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Combinatorial formulas for some generalized Ekeland-Hofer-Zehnder capacities of convex polytopes

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Dedicated to Professor Claude Viterbo on the occasion of his sixtieth birthday.

Abstract. Motivated by Pazit Haim-Kislev's combinatorial formula for the Ekeland-Hofer-Zehnder capacities of convex polytopes, we give corresponding formulas for Ψ-Ekeland-Hofer-Zehnder and coisotropic Ekeland-Hofer-Zehnder capacities of convex polytopes introduced by the second named author and others recently. Contrary to Pazit Haim-Kislev's subadditivity result for the Ekeland-Hofer-Zehnder capacities of convex domains, we show that the coisotropic Hofer-Zehnder capacities satisfy the superadditivity for suitable hyperplane cuts of two-dimensional convex domains.

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Contents

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1. Introduction and results

Symplectic capacities are important invariants in studies of symplectic topology. Different symplectic capacities measure the "symplectic size" of sets from different views. Precise computations of them are usually difficult.

For a compact convex domain K with smooth boundary $\mathcal{S} = \partial K$ in the standard symplectic Euclidean space $(\mathbb{R}^{2n}, \omega_0)$, Ekeland-Hofer [6] (see also [16]) and Hofer-Zehnder [8] showed, respectively, that its Ekeland-Hofer capacity $c_{\mathrm{EH}}(K)$ and Hofer-Zehnder capacity $c_{\mathrm{HZ}}(K)$ were equal to

$$c_{\mathrm{EHZ}}(K) := \min\{A(x) > 0 \,|\, x \text{ is a closed characteristic on } \mathcal{S}\} \tag{1.1}$$

(called the Ekeland-Hofer-Zehnder capacity below), where by a closed characteristic on $\mathcal{S}$ we mean a C^1 embedding z from $S^1 = [0,T]/\{0,T\}$ into $\mathcal{S}$ satisfying $\dot{z}(t) \in (\mathcal{L}_{\mathcal{S}})_{z(t)}$ for all $t \in [0,T]$, where

$$\mathcal{L}_{\mathcal{S}} = \{(x,\xi) \in T\mathcal{S} \mid \omega_{0x}(\xi,\eta) = 0, \forall \eta \in T_x\mathcal{S}\}$$

and the action of a path $z \in W^{1,2}([0,T], \mathbb{R}^{2n})$ is defined by

$$A(z) = \frac{1}{2}\int_0^T \langle -J\dot{z}, z\rangle dt \tag{1.2}$$

with $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, where $z \in W^{1,2}([0,T], \mathbb{R}^{2n})$ if z is absolutely continuous and

$$\int_0^T \|z(t)\|^2 dt < \infty \quad \text{and} \quad \int_0^T \|\dot{z}(t)\|^2 dt < \infty.$$

We equip $H^1([0,T], \mathbb{R}^{2n}) := W^{1,2}([0,T], \mathbb{R}^{2n})$ the natural Sobolev norm:

$$\|z\|_{W^{1,2}} := \left(\int_0^T \|z(t)\|^2 + \|\dot{z}(t)\|^2 dt\right)^{\frac{1}{2}}.$$

When the smoothness assumption of the boundary $\mathcal{S}$ is thrown away, then (1.1) is still true if "closed characteristic" in the right side of (1.1) may be replaced by "generalized closed characteristic", where a generalized closed characteristic on $\mathcal{S}$ is a T-periodic nonconstant absolutely continuous curve $z : \mathbb{R} \to \mathbb{R}^{2n}$ (for some $T > 0$) such that $z(\mathbb{R}) \subset \mathcal{S}$ and $\dot{z}(t) \in JN_{\mathcal{S}}(z(t))$ a.e., where $N_{\mathcal{S}}(x) = \{y \in \mathbb{R}^{2n} \mid \langle u - x, y\rangle \leqslant 0, \forall u \in K\}$ is the normal cone to K at $x \in \mathcal{S}$. The action of such a generalized closed characteristic $x : [0,T] \to \mathcal{S}$ is still defined by (1.2).

In general, it is difficult to compute $c_{\mathrm{EHZ}}(K)$ by finding minimal closed characteristics with (1.1). If K is a convex polytope with $(2n-1)$-dimensional facets $\{F_i\}_{i=1}^{\mathbf{F}_K}$, n_i is the unit outer normal to F_i, and $h_i = h_K(n_i)$ the "**oriented height**" of F_i given by the support function of K, $h_K(y) := \sup_{x \in K}$

$\langle x, y\rangle$, starting from (1.1) Pazit Haim-Kislev [15] recently established the following beautiful combinatorial formula for $c_{\rm EHZ}(K)$:

$$c_{\rm EHZ}(K) = \frac{1}{2}\left[\max_{\sigma\in S_{\mathbf{F}_K}, (\beta_i)\in M(K)} \sum_{1\le j<i\le \mathbf{F}_K} \beta_{\sigma(i)}\beta_{\sigma(j)}\omega_0(n_{\sigma(i)}, n_{\sigma(j)})\right]^{-1}, \tag{1.3}$$

where $S_{\mathbf{F}_K}$ is the symmetric group on $\mathbf{F}_K$ letters and

$$M(K) = \left\{(\beta_i)_{i=1}^{\mathbf{F}_K}\,\middle|\, \beta_i \ge 0, \sum_{i=1}^{\mathbf{F}_K}\beta_i h_i = 1, \sum_{i=1}^{\mathbf{F}_K}\beta_i n_i = 0\right\}.$$

As an important application, Pazit Haim-Kislev [15] proved a subadditivity property of the capacity $c_{\rm EHZ}$ for hyperplane cuts of arbitrary convex domains, which solved a special case of the subadditivity conjecture for capacities ([2]).

Recently, motivated by Clarke [3,4] and Ekeland [7] Rongrong Jin and the second named author introduced relative versions (or generalizations) of the Ekeland-Hofer capacity and the Hofer-Zehnder capacity in [10]. Precisely, for a symplectic manifold (M, ω) and for a $\Psi \in {\rm Symp}(M, \omega)$ with ${\rm Fix}(\Psi) \ne \emptyset$, we defined a relative version of the Hofer-Zehnder capacity $c_{\rm HZ}(M, \omega)$ of (M, ω) with respect to Ψ, $c^{\Psi}_{\rm HZ}(M, \omega)$, which becomes $c_{\rm HZ}(M, \omega)$ if $\Psi = id_M$. For a symplectic matrix $\Psi \in {\rm Sp}(2n, \mathbb{R})$ with ${\rm Fix}(\Psi) \ne \emptyset$, and for each $B \subset \mathbb{R}^{2n}$ such that $B \cap {\rm Fix}(\Psi) \ne \emptyset$, we also introduced a relative version of the Ekeland-Hofer capacity $c_{\rm EH}(B)$ of B with respect to Ψ, $c^{\Psi}_{\rm EH}(B)$, which becomes $c_{\rm EH}(B)$ if $\Psi = I_{2n}$. If a compact convex domain $K \subset \mathbb{R}^{2n}$ with boundary $\mathcal{S} = \partial K$ contains a fixed point of $\Psi \in {\rm Sp}(2n, \mathbb{R})$ in the interior of it, we proved in [10]:

$$\begin{aligned} c^{\Psi}_{\rm EH}(K) &= c^{\Psi}_{\rm HZ}(K)\\ &= \min\{A(x) > 0\,|\, x \text{ is a generalized } \quad \Psi-\text{characteristic on } \mathcal{S}\}, \end{aligned} \tag{1.4}$$

where a generalized Ψ-characteristic on $\mathcal{S}$ is a nonconstant absolutely continuous curve $z : [0, T] \to \mathbb{R}^{2n}$ (for some $T > 0$) such that $z([0, T]) \subset \mathcal{S}$, $z(T) = \Psi z(0)$ and $\dot z(t) \in JN_{\mathcal{S}}(z(t))$ a.e., where $N_{\mathcal{S}}(x)$ is the normal cone to K at $x \in \mathcal{S}$ as above, and the action $A(z)$ of z is still defined by (1.2). (If $\mathcal{S}$ is $C^{1,1}$-smooth, "generalized closed characteristic" in the right side of (1.4) may be replaced by "closed characteristic", where a Ψ-characteristic on $\mathcal{S}$ is a C^1 embedding z from $[0, T]$ (for some $T > 0$) into $\mathcal{S}$ such that $z(T) = \Psi z(0)$ and $\dot z \in (\mathcal{L}_{\mathcal{S}})_{z(t)}$ for all $t \in [0, T]$). Our first result is an analogue of (1.3) for $c^{\Psi}_{\rm EHZ}(K) := c^{\Psi}_{\rm EH}(K) = c^{\Psi}_{\rm HZ}(K)$.

Theorem 1.1. *Let K be a convex polytope as above (1.3). Suppose that $\Psi \in {\rm Sp}(2n, \mathbb{R})$ has a fixed point sitting in the interior of K. Then*

$$c^{\Psi}_{\rm EHZ}(K) = \min_{\left((\beta_i)_{i=1}^{\mathbf{F}_K}, v, \sigma\right)\in M_{\Psi}(K)} \frac{2}{4\sum_{1\leqslant j<i\leqslant \mathbf{F}_K}\beta_{\sigma(i)}\beta_{\sigma(j)}\omega_0(n_{\sigma(j)}, n_{\sigma(i)}) - \omega_0(\Psi v, v)},$$

where

$$M_\Psi(K) = \left\{ ((\beta_i)_{i=1}^{\mathbf{F}_K}, v, \sigma) \,\middle|\, \begin{array}{l} \sigma \in S_{\mathbf{F}_K},\ \beta_i \geqslant 0,\ \sum_{i=1}^{\mathbf{F}_K} \beta_i h_i = 1,\ \sum_{i=1}^{\mathbf{F}_K} 2\beta_i J n_i = \Psi v - v, \\ 4\sum_{1\leqslant j<i\leqslant \mathbf{F}_K} \beta_{\sigma(i)}\beta_{\sigma(j)}\omega_0(n_{\sigma(j)}, n_{\sigma(i)}) > \omega_0(\Psi v, v),\ v \in E_\Psi \end{array} \right\}$$

with E_Ψ *being the orthogonal complement of* $\mathrm{Ker}(\Psi - I_{2n})$ *in* $\mathbb{R}^{2n}$.

Note: Under our convention $\langle x, y\rangle = \omega_0(x, Jy)$, $\omega_0(n_{\sigma(i)}, n_{\sigma(j)})$ in (1.3) should be changed into $\omega_0(n_{\sigma(j)}, n_{\sigma(i)})$.

Lisi and Rieser [13] introduced the notion of a coisotropic capacity and constructed a coisotropic Hofer-Zehnder capacity, which is a relative version of the Hofer-Zehnder capacity with respect to a coisotropic submanifold. Rongrong Jin and the second named author recently constructed a relative version of the Ekeland-Hofer capacity with respect to a special class of coisotropic subspaces in [12]. Consider coisotropic subspaces of $(\mathbb{R}^{2n}, \omega_0)$,

$$\mathbb{R}^{n,k} = \{x \in \mathbb{R}^{2n} | x = (q_1, \cdots, q_n, p_1, \cdots, p_k, 0, \cdots, 0)\}, \quad k = 0, \cdots, n.$$

The isotropic leaf through $x \in \mathbb{R}^{n,k}$ is $x + V_0^{n,k}$, where

$$V_0^{n,k} = \{x \in \mathbb{R}^{2n} \,|\, x = (0, \cdots, 0, q_{k+1}, \cdots, q_n, 0, \cdots, 0)\}.$$

The leaf relation $\sim$ on $\mathbb{R}^{n,k}$ is that $x \sim y$ if and only if $y \in x + V_0^{n,k}$. From now on we fix an integer $0 \leqslant k < n$ and assume that $K \subset \mathbb{R}^{2n}$ is a compact convex domain with $C^{1,1}$-smooth boundary $\mathcal{S} = \partial K$ and satisfying $\mathrm{Int}(K) \cap \mathbb{R}^{n,k} \neq \emptyset$. A nonconstant absolutely continuous curve $z : [0, T] \to \mathbb{R}^{2n}$ (for some $T > 0$) is called a generalized leafwise chord (abbreviated GLC) on $\mathcal{S}$ for $\mathbb{R}^{n,k}$ if $z([0, T]) \subset \mathcal{S}$, $\dot{z}(t) \in JN_{\mathcal{S}}(z(t))$ a.e., $z(0), z(T) \in \mathbb{R}^{n,k}$ and $z(0) - z(T) \in V_0^{n,k}$. The action $A(z)$ of such a chord is still defined by (1.2). In [11,12] Rongrong Jin and the second named author proved respectively that the coisotropic Hofer-Zehnder capacity $c_{\mathrm{LR}}(K, K \cap \mathbb{R}^{n,k})$ of K relative to $\mathbb{R}^{n,k}$ and the coisotropic Ekeland-Hofer capacity $c^{n,k}(K)$ of K relative to $\mathbb{R}^{n,k}$ satisfy

$$c_{\mathrm{LR}}(K, K \cap \mathbb{R}^{n,k}) = c^{n,k}(K) = \min\{A(x) > 0 \mid x \text{ is a GLC on } \mathcal{S} \text{ for } \mathbb{R}^{n,k}\}. \tag{1.5}$$

Here is our second result.

Theorem 1.2. *Let* K *be a convex polytope as above (1.3). Suppose* $K \cap \mathbb{R}^{n,k} \neq \emptyset$. *Then*

$$c_{\mathrm{LR}}(K, K \cap \mathbb{R}^{n,k}) = \frac{1}{2} \min_{((\beta_i)_{i=1}^{\mathbf{F}_K}, \sigma) \in M(K)} \frac{1}{\sum_{1\leqslant j<i\leqslant \mathbf{F}_K} \beta_{\sigma(i)}\beta_{\sigma(j)}\omega_0(n_{\sigma(j)}, n_{\sigma(i)})},$$

where

$$M(K) = \left\{ ((\beta_i)_{i=1}^{\mathbf{F}_K}, \sigma) \,\middle|\, \begin{array}{l} \beta_i \geqslant 0,\ \sum_{i=1}^{\mathbf{F}_K} \beta_i h_i = 1,\ \sum_{i=1}^{\mathbf{F}_K} \beta_i J n_i \in V_0^{n,k}, \\ \sum_{1\leqslant j<i\leqslant \mathbf{F}_K} \beta_{\sigma(i)}\beta_{\sigma(j)}\omega_0(n_{\sigma(j)}, n_{\sigma(i)}) > 0,\ \sigma \in S_{\mathbf{F}_K} \end{array} \right\}. \tag{1.6}$$

Unlike Ekeland-Hofer-Zehnder capacity, one cannot expect that the coisotropic Hofer-Zehnder capacity satisfies the subadditivity as stated in

[15, Theorem 1.8] in general. In fact, when $n = 1$ and $k = 0$, our following result is opposite to the expected one.

Theorem 1.3. *Let $D \subset \mathbb{R}^2$ be a convex domain satisfying $D \cap \mathbb{R}^{1,0} \neq \emptyset$, and let $L \subset \mathbb{R}^2$ be a straight line through D such that $L \neq \mathbb{R}^{1,0}$ and $D \cap L \cap \mathbb{R}^{1,0} \neq \emptyset$. Denote by D_1 and D_2 the two parts divided by L. Then*

$$c_{\mathrm{LR}}(D, D \cap \mathbb{R}^{1,0}) \geq c_{\mathrm{LR}}(D_1, D_1 \cap \mathbb{R}^{1,0}) + c_{\mathrm{LR}}(D_2, D_2 \cap \mathbb{R}^{1,0}). \tag{1.7}$$

Remark 1.4. Inequality (1.7) is sharp, and it can be strict in some cases. Consider the following example. Let $P = \{(x, y) \,|\, |x| \leqslant 1, |y| \leqslant 1\}$ and $L = \{(x, x) \,|\, x \in \mathbb{R}\}$. Then L divides P into two parts $P_1 := \{(x, y) \,|\, x \leqslant y\} \cap P$ and $P_2 := \{(x, y) \,|\, x \geqslant y\} \cap P$. Using Theorem 1.2, we can easily compute $c_{\mathrm{LR}}(P, P \cap \mathbb{R}^{1,0}) = 2$, $c_{\mathrm{LR}}(P_1, P_1 \cap \mathbb{R}^{1,0}) = c_{\mathrm{LR}}(P_2, P_2 \cap \mathbb{R}^{1,0}) = \frac{1}{2}$. Thus

$$c_{\mathrm{LR}}(P, P \cap \mathbb{R}^{1,0}) > c_{\mathrm{LR}}(P_1, P_1 \cap \mathbb{R}^{1,0}) + c_{\mathrm{LR}}(P_2, P_2 \cap \mathbb{R}^{1,0}).$$

Moreover, for any $t \in (-1, 1)$, the line $L_t := \{(t, y) \,|\, y \in \mathbb{R}\}$ divides P into two parts

$$P_+ := \{(x, y) \in P \,|\, x \geq t\} \quad \text{and} \quad P_- := \{(x, y) \in P \,|\, x \leq t\}.$$

It is easily computed that

$$c_{\mathrm{LR}}(P_+, P_+ \cap \mathbb{R}^{1,0}) = 1 - t \quad \text{and} \quad c_{\mathrm{LR}}(P_-, P_- \cap \mathbb{R}^{1,0}) = 1 + t,$$

and hence $c_{\mathrm{LR}}(P_+, P_+ \cap \mathbb{R}^{1,0}) + c_{\mathrm{LR}}(P_-, P_- \cap \mathbb{R}^{1,0}) = c_{\mathrm{LR}}(P, P \cap \mathbb{R}^{1,0})$.

In higher dimensions, we have $c_{\mathrm{LR}}(G, G \cap \mathbb{R}^{n,n}) = c_{\mathrm{EHZ}}(G)$ for any nonempty convex domain $G \subset \mathbb{R}^{2n}$. Thus some coisotropic Hofer-Zehnder capacities of higher dimensions have subadditivity because of the subadditivity of c_{EHZ} under the conditions of [15, Theorem 1.8]. There is no nice result in more general case yet.

For the symmetrical Hofer-Zehnder symplectic capacity of a symmetric convex domain in $\mathbb{R}^{2n}$ introduced by Liu and Wang [14], using a representation formula of it given by Rongrong Jin and the second named author in [9] one is able to generalize the formula in [15], but this is outside the scope of this paper and would appear elsewhere.

This paper is organized as follows. In the next section we collect detailed conclusions coming from [10, §4.1] and [11, §3.1] about proofs of representation formulas of the Ψ-Ekeland-Hofer-Zehnder capacity and the coisotropic Ekeland-Hofer-Zehnder capacity for convex bodies in $\mathbb{R}^{2n}$, respectively. Then we generalize some results on piecewise affine loops in [15, §3] to piecewise affine paths in Sect. 3. Theorem 1.1 will be proved in Sect. 4. Finally, in Sect. 5 we prove Theorems 1.2, 1.3.

2. Preliminaries

For simplicity of the reader's convenience we list two results, which come from [10, Section. 4.1] and [11, Section 3.1], respectively.

Let $K \subset \mathbb{R}^{2n}$ be a compact convex domain K with boundary $\mathcal{S} = \partial K$ and with $0 \in \mathrm{Int}(K)$. Denote by $H_K = (j_K)^2$ the square of the Minkowski functional j_K of K, and by H_K^* the Legendre transformation of H_K defined by

$$H_K^*(w) = \max_{\xi\in\mathbb{R}^{2n}}(\langle x,\xi\rangle - H_K(\xi)).$$

Then $h_K^2 = 4H_K^*$ (see e.g.[1]).

Given $\Psi \in \mathrm{Sp}(2n,\mathbb{R})$ let E_Ψ be the orthogonal complement of $\mathrm{Ker}(\Psi - I_{2n}) \subset \mathbb{R}^{2n}$ with respect to the standard inner product in $\mathbb{R}^{2n}$. (In [10] we wrote $\mathrm{Ker}(\Psi - I_{2n})$ and E_Ψ as E_1 and $E_1^\perp$, respectively.) Define

$$\mathcal{F}_\Psi = \{x \in W^{1,2}([0,1],\mathbb{R}^{2n})\,|\,x(1) = \Psi x(0) \text{ and } x(0) \in E_\Psi\},$$

which was denoted by $\mathcal{F}$ in [10]. If $\dim E_\Psi = 0$, the problem reduces to the periodic case. So we only consider the non-periodic case in which $\dim E_\Psi \geqslant 1$. Define

$$\mathcal{A}_\Psi = \{x \in \mathcal{F}_\Psi\,|\,A(x) = 1\},$$

where $A(x)$ is defined by (1.2) with $T = 1$, and

$$I_K : \mathcal{F}_\Psi \to \mathbb{R},\ x \mapsto \int_0^1 H_K^*(-J\dot{x}).$$

By Theorems 1.8, 1.9, Remark 1.10 and arguments in [10, §4.1] we have

Theorem 2.1. *Under the above assumptions, I_K attains its minimum $\min_{x\in\mathcal{A}_\Psi} I_K(x)$ over $\mathcal{A}_\Psi$, which is positive. For each minimier u of I_K over $\mathcal{A}_\Psi$, there exists $a_0 \in \mathrm{Ker}(\Psi - I_{2n})$ such that the $W^{1,2}$-path*

$$[0, I_K(u)] \ni t \mapsto x^*(t) = \sqrt{I_K(u)}u(t/I_K(u)) + a_0/\sqrt{I_K(u)} \tag{2.1}$$

satifies $A(x^*) = I_K(u) = c^\Psi_{\mathrm{EHZ}}(K)$ and

$$\begin{cases} -J\dot{x}^*(t) \in \partial H_K(x^*(t)), \text{ a.e.,} \\ x^*(T) = \Psi x^*(0) \quad \textit{and} \quad x^*([0,T]) \subset \partial K; \end{cases} \tag{2.2}$$

in particular x^* is a generalized Ψ-characteristic on ∂K because

$$\partial H_K(x) = \{v \in N_{\partial K}(x)\,|\,\langle x,v\rangle = 2\} \quad \forall x \in \partial K. \tag{2.3}$$

(cf. Lemma 2 of [5, Chap. V, §1]). Conversely, if $z : [0,T] \to \partial K$ is a generalized Ψ-characteristic on ∂K with action $A(z) = c^\Psi_{\mathrm{EHZ}}(K)$, then (by [10, Lemma 4.2]) there is a differentiable homeomorphism $\varphi : [0,T] \to [0,T]$ with an absolutely continuous inverse $\psi : [0,T] \to [0,T]$ such that $z^* = z\circ\varphi$ is a $W^{1,\infty}$-map with action $A(z^*) = A(z) = T$ and satisfying (2.2); moreover we can choose $b \in \mathrm{Ker}(\Psi - I_{2n})$ so that the path $u : [0,1] \to \mathbb{R}^{2n}$ defined by $u(t) = z^*(Tt)/\sqrt{T} + b$ belongs to $\mathcal{A}_\Psi$ and satisfies $I_K(u) = T$, i.e., u is a minimier u of I_K over $\mathcal{A}_\Psi$. When this K is also a convex polytope as above (1.3), then there holds

$$\dot{u}(t) = \sqrt{T}\dot{z}^*(Tt) \in \sqrt{T}\mathrm{conv}\{p_i\,|\,\sqrt{T}(u(t) - b) \in F_i\}, \text{ a.e.} \tag{2.4}$$

where $p_i = \frac{2}{h_i}Jn_i$.

To see the final claim, note that for each $i=1,\cdots,\mathbf{F}_K$, H_K is smooth at each relative interior point x of F_i and the subdifferential $\partial H_K(x)=\{\nabla H_K(x)\}=\{\frac{2}{h_i}n_i\}$. For any $x\in\partial K$ we have $\partial H_K(x)=\mathrm{conv}\{\frac{2}{h_i}n_i\,|\,x\in F_i\}$ (cf. [15, page 445]), and therefore $J\partial H_K(x)=\mathrm{conv}\{p_i\,|\,x\in F_i\}$. (The outward normal cone of K at $x\in\partial K$, $N_{\partial K}(x)$, is equal to $\mathbb{R}_+\mathrm{conv}\{n_i: x\in F_i\}$.)

Fix an integer $0\le k<n$. Following [11] consider the Hilbert subspace of $W^{1,2}([0,1],\mathbb{R}^{2n})$,

$$\mathscr{F}_2:=\left\{x\in W^{1,2}([0,1],\mathbb{R}^{2n})\,\middle|\,x(0),x(1)\in\mathbb{R}^{n,k},\ x(1)\sim x(0),\ \int_0^1 x(t)dt\in JV_0^{n,k}\right\}$$

(where $x(1)\sim x(0)$ means $x(1)-x(0)\in V_0^{n,k}$), its subset $\mathcal{A}_2=\{x\in\mathscr{F}_2\,|\,A(x)=1\}$, and the related convex functional

$$I_2:\mathscr{F}_2\to\mathbb{R},\ x\mapsto\int_0^1 H_K^*(-J\dot{x}(t))dt.$$

From [11, §3.1], we obtain the following corresponding result of Theorem 2.1.

Theorem 2.2. *Under the above assumptions, I_2 attains its minimum $\min_{x\in\mathcal{A}_2} I_2(x)$ over $\mathcal{A}_2$, which is positive. For each minimier u of I_2 over $\mathcal{A}_2$, there exists $\mathbf{a}_0\in\mathbb{R}^{n,k}$ such that the $W^{1,2}$-path*

$$[0,1]\ni t\mapsto x^*(t):=\sqrt{I_2(u)}u(t)+\mathbf{a}_0/\sqrt{I_2(u)} \tag{2.5}$$

satisfies $A(x^*)=I_2(u)=c_{\mathrm{LR}}(K,K\cap\mathbb{R}^{n,k})=c^{n,k}(K)$ and

$$\begin{cases}-J\dot{x}^*(t)=\partial H_K(x^*(t)),\ \text{a.e.},\ x^*(0),x^*(1)\in\mathbb{R}^{n,k},\\ x^*(1)-x^*(0)\in V_0^{n,k}\quad\textit{and}\quad x^*([0,1])\subset\partial K;\end{cases} \tag{2.6}$$

in particular x^* is a generalized leafwise chord on ∂K for $\mathbb{R}^{n,k}$ because of (2.3). Conversely, if $z:[0,T]\to\partial K$ is a generalized leafwise chord on ∂K with action $A(z)=c^{n,k}(K)$ for $\mathbb{R}^{n,k}$, then (by [10, Lemma 4.2]) there is a differentiable homeomorphism $\varphi:[0,T]\to[0,T]$ with an absolutely continuous inverse $\psi:[0,T]\to[0,T]$ such that $z^*=z\circ\varphi$ is a $W^{1,\infty}$-map with action $A(z^*)=A(z)=T$ and satisfying

$$\begin{cases}-J\dot{z}^*(t)=\partial H_K(z^*(t)),\ \text{a.e.},\ z^*(0),z^*(T)\in\mathbb{R}^{n,k},\\ z^*(T)-z^*(0)\in V_0^{n,k}\quad\textit{and}\quad z^*([0,T])\subset\partial K;\end{cases} \tag{2.7}$$

moreover the path $u:[0,1]\to\mathbb{R}^{2n}$ defined by

$$u(t)=\frac{1}{\sqrt{T}}z^*(Tt)-\frac{1}{\sqrt{T}}P_{n,k}\int_0^1 z^*(Tt)dt \tag{2.8}$$

where $P_{n,k}:\mathbb{R}^{2n}=JV_0^{n,k}\oplus\mathbb{R}^{n,k}\to\mathbb{R}^{n,k}$ is the orthogonal projection, belongs to $\mathcal{A}_2$ and satisfies $I_2(u)=T$, i.e., u is a minimier u of I_2 over $\mathcal{A}_2$. When this K is also a convex polytope as above (1.3), there holds

$$\dot{u}(t)=\sqrt{T}\dot{z}^*(Tt)\in\sqrt{T}\mathrm{conv}\{p_i\,|\,\sqrt{T}(u(t)-b)\in F_i\},\ \text{a.e.}$$

where $p_i=\frac{2}{h_i}Jn_i$ and $b=-\frac{1}{\sqrt{T}}P_{n,k}\int_0^1 z^*(Tt)dt$.

The final claim is obtained as below Theorem 2.1.

3. Piecewise affine paths

In this section we will generalize some results on piecewise affine loops in [15, §3] to piecewise affine paths.

Recall in [15, Definition 3.2] that a finite sequence of disjoint open intervals $(I_i)_{i=1}^m$ is called a partition of $[0,1]$ if there exists an increasing sequence of numbers $0=\tau_0\le\tau_1\le\ldots\le\tau_m=1$ with $I_i=(\tau_{i-1},\tau_i)$. (Note that the open interval I_i may be empty!) As usual let χ_I denote the characteristic function of a subset $I\subset\mathbb{R}$. A path $z\in H^1([0,1],\mathbb{R}^{2n})$ is said to be piecewise affine if $\dot z$ can be written as $\dot z(t)=\sum_{j=1}^m\chi_{I_j}(t)w_j$ for almost every $t\in[0,1]$, where $(I_j)_{j=1}^m$ is a partition of $[0,1]$ and $(w_j)_{j=1}^m\in\mathbb{R}^{2n}$ is a finite sequence of vectors.

Lemma 3.1. *([15, Lemma 3.1]) Fix a set of vectors* $v_1,\cdots,v_k\in\mathbb{R}^{2n}$. *Suppose* $z\in H^1([0,1],\mathbb{R}^{2n})$ *satisfies that for almost every* $t\in[0,1]$, *one has* $\dot z(t)\in\operatorname{conv}\{v_1,\cdots,v_k\}$. *Then for every* $\varepsilon>0$, *there exists a piecewise affine path* ς *with* $\|z-\varsigma\|_{W^{1,2}}<\varepsilon$, *and so that* $\dot\varsigma$ *is composed of vectors from the set* $\operatorname{conv}\{v_1,\cdots,v_k\}$, *and* $\varsigma(0)=z(0),\varsigma(1)=z(1)$.

The following is an analouge of [15, Proposition 3.3].

Proposition 3.2. *If a path* $z\in H^1([0,1],\mathbb{R}^{2n})$ *is such that* $\dot z(t)=\sum_{i=1}^m\chi_{I_i}(t)w_i$ *almost everywhere, where* $(I_i=(\tau_{i-1},\tau_i))_{i=1}^m$ *is a partition of* $[0,1]$, *and* $w_1,\cdots,w_m\in\mathbb{R}^{2n}$, *then*

$$\int_0^1\langle -J\dot z,z\rangle dt=\sum_{i=1}^m\sum_{j=1}^{i-1}|I_j||I_i|\omega_0(w_j,w_i)+\omega_0(z(0),z(1)).\tag{3.1}$$

As usual $\sum_{j=1}^{i-1}|I_j||I_i|\omega_0(w_j,w_i)$ *for* $i=1$ *is understood as zero.*

Proof. The case $m=1$ is clear. Now we assume $m>1$. Since

$$\int_0^1\langle -J\dot z(t),z(0)\rangle dt=-\langle Jz(1),z(0)\rangle=-\omega_0(Jz(1),Jz(0))=\omega_0(z(0),z(1))$$

we deduce

$$\begin{aligned}&\int_0^1\langle -J\dot z,z\rangle dt\\&=\int_0^1\langle -J\dot z,z(0)+\int_0^t\dot z(s)ds\rangle dt\\&=\int_0^1\langle -J\dot z(t),z(0)\rangle dt\\&\quad+\sum_{i=1}^m\int_{I_i}\langle -J\sum_{l=1}^m\chi_{I_l}(t)w_l,\int_0^{\tau_{i-1}}\sum_{l=1}^m\chi_{I_l}(s)w_lds+\int_{\tau_{i-1}}^t w_ids\rangle dt\end{aligned}$$

$$= \omega_0(z(0), z(1)) + \sum_{i=1}^{m} \int_{I_i} \langle -Jw_i, \sum_{j<i} \int_{I_j} \sum_{l=1}^{m} \chi_{I_l}(s) w_l ds + (t - \tau_{i-1}) w_i \rangle dt$$

$$= \omega_0(z(0), z(1)) + \sum_{i=1}^{m} \int_{I_i} \langle -Jw_i, \sum_{j<i} \int_{I_j} w_j ds \rangle dt$$

$$= \omega_0(z(0), z(1)) + \sum_{i=1}^{m} \sum_{j<i} |I_i||I_j| \omega_0(w_j, w_i).$$

□

Following the proof ideas of [15, Lemma 3.1] we can obtain:

Lemma 3.3. *Given a set of vectors, $v_1, \ldots, v_k \in \mathbb{R}^{2n}$, for any piecewise affine path $z \in H^1([0,1], \mathbb{R}^{2n})$ with $\dot{z}(t) \in conv\{v_1, \ldots, v_k\}$ for almost every $t \in [0,1]$, there exists another piecewise affine path $z' \in H^1([0,1], \mathbb{R}^{2n})$ so that $z'(0) = z(0), z'(1) = z(1), \dot{z}'(t) \in \{v_1, \ldots, v_k\}$ for almost every t, and*

$$\int_0^1 \langle -J\dot{z}', z' \rangle dt \geq \int_0^1 \langle -J\dot{z}, z \rangle dt.$$

Proof. Write $\dot{z}(t) = \sum_{j=1}^{m} \chi_{I_j}(t) w_j$, where $w_j \in \mathrm{conv}\{v_1, \ldots, v_k\}$ for each j, and $(I_j)_{j=1}^{m}$ is a partition of $[0,1]$. Clearly, there exists $l = l(i) \in \mathbb{N}$ such that $w_i = \sum_{j=1}^{l} a_{i_j} v_{i_j}$, where $a_{i_j} > 0$, $i_j \in \{1, \ldots, k\}$, and $\sum_{j=1}^{l} a_{i_j} = 1$. Consider the partition of I_i to disjoint subintervals, $\{I_{i_j}\}_{j=1}^{l}$, where the length of I_{i_j} is $|I_{i_j}| = a_{i_j} |I_i|$. Define

$$\dot{z}_*(t) = \sum_{j<i} \chi_{I_j}(t) w_j + \sum_{j=1}^{l} \chi_{I_{i_j}}(t) v_{i_j} + \sum_{j>i} \chi_{I_j}(t) w_j \tag{3.2}$$

and $z_*(t) = z(0) + \int_0^t \dot{z}_*(s) ds$ for $t \in [0,1]$. Since $\int_0^1 \dot{z}_*(t) dt = \int_0^1 \dot{z}(t) dt$, we deduce $z(0) = z_*(0)$ and $z(1) = z_*(1)$. Then Proposition 3.2 leads to

$$\begin{aligned}
&\int_0^1 \langle -J\dot{z}_*, z_* \rangle dt \\
&\quad = \omega_0(z_*(0), z_*(1)) \\
&\qquad + \sum_{\substack{r<s \\ r,s \neq i}} |I_r||I_s| \omega_0(w_r, w_s) + \sum_{j=1}^{l} \sum_{r<i} |I_r||I_i| a_{i_j} \omega_0(w_r, v_{i_j}) \\
&\qquad + \sum_{j=1}^{l} \sum_{r>i} |I_r||I_i| a_{i_j} \omega_0(v_{i_j}, w_r) + \sum_{1 \leq r < s \leq l} |I_i|^2 a_{i_r} a_{i_s} \omega_0(v_{i_r}, v_{i_s})
\end{aligned}$$

$$= \omega_0(z(0), z(1)) + \sum_{\substack{r<s \\ r,s\neq i}} |I_r||I_s|\omega_0(w_r, w_s) + \sum_{r<i} |I_r||I_i|\omega_0(w_r, w_i)$$
$$+ \sum_{r>i} |I_r||I_i|\omega_0(w_i, w_r) + \sum_{1\le r<s\le l} |I_i|^2 a_{i_r} a_{i_s} \omega_0(v_{i_r}, v_{i_s})$$
$$= \int_0^1 \langle -J\dot{z}, z\rangle dt + |I_i|^2 \sum_{1\le r<s\le l} a_{i_r} a_{i_s} \omega_0(v_{i_r}, v_{i_s}).$$

Define $b_{i_j} = a_{i_{l+1-j}}$ and $u_{i_j} = v_{i_{l+1-j}}$ for $j = 1, \cdots, l$, and

$$\hat{I}_j = I_j \text{ for } j < i \text{ or } j > i, \quad \hat{I}_{i_j} = I_{i_{l+1-j}} \text{ for } j = 1, \cdots, l.$$

As above we may show that $z_{**}(t) = z(0) + \int_0^t \dot{z}_{**}(s)ds$ for $t \in [0, 1]$, where

$$\dot{z}_{**}(t) = \sum_{j<i} \chi_{\hat{I}_j}(t) w_j + \sum_{j=1}^{l} \chi_{\hat{I}_{i_j}}(t) u_{i_j} + \sum_{j>i} \chi_{\hat{I}_j}(t) w_j,$$

satisfies $z(0) = z_{**}(0)$, $z(1) = z_{**}(1)$ and

$$\int_0^1 \langle -J\dot{z}_{**}, z_{**}\rangle dt = \int_0^1 \langle -J\dot{z}, z\rangle dt + |I_i|^2 \sum_{1\le r<s\le l} b_{i_r} b_{i_s} \omega_0(u_{i_r}, u_{i_s}).$$

A straightforward computation as above gives rise to

$$\sum_{1\le r<s\le l} b_{i_r} b_{i_s} \omega_0(u_{i_r}, u_{i_s}) = - \sum_{1\le r<s\le l} a_{i_r} a_{i_s} \omega_0(v_{i_r}, v_{i_s}).$$

Hence we can always choose $u \in \{z_*, z_{**}\}$ so that

$$\int_0^1 \langle -J\dot{u}, u\rangle dt \ge \int_0^1 \langle -J\dot{z}, z\rangle dt. \tag{3.3}$$

Now starting from z and choosing $i = 1$ we get a path z_1 as above, Then starting from z_1 and choosing $i = 2$ we get a path z_2 again. Continuing this progress we obtain $z_1, z_2, \cdots, z_m$. Then $z' := z_m$ satisfies the requirements of the lemma. □

Suitably modifying the proof of [15, Lemma 3.5], we can get the following analogues of it.

Lemma 3.4. *Given a finite sequence of pairwise distinct vectors* $(v_1, \cdots, v_k)$, *if* $z \in H^1([0, 1], \mathbb{R}^{2n})$ *is a piecewise affine path such that* $\dot{z}(t) = \sum_{i=1}^m \chi_{I_i}(t) w_i$ *with* $w_i \in \{v_1, \cdots, v_k\}$ *for each* i, *where* $(I_i = (\tau_{i-1}, \tau_i))_{i=1}^m$ *is a partition of* $[0, 1]$, *then there exists another piecewise affine path* z' *such that* $\dot{z}'(t) \in \{v_1, \cdots, v_k\}$ *for almost every* t, $z'(0) = z(0)$, $z'(1) = z(1)$, *and* $\{t : \dot{z}'(t) = v_j\}$ *is connected for every* $j = 1, \cdots, k$. *In addition,*

$$\int_0^1 \langle -J\dot{z}', z'\rangle dt \ge \int_0^1 \langle -J\dot{z}, z\rangle dt. \tag{3.4}$$

Proof. Assume $w_r = w_s$ for some $r < s$. Consider a rearrangement of the intervals I_i by deleting the intervals I_s and increasing the length of the interval I_r by $|I_s| = \tau_s - \tau_{s-1}$, that is,

$$I_i^* = \begin{cases} (\tau_{i-1}, \tau_i), & i < r, \\ (\tau_{i-1}, \tau_i + \tau_s - \tau_{s-1}), & i = r, \\ (\tau_{i-1} + \tau_s - \tau_{s-1}, \tau_i + \tau_s - \tau_{s-1}), & r < i < s, \\ \emptyset, & i = s, \\ (\tau_{i-1}, \tau_i), & i > s. \end{cases}$$

Define z_* by $z_*(t) = z(0) + \int_0^t \dot{z}_*(s)ds$, where $\dot{z}_*(t) = \sum_{i=1}^m \chi_{I_i^*}(t)w_i$. Then

$$\int_0^1 \dot{z}_* dt = \sum_{i=1}^m |I_i^*| w_i = \sum_{i=1}^m |I_i| w_i = \int_0^1 \dot{z} dt$$

and thus $z_*(0) = z(0)$ and $z_*(1) = z(1)$. Since $I_i^* = I_i$ for $i < r$ or $i > s$, by Proposition 3.2, one can get

$$\int_0^1 \langle -J\dot{z}_*, z_* \rangle dt - \int_0^1 \langle -J\dot{z}, z \rangle dt = \sum_{i=r+1}^{s-1} 2|I_s||I_i|\omega_0(w_s, w_i).$$

Similarly, by erasing I_r and increasing the length of I_s by $|I_r|$, we get a z_{**} such that

$$\int_0^1 \langle -J\dot{z}_{**}, z_{**} \rangle dt - \int_0^1 \langle -J\dot{z}, z \rangle dt = \sum_{i=r+1}^{s-1} 2|I_r||I_i|\omega_0(w_i, w_r).$$

It follows that either z_* or z_{**} satisfies (3.4). Denote by $z_1 \in \{z_*, z_{**}\}$ satisfying (3.4). Then

$$z_1(t) = z(0) + \int_0^t \dot{z}_1(s)ds \quad \text{with } \dot{z}_1(t) = \sum_{i=1}^m \chi_{I_i^1}(t)w_i.$$

Repeating this methods for different disjoint nonempty interval I_r^1, I_s^1 whenever $w_r = w_s$ we get a z_2 again. Proceeding with this progress for z_2, after finite steps we get a z' with the expected properties. □

Having the above lemmas we have the following corresponding result with [15, Proposition 3.5], which may be proved by repeating the arguments therein because $H_K^* = \frac{1}{4}h_K^2$.

Proposition 3.5. *For a convex polytope $K \subset \mathbb{R}^{2n}$ containing 0 in the interior of it, let $\{F_i\}_{i=1}^{\mathbf{F}_K}$ be the $(2n-1)$-dimensional facets of it, let n_i be the unit outer normal to F_i, let $p_i = J\partial H_K|_{F_i} = \frac{2}{h_i}Jn_i$, where $h_i := h_K(n_i)$ and $h_K(x) = \sup\{\langle y, x\rangle \,|\, y \in K\}$. Let $c > 0$ be a constant and let $z \in H^1([0,1], \mathbb{R}^{2n})$ satisfies that for almost every t, there is a non-empty face of K, $F_{j_1} \cap \cdots \cap F_{j_l} \neq \emptyset$, with $\dot{z}(t) \in c \cdot conv\{p_{j_1}, \cdots, p_{j_l}\}$. Then*

$$\int_0^1 H_K^*(-J\dot{z}(t))dt = c^2.$$

4. Proof of Theorem 1.1

We begin with a similar result to [15, Theorem 1.5].

Theorem 4.1. *Let K be a convex polytope as above (1.3). Suppose $0 \in \mathrm{Int}(K)$. Then for any $\Psi \in \mathrm{Sp}(2n, \mathbb{R})$ there exists a generalized Ψ-characteristic $\gamma : [0,1] \to \partial K$ with action*

$$A(\gamma) = \min\{A(x) > 0 \,|\, x \text{ is a generalized } \Psi\text{-characteristic on } \partial K\}$$

such that $\dot{\gamma}$ is piecewise constant and is composed of a finite sequence of vectors, i.e. there exists a sequence of vectors $(w_1, \ldots, w_m)$, and a sequence $(0 = \tau_0 < \cdots < \tau_{m-1} < \tau_m = 1)$ so that $\dot{\gamma}(t) = w_i$ for $\tau_{i-1} < t < \tau_i$. Moreover, for each $j \in \{1, \cdots, m\}$ there exists $i \in \{1, \cdots, \mathbf{F}_K\}$ so that $w_j = C_j J n_i$ for some $C_j > 0$, and for each $i \in \{1, \cdots, \mathbf{F}_K\}$ and for every $C > 0$ the set $\{t \in [0,1] \,|\, \dot{\gamma}(t) = CJn_i\}$ is either empty or connected, i.e. for every i there is at most one $j \in \{1, \ldots, m\}$ with $w_j = C_j J n_i$. Hence $\dot{\gamma}$ has at most $\mathbf{F}_K$ discontinuous points, and γ visits the interior of each facet at most once.

Proof. Let $z : [0, T] \to \partial K$ be a generalized Ψ-characteristic with action $A(z) = c^{\Psi}_{\mathrm{EHZ}}(K) = T$. By Theorem 2.1 we have $b \in \mathrm{Ker}(\Psi - I_{2n})$ and the $W^{1,2}$-path $u \in \mathcal{A}_\Psi$ satisfying $I_K(u) = T$ and (2.4). Thus we obtain $\int_0^1 H_K^*(-J\dot{u}(t))dt = T$ by Proposition 3.5. For convenience let $c = T^{1/2}$. The next argument is the same as the proof of [15, Theorem 1.5], we write it for completeness.

For every $N \in \mathbb{N}$, Lemma 3.1 yields a piecewise affine path ζ_N such that

$$\| u - \zeta_N \|_{W^{1,2}} \leqslant \frac{1}{N} \quad \text{and} \quad \dot{\zeta}_N(t) \in c \cdot \mathrm{conv}\{p_1, \cdots, p_{\mathbf{F}_K}\}$$

for almost every t, $\zeta_N(0) = u(0), \zeta_N(1) = u(1)$. By applying Lemma 3.3 with $v_i = cp_i, i = 1, \cdots, \mathbf{F}_K$ to ζ_N, we get a piecewise affine path $\zeta'_N \in W^{1,2}([0,1], \mathbb{R}^{2n})$ such that

$$\zeta'_N(0) = u(0), \;\; \zeta'_N(1) = u(1), \;\; \dot{\zeta}'_N(t) \in \{v_1, \ldots, v_{\mathbf{F}_K}\} \text{ a.e., and } A(\zeta'_N) \geq A(\zeta_N).$$

Applying Lemma 3.4 to ζ'_N again, we get a piecewise affine path $u_N : [0,1] \to \mathbb{R}^{2n}$ from $u(0)$ to $u(1)$ such that

$$\dot{u}_N(t) = \sum_{i=1}^{m_N} \chi_{I_i^N}(t) v_i^N$$

where $v_i^N = v_j$ for some $j \in \{1, \cdots, \mathbf{F}_K\}$ and for every j there is at most one such i, and that

$$A_N := \sqrt{A(u_N)} \geqslant \sqrt{A(\zeta_N)}.$$

Define $u'_N := \frac{u_N}{A_N} \in \mathcal{A}_\Psi$ and $c_N =: \frac{c}{A_N}$. Write $w_i^N := \frac{v_i^N}{A_N}$ for the velocities of u'_N, which sits in the set $\frac{c}{A_N} \cdot \{p_1, \cdots, p_{\mathbf{F}_K}\}$. Since $\| u - \zeta_N \|_{W^{1,2}} \leqslant \frac{1}{N}$ we deduce that $A(\zeta_N) \to 1$ as $N \to \infty$. Hence $\underline{\lim}_{N\to\infty} A_N \geqslant 1$, and $\overline{\lim}_{N\to\infty} c_N \leqslant c$. Moreover Proposition 3.5 and the minimality of $I_K(u)$ imply $c_N^2 = I_K(u'_N) \geqslant I_K(u) = c^2$. We deduce $\lim_{N\to\infty} c_N = c$ and thus $\lim_{N\to\infty} A_N = 1$.

Let $\mathcal{A}^1$ consist of $z \in H^1([0,1], \mathbb{R}^{2n})$ for which there exist $C > 0$ and an increasing sequence of numbers $0 = \tau_0 \le \tau_1 \le \ldots \le \tau_{\mathbf{F}_K} = 1$ such that

$$\dot{z}(t) = \sum_{i=1}^{\mathbf{F}_K} \chi_{I_i}(t) C \cdot p_{\sigma(i)}$$

with $I_i = (\tau_{i-1}, \tau_i)$, where $\sigma \in S_{\mathbf{F}_K}$ is the permutations on $\{1, \cdots, \mathbf{F}_K\}$. Define a map

$$\Phi : \mathcal{A}^1 \to S_{\mathbf{F}_K} \times \mathbb{R}^{\mathbf{F}_K},\ z \mapsto (\sigma, (|I_1|, \cdots, |I_{\mathbf{F}_K}|)). \tag{4.1}$$

Clearly, the image $\mathrm{Im}(\Phi)$ is contained in the compact subset of $S_{\mathbf{F}_K} \times \mathbb{R}^{\mathbf{F}_K}$,

$$S_{\mathbf{F}_K} \times \left\{ (t_1, \cdots, t_{\mathbf{F}_K}) \in \mathbb{R}^{\mathbf{F}_K} \,\middle|\, t_i \geqslant 0\ \forall i,\ \sum_{i=1}^{\mathbf{F}_K} t_i = 1 \right\}.$$

Since $u'_N \in \mathcal{A}^1$ with $C = c_N$, we can write $\Phi(u'_N) = (\sigma^N, (t_1^N, \cdots, t_{\mathbf{F}_K}^N))$. After passing to a subsequence, we can assume that $\sigma^N = \sigma$ is constant, and $(t_1^N, \cdots, t_{\mathbf{F}_K}^N)$ converges to a vector $(t_1^\infty, \cdots, t_{\mathbf{F}_K}^\infty)$. Define

$$\tau_0^\infty = 0,\ \tau_1^\infty = \tau_0^\infty + t_1^\infty,\ \tau_j^\infty = \tau_0^\infty + \sum_{i=1}^{j} t_i^\infty,\ j = 2, \cdots, \mathbf{F}_K,$$
$$I_i^\infty = (\tau_{i-1}^\infty, \tau_i^\infty),\ i = 1, \cdots, \mathbf{F}_K$$

and the piecewise affine path $u'_\infty(t) := u(0) + \int_0^t \dot{u}'_\infty(s)ds$ with

$$\dot{u}'_\infty(t) = \sum_{i=1}^{\mathbf{F}_K} \chi_{I_i^\infty}(t) c \cdot p_{\sigma(i)}.$$

Let $\mathcal{T}^N = \{t \in [0,1] \,|\, \dot{u}'_N(t) = \frac{c}{c_N} \dot{u}'_\infty(t)\}$. Then

$$\int_{\mathcal{T}^N} \| \dot{u}'_N(t) - \dot{u}'_\infty(t) \|^2 \, dt \to 0 \quad \text{as} \quad N \to \infty.$$

Since $\| \dot{u}'_N(t) - \dot{u}'_\infty(t) \|^2$ is bounded on $\{t \in [0,1] \,|\, \dot{z}'_N(t)$ and $\dot{z}'_\infty(t)$ are defined$\}$, as $N \to \infty$ we get $|\mathcal{T}^N| \to 1$ and therefore

$$\int_{[0,1] \setminus \mathcal{T}^N} \| \dot{u}'_N(t) - \dot{u}'_\infty(t) \|^2 \, dt \to 0.$$

Observe that $\lim_{N \to \infty} \int_0^1 \dot{u}'_N(t)dt = \int_0^1 \dot{u}(t)dt$ implies $\int_0^1 \dot{u}'_\infty(t)dt = \int_0^1 \dot{u}(t)dt$. We deduce

$$u'_\infty(1) = u'_\infty(0) + \int_0^1 \dot{u}'_\infty(t)dt = u(0) + \int_0^1 \dot{u}(t)dt = u(1)$$

and so $u'_\infty(1) = \Psi u'_\infty(0)$. Moreover

$$\begin{aligned}|A(u'_\infty) - 1| =& |A(u'_\infty) - A(u'_N)| \\ =& \left|\frac{1}{2}\int_0^1 \langle -J\dot{u}'_\infty(t), u'_\infty(t)\rangle - \langle -J\dot{u}'_N(t), u'_N(t)\rangle dt\right| \\ \leqslant& \left|\frac{1}{2}\int_0^1 \langle -J(\dot{u}'_\infty(t) - \dot{u}'_N(t)), u'_\infty(t)\rangle dt\right| \\ &+ \left|\frac{1}{2}\int_0^1 \langle -J\dot{u}'_N(t), u'_\infty(t) - u'_N(t)\rangle dt\right| \\ \leqslant& \frac{1}{2}\int_0^1 |\dot{u}'_\infty(t) - \dot{u}'_N(t)||u'_\infty(t)|dt \\ &+ \frac{1}{2}\int_0^1 |\dot{u}'_N(t)||u'_\infty(t) - u'_N(t)|dt \to 0\end{aligned}$$

because $\dot{u}'_N$ and u'_∞ are bounded. Then $A(u'_\infty) = 1$, and thus $u'_\infty \in \mathcal{A}_\Psi$ and

$$I_K(u'_\infty) = \lim_{N\to\infty} I_K(u'_N) = \lim_{N\to\infty} c_N^2 = c^2 = T = c^\Psi_{\mathrm{EHZ}}(K).$$

By Theorem 2.1 we have $a_0 \in \mathrm{Ker}(\Psi - I_{2n})$ such that the $W^{1,2}$-path

$$[0,T] \ni t \mapsto \gamma^*(t) = \sqrt{T}u'_\infty(t/T) + a_0/\sqrt{T} \tag{4.2}$$

is a piecewise affine generalized Ψ-characteristic on ∂K with action $A(\gamma^*) = c^\Psi_{\mathrm{EHZ}}(K)$. Then the generalized Ψ-characteristic on ∂K, $[0,1] \ni t \mapsto \gamma(t) := \gamma^*(Tt)$, has action $A(\gamma) = c^\Psi_{\mathrm{EHZ}}(K)$ and satisfies $\dot{\gamma}(t) \in T\cdot\{p_1, \cdots, p_{\mathbf{F}_K}\}$ for almost every $t \in [0,1]$ and that the set $\{t : \dot{\gamma}(t) = p_i\}$ is connected for every i. Recall $p_i = \frac{2}{h_i}Jn_i$. Theorem 4.1 is proved. □

Proof of Theorem 1.1. **Step 1**. *Case* $0 \in \mathrm{Int}(K)$. Let $\mathcal{A}^0_\Psi$ consist of $z \in \mathcal{A}_\Psi$ for which there exist $C > 0$ and an increasing sequence of numbers $0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_{\mathbf{F}_K} = 1$ such that

$$\dot{z}(t) = \sum_{i=1}^{\mathbf{F}_K} \chi_{I_i}(t) C \cdot p_{\sigma(i)} \tag{4.3}$$

with $I_i = (\tau_{i-1}, \tau_i)$, where $\sigma \in S_{\mathbf{F}_K}$ is a permutation on $\{1, \cdots, \mathbf{F}_K\}$. Then u'_∞ in the proof of Theorem 4.1 belongs to $\mathcal{A}^0_\Psi$ and satisfies $I_K(u'_\infty) = c^\Psi_{\mathrm{EHZ}}(K)$. Thus

$$c^\Psi_{\mathrm{EHZ}}(K) = \min\{I_K(z)\,|\, z \in \mathcal{A}_\Psi\} = \min\{I_K(z)\,|\, z \in \mathcal{A}^0_\Psi\}. \tag{4.4}$$

For any $z \in \mathcal{A}^0_\Psi$, $\dot{z}$ has the form of (4.3) and hence

$$z(1) - z(0) = \int_0^1 \dot{z}(t)dt = C\sum_{i=1}^{\mathbf{F}_K} T_i p_{\sigma(i)}$$

where $T_i = |I_i|$, and Proposition 3.2 yields

$$1 = \frac{1}{2}\int_0^1 \langle -J\dot{z}, z\rangle dt = \frac{1}{2}C^2 \sum_{1\leqslant j<i\leqslant \mathbf{F}_K} T_iT_j\omega_0(p_{\sigma(j)}, p_{\sigma(i)}) + \frac{1}{2}\omega_0(z(0), z(1)).$$

Let $v = z(0)/C$. The above two formulas become, respectively, $\Psi v - v = \sum_{i=1}^{\mathbf{F}_K} T_i p_{\sigma(i)}$ and

$$1 = \frac{1}{2}\int_0^1 \langle -J\dot{z}, z\rangle dt = \frac{1}{2}C^2 \sum_{1\leqslant j<i\leqslant \mathbf{F}_K} T_iT_j\omega_0(p_{\sigma(j)}, p_{\sigma(i)}) + C^2\frac{1}{2}\omega_0(v, \Psi v).$$

By Proposition 3.5 we have $I_K(z) = C^2$, and thus

$$I_K(z) = \frac{2}{\sum_{1\leqslant j<i\leqslant \mathbf{F}_K} T_iT_j\omega_0(p_{\sigma(j)}, p_{\sigma(i)}) - \omega_0(\Psi v, v)} > 0. \tag{4.5}$$

With E_Ψ defined as in Theorem 1.1 let

$$M_\Psi^*(K) = \left\{ ((T_i)_{i=1}^{\mathbf{F}_K}, v, \sigma) \middle| \begin{array}{l} \sigma \in S_{\mathbf{F}_K},\ T_i \geqslant 0, \sum_{i=1}^{\mathbf{F}_K} T_i = 1, \sum_{i=1}^{\mathbf{F}_K} T_i p_{\sigma(i)} = \Psi v - v, \\ \sum_{1\leqslant j<i\leqslant \mathbf{F}_K} T_iT_j\omega_0(p_{\sigma(j)}, p_{\sigma(i)}) > \omega_0(\Psi v, v),\ v \in E_\Psi \end{array} \right\},$$

For every triple $((T_i)_{i=1}^{\mathbf{F}_K}, v, \sigma) \in M_\Psi^*(K)$, as the construction of u'_∞ in the proof of Theorem 4.1 we can use it to construct a $z \in \mathcal{A}_\Psi^0$ such that (4.5) holds. It follows from these and (4.4) that

$$c_{\mathrm{EHZ}}^\Psi(K) = \min_{((T_i)_{i=1}^{\mathbf{F}_K}, v, \sigma)\in M_\Psi^*(K)} \frac{2}{\sum_{1\leqslant j<i\leqslant \mathbf{F}_K} T_iT_j\omega_0(p_{\sigma(j)}, p_{\sigma(i)}) - \omega_0(\Psi v, v)},$$

Let $\beta_{\sigma(i)} = \frac{T_i}{h_{\sigma(i)}}$. Since $p_i = \frac{2}{h_i}Jn_i$, we get

$$c_{\mathrm{EHZ}}^\Psi(K) = \min_{((\beta_i)_{i=1}^{\mathbf{F}_K}, v, \sigma)\in M_\Psi(K)} \frac{2}{4\sum_{1\leqslant j<i\leqslant \mathbf{F}_K} \beta_{\sigma(i)}\beta_{\sigma(j)}\omega_0(n_{\sigma(j)}, n_{\sigma(i)}) - \omega_0(\Psi v, v)},$$

where $M_\Psi(K)$ is as in Theorem 1.1.

Step 2. *General case.* Let $p \in \mathrm{Int}(K)$ be a fixed point of Ψ. Consider the symplectomorphism

$$\phi : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0), x \mapsto x - p. \tag{4.6}$$

Since $\Psi(p) = p$, $\phi \circ \Psi = \Psi \circ \phi$ and thus $c_{\mathrm{EHZ}}^\Psi(K) = c_{\mathrm{EHZ}}^\Psi(\phi(K))$ by the arguments below Proposition 1.2 of [10]. Let us write $\hat{K} = \phi(K)$ for convenience. Denote all $(2n-1)$-dimensional facets of it by $\{\hat{F}_i\}_{i=1}^{\mathbf{F}_{\hat{K}}}$, the unit outer normal to $\hat{F}_i$ by $\hat{n}_i$, the support function of $\hat{K}$ by $h_{\hat{K}}$. Then $\mathbf{F}_{\hat{K}} = \mathbf{F}_K$, $\hat{F}_i = F_i - p$ and $\hat{n}_i = n_i$ for $i = 1, \cdots, \mathbf{F}_K$, and $h_{\hat{K}}(y) = h_K(y) - \langle p, y\rangle$. By Step 1 we get

$$c_{\mathrm{EHZ}}^\Psi(\hat{K}) = \min_{\left((\beta_i)_{i=1}^{\mathbf{F}_K}, v, \sigma\right)\in M_\Psi(\hat{K})} \frac{2}{4\sum_{1\leqslant j<i\leqslant \mathbf{F}_K} \beta_{\sigma(i)}\beta_{\sigma(j)}\omega_0(n_{\sigma(j)}, n_{\sigma(i)}) - \omega_0(\Psi v, v)},$$

where with $\hat{h}_i = \hat{h}_{\hat{K}}(n_i) = h_K(n_i) - \langle p, n_i\rangle = h_i - \langle p, n_i\rangle$ for $i = 1, \cdots, \mathbf{F}_K$,

$$M_\Psi(\hat{K}) = \left\{ ((\beta_i)_{i=1}^{\mathbf{F}_K}, v, \sigma) \middle| \begin{array}{l} \sigma \in S_{\mathbf{F}_K},\ \beta_i \geqslant 0,\ \sum_{i=1}^{\mathbf{F}_K} \beta_i\hat{h}_i = 1,\ \sum_{i=1}^{\mathbf{F}_K} 2\beta_i Jn_i = \Psi v - v, \\ 4\sum_{1\leqslant j<i\leqslant \mathbf{F}_K} \beta_{\sigma(i)}\beta_{\sigma(j)}\omega_0(n_{\sigma(j)}, n_{\sigma(i)}) > \omega_0(\Psi v, v),\ v \in E_\Psi \end{array} \right\}.$$

Clearly, it remains to prove $M_\Psi(\hat{K}) = M_\Psi(K)$. In fact, for any $\left((\beta_i)_{i=1}^{\mathbf{F}_K}, v, \sigma\right) \in M_\Psi(\hat{K})$, since

$$1 = \sum_{i=1}^{\mathbf{F}_K} \beta_i\hat{h}_i = \sum_{i=1}^{\mathbf{F}_K} \beta_i h_i - \langle p, \sum_{i=1}^{\mathbf{F}_K} \beta_i n_i\rangle,$$

it suffices to prove $\langle p, \sum_{i=1}^{\mathbf{F}_K} \beta_i n_i\rangle = 0$. Note that $\sum_{i=1}^{\mathbf{F}_K} 2\beta_i J n_i = \Psi v - v$, $v \in E_\Psi$. We have

$$\langle p, \sum_{i=1}^{\mathbf{F}_K} \beta_i n_i\rangle = \omega_0(p, \sum_{i=1}^{\mathbf{F}_K} \beta_i J n_i) = \frac{1}{2}\omega_0(p, \Psi v - v) = \frac{1}{2}(\omega_0(p, \Psi v) - \omega_0(p, v)) = 0$$

because $\omega_0(p, \Psi v) = \omega_0(\Psi p, \Psi v) = \omega_0(p, v)$. Hence $M_\Psi(\hat{K}) \subset M_\Psi(K)$, and hence $M_\Psi(K) \subset M_\Psi(\hat{K})$ since $K = \hat{K} - (-p)$ and $\Psi(-p) = -p$. □

5. Proofs of Theorems 1.2, 1.3

We have an analogue of Theorem 4.1:

Theorem 5.1. *Let K be a convex polytope as above (1.3). If $0 \in \mathrm{Int}(K)$, there exists a generalized leafwise chord on ∂K for $\mathbb{R}^{n,k}$: $\gamma : [0,1] \to \partial K$ with $A(z) = \min\{A(x) | x$ is a generalized leafwise chord on ∂K for $\mathbb{R}^{n,k}\}$ such that $\dot{\gamma}$ is piecewise constant and is composed of a finite sequence of vectors, i.e. there exists a sequence of vectors $(w_1, \ldots, w_m)$, and a sequence $(0 = \tau_0 < \cdots < \tau_{m-1} < \tau_m = 1)$ so that $\dot{\gamma}(t) = w_i$ for $\tau_{i-1} < t < \tau_i$. Moreover, for each $j \in \{1, \cdots, m\}$ there exists $i \in \{1, \cdots, \mathbf{F}_K\}$ so that $w_j = C_j J n_i$, for some $C_j > 0$, and for each $i \in \{1, \cdots, \mathbf{F}_K\}$, the set $\{t : \exists C > 0, \dot{\gamma}(t) = CJn_i\}$ is connected, i.e. for every i there is at most one $j \in \{1, \ldots, m\}$ with $w_j = C_j J n_i$. Hence there are at most $\mathbf{F}_K$ points of discontinuity in $\dot{\gamma}$, and γ visits the interior of each facet at most once.*

Proof. Let $z : [0, T] \to \partial K$ be a generalized leafwise chord with action $A(z) = c_{\mathrm{LR}}(K, K \cap \mathbb{R}^{n,k}) = c^{n,k}(K)$ for $\mathbb{R}^{n,k}$. By Theorem 2.2 we can assume it to satisfy (2.7) (by a reparametrization if necessary), and obtain that the path

$$u : [0,1] \to \mathbb{R}^{2n},\ t \mapsto \frac{1}{\sqrt{T}} z(Tt) - \frac{1}{\sqrt{T}} P_{n,k} \int_0^1 z(Tt)dt$$

belongs to $\mathcal{A}_2$ and satisfies $I_2(u) = T = c^{n,k}(K)$. Moreover

$$\dot{u}(t) = \sqrt{T}\dot{z}(Tt) \in \sqrt{T}\mathrm{conv}\{p_i \,|\, \sqrt{T}(u(t) - b) \in F_i\} \subset T^{1/2} \cdot \mathrm{conv}\{p_1, \cdots, p_{\mathbf{F}_K}\}$$

with $b = -\frac{1}{\sqrt{T}} P_{n,k} \int_0^1 z(Tt)dt$ and with $c = T^{1/2}$, and so $I_2(u) = c^2$ by Proposition 3.5.

For every $N \in \mathbb{N}$, Lemma 3.1 yields a piecewise affine path ζ_N such that

$$\| u - \zeta_N \|_{W^{1,2}} \leqslant \frac{1}{N}, \quad \zeta_N(0) = u(0), \quad \zeta_N(1) = u(1) \quad \text{and}$$
$$\dot{\zeta}_N(t) \in c \cdot \mathrm{conv}\{p_1, \cdots, p_{\mathbf{F}_K}\}$$

for almost every t. By applying Lemma 3.3 with $v_i = cp_i, i = 1, \cdots, \mathbf{F}_K$ to ζ_N, we get a piecewise affine path $\zeta'_N \in W^{1,2}([0,1], \mathbb{R}^{2n})$ such that

$$A(\zeta'_N) \geq A(\zeta_N), \quad \zeta'_N(0) = u(0), \quad \zeta'_N(1) = u(1), \quad \dot{\zeta}'_N(t) \in \{v_1, \ldots, v_{\mathbf{F}_K}\}$$

for almost every t. Applying Lemma 3.4 to ζ_N' again, we can obtain a piecewise affine path $u_N : [0,1] \to \mathbb{R}^{2n}$ from $u(0)$ to $u(1)$ such that

$$\dot{u}_N(t) = \sum_{i=1}^{m_N} \chi_{I_i^N}(t) v_i^N$$

where $v_i^N = v_j$ for some $j \in \{1, \cdots, \mathbf{F}_K\}$ and for every j there is at most one such i, and that

$$A_N := \sqrt{A(u_N)} \geqslant \sqrt{A(\zeta_N)}.$$

Define $u_N' := \frac{u_N}{A_N}$ and $c_N =: \frac{c}{A_N}$. Notice that $\int_0^1 u_N'(t)dt$ may not belong to $JV_0^{n,k}$ and u_N' may not belong to $\mathscr{F}_2$. Recall that $P_{n,k} : \mathbb{R}^{2n} = JV_0^{n,k} \oplus \mathbb{R}^{n,k} \to \mathbb{R}^{n,k}$ is the orthogonal projection. Define

$$y_N := u_N' - P_{n,k}\left(\int_0^1 u_N'(t)dt\right).$$

Then $\int_0^1 y(t)dt \in JV_0^{n,k}$ and

$$\begin{aligned} A(y_N) &= \int_0^1 \Big\langle -J\dot{u}_N', u_N'(t) - P_{n,k}\Big(\int_0^1 u_N'(t)dt\Big)\Big\rangle dt \\ &= A(u_N') - \Big\langle J(u_N'(1) - u_N'(0)), P_{n,k}\Big(\int_0^1 u_N'(t)dt\Big)\Big\rangle. \end{aligned}$$

Since $u_N'(1) - u_N'(0) \in V_0^{n,k}$, $A(y_N) = A(u_N') = 1$. Thus, $y_N \in \mathcal{A}_2$. Write $w_i^N := \frac{v_i^N}{A_N}$ for the velocities of y_N, which sits in the set $\frac{c}{A_N} \cdot \{p_1, \cdots, p_{\mathbf{F}_K}\}$. Since $\| u - \zeta_N \|_{W^{1,2}} \leqslant \frac{1}{N}$ we deduce that $A(\zeta_N) \to 1$ as $N \to \infty$. Hence $\underline{\lim}_{N\to\infty} A_N \geqslant 1$, and $\overline{\lim}_{N\to\infty} c_N \leqslant c$. Moreover Proposition 3.5 and the minimality of $I_2(u)$ imply that $c_N^2 = I_K(y_N) \geqslant I_K(u) = c^2$. Then $\lim_{N\to\infty} c_N = c$ and thus $\lim_{N\to\infty} A_N = 1$.

Recall that the set $\mathcal{A}^1$ is defined as above (4.1) and that the map Φ is as in (4.1). By the proof of Theorem 4.1, the image $\mathrm{Im}(\Phi)$ is contained in the compact subset of $S_{\mathbf{F}_K} \times \mathbb{R}^{\mathbf{F}_K}$,

$$S_{\mathbf{F}_K} \times \{(t_1, \cdots, t_{\mathbf{F}_K}) \in \mathbb{R}^{\mathbf{F}_K} \mid t_i \geqslant 0 \ \forall i, \ \sum_{i=1}^{\mathbf{F}_K} t_i = 1\}.$$

Since $y_N \in \mathcal{A}^1$ with $C = c_N$, we can write $\Phi(y_N) = (\sigma^N, (t_1^N, \cdots, t_{\mathbf{F}_K}^N))$. After passing to a subsequence, we can also assume that $\sigma^N = \sigma$ is constant, and $(t_1^N, \cdots, t_{\mathbf{F}_K}^N)$ converges to a vector $(t_1^\infty, \cdots, t_{\mathbf{F}_K}^\infty)$. Define

$$\begin{aligned} &\tau_0^\infty = 0, \quad \tau_1^\infty = \tau_0^\infty + t_1^\infty, \quad \tau_j^\infty = \tau_0^\infty + \sum_{i=1}^{j} t_i^\infty, \quad j = 2, \cdots, \mathbf{F}_K, \\ &I_i^\infty = (\tau_{i-1}^\infty, \tau_i^\infty), \quad i = 1, \cdots, \mathbf{F}_K \end{aligned}$$

and the piecewise affine path $u_\infty'(t) = u(0) + \int_0^t \dot{u}_\infty'(s)ds$ with

$$\dot{u}_\infty'(t) = \sum_{i=1}^{\mathbf{F}_K} \chi_{I_i^\infty}(t) c \cdot p_{\sigma(i)}.$$

Similar to the proof of Theorem 4.1, one gets u'_∞ satisfying $u'_\infty(0) = u(0), u'_\infty(1) = u(1)$, $A(u'_\infty) = 1$ and $I_2(u'_\infty) = c^2$. Define

$$u_\infty := u'_\infty - P_{n,k}\Big(\int_0^1 u'_\infty(t)dt\Big).$$

Then $u_\infty \in \mathcal{A}_2$ and $I_2(u_\infty) = T = c^{n,k}(K)$. By Theorem 2.2 we have $\mathbf{a}_0 \in \mathbb{R}^{n,k}$ such that

$$[0,1] \ni t \mapsto \gamma(t) := \sqrt{T}u_\infty(t) + \mathbf{a}_0/\sqrt{T}$$

is a piecewise affine generalized leafwise chord on ∂K for $\mathbb{R}^{n,k}$ with action

$$A(\gamma) = I_2(u) = c_{\mathrm{LR}}(K, K \cap \mathbb{R}^{n,k})$$

and satisfying $\dot\gamma(t) \in T\cdot\{p_1, \cdots, p_{\mathbf{F}_K}\}$ for almost every $t \in [0,1]$ and that the set $\{t : \dot\gamma(t) = p_i\}$ is connected for every i. Recall $p_i = \frac{2}{h_i}Jn_i$. Theorem 5.1 is proved. □

Proof of Theorem 1.2. **Step 1**. *Case* $0 \in \mathrm{Int}(K)$. Let $\mathcal{A}_2^0$ consist of $z \in \mathcal{A}_2$ for which there exist $C > 0$ and an increasing sequence of numbers $0 = \tau_0 \le \tau_1 \le \ldots \le \tau_{\mathbf{F}_K} = 1$ such that

$$\dot z(t) = \sum_{i=1}^{\mathbf{F}_K} \chi_{I_i}(t)C\cdot p_{\sigma(i)} \tag{5.1}$$

with $I_i = (\tau_{i-1}, \tau_i)$, where $\sigma \in S_{\mathbf{F}_K}$ is the permutation on $\{1, \cdots, \mathbf{F}_K\}$. Then u'_∞ in the proof of Theorem 5.1 belongs to $\mathcal{A}_2^0$ and satisfies $I_K(u'_\infty) = c_{\mathrm{LR}}(K, K\cap\mathbb{R}^{n,k})$. Thus

$$c_{\mathrm{LR}}(K, K\cap\mathbb{R}^{n,k}) = \min\{I_2(z)\,|\, z\in\mathcal{A}_2\} = \min\{I_2(z)\,|\,z\in\mathcal{A}_2^0\}. \tag{5.2}$$

For any $z \in \mathcal{A}_2^0$, we have $z(0), z(1) \in \mathbb{R}^{n,k}$, $\dot z$ has the form of (5.1) and hence

$$V_0^{n,k} \ni z(1) - z(0) = \int_0^1 \dot z(t)dt = C\sum_{i=1}^{\mathbf{F}_K} T_i p_{\sigma(i)}$$

where $T_i = |I_i|$, and Proposition 3.2 yields

$$1 = \frac{1}{2}\int_0^1 \langle -J\dot z, z\rangle dt = \frac{1}{2}C^2 \sum_{1\leqslant j<i\leqslant \mathbf{F}_K} T_iT_j\omega_0(p_{\sigma(j)}, p_{\sigma(i)}) + \frac{1}{2}\omega_0(z(0), z(1)).$$

Note that $\omega_0(z(0), z(1)) = \omega_0(z(0), z(1) - z(0)) = 0$, and $I_2(z) = C^2$ by Proposition 3.5. Then

$$I_2(z) = \frac{2}{\sum_{1\leqslant j<i\leqslant\mathbf{F}_K} T_iT_j\omega_0(p_{\sigma(j)}, p_{\sigma(i)})} > 0. \tag{5.3}$$

Let

$$M^*(K) = \left\{((T_i)_{i=1}^{\mathbf{F}_K}, \sigma) \,\middle|\, \begin{array}{l} \sigma\in S_{\mathbf{F}_K}, T_i \geqslant 0, \sum_{i=1}^{\mathbf{F}_K} T_i = 1, \sum_{i=1}^{\mathbf{F}_K} T_i p_{\sigma(i)} \in V_0^{n,k} \\ \sum_{1\leqslant j<i\leqslant\mathbf{F}_K} T_iT_j\omega_0(p_{\sigma(j)}, p_{\sigma(i)}) > 0 \end{array}\right\}.$$

For every pair $((T_i)_{i=1}^{\mathbf{F}_K},\sigma)\in M^*(K)$, as in the construction of u'_∞ in the proof of Theorem 4.1 we can use $((T_i)_{i=1}^{\mathbf{F}_K},\sigma)$ to construct a $z\in\mathcal{A}_2^0$ such that (5.3) holds. It follows that

$$c_{\rm LR}(K,K\cap\mathbb{R}^{n,k})=\min_{((T_i)_{i=1}^{\mathbf{F}_K},\sigma)\in M^*(K)}\frac{2}{\sum_{1\leqslant j<i\leqslant\mathbf{F}_K}T_iT_j\omega_0(p_{\sigma(j)},p_{\sigma(i)})},$$

Define $\beta_{\sigma(i)}:=\frac{T_i}{h_{\sigma(i)}}$. Since $p_i=\frac{2}{h_i}Jn_i$, The above two formulas give the desired formula in this case.

Step 2. *General case.* Let $p\in\mathrm{Int}(K)\cap\mathbb{R}^{n,k}$. Then the symplectomorphism ϕ defined by (4.6) satisfies $c_{\rm LR}(\phi(K),\phi(K)\cap\mathbb{R}^{n,k})=c_{\rm LR}(K,K\cap\mathbb{R}^{n,k})$ by the arguments at the beginning of [11, §3]. As in Step 2 of the proof of Theorem 1.1 let $\hat{K}=\phi(K)$. By Step 1 we obtain

$$c_{\rm LR}(\hat{K},\hat{K}\cap\mathbb{R}^{n,k})=\frac{1}{2}\min_{((\beta_i)_{i=1}^{\mathbf{F}_K},\sigma)\in M(\hat{K})}\frac{1}{\sum_{1\leqslant j<i\leqslant\mathbf{F}_K}\beta_{\sigma(i)}\beta_{\sigma(j)}\omega_0(n_{\sigma(j)},n_{\sigma(i)})},$$

where

$$M(\hat{K})=\left\{((\beta_i)_{i=1}^{\mathbf{F}_K},\sigma)\;\middle|\;\begin{array}{l}\beta_i\geqslant 0,\ \sum_{i=1}^{\mathbf{F}_K}\beta_i\hat{h}_i=1,\ \sum_{i=1}^{\mathbf{F}_K}\beta_iJn_i\in V_0^{n,k},\\ \sum_{1\leqslant j<i\leqslant\mathbf{F}_K}\beta_{\sigma(i)}\beta_{\sigma(j)}\omega_0(n_{\sigma(j)},n_{\sigma(i)})>0,\ \sigma\in S_{\mathbf{F}_K}\end{array}\right\}.$$

Now we are in position to prove that $M(\hat{K})$ is equal to $M(K)$ in (1.6). We only need to prove $M(\hat{K})\subset M(K)$ because of obvious reasons. Since $((\beta_i)_{i=1}^{\mathbf{F}_K},\sigma)\in M(\hat{K})$ satisfies

$$1=\sum_{i=1}^{\mathbf{F}_K}\beta_i\hat{h}_i=\sum_{i=1}^{\mathbf{F}_K}\beta_ih_i-\Big\langle p,\sum_{i=1}^{\mathbf{F}_K}\beta_in_i\Big\rangle,$$

it suffices to prove $\langle p,\sum_{i=1}^{\mathbf{F}_K}\beta_in_i\rangle=0$. Note that $\sum_{i=1}^{\mathbf{F}_K}\beta_iJn_i\in V_0^{n,k}$. We have

$$\Big\langle p,\sum_{i=1}^{\mathbf{F}_K}\beta_in_i\Big\rangle=\omega_0\Big(p,\sum_{i=1}^{\mathbf{F}_K}\beta_iJn_i\Big)=0$$

because $\mathbb{R}^{n,k}$ and $V_0^{n,k}$ are ω_0-orthogonal. Hence $M(\hat{K})\subset M(K)$. □

Proof of Theorem 1.3. Let $p\in D\cap L\cap\mathbb{R}^{1,0}$, define $\phi:\mathbb{R}^2\to\mathbb{R}^2, x\mapsto x-p$. As in [11, §3] we have $c_{\rm LR}(D,D\cap\mathbb{R}^{1,0})=c_{\rm LR}(\phi(D),\phi(D)\cap\mathbb{R}^{1,0})$ and

$$c_{\rm LR}(D_1,D_1\cap\mathbb{R}^{1,0})=c_{\rm LR}(\phi(D_1),\phi(D_1)\cap\mathbb{R}^{1,0}),$$
$$c_{\rm LR}(D_2,D_2\cap\mathbb{R}^{1,0})=c_{\rm LR}(\phi(D_2),\phi(D_2)\cap\mathbb{R}^{1,0}).$$

Thus we can assume $0\in D\cap L\cap\mathbb{R}^{1,0}$ below.

Let $H^+:=\{(x,y)\in\mathbb{R}^2\,|\,y\geqslant 0\}$, $H^-:=\{(x,y)\in\mathbb{R}^2\,|\,y\leqslant 0\}$, and write $K^+=H^+\cap K$ and $K^-=H^-\cap K$ for any subset $K\subset\mathbb{R}^2$. On each of ∂D, ∂D_1 and ∂D_2 there only exist two generalized leafwise chords for $\mathbb{R}^{1,0}$, that is, $(\partial D)^+$ and $(\partial D)^-$ on ∂D, $(\partial D_1)^+$ and $(\partial D_1)^-$ on ∂D_1, $(\partial D_2)^+$ and $(\partial D_2)^-$ on ∂D_2. Note that a GLC x on ∂D for $\mathbb{R}^{1,0}$ and the line segment $D\cap\mathbb{R}^{1,0}$ form a loop γ and that $\langle -J\dot{z},z\rangle$ vanishes along the line segment $D\cap\mathbb{R}^{1,0}$.

Using these and Stokes theorem we deduce that $A(x) = \int_x qdp = \int_\gamma qdp$ is equal to the symplectic area of the domain surrounded by γ. Hence

$$\begin{aligned} c_{\rm LR}(D, D\cap\mathbb{R}^{1,0}) &= \min\{{\rm Area}(D^+), {\rm Area}(D^-)\},\\ c_{\rm LR}(D_1, D_1\cap\mathbb{R}^{1,0}) &= \min\{{\rm Area}(D_1^+), {\rm Area}(D_1^-)\},\\ c_{\rm LR}(D_2, D_2\cap\mathbb{R}^{1,0}) &= \min\{{\rm Area}(D_2^+), {\rm Area}(D_2^-)\}. \end{aligned}$$

Assume without loss of generality that $c_{\rm LR}(D, D\cap\mathbb{R}^{1,0}) = {\rm Area}(D^+)$. Then

$$\begin{aligned} c_{\rm LR}(D_1, D_1\cap\mathbb{R}^{1,0}) + c_{\rm LR}(D_2, D_2\cap\mathbb{R}^{1,0}) &\leqslant {\rm Area}(D_1\cap D^+) + {\rm Area}(D_2\cap D^+)\\ &= {\rm Area}(D^+) = c_{\rm LR}(D, D\cap\mathbb{R}^{1,0}). \end{aligned}$$

□

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