k}\left(\bar{\rho}_{i-1}+\frac{\bar{\rho}_i-\bar{\rho}_{i-1}}{\rho_i^n-\rho_{i-1}^n}(\rho-\rho_{i-1}^n)\right), & i=1,\dots,k,\\ t_{k+1}^n\xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}\left(\frac{\bar{\rho}_k}{\rho_k^n}\rho\right), & i=k+1,\end{cases}$$

where $(\xi_1^{\boldsymbol{\rho}_k^n},\dots,\xi_{k+1}^{\boldsymbol{\rho}_k^n})$ and $(\xi_1^{\bar{\boldsymbol{\rho}}_k},\dots,\xi_{k+1}^{\bar{\boldsymbol{\rho}}_k})$ are minimizers of $J_b|_{\mathcal{M}_k^{\boldsymbol{\rho}_k^n}}$ and $J_b|_{\mathcal{M}_k^{\bar{\boldsymbol{\rho}}_k}}$ respectively. $(t_1^n,\dots,t_{k+1}^n)$ is the unique $(k+1)$ tuple of positive numbers such that $(v_1^{\boldsymbol{\rho}_k^n},\dots,v_{k+1}^{\boldsymbol{\rho}_k^n})\in\mathcal{M}_k^{\boldsymbol{\rho}_k^n}$. By the definition of $(\xi_1^{\boldsymbol{\rho}_k^n},\dots,\xi_{k+1}^{\boldsymbol{\rho}_k^n})$, we know that

$$J_b(v_1^{\boldsymbol{\rho}_k^n},\dots,v_{k+1}^{\boldsymbol{\rho}_k^n})\ge J_b(\xi_1^{\boldsymbol{\rho}_k^n},\dots,\xi_{k+1}^{\boldsymbol{\rho}_k^n})=d(\boldsymbol{\rho}_k^n). \tag{3.5}$$

Since $\boldsymbol{\rho}_k^n \to \bar{\boldsymbol{\rho}}_k \in \Gamma_k$, we can easily get the following equations,

$$\int_{B_i^{\boldsymbol{\rho}_k^n}} |v_i^{\boldsymbol{\rho}_k^n}|^2 = (t_i^n)^2 \int_{B_i^{\bar{\boldsymbol{\rho}}_k}} |\xi_i^{\bar{\boldsymbol{\rho}}_k}|^2 \mathrm{d}x + o(1)$$

$$\|v_i^{\boldsymbol{\rho}_k^n}\|_i^2 = (t_i^n)^2 \|\xi_i^{\bar{\boldsymbol{\rho}}_k}\|_i^2 + o(1)$$

$$\int_{B_i^{\boldsymbol{\rho}_k^n}} |v_i^{\boldsymbol{\rho}_k^n}|^q \mathrm{d}x = (t_i^n)^q \int_{B_i^{\bar{\boldsymbol{\rho}}_k}} |\xi_i^{\bar{\boldsymbol{\rho}}_k}|^q \mathrm{d}x + o(1)$$

$$\int_{B_i^{\boldsymbol{\rho}_k^n}} \int_{B_j^{\boldsymbol{\rho}_k^n}} \frac{|v_i^{\boldsymbol{\rho}_k^n}(x)|^p |v_j^{\boldsymbol{\rho}_k^n}(y)|^p}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y$$

$$= (t_i^n)^p (t_j^n)^p \int_{B_i^{\bar{\boldsymbol{\rho}}_k}} \int_{B_j^{\bar{\boldsymbol{\rho}}_k}} \frac{|\xi_i^{\bar{\boldsymbol{\rho}}_k}(x)|^p |\xi_j^{\bar{\boldsymbol{\rho}}_k}(y)|^p}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y + o(1),$$

and

$$\int_{B_i^{\boldsymbol{\rho}_k^n}} |\nabla v_i^{\boldsymbol{\rho}_k^n}(x)|^2 \mathrm{d}x \int_{B_j^{\boldsymbol{\rho}_k^n}} |\nabla v_j^{\boldsymbol{\rho}_k^n}(x)|^2 \mathrm{d}x$$

$$= (t_i^n)^2 (t_j^n)^2 \int_{B_i^{\bar{\boldsymbol{\rho}}_k}} |\nabla \xi_i^{\bar{\boldsymbol{\rho}}_k}(x)|^2 \mathrm{d}x \int_{B_j^{\bar{\boldsymbol{\rho}}_k}} |\nabla \xi_j^{\bar{\boldsymbol{\rho}}_k}(x)|^2 \mathrm{d}x + o(1).$$

Since $(v_1^{\boldsymbol{\rho}_k^n}, \ldots, v_{k+1}^{\boldsymbol{\rho}_k^n}) \in \mathcal{M}_k^{\boldsymbol{\rho}_k^n}$ and $(\xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, \xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}) \in \mathcal{M}_k^{\bar{\boldsymbol{\rho}}_k}$, there holds that

$$\|\xi_i^{\bar{\boldsymbol{\rho}}_k}\|_i^2 + b \sum_{j=1}^{k+1} \int_{B_i^{\bar{\boldsymbol{\rho}}_k}} |\nabla \xi_i^{\bar{\boldsymbol{\rho}}_k}(x)|^2 \mathrm{d}x \int_{B_j^{\bar{\boldsymbol{\rho}}_k}} |\nabla \xi_j^{\bar{\boldsymbol{\rho}}_k}(x)|^2 \mathrm{d}x$$

$$- \sum_{j=1}^{k+1} \int_{B_i^{\bar{\boldsymbol{\rho}}_k}} \int_{B_j^{\bar{\boldsymbol{\rho}}_k}} \frac{|\xi_i^{\bar{\boldsymbol{\rho}}_k}(x)|^p |\xi_j^{\bar{\boldsymbol{\rho}}_k}(y)|^p}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y - \int_{B_i^{\bar{\boldsymbol{\rho}}_k}} |\xi_i^{\bar{\boldsymbol{\rho}}_k}|^q \mathrm{d}x = 0,$$

and

$$(t_i^n)^2 \|\xi_i^{\bar{\boldsymbol{\rho}}_k}\|_i^2 + b (t_i^n)^2 (t_j^n)^2 \sum_{j=1}^{k+1} \int_{B_i^{\bar{\boldsymbol{\rho}}_k}} |\nabla \xi_i^{\bar{\boldsymbol{\rho}}_k}(x)|^2 \mathrm{d}x \int_{B_j^{\bar{\boldsymbol{\rho}}_k}} |\nabla \xi_j^{\bar{\boldsymbol{\rho}}_k}(x)|^2 \mathrm{d}x$$

$$- (t_i^n)^p (t_j^n)^p \sum_{j=1}^{k+1} \int_{B_i^{\bar{\boldsymbol{\rho}}_k}} \int_{B_j^{\bar{\boldsymbol{\rho}}_k}} \frac{|\xi_i^{\bar{\boldsymbol{\rho}}_k}(x)|^p |\xi_j^{\bar{\boldsymbol{\rho}}_k}(y)|^p}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y - (t_i^n)^q \int_{B_i^{\bar{\boldsymbol{\rho}}_k}} |\xi_i^{\bar{\boldsymbol{\rho}}_k}|^q \mathrm{d}x = o(1).$$

This combined with Lemma 3.1, we have $\lim_{n\to\infty} t_i^n = 1$ for all i. Hence, from (3.5) we can see that

$$d(\bar{\boldsymbol{\rho}}_k) = J_b(\xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, \xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}) = \limsup_{n\to\infty} J_b(v_1^{\boldsymbol{\rho}_k^n}, \ldots, v_{k+1}^{\boldsymbol{\rho}_k^n}) \geq \limsup_{n\to\infty} d(\boldsymbol{\rho}_k^n). \tag{3.6}$$

Next, we prove that $d(\bar{\boldsymbol{\rho}}_k) \leq \liminf_{n\to\infty} d(\boldsymbol{\rho}_k^n)$. By the same argument as former case, let $u_i^{\boldsymbol{\rho}_k^n} = [\bar{\rho}_{i-1}, \bar{\rho}_i] \to \mathbb{R}$ be defined by

$$u_i^{\boldsymbol{\rho}_k^n}(\rho) = \begin{cases} s_i^n \xi_i^{\boldsymbol{\rho}_k^n} \left(\rho_{i-1}^n + \frac{\rho_i^n - \rho_{i-1}^n}{\bar{\rho}_i - \bar{\rho}_{i-1}} (\rho - \bar{\rho}_{i-1}) \right), & \text{if } \ i = 1, \ldots, k, \\ s_{k+1}^n \xi_{k+1}^{\boldsymbol{\rho}_k^n} \left(\frac{\rho_k^n}{\bar{\rho}_k} \rho \right), & \text{if } \ i = k+1, \end{cases}$$

where $(s_1^n, \ldots, s_{k+1}^n) \in (\mathbb{R}_+)^{k+1}$ such that $(u_1^{\boldsymbol{\rho}_k^n}, \ldots, u_{k+1}^{\boldsymbol{\rho}_k^n}) \in \mathcal{M}_k^{\bar{\boldsymbol{\rho}}_k}$.

By the same arguments, we can deduce $s_i^n \to 1$ as $n \to \infty$ for all $i = 1, \ldots, k+1$. Thus

$$\begin{aligned} d(\bar{\boldsymbol{\rho}}_k) &= J_b(\xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, \xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}) \\ &\le \liminf_{n\to\infty} J_b(u_1^{\boldsymbol{\rho}_k^n}, \ldots, u_{k+1}^{\boldsymbol{\rho}_k^n}) = \liminf_{n\to\infty} J_b(\xi_1^{\boldsymbol{\rho}_k^n}, \ldots, \xi_{k+1}^{\boldsymbol{\rho}_k^n}) = \liminf_{n\to\infty} d(\rho_k^n). \end{aligned}$$

This combined with (3.6) yields that d is continuous in Γ_k. Furthermore, this combined with (i), (ii), we know that there is a $\bar{\boldsymbol{\rho}}_k \in \Gamma_k$ such that $d(\bar{\boldsymbol{\rho}}_k) = \inf_{\boldsymbol{\rho}_k \in \Gamma_k} d(\boldsymbol{\rho}_k)$. Hence, (iii) holds. □

Proof of Theorem 1.1. By Lemmas 3.4 and 3.5, there exist $\bar{\boldsymbol{\rho}}_k \in \Gamma_k$ and $(\xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, \xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}) \in \mathcal{M}_k^{\bar{\boldsymbol{\rho}}_k}$ with $(-1)^{i+1}\xi_i^{\boldsymbol{\rho}_k} > 0$ in $B_i^{\boldsymbol{\rho}_k}$ such that

$$J_b(\xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, \xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}) = d(\bar{\boldsymbol{\rho}}_k) = \inf_{\boldsymbol{\rho}_k \in \Gamma_k} d(\boldsymbol{\rho}_k).$$

This implies that

$$c_k = d(\bar{\boldsymbol{\rho}}_k) = I_b\left(\sum_{i=1}^{k+1} \xi_i^{\bar{\boldsymbol{\rho}}_k}\right).$$

□

We claim that $u_k^b = \sum_{i=1}^{k+1} \xi_i^{\bar{\boldsymbol{\rho}}_k}$ is a solution of (1.3). Suppose by contradiction that the claim does not hold, that is, u_k^b is not a weak solution of (1.3). Then by the density argument, there is a radial function $\phi \in \mathbb{C}_0^\infty(\mathbb{R}^3)$ such that

$$I_b'\left(\sum_{i=1}^{k+1} \xi_i^{\bar{\boldsymbol{\rho}}_k}\right)\phi = -2. \tag{3.7}$$

For $\boldsymbol{s} = (s_1, \ldots, s_{k+1})$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^{k+1}$, we define function $g := \mathbb{R}^{k+1} \times \mathbb{R} \to H^1(\mathbb{R}^3)$ by

$$g(\boldsymbol{s}, \epsilon) := \sum_{i=1}^{k+1} s_i^{\frac{1}{p}} \xi_i^{\bar{\boldsymbol{\rho}}_k} + \epsilon\phi.$$

Since $\sum_{i=1}^{k+1} \xi_i^{\bar{\boldsymbol{\rho}}_k}$ is continuous and has k nodes, we know that there exists a neighborhood $B_\tau(\mathbf{1}) := \left\{\boldsymbol{s} \in \mathbb{R}^{k+1} : |\boldsymbol{s} - \mathbf{1}| < \tau\right\}$ such that $g(\boldsymbol{s}, \tau)$ also changes signs exactly k times and

$$J_b'(g(\boldsymbol{s}, \epsilon))\phi < -1 \quad \forall (\boldsymbol{s}, \epsilon) \in B_\tau(\mathbf{1}) \times [0, \tau] \tag{3.8}$$

for all $(\boldsymbol{s}, \epsilon) \in B_\tau(\mathbf{1}) \times [0, \tau]$.

Let $\eta \in \mathbb{C}^\infty(\mathbb{R}^3)$, $0 \le \eta \le 1$ with $\eta(\boldsymbol{s}) = 1$ if $s \in \overline{B_{\frac{\tau}{4}}(1, \ldots, 1)}$ and $\eta(\boldsymbol{s}) = 0$ if $\boldsymbol{s} \notin B_{\frac{\tau}{2}}(1, \ldots, 1)$. We define another continuous function $\bar{g} : \mathbb{R}^{k+1} \to \mathbb{H}_V$

by

$$\bar{g}(\boldsymbol{s}) = \sum_{i=1}^{k+1} s_i^{\frac{1}{p}} \xi_i^{\bar{\boldsymbol{\rho}}_k} + \tau\eta(\boldsymbol{s})\phi.$$

Obviously, for any $\boldsymbol{s} \in B_\tau(\mathbf{1})$, $\bar{g}(\boldsymbol{s})$ also changes signs exactly k times and has k nodes $0 < \rho_1(\boldsymbol{s}) < \cdots < \rho_k(\boldsymbol{s}) < \infty$. Moreover,

$$J_b'(g(\boldsymbol{s}, \epsilon))\phi < -1 \quad \forall (\boldsymbol{s}, \epsilon) \in B_\tau(\mathbf{1}) \times [0, \tau].$$

Next, we will prove that there exists $\bar{\boldsymbol{s}} \in B_{\frac{\tau}{2}}(\mathbf{1})$ such that $\bar{g}(\bar{\boldsymbol{s}}) \in \mathcal{N}_k$ changing sign k times. Denote

$$G(\mathbf{s}) = J_b\left(s_1^{\frac{1}{p}} \xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, s_{k+1}^{\frac{1}{p}} \xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}\right).$$

For any $\boldsymbol{s} \in \partial B_{\frac{\tau}{2}}(\mathbf{1})$, we have

$$\begin{aligned}
\nabla G(\mathbf{s})(\mathbf{1} - \boldsymbol{s}) =& \nabla J_b\left(s_1^{\frac{1}{p}} \xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, s_{k+1}^{\frac{1}{p}} \xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}\right)(\mathbf{1} - \boldsymbol{s}) \\
=& \sum_{i=1}^{k+1} \frac{1}{p} s_i^{\frac{1}{p}-1} \left\langle \partial_{u_i} J_b\left(s_1^{\frac{1}{p}} \xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, s_{k+1}^{\frac{1}{p}} \xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}\right), \xi_i^{\bar{\boldsymbol{\rho}}_k} \right\rangle (1 - s_i) \\
=& \sum_{i=1}^{k+1} \frac{1}{p s_i}(1 - s_i) \left\langle \partial_{u_i} J_b(s_1^{\frac{1}{p}} \xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, s_{k+1}^{\frac{1}{p}} \xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}), s_i^{\frac{1}{p}} \xi_i^{\bar{\boldsymbol{\rho}}_k} \right\rangle.
\end{aligned}$$

By Lemma 3.1 and Corollary 3.2, we obtain $G(\mathbf{s})$ is strictly concave function in $\mathbb{R}_+^{k+1}$ and attains its unique global maximum point at $\mathbf{1}$. By the Taylor expansion at $\boldsymbol{s} \neq \mathbf{1}$ in $[0, \infty)^{k+1}$, and the strictly concavity, we have

$$0 < \varphi_b(\mathbf{1}) - \varphi_b(\mathbf{s}) - D^2\varphi_b(\boldsymbol{s})(\mathbf{1} - \mathbf{s})^2 = \nabla G(\mathbf{s})(\mathbf{1} - \mathbf{s}),$$

that is

$$\nabla G(\mathbf{s})(\mathbf{1} - \mathbf{s}) > 0.$$

Set $\widetilde{G}_i(\mathbf{s}) = \frac{1}{p} s_i^{-1} \left\langle \partial_{u_j} J_b(s_1^{\frac{1}{p}} \xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, s_{k+1}^{\frac{1}{p}} \xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}), s_i^{\frac{1}{p}} \xi_i^{\bar{\boldsymbol{\rho}}_k} \right\rangle$ and $\widetilde{G}(\mathbf{s}) = (\widetilde{G}_1(\mathbf{s}), \ldots, \widetilde{G}_{k+1}(\mathbf{s}))$. Define a map $F(\theta, \mathbf{s}) : [0,1] \times \bar{B}_{\frac{\tau}{2}}(\mathbf{1}) \to \mathbb{R}^{k+1}$ by

$$F(\theta, \mathbf{s}) = \theta \widetilde{G}(\mathbf{s}) + (1 - \theta)(\mathbf{1} - \mathbf{s}).$$

Obviously, $F(0, \mathbf{s}) = \mathbf{1} - \mathbf{s}$, $F(1, \mathbf{s}) = \widetilde{G}(\mathbf{s})$. Thus, $\mathbf{1} - \mathbf{s}$ and $\widetilde{G}(\mathbf{s})$ are homotopy. Moreover, for any $\theta \in [0,1]$ and $\mathbf{s} \in B_{\frac{\tau}{2}}(\mathbf{1})$, we obtain $F(\theta, \mathbf{s}) \cdot (\mathbf{1} - \mathbf{s}) > 0$. Thus, $F(\theta, \mathbf{s}) \neq \mathbf{0}$. By the Brouwer degree theory, we have

$$\deg(\widetilde{G}, B_{\frac{\tau}{2}}(\mathbf{1}), 0) = \deg(1 - id, B_{\frac{\tau}{2}}(\mathbf{1}), 0) = (-1)^{k+1} \neq 0.$$

Therefore, there exists some $\bar{\boldsymbol{s}} \in B_{\frac{\tau}{2}}(\mathbf{1})$ such that $\bar{g}(\bar{\boldsymbol{s}}) \in \mathcal{N}_k$. The claim follows.

According to the claim, we have

$$J_b(\bar{g}(\bar{\boldsymbol{s}})) \geq c_k. \tag{3.9}$$

On the other hand, by the mean value theorem and (3.8), we have

$$J_b(\bar g(\bar{\boldsymbol{s}})) = I_b\left(\sum_{i=1}^{k+1} \bar s_i^{\frac{1}{p}}\xi_i^{\bar{\boldsymbol{\rho}}_k}\right) + \int_0^1 \left\langle I_b'\left(\sum_{i=1}^{k+1} s_i^{\frac{1}{p}}\xi_i^{\bar{\boldsymbol{\rho}}_k} + \theta\tau\eta(\bar{\boldsymbol{s}})\phi\right), \tau\eta(\bar{\boldsymbol{s}})\phi\right\rangle d\theta$$
$$\le I_b\left(\sum_{i=1}^{k+1} \bar s_i^{\frac{1}{p}}\xi_i^{\bar{\boldsymbol{\rho}}_k}) - \tau\eta(\bar{\boldsymbol{s}}\right).$$

If $\boldsymbol{s} \in B_{\frac{\tau}{2}}(\mathbf{1})$ for each i, then $\eta(\bar{\boldsymbol{s}}) > 0$, by Corollary 3.2

$$J_b(\bar g(\bar{\boldsymbol{s}})) < I_b\left(\sum_{i=1}^{k+1} \bar s_i^{\frac{1}{p}}\xi_i^{\bar{\boldsymbol{\rho}}_k}\right) = J_b(\bar s_1^{\frac{1}{p}}\xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, \bar s_{k+1}^{\frac{1}{p}}\xi_{k+1}^{\bar{\boldsymbol{\rho}}_k})$$
$$\le J_b(\xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, \xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}) = d(\bar{\boldsymbol{\rho}}_k) = c_k,$$

and if $\boldsymbol{s} \notin B_{\frac{\tau}{2}}(\mathbf{1})$, then $\eta(\bar{\boldsymbol{s}}) = 0$, by corollary 3.2

$$J_b(\bar g(\bar{\boldsymbol{s}})) = I_b\left(\sum_{j=1}^{k+1} \bar s_j^{\frac{1}{p}}\xi_j^{\bar{\boldsymbol{\rho}}_k}\right) = J_b\left(\bar s_1^{\frac{1}{p}}\xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, \bar s_{k+1}^{\frac{1}{p}}\xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}\right)$$
$$< J_b(\xi_1^{\bar{\boldsymbol{\rho}}_k}, \ldots, \xi_{k+1}^{\bar{\boldsymbol{\rho}}_k}) = d(\bar{\boldsymbol{\rho}}_k) = c_k,$$

which also contradicts to (3.9). Therefore, the function u_k^b is a solution of (1.3), such that $J_b(u_k^b) = c_k$. The proof of Theorem 1.1 is complete.

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