J. Fixed Point Theory Appl. (2022) 24:6
https://doi.org/10.1007/s11784-021-00924-7
Published online December 18, 2021

Journal of Fixed Point Theory and Applications

Singular solutions for nonlinear elliptic equations on bounded domains

Sidi Hamidou Jah and Lotfi Riahi

Abstract. We consider the nonlinear elliptic equation $\Delta u + V(x)u + f(x, u(x)) = 0$ on $D \setminus \{0\}$, where D is a bounded domain containing 0 in $\mathbb{R}^n$, $n \geq 2$, and V and f are Borel measurable functions. Under general conditions on the functions V and f, we prove the existence of positive singular solutions globally comparable to the Dirichlet Green's function of the Laplacian with pole at the origin. Our result applies to various types of semilinear equations, in particular to $\Delta u + W(x)u^p = 0$ for all real exponent p which was extensively studied for the range $p > 1$. Moreover for this equation with sign-unchanging function W our condition is the optimal one.

Mathematics Subject Classifications. 35J60, 35B40.

Keywords. Nonlinear elliptic equation, Dirichlet boundary condition, positive solution, singular solution, Kato class, asymptotic behavior.

1. Introduction

There is a vast literature on the topic of isolated singularities for second order nonlinear elliptic equations (see [1–4, 6–8, 11–13, 17–21] and the references therein). In particular complete classification of singularities of positive solutions has been established for various types of semilinear elliptic equations. Véron [19] and Brezis and Véron [4] studied the equation with absorption term $\Delta u = u^p$ in $\Omega \setminus \{0\}$, where Ω is a bounded domain containing 0 in $\mathbb{R}^n$, $n \geq 1$, in the superlinear case $p > 1$ which appears in the modeling of many physical phenomena. They proved that, for $p \geq n/(n-2)$, $n \geq 3$, any singularity is removable, i.e. u is a classical solution on the whole domain Ω; and for $1 < p < n/(n-2)$, $n \geq 3$, we have a trichotomy of nonnegative solutions (u is either with removable singularity or $u(x) \sim E(x)$ near 0, i.e. u blows up at the origin with the same speed as the fundamental solution $E(x)$ of $-\Delta$ (weak singularity) or $u(x) \sim |x|^{-2/(p-1)}$ near 0 (strong singularity)). See also Brezis and Oswald [3] for a simpler proof of the result. The equation with source term $-\Delta u = u^p$ in $\Omega \setminus \{0\}$ which behaves differently has been studied by Lions [12], Gidas and Spruck [8] and Aviles [1], and all possible singularities are described. For the two equations, with source or absorption term, solutions with weak singularity do not exist in the case $p \geq n/(n-2)$, $n \geq 3$. Ni [13] also proved that the equation $\Delta u + K(x)u^p = 0$ on $\Omega \setminus \{0\}$ with K satisfying some integrability condition and $p > 1$, does not possess any positive solution with an isolated singularity at the origin. His result covers the case where $C_1|x|^\sigma \leq K(x) \leq C_2|x|^\sigma$ for some positive constants C_1, C_2 and $\sigma \leq -2$. Vazquez and Véron [18] studied the equation $-\Delta u + g(u) = 0$ in $\Omega \setminus \{0\}$ where g is a nondecreasing real-valued function and they proved that solutions with weak singularity exist if and only if g satisfies for some $\alpha > 0$, $\int_\alpha^\infty g(t)t^{-2(n-1)/(n-2)}dt < \infty$ (see also Véron [20]). Existence of weak and strong singularities is also proved by Cîrstea and Du [6] for the equation $\Delta u = h(u)$ in $\Omega \setminus \{0\}$, where h is a positive locally Lipschitz continuous function on $[0, \infty)$ which is regularly varying at infinity of index $q \in (1, n/(n-2))$. Necessary and sufficient conditions for the existence of solutions with weak singularity are also proved by Brandolini et al. [2] for the equation $-\mathrm{div}(A(|x|)\nabla u) + u^p = 0$ in $B_1(0) \setminus \{0\}$ where $p > 1$. Recall also that the existence of positive solutions with weak singularity subject to a Dirichlet boundary condition is investigated in [2,3,7,11,17,21]. In particular, Zhang and Zhao [21] proved the existence of solutions, which are globally comparable to the Dirichlet Green's function of the Laplacian, for the equation $\Delta u + W(x)u^p = 0$ on $D \setminus \{0\}$, where D is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$, $p > 1$ and W is in the Kato class K_n. Their result was later extended by the second author in [17] to a more general class of functions W and a more general non-smooth domain D (NTA-domain) in $\mathbb{R}^n$, $n \geq 2$. In this paper, we aim to prove the existence of solutions with the same type of singularity for the more general nonlinear problem:

$$\begin{cases} \Delta u(x) + V(x)u(x) + f(x, u(x)) = 0, \quad x \in D \setminus \{0\}, \\ u(x) > 0, \quad x \in D \setminus \{0\}, \\ u(x) \sim G(x, 0) \quad \text{near } x = 0, \\ u(x) = 0, \quad x \in \partial D, \end{cases} \tag{1.1}$$

where Δ is the Laplace operator and D is a bounded domain containing the origin point 0 in $\mathbb{R}^n$, $n \geq 2$ which is assumed to be regular for the Dirichlet problem with Green's function G. In particular, this covers smooth domains and a wide range of nonsmooth domains. The functions V and f are Borel measurable satisfying some general conditions related to a functional class $K_n(D)$ defined below and which properly contains the Kato class K_n used in [21]. Solutions of problem (1.1) are understood in the distribution sense. Under our conditions to be specified later, these solutions are continuous except at $x = 0$. Solutions with weak singularity fail to exist in many situations as it is explained before, for the special equations $\Delta u = \pm u^p$ with $p \geq n/(n-2)$, $n \geq 3$; and for the equation $\Delta u + |x|^\sigma u^p = 0$ in $B_1(0) \setminus \{0\}$ with either $1 < (n+\sigma)/(n-2) < p < (n+2)/(n-2)$, $n \geq 3$ and $-2 < \sigma < 2$ or $p > 1$ and $\sigma \leq -2$ (see [8] and [13]). See also examples in Sect. 4 where solutions with strong singularity exist and solutions with weak singularity do not exist. Our approach is based on arguments from potential theory and the Schauder fixed point theorem. Our results apply to various types of nonlinear equations, in particular to $\Delta u + W(x)u^p = 0$ for all exponent $p \in \mathbb{R}$ (not restricted to the superlinear case $p > 1$ as in the above mentioned papers). Moreover, for this equation with sign-unchanging function W our condition is the optimal one.

To state our main result, we first define the functional class $K_n(D)$.

Definition 1.1. A function $V \in L^1_{\text{loc}}(D)$ is said to be in the class $K_n(D)$ if it satisfies

$$\lim_{r\to 0} \sup_{x\in\overline{D}} \int_{D\cap(|y-x|<r)} \frac{G(x,y)G(y,0)}{G(x,0)} |V(y)|dy = 0, \tag{1.2}$$

where $G(x,y)$ is the Green's function of the operator $-\Delta$ with the homogeneous Dirichlet boundary condition on D (note that for each $y \in D$ the function $x \to G(x,y)/G(x,0) = G(y,x)/G(0,x)$ has a Hölder continuous extension to $\overline{D}$ and its value at $\xi \in \partial D$ is given by $M(y,\xi)$, where $M(.,\xi)$ is the Martin kernel at ξ).

Clearly, since D is bounded, by a compactness argument if $V \in K_n(D)$, then

$$\|V\| := \sup_{x\in\overline{D}} \int_D \frac{G(x,y)G(y,0)}{G(x,0)} |V(y)|dy < \infty. \tag{1.3}$$

Throughout this work, the function V belongs to the class $K_n(D)$ and according to our purpose the nonlinear term f is assumed to satisfy (H_1) and (H_2) or (H_1) and (H_2') below.

(H_1) $f : D \times (0,\infty) \to \mathbb{R}$, $(x,t) \to f(x,t)$ is Borel measurable and continuous with respect to t, and

(H_2) there is a Borel measurable function $\psi : D \times (0,\infty) \to [0,\infty)$ such that

- for each x, $t \to \psi(x,t)$ is nondecreasing on $(0,\infty)$ and $\lim_{t\to 0^+} \psi(x,t) = 0$,
- $|f(x,t)| \le t\psi(x,t), \quad$ for all $t \ge 0,\ x \in D$,
- $\psi(x, G(x,0)) \in K_n(D)$.

(H_2') there is a Borel measurable function $\psi : D \times (0,\infty) \to [0,\infty)$ such that

- for each x, $t \to \psi(x,t)$ is decreasing on $(0,\infty)$ and $\lim_{t\to\infty} \psi(x,t) = 0$,
- $|f(x,t)| \le t\psi(x,t), \quad$ for all $t \ge 0,\ x \in D$,
- $\psi(x, G(x,0)) \in K_n(D)$.

Examples. 1. Let $W \in L^1_{\text{loc}}(D)$ and $p \in \mathbb{R}$ such that $W(x)(G(x,0))^{p-1} \in K_n(D)$. Then $f(x,t) = W(x)t^p g(t)$, where g is a bounded continuous function on $(0,\infty)$, satisfies (H_1) and (H_2) when $p > 1$, and it satisfies (H_1) and (H_2') when $p < 1$. The power nonlinearity $f(x,t) = W(x)t^p$ which extensively studied corresponds to $g = 1$. Very complicated examples such as $g(t) = \cos(h(t))$ or $g(t) = \sin(h(t))$ with h any continuous function on $(0,\infty)$, are also allowed.

2. Let $W \in L^1_{\rm loc}(D)$ and $p \in \mathbb{R}$ such that $W(x)(G(x,0))^{p+\alpha-1} \in K_n(D)$, $\alpha \geq 0$. Then $f(x,t) = W(x)t^p(\ln(1+t))^\alpha$ satisfies (H_1) and (H_2) when $p+\alpha > 1$, and it satisfies (H_1) and (H_2') when $p+\alpha < 1$.
Our main result is the following.

Theorem 1.1. *Assume that $V \in K_n(D)$ with $\|V\| < 1/3$ and f satisfies (H_1) and (H_2) (resp. (H_1) and (H_2')). Then there exists a number $\lambda_0 \in (0,1)$ (resp. $\lambda_0 > 1$) such that for $0 < \lambda < \lambda_0$ (resp. $\lambda > \lambda_0$), problem (1.1) has a positive solution u satisfying*

$$\frac{u}{G(.,0)} \in C(\overline{D}), \qquad \lambda/2 \leq \frac{u}{G(.,0)} \leq 3\lambda/2 \qquad \text{and} \quad \lim_{x\to 0} \frac{u(x)}{G(x,0)} = \lambda. \tag{1.4}$$

Remark 1. By Theorem 1.1, problem (1.1) with the power nonlinearity $f(x,u) = W(x)u^p$ with $W \in L^1_{\rm loc}(D)$ and $p \in \mathbb{R}$, has a positive solution u satisfying (1.4) if $WG^{p-1}(.,0)$ is in the class $K_n(D)$. The following theorem shows that this condition is the optimal one for the existence of such solutions, whenever W is sign-unchanging.

Theorem 1.2. *Let $W \in L^1_{\rm loc}(D)$ which it does not change sign. Assume that problem (1.1) with $V = 0$ and $f(x,u) = W(x)u^p$ has a positive solution u satisfying (1.4). Then the function $WG^{p-1}(.,0)$ is in the class $K_n(D)$.*

Theorems 1.1 and 1.2 together with Lemma 2.3 below yield the following result on smooth domains.

Corollary 1.3. *Consider the problem:*

$$\begin{cases} \Delta u(x) + V(x)u \pm u^p = 0, & x \in D\setminus\{0\}, \\ u(x) > 0, & x \in D\setminus\{0\}, \\ u(x) \sim G(x,0) & \text{near } x = 0, \\ u(x) = 0, & x \in \partial D, \end{cases} \tag{1.5}$$

where D is a bounded $C^{1,\gamma}$-domain with $\gamma \in]0,1]$ in $\mathbb{R}^n$, $n \geq 2$, $p \in \mathbb{R}$, and $V \in K_n(D)$ with $\|V\| < 1/3$. We have the following.

(1) *Assume $n \geq 3$. Then for $1 < p < n/(n-2)$ (resp. $-1 < p < 1$) there exists a number $\lambda_0 \in (0,1)$ (resp. $\lambda_0 > 1$) such that for any $0 < \lambda < \lambda_0$ (resp. $\lambda > \lambda_0$), problem (1.5) has a positive solution u satisfying (1.4). For either $p \leq -1$ or $n/(n-2) \leq p$, problem (1.5) has no positive solution satisfying (1.4).*

(2) *Assume $n = 2$. Then for $1 < p$ (resp. $-1 < p < 1$) there exists a number $\lambda_0 \in (0,1)$ (resp. $\lambda_0 > 1$) such that for any $0 < \lambda < \lambda_0$ (resp. $\lambda > \lambda_0$), problem (1.5) has a positive solution u satisfying (1.4). For $p \leq -1$, problem (1.5) has no positive solution satisfying (1.4).*

In Sect. 2, we give some preliminary results that will be needed to prove our results. In Sect. 3, we prove Theorems 1.1 and 1.2. In Sect. 4, we give some interesting examples of problems having singular solutions with different behaviors near the singularity.

2. Preliminary results

Throughout this work D is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, containing the origin point 0 which is regular for the Dirichlet problem. Let $G(x, y)$ be the Green's function of the operator $-\Delta$ with the Dirichlet boundary condition on D.

Lemma 2.1. *Let $V \in K_n(D)$. Then*

$$\int_D G(y,0)|V(y)|dy \leq \|V\|, \tag{2.1}$$

and the family of functions $\left\{\frac{G(x,y)G(y,0)}{G(x,0)}|V(y)|\right\}_{x\in\overline{D}}$ is uniformly integrable over D.

Proof. By [14, Theorem 2.1 in chapter 6] for any nonnegative superharmonic function s on D, there is an increasing sequence $(\mu_n)_n$ of measures on D such that

$$s(y) = \sup_n \int_D G(x,y)\mu_n(dx) \quad \text{for all } y \in D.$$

Moreover, by (1.3), for all $x \in D$,

$$\int_D G(x,y)G(y,0)V(y)dy \leq \|V\|G(x,0).$$

So multiplying by $\mu_n(dx)$ this inequality and integrating with respect to x, we get

$$\int_D s(y)G(y,0)V(y)dy \leq \|V\|s(0).$$

Take $s = 1$ we get (2.1). By (1.2) for a given $\varepsilon > 0$, there is a number $r > 0$ such that

$$\sup_{x\in\overline{D}} \int_{D\cap(|x-y|<r)} \frac{G(x,y)G(y,0)}{G(x,0)}|V(y)|dy < \varepsilon.$$

For a Borel set $A \subset D$ we then have

$$\begin{aligned}\int_A \frac{G(x,y)G(y,0)}{G(x,0)}|V(y)|dy &= \int_{A\cap(|x-y|<r)} ...dy + \int_{A\cap(|x-y|\geq r)} ...dy \\ &\leq \varepsilon + \frac{C}{r^n}\int_A G(y,0)|V(y)|dy.\end{aligned}$$

The last inequality holds using the estimate $G(x,y)/G(x,0) \leq C/|x-y|^n$ for all $x, y \in D$ for some constant $C = C(D,n) > 0$. By (2.1), it follows that

$$\lim_{m(A)\to 0} \sup_{x\in\overline{D}} \int_A \frac{G(x,y)G(y,0)}{G(x,0)}|V(y)|dy = 0,$$

where $m(A)$ denotes the Lebesgue measure of A.

Proposition 2.2. *Let $V \in K_n(D)$, $V \geq 0$. Let $\mathcal{F}$ be the collection of all Borel measurable functions $\Phi : D \to \mathbb{R}$, $|\Phi| \leq V$. For $\Phi \in \mathcal{F}$, let*

$$H_\Phi(x) = \int_D \frac{G(x,y)G(y,0)}{G(x,0)} \Phi(y)dy.$$

Then the family $\{H_\Phi,\, \Phi \in \mathcal{F}\}$ is uniformly bounded, equicontinuous on $\overline{D}$ and $H_\Phi(0) = 0$.

Proof. We have

$$|H_\Phi(x)| \leq \int_D \frac{G(x,y)G(y,0)}{G(x,0)} V(y)dy \leq \|V\|$$

and so $\{H_\Phi,\, \Phi \in \mathcal{F}\}$ is uniformly bounded. By Lemma 2.1, the family of functions $\left\{\frac{G(x,y)G(y,0)}{G(x,0)}|V(y)|\right\}_{x\in\overline{D}}$ is uniformly integrable over D. Moreover, for each $y \in D\setminus\{0\}$, the function $x \to \frac{G(x,y)G(y,0)}{G(x,0)}$ is continuous on $\overline{D}$. So by the Vitali convergence theorem we have, for all $x_0 \in \overline{D}$,

$$\lim_{x\to x_0} \int_D \left|\frac{G(x,y)}{G(x,0)} - \frac{G(x_0,y)}{G(x_0,0)}\right| G(y,0)V(y)dy = 0,$$

and so the family $\{H_\Phi,\, \Phi \in \mathcal{F}\}$ is equicontinuous on $\overline{D}$. We prove that $H_\Phi(0) = 0$. By (1.2), for each $\varepsilon > 0$ there is a number $r > 0$ such that

$$\sup_{x\in\overline{D}} \int_{D\cap(|y-x|<r)} \frac{G(x,y)G(y,0)}{G(x,0)} V(y)dy < \varepsilon.$$

Using the estimate $G(x,y) \leq C/|x-y|^n$ for all $x, y \in D$ for some constant $C = C(D,n) > 0$ and (2.1), it follows that

$$\begin{aligned}
|H_\Phi(x)| &\leq \int_D \frac{G(x,y)G(y,0)}{G(x,0)} V(y)dy \\
&= \int_{D\cap(|y-x|<r)} \frac{G(x,y)G(y,0)}{G(x,0)} V(y)dy \\
&\quad + \int_{D\cap(|y-x|\geq r)} \frac{G(x,y)G(y,0)}{G(x,0)} V(y)dy \\
&\leq \varepsilon + \frac{C}{r^n G(x,0)} \int_D G(y,0)V(y)dy \\
&\leq \varepsilon + \frac{C\|V\|}{r^n G(x,0)}.
\end{aligned}$$

Since $G(x,0) \to \infty$ as $x \to 0$, we obtain $H_\Phi(0) = 0$. □

The following lemma together with Theorems 1.1 and 1.2 yield Corollary 1.3.

Lemma 2.3. *Assume that D is a bounded $C^{1,\gamma}$-domain with $\gamma \in]0,1]$ in $\mathbb{R}^n$, $n \geq 2$ and $p \in \mathbb{R}$. Then we have the following.*

1. *For $n \geq 3$, $G^{p-1}(y,0) \in K_n(D)$ if and only if $-1 < p < n/(n-2)$.*
2. *For $n = 2$, $G^{p-1}(y,0) \in K_n(D)$ if and only if $-1 < p$.*

Proof. In what follows for two functions f and g we use the notation $f \approx g$ to mean that there is a constant $C > 1$ such that $C^{-1}g \leq f \leq Cg$.

(1) Assume that $n \geq 3$. By [16, Corollary 4.5] the Green's function G on the bounded $C^{1,\gamma}$-domain D in $\mathbb{R}^n$, $n \geq 3$, satisfies the following global estimates:

$$G(x,y) \approx \min\left\{1, \frac{\delta(x)\delta(y)}{|x-y|^2}\right\}\frac{1}{|x-y|^{n-2}},\ x,y \in D, \tag{2.2}$$

with a constant of comparison $C = C(D,n) > 0$, where $\delta(x) = d(x,\partial D)$ is the Euclidean distance from x to the boundary ∂D of D. Assume that $G^{p-1}(y,0) \in K_n(D)$. Then

$$\lim_{r\to 0}\sup_{x\in D}\int_{D\cap(|y-x|<r)} \frac{G(x,y)G^p(y,0)}{G(x,0)}dy = 0. \tag{2.3}$$

Since by (2.2)

$$\sup_{2r<|x|<3r}\int_{D\cap(|y-x|<r)} \frac{G(x,y)G^p(y,0)}{G(x,0)}dy \approx r^{n-(n-2)p}, \tag{2.4}$$

(2.3) yields $p < n/(n-2)$. Moreover, take $\xi \in \partial D$, let $x \to \xi$ in (2.3) and use (2.2), we get by Fatou's lemma,

$$\int_{D\cap(|y-\xi|<r)} \frac{\delta^{p+1}(y)}{|\xi-y|^n}dy < \infty.$$

Since D is a Lipschitz domain, there is a cone $V_\xi(\theta)$ with vertex ξ and aperture $\theta \in (0,\pi/2)$ congruent to $V_0(\theta) = \{y : y_n > |y|\cos\theta\}$ such that $V_\xi(\theta) \cap B(\xi,r) \subset D$ for r small. We have $|\xi-y|\sin(\theta/2) \leq \delta(y) \leq |\xi-y|$ for all $y \in V_\xi(\theta/2) \cap B(\xi,r)$. It follows that

$$\int_{V_\xi(\theta/2)\cap(|y-\xi|<r)} \frac{dy}{|\xi-y|^{n-p-1}} < \infty,$$

and so $-1 < p$. Conversely, assume that $-1 < p < n/(n-2)$. To prove $G^{p-1}(x,0) \in K_n(D)$ we treat two cases.

Case 1: $1 \leq p < n/(n-2)$.

Put $\rho = \delta(0) > 0$ and let $0 < r < \rho/8$. Using the global bounds (2.2) we obtain, for some positive constant $C = C(D,n,p)$,

$$\begin{aligned}\sup_{x\in\overline{D}}&\int_{D\cap(|y-x|<r)} \frac{G(x,y)G^p(y,0)}{G(x,0)}dy\\ &\leq \sup_{|x|\leq\rho/2}\int_{D\cap(|y-x|<r)} \frac{G(x,y)G^p(y,0)}{G(x,0)}dy\\ &\quad+\sup_{|x|>\rho/2}\int_{D\cap(|y-x|<r)} \frac{G(x,y)G^p(y,0)}{G(x,0)}dy\end{aligned}$$

$$\leq C \sup_{|x|\leq\rho/2}\int_{|y-x|<r}\frac{dy}{|y-x|^{(n-2)p}} + C\sup_{|x|>\rho/2}\int_{|y-x|<r}\frac{dy}{|y-x|^{n-1}}$$
$$\leq C(r^{n-(n-2)p}+r),$$

and so (1.2) is satisfied.

Case 2: $-1<p<1$.

Put $\rho=\delta(0)>0$ and let $0<r<\rho/8$. Using the global bounds (2.2) we obtain, for some positive constant $C=C(D,n,p)$,

$$\sup_{x\in\overline{D}}\int_{D\cap(|y-x|<r)}\frac{G(x,y)G^p(y,0)}{G(x,0)}dy$$
$$\leq \sup_{|x|\leq\rho/2}\int_{D\cap(|y-x|<r)}\frac{G(x,y)G^p(y,0)}{G(x,0)}dy$$
$$+\sup_{|x|>\rho/2}\int_{D\cap(|y-x|<r)}\frac{G(x,y)G^p(y,0)}{G(x,0)}dy$$
$$\leq C\sup_{|x|\leq\rho/2}\int_{|y-x|<r}\frac{dy}{|y-x|^{(n-2)p}} + C\sup_{|x|>\rho/2}\int_{|y-x|<r}\frac{dy}{|y-x|^{(n-2)p+2}}$$
$$\leq C(r^{n-(n-2)p}+r^{(n-2)(1-p)}),$$

and so (1.2) is satisfied.

(2) Assume that $n=2$. The $C^{1,\gamma}$-domain D is a Dini-smooth domain in $\mathbb{R}^2$ and so by [15] its Green's function G satisfies the following global estimates:

$$G(x,y)\approx \ln\Big(1+\frac{\delta(x)\delta(y)}{|x-y|^2}\Big),\ x,y\in D, \tag{2.5}$$

with a constant of comparison $C=C(D)>0$, where $\delta(x)=d(x,\partial D)$ is the Euclidean distance from x to the boundary ∂D of D. Assume that $G^{p-1}(y,0)\in K_2(D)$. Then

$$\lim_{r\to0}\sup_{x\in D}\int_{D\cap(|y-x|<r)}\frac{G(x,y)G^p(y,0)}{G(x,0)}dy=0. \tag{2.6}$$

Moreover, take $\xi\in\partial D$, let $x\to\xi$ in (2.6) and use (2.5) we get by Fatou's lemma,

$$\int_{D\cap(|y-\xi|<r)}\frac{\delta^{p+1}(y)}{|\xi-y|^2}dy<\infty,$$

which yields $-1<p$ as in (1). Conversely, assume that $-1<p$. To prove $G^{p-1}(x,0)\in K_2(D)$ we treat two cases.

Case 1: $1\leq p$.

Put $\rho=\delta(0)>0$ and let $0<r<\rho/8$. Using the global bounds (2.5) we obtain, for some positive constant $C=C(D,p)$,

$$\sup_{x\in\overline{D}}\int_{D\cap(|y-x|<r)}\frac{G(x,y)G^p(y,0)}{G(x,0)}dy$$
$$\leq \sup_{|x|\leq\rho/2}\int_{D\cap(|y-x|<r)}\frac{G(x,y)G^p(y,0)}{G(x,0)}dy$$

$$+ \sup_{|x|>\rho/2} \int_{D\cap(|y-x|<r)} \frac{G(x,y)G^p(y,0)}{G(x,0)} dy$$
$$\leq C \sup_{|x|\leq\rho/2} \int_{|y-x|<r} (\ln(\frac{1}{|x-y|}))^{p+2} dy + C \sup_{|x|>\rho/2} \int_{|y-x|<r} \frac{dy}{|y-x|}$$
$$\leq C(r^2(\ln r)^{p+3} + r),$$

and so (1.2) is satisfied.
Case 2: $-1 < p < 1$.
Put $\rho = \delta(0) > 0$ and let $0 < r < \rho/8$. Using the global bounds (2.5) we obtain, for some positive constant $C = C(D,p)$,

$$\sup_{x\in\overline{D}} \int_{D\cap(|y-x|<r)} \frac{G(x,y)G^p(y,0)}{G(x,0)} dy$$
$$\leq \sup_{|x|\leq\rho/2} \int_{D\cap(|y-x|<r)} \frac{G(x,y)G^p(y,0)}{G(x,0)} dy$$
$$+ \sup_{|x|>\rho/2} \int_{D\cap(|y-x|<r)} \frac{G(x,y)G^p(y,0)}{G(x,0)} dy$$
$$\leq C \sup_{|x|\leq\rho/2} \int_{|y-x|<r} \frac{dy}{|y-x|} + C \sup_{|x|>\rho/2} \int_{|y-x|<r} \frac{dy}{|y-x|^{(3-p)/2}}$$
$$\leq C(r + r^{(1+p)/2}),$$

and so (1.2) is satisfied. □

In what follows we prove that our class $K_n(D)$ is more general than the class $\mathcal{K}(D)$ considered in [16] and [17] in some special domains. First recall that in smooth domains and a wide range of nonsmooth domains the Green's function G satisfies the following $3G$-inequality (this includes bounded Lipschitz domains (see [16]) or more generally non-tangentially accessible (NTA for short) domains in the sense of [10] in $\mathbb{R}^n$, $n \geq 3$ (see [9]), and bounded Jordan domains in the plane $\mathbb{R}^2$ as defined in [5], i.e. domains with boundary ∂D consisting of finitely many disjoint closed Jordan curves (see [15])).

The $3G$-inequality: There exists a constant $C_0 = C_0(n,D) > 0$ such that for all $x,y,z \in D$,

$$\frac{G(x,y)G(y,z)}{G(x,z)} \leq C_0\Big[\frac{\varphi(y)}{\varphi(x)}G(x,y) + \frac{\varphi(y)}{\varphi(z)}G(y,z)\Big], \tag{2.7}$$

where $\varphi(x) = \min\{1, G(x,0)\}$, $x \in D$.

The class $\mathcal{K}(D)$ is defined as the set of all functions $V \in L^1_{\text{loc}}(D)$ satisfying:

$$\lim_{r\to 0} \sup_{x\in\overline{D}} \int_{D\cap(|y-x|<r)} \frac{\varphi(y)}{\varphi(x)} G(x,y)|V(y)|dy = 0. \tag{2.8}$$

Clearly by the compactness of $\overline{D}$, if $V \in \mathcal{K}(D)$, then

$$\|V\|_\varphi := \sup_{x\in\overline{D}} \int_D \frac{\varphi(y)}{\varphi(x)} G(x,y)|V(y)|dy < \infty. \tag{2.9}$$

We have the following.

Proposition 2.4. *Assume that the* $3G$*-inequality (2.7) is satisfied on the domain* D*. Then* $V \in \mathcal{K}(D)$ *if and only if for any positive* Δ*-superharmonic function* s *on* D*, we have*

$$\lim_{r\to 0} \sup_{x\in \overline{D}} \int_{D\cap(|y-x|<r)} \frac{s(y)}{s(x)} G(x,y)|V(y)|dy = 0. \tag{2.10}$$

Proof. The condition is sufficient since φ is a positive Δ-superharmonic function on D. Assume that $V \in \mathcal{K}(D)$. Let s be a positive Δ-superharmonic function. By [14, Theorem 2.1 in chapter 6], there is an increasing sequence $(\mu_n)_n$ of measures on D such that $s(x) = \sup_n \int_D G(x,z)\mu_n(dz)$ for all $x \in D$. Multiplying the $3G$-inequality by $|V(y)|\mu_n(dz)$ and integrating with respect to y and next with respect to z, it follows that

$$\begin{aligned} &\sup_{x\in \overline{D}} \int_{D\cap(|y-x|<r)} \frac{s(y)}{s(x)} G(x,y)|V(y)|dy \\ &\quad \le 2C_0 \sup_{x,z\in \overline{D}} \int_{D\cap(|y-x|<r)} \frac{\varphi(y)}{\varphi(z)} G(y,z)|V(y)|dy. \end{aligned} \tag{2.11}$$

Moreover, as in Lemma 2.1, by (2.8) and (2.9) the family $\left\{\frac{\varphi(y)}{\varphi(x)} G(x,y)|V(y)|\right\}_{x\in\overline{D}}$ is uniformly integrable over D. Thus, the right-hand side member in (2.11) vanishes as $r \to 0$ and so (2.10) holds. □

By (1.2) and (2.10), we have the following.

Corollary 2.5. *Assume that the* $3G$*-inequality (2.7) is satisfied on the domain* D*. Then* $\mathcal{K}(D) \subset K_n(D)$*.*

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. As in [21] we convert the problem (1.1) into an integral equation and we use the Schauder fixed point theorem.

(1) Assume that $V \in K_n(D)$ with $\|V\| < 1/3$ and f satisfies (H_1) and (H_2). We will show that there exists a number $\lambda_0 \in (0,1)$ such that for any $\lambda \in (0,\lambda_0)$, there is a function u satisfying the integral equation

$$u(x) = \lambda G(x,0) + \int_D G(x,y)(V(y)u(y) + f(y,u(y)))dy, \tag{3.1}$$

and (1.4).
Indeed, if u satisfies (3.1), then consider a test function $\phi \in C^\infty(D)$ with the closure of suppϕ be a subset of $D \setminus \{0\}$, it is easy to see that u is a solution, in the distribution sense, of the equation $\Delta u + V(x)u + f(x,u) = 0$ in $D \setminus \{0\}$.

Let $C(\overline{D})$ be the space of all real functions continuous on $\overline{D}$. For $\lambda \in (0,1/3)$, consider the set

$$F_\lambda = \{w \in C(\overline{D}) : \lambda/2 \le w \le 3\lambda/2\}.$$

Clearly F_λ is a nonempty bounded, convex and closed set in the Banach space $(C(\overline{D}), \|.\|_\infty)$, where $\|w\|_\infty = \sup\limits_{x\in D} |w(x)|$. Define the integral operator T_λ on F_λ by

$$T_\lambda w(x) = \lambda + \int_D \frac{G(x,y)G(y,0)}{G(x,0)} \Big[V(y)w(y) + \frac{f(y, G(y,0)w(y))}{G(y,0)}\Big] dy, \tag{3.2}$$

for $x \in \overline{D}$. We will use Schauder's theorem to show that T_λ has a fixed point in F_λ for $\lambda \in (0, \lambda_0)$ with λ_0 small. So we need to check the following.

(i) $T_\lambda(F_\lambda) \subset F_\lambda$.

(ii) $T_\lambda(F_\lambda)$ has a compact closure in F_λ.

(iii) T_λ is a continuous operator.

We prove (i). Let $w \in F_\lambda$. For simplicity, write

$$\Phi(y) = V(y)w(y) + \frac{f(y, G(y,0)w(y))}{G(y,0)}. \tag{3.3}$$

We have

$$T_\lambda w(x) = \lambda + \int_D \frac{G(x,y)G(y,0)}{G(x,0)} \Phi(y) dy. \tag{3.4}$$

Since by (H_2),

$$\begin{aligned} |\Phi(y)| &\le w(y)(|V(y)| + \psi(y, G(y,0)w(y))) \\ &\le \frac{3\lambda}{2}(|V(y)| + \psi(y, 3\lambda G(y,0)/2)) \\ &\le |V(y)| + \psi(y, G(y,0)) \in K_n(D), \end{aligned} \tag{3.5}$$

by Proposition 2.3, it follows that $T_\lambda w \in C(\overline{D})$. Moreover by (3.3)–(3.5) and (1.3),

$$\begin{aligned} |T_\lambda w(x) - \lambda| &\le \frac{3\lambda}{2} \int_D \frac{G(x,y)G(y,0)}{G(x,0)} (|V(y)| + \psi(y, 3\lambda G(y,0)/2)) dy \\ &\le \frac{3\lambda}{2}(\|V\| + \|\psi(., 3\lambda G(.,0)/2)\|). \end{aligned} \tag{3.6}$$

Note that by the assumption (H_2) and the Dini theorem $\lim\limits_{\lambda\downarrow 0} \|\psi(., 3\lambda G(.,0)/2)\| = 0$, and so since $\|V\| < 1/3$, there is a number $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$,

$$\|\psi(., 3\lambda G(.,0)/2)\| < 1/3 - \|V\|.$$

By (3.6) we obtain $\lambda/2 \le T_\lambda w \le 3\lambda/2$, and so $T_\lambda w \in F_\lambda$ for $\lambda \in (0, \lambda_0)$. Thus (i) is proved.

By Proposition 2.2 and Ascoli's theorem $T_\lambda(F_\lambda)$ has a compact closure in F_λ which proves (ii).

We prove (iii). Let (w_m) be a sequence in F_λ converging to w with respect to $\|.\|_\infty$. Using (3.2), the inequality

$$|V(y)w_m(y) + f(y, G(y,0)w_m(y))/G(y,0)| \le |V(y)| + \psi(y, G(y,0)) \in K_n(D)$$

and the dominated convergence theorem, it follows that $T_\lambda w_m(x) \to T_\lambda w(x)$ as $m \to \infty$, for all $x \in D$. Since $T_\lambda(F_\lambda)$ has a compact closure in F_λ, it follows that

$$\|T_\lambda w_m - T_\lambda w\|_\infty \to \text{ as } m \to \infty,$$

and so the operator T_λ is continuous. By the Schauder fixed point theorem, for any $\lambda \in (0, \lambda_0)$ there is $w \in F_\lambda$ such that $T_\lambda w = w$. The function $u(x) = G(x,0)w(x)$ satisfies the integral equation (3.1) and (1.4).

(2) Assume that $V \in K_n(D)$ with $\|V\| < 1/3$ and f satisfies (H_1) and (H_2'). For $\lambda > 2$ we consider the set F_λ and the operator T_λ as in (1). Since by (H_2')

$$\begin{aligned} |\Phi(y)| &\leq w(y)(|V(y)| + \psi(y, G(y,0)w(y))) \\ &\leq \frac{3\lambda}{2}(|V(y)| + \psi(y, \lambda G(y,0)/2)) \\ &\leq \frac{3\lambda}{2}(|V(y)| + \psi(y, G(y,0))) \in K_n(D), \end{aligned} \tag{3.7}$$

by Proposition 2.2, it follows that $T_\lambda w \in C(\overline{D})$. Moreover by (3.2), (3.3), (3.7) and (1.3),

$$\begin{aligned} |T_\lambda w(x) - \lambda| &\leq \frac{3\lambda}{2}\int_D \frac{G(x,y)G(y,0)}{G(x,0)}(|V(y)| + \psi(y, \lambda G(y,0)/2))dy \\ &\leq \frac{3\lambda}{2}(\|V\| + \|\psi(., \lambda G(.,0)/2)\|). \end{aligned} \tag{3.8}$$

By the assumption (H_2') and the Dini theorem $\lim_{\lambda \uparrow \infty} \|\psi(., \lambda G(.,0)/2)\| = 0$, and so since $\|V\| < 1/3$, there is a number $\lambda_0 > 1$ such that for $\lambda > \lambda_0$,

$$\|\psi(., \lambda G(.,0)/2)\| < 1/3 - \|V\|.$$

By (3.8) we obtain $\lambda/2 \leq T_\lambda w \leq 3\lambda/2$, and so $T_\lambda w \in F_\lambda$ for $\lambda > \lambda_0$. Thus (i) is proved. The remainder of the proof is the same as in (1). □

Proof of Theorem 1.2. Assume that $W \in L^1_{\text{loc}}(D)$ is nonnegative and u is a solution, in the distribution sense, of the problem (1.1) with $V = 0$ and $f(x,u) = W(x)u^p$ satisfying (1.4). Then u satisfies the integral equation

$$u(x) = \lambda G(x,0) + \int_D G(x,y)W(y)u^p(y)dy, \quad x \in D \setminus \{0\}.$$

It follows that the function $w(x) = \frac{u(x)}{G(x,0)} \in C(\overline{D})$ and satisfies

$$w(x) = \lambda + \int_D \frac{G(x,y)G(y,0)}{G(x,0)} W(y)G^{p-1}(y,0)w^p(y)dy, \quad x \in D$$

with $\lambda/2 \leq w \leq 3\lambda/2$. So the function

$$p(x) = \int_D \frac{G(x,y)G(y,0)}{G(x,0)} W(y)G^{p-1}(y,0)w^p(y)dy$$

is in $C(\overline{D})$. Since for each $r > 0$ small the function

$$p_r(x) = \int_{D\cap(|x-y|\geq r)} \frac{G(x,y)G(y,0)}{G(x,0)} W(y)G^{p-1}(y,0)w^p(y)dy$$

is in $C(\overline{D})$, we deduce that the function

$$q_r(x) = p(x) - p_r(x) = \int_{D\cap(|x-y|<r)} \frac{G(x,y)G(y,0)}{G(x,0)} W(y)G^{p-1}(y,0)w^p(y)dy$$

is in $C(\overline{D})$, and so by the Dini theorem, we have

$$\lim_{r\to 0} \sup_{x\in\overline{D}} \int_{D\cap(|x-y|<r)} \frac{G(x,y)G(y,0)}{G(x,0)} W(y)G^{p-1}(y,0)w^p(y)dy = 0.$$

Since $\lambda/2 \leq w \leq 3\lambda/2$, we obtain

$$\lim_{r\to 0} \sup_{x\in\overline{D}} \int_{D\cap(|x-y|<r)} \frac{G(x,y)G(y,0)}{G(x,0)} W(y)G^{p-1}(y,0)dy = 0,$$

i.e. $WG^{p-1}(.,0) \in K_n(D)$. □

4. Examples

Example 1. Assume that $n \geq 3$. For $-1 < p < 1$ and $2 < \alpha < 2+(n-2)(1-p)$, the function

$$u_\alpha(x) = \frac{[(\alpha-2)(2+n(1-p)-\alpha)/(p-1)^2]^{1/(p-1)}}{|x|^{(\alpha-2)/(1-p)}}$$

is a solution of the problem

$$\begin{cases} u \in C^2(B(0,R)\setminus\{0\}), \\ \Delta u(x) + \frac{1}{|x|^\alpha}u^p = 0, \quad x \in B(0,R)\setminus\{0\}, \\ u(x) > 0, \quad x \in B(0,R)\setminus\{0\}, \\ \lim\limits_{x\to 0} u(x) = \infty. \end{cases} \tag{4.1}$$

Moreover, since $\frac{G^{p-1}(x,0)}{|x|^\alpha}$ is in the class $K_n(B(0,R))$, by Theorem 1.1 problem (4.1) has a positive solution $u \in C(B(0,R)\setminus\{0\})$ with $u(x) \sim \frac{1}{|x|^{n-2}}$ near $x = 0$ and $u = 0$ on $\partial B(0,R)$. Since u is locally bounded on $B(0,R)\setminus\{0\}$, by Theorem 6.6 in [14] $u \in C^2(B(0,R)\setminus\{0\})$. It is clear that the singularity of u near 0 is stronger than the singularity of u_α.

Example 2. Assume that $n = 2$. For $-1 < p < 1$, the function

$$v(x) = \left(\frac{(1-p)^2}{2(1+p)}\right)^{1/(1-p)} \left(\ln\left(\frac{1}{|x|}\right)\right)^{2/(1-p)}$$

is a solution of the problem

$$\begin{cases} u \in C^2(B(0,1) \setminus \{0\}), \\ \Delta u(x) - \frac{1}{|x|^2} u^p = 0, \quad x \in B(0,1) \setminus \{0\}, \\ u(x) > 0, \quad x \in B(0,1) \setminus \{0\}, \\ \lim_{x \to 0} u(x) = \infty, \\ u(x) = 0, \quad x \in \partial B(0,1). \end{cases} \tag{4.2}$$

Since $\frac{G^{p-1}(x,0)}{|x|^2}$ does not belong to the class $K_2(B(0,1))$, by Theorem 1.2 problem (4.2) does not have a positive solution $u \in C(B(0,1) \setminus \{0\})$ satisfying $u(x) \sim \ln(\frac{1}{|x|})$ near $x = 0$.

Example 3. Assume that $n \geq 4$. For $p < 1$ and $\alpha = (n-1)(1-p)/2 + 2$, the function

$$u(x) = \Big(\frac{(n-3)(n-1)}{4}\Big)^{1/(p-1)} \frac{1-|x|}{|x|^{(n-1)/2}}$$

is a solution of the problem

$$\begin{cases} u \in C^2(B(0,1) \setminus \{0\}), \\ \Delta u(x) + \frac{(1-|x|)^{1-p}}{|x|^\alpha} u^p = 0, \quad x \in B(0,1) \setminus \{0\}, \\ u(x) > 0, \quad x \in B(0,1) \setminus \{0\}, \\ \lim_{x \to 0} u(x) = \infty, \\ u(x) = 0, \quad x \in \partial B(0,1). \end{cases} \tag{4.3}$$

Since $\frac{(1-|x|)^{1-p}}{|x|^\alpha} G^{p-1}(x,0) \in K_2(B(0,1))$, by Theorem 1.1 problem (4.3) has a positive solution $v \in C(B(0,1) \setminus \{0\})$ satisfying $v(x) \sim \frac{1}{|x|^{n-2}}$ near $x = 0$. Clearly, the singularity of v near 0 is stronger than the singularity of u.

In the next example, we discuss the existence of singular solutions in the one-dimensional case.

Example 4. In the case $n = 1$, consider, for $p \in \mathbb{R}$, the problem

$$\begin{cases} u \in C^2(]-1,1[\setminus\{0\}), \\ u''(x) + u^p(x) = 0, \quad x \in]-1,1[\setminus\{0\}, \\ u(x) > 0, \quad x \in]-1,1[\setminus\{0\}, \\ \lim_{x \to 0^+} u(x) = +\infty. \end{cases} \tag{4.4}$$

Clearly (4.4) has no solution since any solution u should be concave on $]0,1[$.

Now consider, for $p \in \mathbb{R}$, the problem

$$\begin{cases} u \in C^2(]-1,1[\setminus\{0\}), \\ u''(x) - u^p(x) = 0, \quad x \in]-1,1[\setminus\{0\}, \\ u(x) > 0, \quad x \in]-1,1[\setminus\{0\}, \\ \lim_{x \to 0^+} u(x) = +\infty. \end{cases} \tag{4.5}$$

For $p > 1$ the function

$$u(x) = \left(\frac{2(p+1)}{(p-1)^2}\right)^{1/(p-1)} \frac{1}{|x|^{2/(p-1)}}$$

is a singular solution of (4.5).

Assume $p \le 1$ with $p \ne -1$. If (4.5) has a solution, then clearly there is a constant c such that

$$\frac{(u')^2}{2} = \frac{u^{p+1}}{p+1} + c \quad \text{on }]0,1[.$$

By the condition $\lim_{x \to 0^+} u(x) = +\infty$, it follows that

$$(u')^2 \le Cu^2 \quad \text{on }]0,\varepsilon[,$$

for some constants $C > 0$ and $\varepsilon > 0$. We then have

$$|\frac{u'}{u}| \le C \quad \text{on }]0,\varepsilon[,$$

which yields

$$0 < \ln\left(\frac{\varepsilon}{x}\right) \le C\varepsilon, \quad x \in]0,\varepsilon[,$$

and this is impossible.
Assume $p = -1$. If (4.5) has a solution, then clearly there is a constant c such that

$$\frac{(u')^2}{2} = \ln u + c \quad \text{on }]0,1[.$$

Since $u \to \infty$ as $x \to 0^+$, it follows that

$$|\frac{u'}{u}| \le C \quad \text{on }]0,\varepsilon[,$$

for some constants $C > 0$ and $\varepsilon > 0$, which yields

$$0 < \ln\left(\frac{\varepsilon}{x}\right) \le C\varepsilon, \quad x \in]0,\varepsilon[,$$

and this is impossible.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Aviles, P.: On isolated singularities in some nonlinear partial differential equations. Indiana Univ. Math. J. **32**(5), 773–791 (1983)

[2] Brandolini, B., Chiacchio, F., Cîrstea, F.C., Trombetti, C.: Local behaviour of singular solutions for nonlinear elliptic equations in divergence form. Calc. Var. Partial Diff. Eq. **48**, 367–393 (2013)

[3] Brezis, H., Oswald, L.: Singular solutions of some semilinear elliptic equations. Arch. Rat. Mech. Anal. **89**, 249–259 (1987)

[4] Brezis, H., Véron, L.: Removable singularities for some nonlinear elliptic equations. Arch. Rat. Mech. Anal. **75**, 1–6 (1980)

[5] Chung, K.L., Zhao, Z.: From Brownian motion to Schrödinger's equation. Springer, Berlin (1995)

[6] Cîrstea, F.C., Du, Y.: Asymptotic behavior of solutions of semilinear elliptic equations near an isolated singularity. J. Funct. Anal. **250**, 317–346 (2007)

[7] Cîrstea, F.C.: A complete classification of isolated singularities for nonlinear elliptic equations with inverse square potentials, *Mem. Amer. Math. Soc.*, 227 (2014), no 1068

[8] Gidas, B., Spruk, J.: Global and local behavior of positive solutions of nonlinear elliptic equations. Comm. Pure Appl. Math. **34**, 525–598 (1981)

[9] Hansen, W.: Uniform boundary Harnack principle and generalized triangle property. J. Funct. Anal. **226**, 452–484 (2005)

[10] Jerison, D., Kenig, C.: Boundary behavior of harmonic functions in non-tangentially accessible domains. Adv. Math. **46**, 80–147 (1982)

[11] Li, Y., Santanilla, J.: Existence and nonexistence of positive singular solutions for semilinear elliptic problems with applications in astrophysics. Diff. Int. Eq. **8**, 1369–1383 (1995)

[12] Lions, P.L.: Isolated singularities in semilinear problems. J. Diff. Eq. **38**, 441–450 (1980)

[13] Ni, W.M.: On a singular elliptic equation. Proc. Amer. Math. Soc. **88**(4), 614–616 (1983)

[14] Port, S.C., Stone, C.J.: Brownian motion and classical potential theory. Academic, New York (1978)

[15] Riahi, L.: A $3G$-Theorem for Jordan domains in $\mathbb{R}^2$. Colloq. Math. **101**, 1–7 (2004)

[16] Riahi, L.: The $3G$-inequality for general Schrödinger operators on Lipschitz domains. Manuscripta Math. **116**, 211–227 (2005)

[17] Riahi, L.: Singular solutions of a semilinear elliptic equation on nonsmooth domains. Intern. J. Evol. Eq. **2**(4), 55–64 (2007)

[18] Vazquez, J.L., Véron, L.: Isolated singularities of some semilinear elliptic equations. J. Diff. Eq. **60**, 301–321 (1985)

[19] Véron, L.: Singular solutions of some nonlinear elliptic equations. Nonlinear Anal. **5**, 225–242 (1981)

[20] Véron, L.: Weak and strong singularities of nonlinear elliptic equations, in nonlinear functional analysis and its applications, Part 2. Proc. Symposia Pure Math. **45**, 477–495 (1986)

[21] Zhang, Q.S., Zhao, Z.: Singular solutions of semilinear elliptic and parabolic equations. Math. Ann. **310**, 777–794 (1998)

Sidi Hamidou Jah and Lotfi Riahi
Department of Mathematics, College of Science
Qassim University
P.O. Box 6644
Buraydah 51452
Saudi Arabia
e-mail: jah@qu.edu.sa;
3869@qu.edu.sa

Accepted: December 2, 2021.