J. Fixed Point Theory Appl. (2022) 24:11
https://doi.org/10.1007/s11784-021-00927-4
Published online January 3, 2022

Journal of Fixed Point Theory and Applications

A unified approach to the Nash equilibrium existence in large games from finitely many players to infinitely many players

Zhe Yang and Qingping Song

Abstract. In this paper, we use the property of compact sets to propose a unified approach to the Nash equilibrium existence in large games. Our main technique is to convert the large game into a game with finitely many players. We shall use the method to prove the existence theorems of Nash equilibria in different games with infinitely many players.

Mathematics Subject Classification. 91A06, 91A07, 91A10.

Keywords. Large game, Nash equilibrium, existence, a unified approach.

1. Introduction

The purpose of our paper is to propose a unified approach to the Nash equilibrium existence in large games by aid of the equilibrium existence theorems with finitely many players. Nash [9] first proved the existence of noncooperative equilibria in n-person games by the fixed point theorem. Following the work of Nash [9], Shafer and Sonnenschein [14] showed the existence of equilibria in generalized games with nonordered preferences and feasible-strategy correspondences.

Nash [9] and Shafer and Sonnenschein [14] studied the model with finitely many players. As a development, the existence of Nash equilibria was extended to allow infinitely many players. For the model with infinitely many players, there exist two aspects. First, Aumann [3] introduced the economy with a measure space of players, and Aumann [4] analyzed the existence of competitive equilibria in a market with a continuum of agents. Following the work of Aumann [3,4], by assuming that the set of players is a measure space, the existence theorems of Nash equilibria were studied by Balder [5],

This study is supported by the Fundamental Research Funds for the Central Universities of China (nos. 2020110139, 2019110312) and the Independent Scientific Research Project of Colleges for Shanghai University of Finance and Economics (Nos. 2014110323, 2021110348).

Khan et al. [8], Noguchi [10], Rath [11], Schemeidler [13] and so on. Second, Yannelis and Prabhakar [22] analyzed the maximal element theorem and gave an existence theorem of Nash equilibria in games with a countable number of players. Furthermore, Yuan [24] proved the existence of equilibria in games with any nonempty set of players. The work was also analyzed by Deguire et al. [6]. Note that the proofs used the maximal element theorem and fixed point theorem. Later, Salonen [12] used a new proof method to obtain the existence of Nash equilibria in large games. Salonen [12] assumed that the set of players is an infinite index set and completed the proof using the n-person game as approximations.

Recently, following the work of Weber [16], an argument line was considered to analyze the cooperative equilibria by Askoura [1,2], Yang [17–19], Yang and Yuan [20] and Yang and Zhang [21], where the game with infinitely many players is converted into a game with finitely many players. Inspired by above work, we propose a unified approach to the Nash equilibrium existence in games with infinitely many players. Inspired by Deguire et al. [6], Yannelis and Prabhakar [22] and Yuan [24], we assume that the set of players is a nonempty infinite index set, where the measure or topology is removed. We shall convert the large game into a game with finitely many players, and prove the existence theorems of Nash equilibria by the existence theorems with finitely many players. Note that some Nash equilibrium existence theorems with infinitely many players had been proved in the past. However, our main contribution is to propose a unified approach to the Nash equilibrium existence in large games with aid of the existence results with finitely many players.

The rest of our paper is organized as follows. Section 2 recalls the models and results of games with finitely many players. Section 3 propose a unified approach to the Nash equilibrium existence in large games. Section 4 is the conclusion.

2. The models and results of games with finitely many players

2.1. Normal-form games

A normal-form game is defined by

$$\Gamma = (N, (X_i, u_i)_{i\in N}),$$

where $N = \{1, \ldots, n\}$ is the set of players; X_i is the strategy set of player i,

$$X = \prod_{i\in N} X_i, \quad X_{-i} = \prod_{j\neq i} X_j;$$

$u_i : X \longrightarrow R$ is the utility function of player i.

A Nash equilibrium of $\Gamma = (N, (X_i, u_i)_{i\in N})$ is a point $x^* \in X$ satisfying that for any $i \in N$,

$$u_i(x_i^*, x_{-i}^*) = \max_{y_i\in X_i} u_i(y_i, x_{-i}^*).$$

We give the following result, which is a special case of Theorem 3.2 in [23].

Theorem 2.1. *Suppose that a normal-form game* $\Gamma = (N, (X_i, u_i)_{i\in N})$ *satisfies the following conditions*:

(i) *for any* $i \in N$, X_i *is a nonempty convex compact subset of a Hausdorff topological vector space*;
(ii) *for any* $i \in N$, u_i *is continuous and* $u_i(\cdot, x_{-i})$ *is quasiconcave on* X_i *for any fixed* $x_{-i} \in X_i$.

Then there exists a Nash equilibrium.

2.2. Qualitative games

A qualitative game is defined by

$$\Gamma = (N, (X_i, P_i)_{i\in N}),$$

where N, $(X_i)_{i\in N}$ are defined as Sect. 2.1; $P_i : X \rightrightarrows X_i$ is the preference correspondence of player i.

A Nash equilibrium of the qualitative game $\Gamma = (N, (X_i, P_i)_{i\in N})$ is a point $x^* \in X$ satisfying that

$$P_i(x^*) = \emptyset, \quad \forall i \in N.$$

We next give Theorem 2.2, which satisfies the conditions of Theorem 3 in [6].

Theorem 2.2. *Suppose that a qualitative game* $\Gamma = (N, (X_i, P_i)_{i\in N})$ *satisfies the following conditions*:

(i) *for any* $i \in N$, X_i *is a nonempty convex compact subset of a Hausdorff topological vector space*;
(ii) *for any* $i \in N$, $P_i^{-1}(y_i) = \{x \in X | y_i \in P_i(x)\}$ *is open in* X *for any* $y_i \in X_i$;
(iii) *for any* $i \in N$ *and any* $x \in X$, $P_i(x)$ *is convex and* $x_i \notin P_i(x)$.

Then there exists a Nash equilibrium.

2.3. Generalized games with payoff functions

A generalized game with payoff functions is defined by

$$\Gamma = (N, (X_i, u_i, G_i)_{i\in N}),$$

where N, $(X_i, u_i)_{i\in N}$ are defined as Sect. 2.1; $G_i : X \rightrightarrows X_i$ is the feasible-strategy correspondence of player i.

A Nash equilibrium of the generalized game $\Gamma = (N, (X_i, u_i, G_i)_{i\in N})$ is a point $x^* \in X$ satisfying that for any $i \in N$, $x_i^* \in G_i(x^*)$ and

$$u_i(x_i^*, x_{-i}^*) = \max_{y_i \in G_i(x^*)} u_i(y_i, x_{-i}^*).$$

Theorem 2.3. *Suppose that a generalized game* $\Gamma = (N, (X_i, u_i, G_i)_{i\in N})$ *satisfies the following conditions*:

(i) *for any* $i \in N$, X_i *is a nonempty convex compact subset of a locally convex Hausdorff topological vector space*;
(ii) *for any* $i \in N$, u_i *is continuous and* $u_i(\cdot, x_{-i})$ *is quasiconcave on* X_i *for any* $x_{-i} \in X_{-i}$;
(iii) *for any* $i \in N$, G_i *is continuous with nonempty convex compact values.*

Then there exists a Nash equilibrium.

Theorem 2.3 is an extension of Theorem 4.3.1 in [7] to generalized games with strategy spaces defined on locally convex Hausdorff topological vector spaces. Theorem 2.3 can be proved by replacing Kakutani's fixed-point theorem by Fan–Glicksberg fixed-point theorem in the proof of Theorem 4.3.1 [7].

2.4. Generalized games with nonordered preferences

A generalized game with nonordered preferences is defined by

$$\Gamma = (N, (X_i, P_i, G_i)_{i\in N}),$$

where N, $(X_i, G_i)_{i\in N}$ are defined as Sect. 2.3, and $(P_i)_{i\in N}$ is defined as Sect. 2.2.

A Nash equilibrium of the generalized game $\Gamma = (N, (X_i, P_i, G_i)_{i\in N})$ is a point $x^* \in X$ satisfying that for any $i \in N$, $x_i^* \in G_i(x^*)$ and

$$P_i(x^*) \cap G_i(x^*) = \emptyset.$$

Theorem 2.4. *Suppose that a generalized game* $\Gamma = (N, (X_i, P_i, G_i)_{i\in N})$ *satisfies the following conditions*:

(i) *for any* $i \in N$, X_i *is a nonempty convex compact subset of a locally convex Hausdorff topological vector space*;
(ii) *for any* $i \in N$, P_i *has an open graph with convex values*, *and* $x_i \notin P_i(x)$ *for any* $x \in X$;
(iii) *for any* $i \in N$, G_i *is continuous with nonempty convex compact values.*

Then there exists a Nash equilibrium.

Theorem 2.4 satisfies the conditions of Corollary 5.123.1 in [15], which is an extension of [14] to generalized games with strategy spaces defined on locally convex Hausdorff topological vector spaces.

3. Main results

By the property of compact sets, we propose a unified approach to the Nash equilibrium existence in games with infinitely many players. A formation is defined by a list

$$(I, X, (F_i)_{i\in I}),$$

where I is an infinite index set; X is the strategy space, which is a compact subset of a Hausdorff topological vector space E; $F_i \subset X$ is the set of strategies that cannot be improved upon by the index i.

An equilibrium is a point $x^* \in X$ satisfying that x^* cannot be improved upon by any $i \in I$, that is,

$$x^* \in \bigcap_{i\in I} F_i.$$

To obtain the existence of equilibria, we have the following steps.

(1) For any $i \in I$, we show that F_i is closed in X.

(2) For any finite set $N \subset I$, we construct a game with finitely many players, and use the equilibrium existence theorem with finitely many players to show that there exists $x^* \in X$ such that

$$x^* \in \bigcap_{i \in N} F_i.$$

(3) By applying the compactness of X, we complete the proof, that is,

$$\bigcap_{i \in I} F_i \neq \emptyset.$$

We next apply above technique to give the Nash equilibrium existence theorem in different games with infinitely many players.

Theorem 3.1. *Suppose that a normal-form game* $\Gamma = (N, (X_i, u_i)_{i \in N})$ *satisfies the following conditions*:

(i) N *is an infinite index set*;
(ii) *for any* $i \in N$, X_i *is a nonempty convex compact subset of a Hausdorff topological vector space* E_i;
(iii) *for any* $i \in N$, u_i *is continuous and* $u_i(\cdot, x_{-i})$ *is quasiconcave on* X_i *for any* $x_{-i} \in X_{-i}$.

Then there exists a Nash equilibrium.

Proof. By the unified approach, we establish a list

$$(N, X, (F_i)_{i \in N}),$$

where $X = \prod_{i \in N} X_i$ is a nonempty convex compact subset of a Hausdorff topological vector space, and

$$F_i = \{x \in X | u_i(x) \geq u_i(y_i, x_{-i}),\ \forall y_i \in X_i\}.$$

(1) Since u_i is continuous on X for any $i \in N$, it follows that

$$X \backslash F_i = \bigcup_{y_i \in X_i} \{x \in X | u_i(y_i, x_{-i}) > u_i(x)\}$$

is open in X, implying that F_i is closed in X for any $i \in N$.

(2) For any finite set $\overline{N} = \{i_1, \ldots, i_n\} \subset N$, we shall show that

$$\bigcap_{j=1}^{n} F_{i_j} \neq \emptyset.$$

We first pick and fix a point $\overline{x}_{-\overline{N}} \in X_{-\overline{N}}$ and construct a normal form game

$$\overline{\Gamma} = (\overline{N}, (Y_j, \overline{u}_j)_{j \in \overline{N}})$$

as follows:

(a) for any $j \in \overline{N}$, $Y_j = X_{i_j}$ is a nonempty convex compact subset of a Hausdorff topological vector space E_{i_j}; denote

$$Y = \prod_{l \in \overline{N}} Y_l, \quad Y_{-l} = \prod_{l' \in \overline{N} \backslash \{l\}} Y_{l'};$$

(b) for any $j \in \overline{N}$,

$$\overline{u}_j(x_{i_1}, \ldots, x_{i_n}) = u_{i_j}(x_{i_1}, \ldots, x_{i_n}, \overline{x}_{-\overline{N}}), \quad \forall (x_{i_1}, \ldots, x_{i_n}) \in Y.$$

By (iii), it is easy to verify that for any $j \in \overline{N}$, $\overline{u}_j$ is continuous on Y, and $\overline{u}_j(\cdot, x_{\overline{N}\setminus\{i_j\}})$ is quasiconcave on Y_j for any $x_{\overline{N}\setminus\{i_j\}} \in Y_{-j}$.

Then, $\overline{\Gamma}$ satisfies the conditions of Theorem 2.1. Thus, there exists

$$(x^*_{i_1}, \ldots, x^*_{i_n}) \in Y$$

such that for any $j \in \overline{N}$,

$$u_{i_j}(x^*_{i_1}, \ldots, x^*_{i_n}, \overline{x}_{-\overline{N}}) = \max_{y_{i_j} \in X_{i_j}} u_{i_j}(y_{i_j}, x^*_{\overline{N}\setminus\{i_j\}}, \overline{x}_{-\overline{N}}),$$

implying that

$$(x^*_{i_1}, \ldots, x^*_{i_n}, \overline{x}_{-\overline{N}}) \in \bigcap_{j=1}^{n} F_{i_j}.$$

(3) By the compactness of X, we have

$$\bigcap_{i \in N} F_i \neq \emptyset.$$

This completes the proof. □

Theorem 3.2. *Suppose that a qualitative game* $\Gamma = (N, (X_i, P_i)_{i \in N})$ *satisfies the following conditions*:

(i) N *is an infinite index set*;
(ii) *for any* $i \in N$, X_i *is a nonempty convex compact subset of a Hausdorff topological vector space* E_i;
(iii) *for any* $i \in N$, $P_i^{-1}(y_i) = \{x \in X | y_i \in P_i(x)\}$ *is open in* X *for any* $y_i \in X_i$;
(iv) *for any* $i \in N$ *and any* $x \in X$, $P_i(x)$ *is convex and* $x_i \notin P_i(x)$.

Then there exists a Nash equilibrium, that is, there exists $\widetilde{x} \in X$ *such that* $P_i(\widetilde{x}) = \emptyset$, $\forall i \in N$.

Proof. By the unified method, we establish a list

$$(N, X, (F_i)_{i \in N}),$$

where $X = \prod_{i \in N} X_i$ is a nonempty convex compact subset of a Hausdorff topological vector space $E = \prod_{i \in N} E_i$, and

$$F_i = \{x \in X | P_i(x) = \emptyset\}.$$

(1) For any $i \in N$, by (iii),

$$X \setminus F_i = \bigcup_{y_i \in X_i} P_i^{-1}(y_i)$$

is open in X, implying that F_i is closed in X.

(2) For any finite set $\overline{N} = \{i_1, \ldots, i_n\} \subset N$, we pick and fix a point $\overline{x}_{-\overline{N}} \in X_{-\overline{N}}$. We next construct a qualitative game

$$\overline{\Gamma} = (\overline{N}, (Y_j, \overline{P}_j)_{j \in \overline{N}}),$$

where

(a) for any $j \in \overline{N}$, $Y_j = X_{i_j}$ is a nonempty convex compact subset of a Hausdorff topological vector space E_{i_j};

(b) for any $j \in \overline{N}$, $\overline{P}_j : Y \rightrightarrows Y_j$ is defined by

$$\overline{P}_j(x_{i_1}, \ldots, x_{i_n}) = P_{i_j}(x_{i_1}, \ldots, x_{i_n}, \overline{x}_{-\overline{N}}), \quad \forall (x_{i_1}, \ldots, x_{i_n}) \in Y.$$

It is easy to verify that $\overline{P}_j^{-1}(y_j)$ is open in Y, $\overline{P}_j(x_{i_1}, \ldots, x_{i_n})$ is convex, and $x_{i_j} \notin \overline{P}_j(x_{i_1}, \ldots, x_{i_n})$ for any $(x_{i_1}, \ldots, x_{i_n}) \in Y$ by (iii) and (iv). Then, $\overline{\Gamma}$ satisfies the conditions of Theorem 2.2. Thus, there exists

$$(x^*_{i_1}, \ldots, x^*_{i_n}) \in Y$$

such that

$$P_{i_j}(x^*_{i_1}, \ldots, x^*_{i_n}, \overline{x}_{-\overline{N}}) = \emptyset, \quad \forall j \in \overline{N},$$

implying that

$$(x^*_{i_1}, \ldots, x^*_{i_n}, \overline{x}_{-\overline{N}}) \in \bigcap_{j=1}^{n} F_{i_j}.$$

(3) By the compactness of X, we have

$$\bigcap_{i \in N} F_i \neq \emptyset.$$

This completes the proof. □

Theorem 3.3. *Suppose that a generalized game* $\Gamma = (N, (X_i, u_i, G_i)_{i \in N})$ *satisfies the following conditions*:

(i) N *is an infinite index set*;

(ii) *for any* $i \in N$, X_i *is a nonempty convex compact subset of a locally convex Hausdorff topological vector space* E_i;

(iii) *for any* $i \in N$, u_i *is continuous on* X *and* $u_i(\cdot, x_{-i})$ *is quasiconcave on* X_i *for any* $x_{-i} \in X_{-i}$;

(iv) *for any* $i \in N$, G_i *is continuous with nonempty convex compact values.*

Then there exists a Nash equilibrium.

Proof. By the unified approach, we establish a list

$$(N, X, (F_i)_{i \in N}),$$

where $X = \prod_{i \in N} X_i$ is a nonempty convex compact subset of a locally convex Hausdorff topological vector space, and

$$F_i = \{x \in X | x_i \in G_i(x),\ u_i(x) \geq u_i(y_i, x_{-i}),\ \forall y_i \in G_i(x)\}.$$

(1) Let $x^m \in F_i$ with $x^m \longrightarrow x \in X$. By (iii) and (iv), it follows that $x_i \in G_i(x)$. Since

$$u_i(x^m) = \max_{y_i \in G_i(x^m)} u_i(y_i, x^m_{-i}),$$

by Lemma 2.5 of Yu [23], we have that as $m \longrightarrow +\infty$,

$$u_i(x) = \max_{y_i \in G_i(x)} u_i(y_i, x_{-i}),$$

implying that F_i is closed in X for any $i \in N$.

(2) For any finite set $\overline{N} = \{i_1, \ldots, i_n\} \subset N$, we shall prove that

$$\bigcap_{j=1}^{n} F_{i_j} \neq \emptyset.$$

We first pick and fix a point $\overline{x}_{-\overline{N}} \in X_{-\overline{N}}$, and construct a generalized game

$$\overline{\Gamma} = (N, (Y_j, \overline{u}_j, \overline{G}_j)_{j\in\overline{N}})$$

as follows:

(a) for any $j \in \overline{N}$, $Y_j = X_{i_j}$ is a nonempty convex compact subset of a locally convex Hausdorff topological vector space E_{i_j};

(b) for any $j \in \overline{N}$,

$$\overline{u}_j(x_{i_1}, \ldots, x_{i_n}) = u_{i_j}(x_{i_1}, \ldots, x_{i_n}, \overline{x}_{-\overline{N}}), \quad \forall (x_{i_1}, \ldots, x_{i_n}) \in Y,$$
$$\overline{G}_j(x_{i_1}, \ldots, x_{i_n}) = G_{i_j}(x_{i_1}, \ldots, x_{i_n}, \overline{x}_{-\overline{N}}), \quad \forall (x_{i_1}, \ldots, x_{i_n}) \in Y.$$

By (iii) and (iv), it is easy to verify that for any $j \in \overline{N}$, $\overline{u}_j$ is continuous on Y, $\overline{u}_j(\cdot, x_{\overline{N}\setminus\{i_j\}})$ is quasiconcave for any $x_{\overline{N}\setminus\{i_j\}} \in Y_{-j}$, and $\overline{G}_j$ is continuous with nonempty convex compact values.

Then, $\overline{\Gamma}$ satisfies the conditions of Theorem 2.3. Therefore, there exists

$$(x^*_{i_1}, \ldots, x^*_{i_n}) \in Y$$

such that for any $j \in \overline{N}$,

$$x^*_{i_j} \in G_{i_j}(x^*_{i_1}, \ldots, x^*_{i_n}, \overline{x}_{-\overline{N}}),$$
$$u_{i_j}(x^*_{i_1}, \ldots, x^*_{i_n}, \overline{x}_{-\overline{N}}) = \max_{y_{i_j} \in G_{i_j}(x^*_{i_1}, \ldots, x^*_{i_n}, \overline{x}_{-\overline{N}})} u_{i_j}(y_{i_j}, x^*_{\overline{N}\setminus\{i_j\}}, \overline{x}_{-\overline{N}}),$$

implying that

$$(x^*_{i_1}, \ldots, x^*_{i_n}, \overline{x}_{-\overline{N}}) \in \bigcap_{j=1}^{n} F_{i_j}.$$

(3) By the compactness of X, we have

$$\bigcap_{i\in N} F_i \neq \emptyset$$

This completes the proof. □

Theorem 3.4. *Suppose that a generalized game $\Gamma = (N, (X_i, P_i, G_i)_{i\in N})$ satisfies the following conditions*:

(i) *N is an infinite index set*;

(ii) *for any $i \in N$, X_i is a nonempty convex compact subset of a locally convex Hausdorff topological vector space E_i*;

(iii) *for any $i \in N$, P_i has an open graph with convex values and $x_i \notin P_i(x)$ for all $x \in X$*;

(iv) *for any $i \in N$, G_i is continuous with nonempty convex compact values.*

Then there exists a Nash equilibrium, that is, there exists $\widetilde{x} \in X$ such that for any $i \in N$, $\widetilde{x}_i \in G_i(\widetilde{x})$ and

$$P_i(\widetilde{x}) \cap G_i(\widetilde{x}) = \emptyset.$$

Proof. By the unified approach, we establish a list

$$(N, X, (F_i)_{i\in N}),$$

where $X = \prod_{i\in N} X_i$ is a nonempty convex compact subset of a locally convex Hausdorff topological vector space, and

$$F_i = \{x \in X | x_i \in G_i(x),\ P_i(x) \cap G_i(x) = \emptyset\}.$$

(1) Let $x^m \in F_i$ with $x^m \longrightarrow x \in X$. By (iii) and (iv), it follows that $x_i \in G_i(x)$. By contrast, suppose that there exists $y_i \in X_i$ such that

$$y_i \in P_i(x) \cap G_i(x).$$

Since G_i is continuous, there exists $\{y_i^m\}$ of X_i such that $y_i^m \in G_i(x^m)$ for any sufficiently large m and $y_i^m \longrightarrow y_i$. By the openness of $Graph(P_i)$, for any sufficiently large m, we have

$$y_i^m \in P_i(x^m) \cap G_i(x^m),$$

which contradicts that $x^m \in F_i$. Thus, $x_i \in F_i$, implying that F_i is closed in X for any $i \in N$.

(2) For any finite set $\overline{N} = \{i_1, \dots, i_n\} \subset N$, we pick and fix a point $\overline{x}_{-\overline{N}} \in X_{-\overline{N}}$. We next construct a generalized game

$$\overline{\Gamma} = (\overline{N}, (X_j, \overline{P}_j, \overline{G}_j)_{j\in\overline{N}}),$$

where

(a) for any $j \in \overline{N}$, $Y_j = X_{i_j}$ is a nonempty convex compact subset of a locally convex Hausdorff topological vector space E_{i_j};

(b) for any $j \in \overline{N}$,

$$\overline{P}_j(x_{i_1}, \dots, x_{i_n}) = P_{i_j}(x_{i_1}, \dots, x_{i_n}, \overline{x}_{-\overline{N}}), \quad \forall (x_{i_1}, \dots, x_{i_n}) \in Y,$$
$$\overline{G}_j(x_{i_1}, \dots, x_{i_n}) = G_{i_j}(x_{i_1}, \dots, x_{i_n}, \overline{x}_{-\overline{N}}), \quad \forall (x_{i_1}, \dots, x_{i_n}) \in Y.$$

It is easy to verify that $\overline{P}_j$ has an open graph with convex values and $x_{i_j} \notin \overline{P}_j(x_{i_1}, \dots, x_{i_n})$ for any $(x_{i_1}, \dots, x_{i_n}) \in Y$ and any $j \in \overline{N}$; $\overline{G}_j$ is continuous with nonempty convex compact values for any $j \in \overline{N}$.

Then $\overline{\Gamma}$ satisfies the conditions of Theorem 2.4. Therefore, there exists $(x^*_{i_1}, \dots, x^*_{i_n}) \in Y$ such that for any $j \in \overline{N}$, $x^*_{i_j} \in G_{i_j}(x^*_{i_1}, \dots, x^*_{i_n}, \overline{x}_{-\overline{N}})$ and

$$P_{i_j}(x^*_{i_1}, \dots, x^*_{i_n}, \overline{x}_{-\overline{N}}) \cap G_{i_j}(x^*_{i_1}, \dots, x^*_{i_n}, \overline{x}_{-\overline{N}}) = \emptyset,$$

implying that

$$(x^*_{i_1}, \dots, x^*_{i_n}, \overline{x}_{-\overline{N}}) \in \bigcap_{j=1}^{n} F_{i_j}.$$

(3) By the compactness of X, we have

$$\bigcap_{i\in N} F_i \neq \emptyset.$$

This completes the proof. □

Remark 3.1. Theorem 3.2 is a special case of Theorem 3 in [6]. Theorem 3.3 also satisfies Theorem 4.53 in [15]. When $G_i(x) = X_i$ for any $x \in X$ and any $i \in N$, Theorem 3.3 reduces to Theorem 3.1. Moreover, Theorem 3.4 can be proved by Corollary 5.123.1 in [15].

4. Concluding remarks

We shall end the paper with the following remarks. First, our main contribution is to propose a unified approach to the Nash equilibrium existence in games with infinitely many players. Second, Theorems 3.1 and 3.2 have been analyzed in [6,22–24] by the maximal element theorem and fixed point theorem. In our paper, we follow the results of games with finitely many players and use a unified approach to reprove the Nash equilibrium existence theorems with infinitely many players. Note that we mainly emphasize the process of converting large games to games with finitely many players. Our paper is a development on the work of the Nash equilibrium existence in large games.

References

[1] Askoura, Y.: The weak-core of a game in normal form with a continuum of players. J. Math. Econ. **47**(1), 43–47 (2011)

[2] Askoura, Y.: On the core of normal form games with a continuum of players. Math. Soc. Sci. **89**, 32–42 (2017)

[3] Aumann, R.J.: Markets with a continuum of traders. Econometrica **32**, 39–50 (1964)

[4] Aumann, R.J.: Existence of competitive equilibria in markets with a continuum of traders. Econometrica **34**, 1–17 (1966)

[5] Balder, E.J.: A unifying approach to existence of Nash equilibria. Int. J. Game Theory **24**, 79–94 (1995)

[6] Deguire, P., Tan, K.K., Yuan, X.Z.: The study of maximal elements, fixed points for LS-majorized mappings and their applications to minimax and variational inequalities in product topological spaces. Nonlinear Anal. **37**, 933–951 (1999)

[7] Ichiishi, I.: Game Theory for Economic Analysis. Academic Press, New York (1983)

[8] Khan, M.A., Rath, K.P., Sun, Y.: On the existence of pure strategy equilibria in games with a continuum of players. J. Econ. Theory **76**, 13–46 (1997)

[9] Nash, J.: Non-cooperative games. Ann. Math. **54**, 286–295 (1951)

[10] Noguchi, M.: Existence of Nash equilibria in large games. J. Math. Econ. **45**, 168–184 (2009)

[11] Rath, K.P.: A direct proof of the existence of pure strategy equilibria in games with a continuum of players. Econ. Theory **2**, 427–433 (1992)

[12] Salonen, H.: On the existence of Nash equilibria in large games. Int. J. Game Theory **39**, 351–357 (2010)

[13] Schmeidler, D.: Equilibrium points in nonatomic games. J. Stat. Phys. **7**, 295–300 (1973)

[14] Shafer, W., Sonnenschein, H.: Equilibrium in abstract economies without ordered preferences. J. Math. Econ. **2**(3), 345–348 (1975)

[15] Tarafdar, E.U., Chowdhury, M.S.R.: Topological Methods for Set-Valued Nonlinear Analysis. World Scientific Publishing Co. Pre. Ltd., Singapore (2008)

[16] Weber, S.: Some results on the weak core of a non-side-payment game with infinitely many players. J. Math. Econ. **8**(1), 101–111 (1981)

[17] Yang, Z.: Some infinite-player generalizations of Scarf's theorem: finite-coalition α-cores and weak α-cores. J. Math. Econ. **73**, 81–85 (2017)

[18] Yang, Z.: Some generalizations of Kajii's theorem to games with infinitely many players. J. Math. Econ. **76**, 131–135 (2018)

[19] Yang, Z.: The weak α-core of exchange economies with a continuum of players and pseudo-utilities. J. Math. Econ. **91**, 43–50 (2020)

[20] Yang, Z., Yuan, G.X.: Some generalizations of Zhao's theorem: hybrid solutions and weak hybrid solutions for games with nonordered preferences. J. Math. Econ. **84**, 94–100 (2019)

[21] Yang, Z., Zhang, X.: A weak α-core existence theorem of games with nonordered preferences and a continuum of agents. J. Math. Econ. **94**, 102464 (2021)

[22] Yannelis, N.C., Prabhakar, N.D.: Existence of maximal elements and equilibria in linear topological spaces. J. Math. Econ. **12**, 233–245 (1983)

[23] Yu, J.: Essential equilibria of n-person noncooperative games. J. Math. Econ. **31**(3), 361–372 (1999)

[24] Yuan, X.Z.: The study of equilibria for abstract economics in topological vector spaces-a unified approach. Nonlinear Anal., **37**(4), 409–430 (1999)

Zhe Yang and Qingping Song
School of Economics
Shanghai University of Finance and Economics
Shanghai 200433
People's Republic of China
e-mail: zheyang211@163.com

Zhe Yang
Key Laboratory of Mathematical Economics (SUFE)
Ministry of Education
Shanghai 200433
People's Republic of China

Accepted: December 15, 2021.