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Iterated nonexpansive mappings in Hilbert spaces

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Abstract. In [T. Dominguez Benavides and E. Llorens-Fuster, Iterated nonexpansive mappings, J. Fixed Point Theory Appl. 20 (2018), no. 3, Paper No. 104, 18 pp.], the authors raised the question about the existence of a fixed point free continuous INEA mapping T defined on a closed convex and bounded subset (or on a weakly compact convex subset) of a Banach space with normal structure. Our main goal is to give the affirmative answer to this problem in the very special case of a Hilbert space.

Mathematics Subject Classification. 47H09; 47H10; 47J26.

Keywords. Hilbert space, generalized nonexpansive mapping, asymptotically regular mapping.

1. Introduction and preliminaries

Let C be a weakly compact convex subset of a Banach space. In [1], the following concept of iterated nonexpansive mappings (INE in short) was stated:
The mapping $T: C \to C$ is INE if satisfies

$$\|T(Tx) - Tx\| \leq \|Tx - x\|, \qquad \text{for all } x \in C.$$

It is not surprising that an INE mapping does not have to have fixed points even if it is defined on a subset of a finite-dimensional Hilbert space (see, for instance, [2, Example 1.1]). Thus, it seems natural to raise the question of whether the same mapping must have a fixed point provided it is continuous. Clearly, according to the Klee result (see [3]), in this case we are considering a noncompact domain C, so the space must have infinite dimension (but still being a Hilbert space or a Banach one with normal structure). A Banach space is said to have normal structure if each convex subset C which contains more than one point has a point $x \in C$ which is not a diametral one of C, i.e., the following condition holds: $\sup\{\|x - y\| : \ y \in C\} < \operatorname{diam} C$ (see, for instance, [4]).

The negative answer to this question was given in [2,5], where the authors presented an example of a class of fixed point free mappings which are INE and continuous on any closed convex bounded but noncompact subset of a Banach space. The same is true when the set is convex and weakly compact (and noncompact with respect to the norm topology). The common denominator of these mappings was the fact that all of them satisfy

$$\|Tx - x\| = \|T(Tx) - Tx\|;$$

therefore, they were not asymptotically regular. Let us remind that the self-mapping $T : C \to C$ is asymptotically regular if for each $x \in C$ the sequence $\|T^n x - T^{n+1}x\|$ tends to 0. This condition can be generalized to the case of mappings which have the so-called almost fixed point sequence. A sequence (x_n) in C is called an almost fixed point sequence (a.f.p.s. for short) for the mapping T on C whenever $\|x_n - T(x_n)\| \to 0$. It is well known that if the self-mapping $T : C \to C$ is nonexpansive then T has an a.f.p.s. in C. Combining this fact with the normal structure of a space leads to the existence of fixed points (see, for instance, [6, Theorem 4.1] and [7, Theorem 2.7]). Iterated nonexpansive mappings which have a.f.p.s. are called INEA for short. Since the assumption of the existence of a.f.p.s. seems to play a crucial role, one may ask whether there is any fixed point free continuous INEA self-mapping of a closed convex bounded (or weakly-compact convex) subset C of a Banach space into C. Here we suppose additionally that the Banach space is a Hilbert one or Banach with normal structure.

As it was mentioned before, our main goal is to give an example of a continuous and INEA mapping T defined on a closed convex bounded subset (more precisely, on a closed unit ball) of the Hilbert space into itself for which the set of fixed points is empty. To do it, let us take the Hilbert space $l^2$and let B be its closed unit ball. Further, we will apply two kinds of geometry. The first one is Cartesian geometry based on the standard base of l^2 denoted by $\{e_n : \ n \in \mathbb{N}\}$. Then let $\langle \cdot, \cdot \rangle$ mean the inner product in l^2. Moreover, we denote the unit sphere by S. This set will be very often considered with spherical geometry based on the spherical metric ρ, i.e.,

$$\rho(A, B) = \arccos\langle A, B\rangle$$

for a pair of two elements $A, B \in S$. By the angle between two curves c and $\tilde{c}$ ($c(0) = \tilde{c}(0)$) on the sphere with respect to spherical geometry we mean the Alexandrov angle, defined by

$$\lim_{s,t \to 0^+} \angle_{c(0)}(c(s), \tilde{c}(t)).$$

This limit always exists (see, for instance, [8, p. 16]). More details about spherical geometry can be found in [8,9]).

2. Example

Our example may seem to be rather complicated. So, for the reader's convenience, we divide its description into six steps.

Step 1—construction of the curve

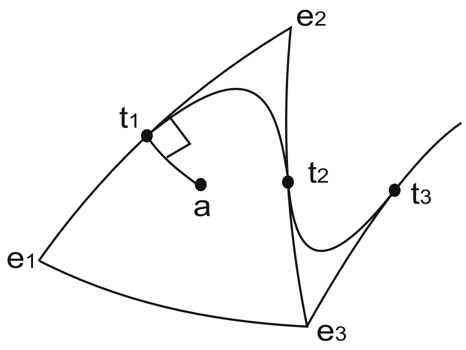

FIGURE 1. The construction of the curve

Let us take an infinite set of points $\{t_n : n \in \mathbb{N}\}$, where

$$t_n = \frac{e_n + e_{n+1}}{\sqrt{2}},$$

and first consider the convex hull of $\{e_1, e_2, e_3\}$ with respect to spherical geometry, i.e., the set

$$\operatorname{conv}_S(\{e_1, e_2, e_3\}) = \{x \in S : \langle x, e_1\rangle, \langle x, e_2\rangle, \langle x, e_3\rangle \geq 0, \ \langle x, e_n\rangle = 0, n > 3\}.$$

Further, we will use the same denotation $\operatorname{conv}_S(A)$ for any nonempty set $A \subset S$.

Now, still with respect to spherical geometry on $\operatorname{conv}_S(\{e_1, e_2, e_3\})$, let us notice that $a = 3^{-1/2}(e_1 + e_2 + e_3) \in \operatorname{conv}_S(\{e_1, e_2, e_3\})$. Moreover, if we join points a and t_1 and points t_1 and e_i, $i = 1, 2$, with the geodesic segments, then the angle between the segments is equal to $\pi/2$. Notice that the same is true for t_2 and e_i, $i = 2, 3$; hence, we can join points t_1 and t_2 with the curve

$$\operatorname{conv}_S(\{e_1, e_2, e_3\}) \cap S(a, \|a - t_1\|),$$

where $S(a, r)$ is a sphere in l^2 with the radius with respect to the norm. The angle between the curve and the segments $[a, t_1]$ or $[a, t_2]$ is also equal to $\pi/2$, so the curve is tangent to $[e_1, e_2]$ and $[e_2, e_3]$ at the points t_1 and t_2, respectively (Fig. 1).

Now, we repeat our construction on each three-dimensional set conv_S $(\{e_n, e_{n+1}, e_{n+2}\})$ and we obtain a smooth curve of infinite length joining all points t_n, $n = 1, 2, \ldots$. Let us notice that points of the curve between t_n and t_{n+1} can be treated as points of the circle centered at a point $\tilde{a} = \frac{\sqrt{2}}{3\sqrt{3}}(e_n + e_{n+1} + e_{n+2})$ with a radius of $r = \sqrt{\frac{11 - 4\sqrt{3}}{9}}$ (now with respect to Cartesian geometry). Then, the angle between radiuses $[\tilde{a}, t_n]$ and $[\tilde{a}, t_{n+1}]$ is independent of n and smaller than π.

Let us denote this angle by α and for all $t \in [0, \infty)$ we define $\varphi(t)$ as a point of the curve. If $t = n\,\alpha + \tau$ then $\varphi(t)$ is located between t_n and t_{n+1} in such a way that $\angle_{\tilde{a}}(t_n, \varphi(t)) = \tau$.

Step 2—definition of the map on the curve

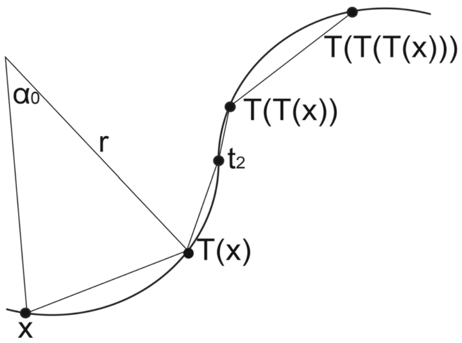

FIGURE 2. The construction of the map

Now, we would like to define the map $T\colon \varphi \to \varphi$. Let us divide the curve between points t_1 and t_2 into 128 equal parts and let $\alpha_0 = \alpha/128$. Then, for points of the form $\varphi(t)$, $t \in [0, \alpha - \alpha_0]$, we take $T(\varphi(t)) = \varphi(t + \alpha_0)$. So far, the map T is an isometry.

To extend the map T on the whole curve, we need to make some calculations because our map must be INEA. To do it, first, let us choose α_1 in such a way that

$$\frac{\alpha_1}{\alpha_0} = \frac{\sin(2\alpha_0)}{2\alpha_0}.$$

Hence, for $t = 127\alpha_0 + \tau$, $\tau \in (0, \alpha_0)$ we define

$$T(\varphi(t)) = \varphi\left(\alpha + \tau \cdot \frac{\alpha_1}{\alpha_0}\right).$$

Next we will show that so far

$$\|T(x) - T(T(x))\| \leq \|x - T(x)\| \tag{1}$$

for $x = \varphi(t)$, $t \leq 127\alpha_0$.

Let us consider $x = \varphi(t)$, where $t \in (126\alpha_0, 127\alpha_0]$. Then $T(x) = \varphi(t + \alpha_0)$ and

$$\|x - T(x)\| = 2r \sin\frac{\alpha_0}{2}, \qquad r = \|\tilde{a} - t_2\|.$$

Simultaneously,

$$\|T(x) - T(T(x))\| \leq \|T(x) - t_2\| + \|t_2 - T(T(x))\| = 2r\sin\frac{s\alpha_0}{2} + 2r\sin\frac{(1-s)\alpha_1}{2},$$

where $s \in [0, 1)$. To prove (1), it is sufficient to notice that

$$\begin{aligned}\sin\frac{\alpha_0}{2} - \sin\frac{s\alpha_0}{2} &= 2\sin\frac{(1-s)\alpha_0}{4}\cos\frac{(1+s)\alpha_0}{4}\\ &= \sin\frac{(1-s)\alpha_0}{2}\cdot\frac{\cos\frac{(1+s)\alpha_0}{4}}{\cos\frac{(1-s)\alpha_0}{4}} \geq \sin\frac{(1-s)\alpha_0}{2}\cdot\cos\frac{\alpha_0}{2}.\end{aligned}$$

On the other hand,

$$\sin\frac{(1-s)\alpha_1}{2} \le \frac{(1-s)\alpha_1}{2} \tag{2}$$

and

$$\sin\frac{(1-s)\alpha_0}{2} \ge \frac{(1-s)\alpha_0}{2}\cdot\frac{\sin\alpha_0}{\alpha_0}. \tag{3}$$

Therefore,

$$\begin{aligned}\sin\frac{(1-s)\alpha_1}{2} &\le \sin\frac{(1-s)\alpha_0}{2}\cdot\frac{\alpha_1}{\alpha_0}\cdot\frac{\alpha_0}{\sin\alpha_0} = \sin\frac{(1-s)\alpha_0}{2}\cdot\frac{\sin(2\alpha_0)}{2\alpha_0}\cdot\frac{\alpha_0}{\sin\alpha_0}\\ &= \sin\frac{(1-s)\alpha_0}{2}\cdot\cos\alpha_0 \le \sin\frac{(1-s)\alpha_0}{2}\cdot\cos\frac{\alpha_0}{2}\end{aligned}$$

and finally

$$2r\sin\frac{(1-s)\alpha_1}{2} + 2r\sin\frac{s\alpha_0}{2} \le 2r\sin\frac{\alpha_0}{2}.$$

Now, using the angle α_1 we define

$$T(\varphi(\alpha+t)) = \varphi(\alpha+t+\alpha_1) \text{ as long as } t \le 2\alpha-\alpha_1.$$

Let us take the same point x as above (see Fig. 2) and notice that

$$\|T(T(x)) - T(T(T(x)))\| = 2r\sin\frac{\alpha_1}{2}.$$

We would like to show that

$$2r\sin\frac{\alpha_1}{2} \le \|T(x) - T(T(x))\|.$$

Let us denote $\angle_{t_2}(T(x), T(T(x))) = \beta$. In the sequel, we will show that

$$\beta > \pi - \alpha_0. \tag{4}$$

Let us notice that, on account of the cosine law, we get

$$\begin{aligned}\|T(x) - T(T(x))\|^2 = &\left(2r\sin\frac{s\alpha_0}{2}\right)^2 + \left(2r\sin\frac{(1-s)\alpha_1}{2}\right)^2\\ &-2\cdot\left(2r\sin\frac{s\alpha_0}{2}\right)\cdot\left(2r\sin\frac{(1-s)\alpha_1}{2}\right)\cdot\cos\beta.\end{aligned}$$

Simultaneously,

$$\begin{aligned}\|T(T(x)) - T(T(T(x)))\| = &\left(2r\sin\frac{s\alpha_1}{2}\right)^2 + \left(2r\sin\frac{(1-s)\alpha_1}{2}\right)^2\\ &-2\cdot\left(2r\sin\frac{s\alpha_1}{2}\right)\cdot\left(2r\sin\frac{(1-s)\alpha_1}{2}\right)\cdot\cos\pi.\end{aligned}$$

So, it is sufficient to prove the following inequality

$$\sin\frac{s\alpha_0}{2}\cdot\cos(\pi-\beta) \ge \sin\frac{s\alpha_1}{2}.$$

We know that (see (2) and (3))

$$\frac{\sin\frac{s\alpha_1}{2}}{\sin\frac{s\alpha_0}{2}} \le \frac{\frac{s\alpha_1}{2}}{\frac{s\alpha_0}{2}}\cdot\frac{\alpha_0}{\sin\alpha_0} = \frac{\sin(2\alpha_0)}{2\alpha_0}\cdot\frac{\alpha_0}{\sin\alpha_0} = \cos\alpha_0.$$

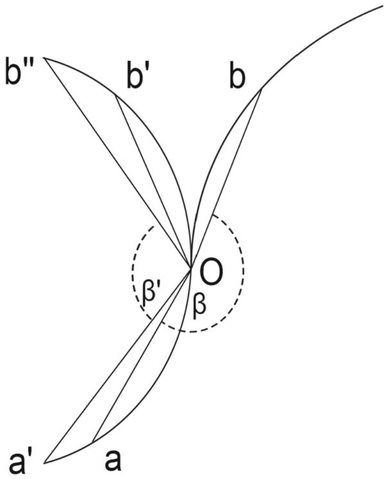

FIGURE 3. Estimation of β

Thus, showing (4) we complete the proof of the inequality

$$\|T(T(x)) - T(T(T(x)))\| \leq \|T(x) - T(T(x))\|.$$

The angle between the arcs $\widehat{t_1 t_2}$ and $\widehat{t_2 t_3}$ is equal to π. Since these two arcs are the subsets of two different two-dimensional spaces, the angle between segments $[T(x), t_2]$ and $[t_2, T(T(x))]$ (with respect to Cartesian geometry) is greater than in the case when all points are located on one two-dimensional space. The same situation can be observed in Fig. 3, where

$$\angle_O(a, b) \geq \angle_O(a, b').$$

Next, if all points are on one two-dimensional space, then the angle between the metric segments is greater than the angle between two metric segments joining points on the same circle and having the same length $2r \sin \frac{\alpha_0}{2}$ greater than $\|T(x) - t_2\|$ and $\|t_2 - T(T(x))\|$. See also points a, a', b' and b'' (Fig. 3). Thus, the angle $\angle_{t_2}(T(x), T(T(x)))$ is not smaller than β'. To estimate β', let us consider the triangle of the sides of length equal to $2r \sin \frac{\alpha_0}{2}$, $2r \sin \frac{\alpha_0}{2}$ and $2r \sin \frac{2\alpha_0}{2}$ (all vertices are located on a circle with radius r). Therefore,

$$\sin \frac{\beta'}{2} = \frac{r \sin \frac{2\alpha_0}{2}}{2r \sin \frac{\alpha_0}{2}} = \cos \frac{\alpha_0}{2},$$

which completes the proof of (4).

To define the map on $\{\varphi(t) : \ t \in (2\alpha - \alpha_1, 2\alpha)\}$, let us choose

$$\alpha_2 = \alpha_1 \cdot \frac{\sin(2\alpha_0)}{2\alpha_0}.$$

Since $\alpha_1 < \alpha_0$ one may repeat considerations from the previous part to show that T is still INEA. We can also define the map on $\{\varphi(t) : \ t \in (\alpha, 3\alpha - \alpha_2)\}$ as a movement along the curve. Repeating all steps with

$$\alpha_{n+1} = \alpha_n \cdot \frac{\alpha_1}{\alpha_0}, \qquad n \in \mathbb{N}$$

one may extend the map T on the whole curve γ. Let us notice that T is also continuous.

Step 3—neighborhood of the curve

In this step, we consider the neighborhood of the curve. Mainly, let us consider the set

$$U = \{x \in B : \ \exists t \in [0, \infty) : \|\varphi(t) - x\| \leq \alpha_{m-1}, \text{ if } t \in [(m-1)\alpha, m\alpha]\}.$$

We will see that this set is closed. Indeed, let us take any Cauchy sequence (x_n), $x_n \in U$. Let $x_n \to \bar{x} \in B$. Since $\|t_n - t_m\| \geq 1 >> \alpha_0$ for $n \neq m$, without loss of generality we may assume that there is a sequence (τ_n) such that $\tau_n \in [m\alpha, (m+1)\alpha]$ and $\|x_n - \varphi(\tau_n)\| \leq \alpha_m$. Then, there is a convergent subsequence (denoting again by (τ_n)) such that $\tau_n \to \bar{\tau}$. If $\bar{\tau} \in (m\alpha, (m+1)\alpha]$, then the same holds for almost all τ_n, so $\|\bar{x} - \varphi(\bar{\tau})\|$ as the limit is also not greater than α_m. If $\bar{\tau} = m\alpha$, then

$$\|x_n - \varphi(\tau_n)\| \leq \alpha_m$$

and so

$$\|\bar{x} - \varphi(\bar{\tau})\| \leq \alpha_m < \alpha_{m-1}$$

and $\bar{x}$ is also an element of U.

Step 4—definition of the hyperplanes

Now, for each point $x \in \varphi$ there is a unique hyperplane

$$H_x = \{y \in l^2 : \ \langle y - x, T(x) - x\rangle = 0\}. \tag{5}$$

We will show that two hyperplanes do not intersect inside U as long as they are determined by points which are not located too far from each other. Let us fix a point $x = \varphi(t_0)$ and let $x' = \varphi(t_0 + \tau)$, where $\tau \in (0, 9\alpha_0)$. Let us also assume that $t_0 \in [m\alpha, (m+1)\alpha)$.

Claim: For all possible positions of $x, x', T(x)$ and $T(x')$, the angle between vectors $\overline{x\, T(x)}$ and $\overline{x'\, T(x')}$ is not greater than τ.

- Case 1. First, we assume that all points are located on the curve between t_m and t_{m+1}. So, the aforementioned vectors $\overline{x\, T(x)}$ and $\overline{x'\, T(x')}$ span one two-dimensional space and it is sufficient to consider only points on this space. Clearly, the intersection of H_x (or $H_{x'}$) with this space is a line—see Fig. 4.

  From the equality $\|x - T(x)\| = \|x' - T(x')\| = \alpha_m$ it follows that

  $$\angle_x(p, \tilde{a}) = \angle_{x'}(p, \tilde{a}),$$

  where p is the projection of x onto the set of common points of H_x and $H_{x'}$. Clearly, p also belongs to the same two-dimensional space. Since $\angle_x(p, T(x)) = \angle_{x'}(p, T(x')) = \pi/2$, we get that the angle between vectors $\overline{x\, T(x)}$ and $\overline{x'\, T(x')}$ is equal to the angle $\angle_{\tilde{a}}(x, x')$, i.e., is equal to τ.

- Case 2. Now, we assume that the three points x, x', $T(x)$ are located on the curve between t_m and t_{m+1} and $T(x')$ is between t_{m+1} and t_{m+2}—see Fig. 5. Without loss of generality we may assume that $\tau < \alpha_m$.
  Let $\tilde{a}$ be the center of the circle containing t_m and t_{m+1} while $\tilde{b}$ is the center of the circle containing t_{m+1} and t_{m+2}. Then, there is a number $s \in (0, 1)$ such that
  $$\angle_{\tilde{a}}(x', t_{m+1}) = (1-s)\alpha_m \quad \text{and} \quad \angle_{\tilde{b}}(t_{m+1}, T(x')) = s\ \alpha_{m+1}.$$
  Let us choose T' on the same circle as t_m and t_{m+1} (Fig. 5) for which
  $$\angle_{\tilde{a}}(x', T') = (1-s)\alpha_m + s\ \alpha_{m+1}.$$
  Since the curve is smooth and $T(x')$ does not belong to the same two-dimensional space as the rest of the points, the following inequalities hold
  $$\|x' - T(x')\| > \|x' - T'\| \quad \text{and} \quad \angle_{x'}(T(x), T') > \angle_{x'}(T(x), T(x')). \tag{6}$$
  Clearly, the angle between vectors $\overline{x\,T(x)}$ and $\overline{x'\,T'}$ is equal to the sum of angles $\angle_{T(x)}(x, x')$ and $\angle_{x'}(T(x), T')$. Moreover, this sum is smaller than τ, because
  $$\|x - T(x)\| > \|x' - T'\|.$$
  However, from (6) and the fact that $T(x')$ does not belong on the same space as the rest of points it follows that the angle between vectors $\overline{x\,T(x)}$ and $\overline{x'\,T(x')}$ is smaller than the angle between vectors $\overline{x\,T(x)}$ and $\overline{x'\,T'}$ and so smaller than τ.
  The proofs for the cases where $\tau \geq \alpha_m$ or three points x', $T(x)$ and $T(x')$ are between t_{m+1} and t_{m+2} go with the same patterns.
- Case 3. Now we assume that x and x' are located between t_m and t_{m+1} while $T(x)$ and $T(x')$ are between points t_{m+1} and t_{m+2}—see Fig. 6
  Let us fix T and T' on the circle containing x and x' in such a way that
  $$\angle_{\tilde{a}}(x, T) = \angle_{\tilde{a}}(x, t_{m+1}) + \angle_{\tilde{b}}(t_{m+1}, T(x))$$
  and
  $$\angle_{\tilde{a}}(x', T') = \angle_{\tilde{a}}(x', t_{m+1}) + \angle_{\tilde{b}}(t_{m+1}, T(x')),$$
  where $\tilde{a}$ and $\tilde{b}$ are defined in the same way as in Case 2.
  We want to show that
  $$\angle_{T(x)}(x, x') < \angle_T(x, x'). \tag{7}$$
  To do it, let us notice that
  $$\|x - T(x)\| > \|x - T\|$$
  while
  $$\angle_{x'}(x, T(x)) > \angle_{x'}(x, T) > \frac{\pi}{2}.$$
  Hence, the inequality (7) follows directly from the sine law.
  In a similar way one may see that
  $$\angle_{x'}(T(x), T(x')) < \angle_{x'}(T, T').$$

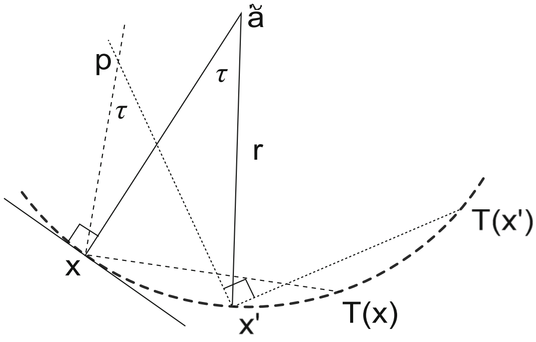

FIGURE 4. Estimation of $\|p - x\|$

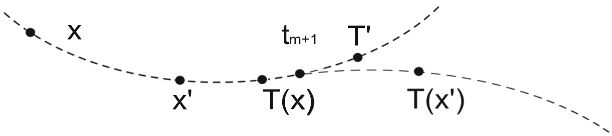

FIGURE 5. Estimation of angle between vectors

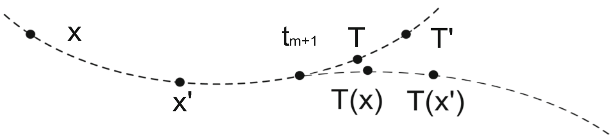

FIGURE 6. Estimation of the angle between vectors

Moreover, since all four points do not belong to the same two-dimensional space, the angle between vectors $\overline{x\,T(x)}$ and $\overline{x'\,T(x')}$ is smaller than the sum of $\angle_{T(x)}(x, x')$ and $\angle_{x'}(T(x), T(x'))$. This completes the proof for Case 3.
The case where two points x and $T(x)$ are between t_m and t_{m+1} is slightly easier.

Now, we consider the projection of x' onto H_x.

First, we want to estimate the angle $\angle_x(x', T(x))$. We may consider three cases as it was done in the previous part when we studied the angle between vectors but here we do not need to make the estimation so precise; therefore, only notice that this angle is smaller than the sum of the angle between the vector $\overline{x\,x'}$ and the curve at the point x and the angle between the vector $\overline{x\,T(x)}$ and the curve at the same point x.

In both cases, the angles are of the largest measure if all points x, x' and $T(x)$ are located between points t_m and t_{m+1}. Hence, using denotations from Fig. 7, we get

$$\angle_x(y, z) = \frac{\pi}{2} - \angle_x(y, \tilde{a}) = \frac{\angle_{\tilde{a}}(x, y)}{2}.$$

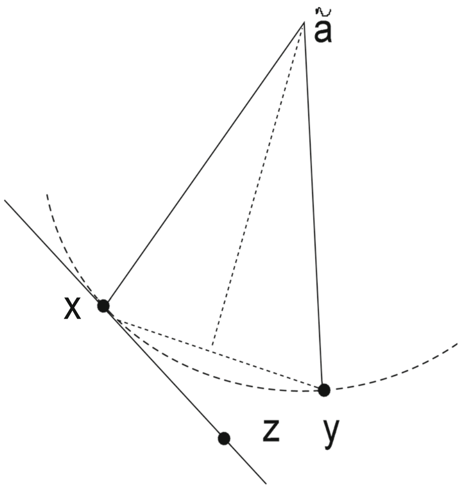

Figure 7. Measure of the angle $\angle_x y, z$

Since $\angle_{\tilde{a}}(x, T(x)) \leq \alpha_m$ and $\angle_{\tilde{a}}(x, x') = \tau \leq 9\alpha_0$, it follows finally that

$$\angle_x(x', T(x)) \leq 5\alpha_0.$$

Now we find the projection of x' onto the hyperplane H_x and denote this by p_x. Since H_x is determined by the vector $\overline{x\,T(x)}$ and

$$\overline{x\,T(x)} \parallel \overline{p_x\,x'},$$

we obtain two estimations:

$$\|x' - p_x\| \geq \|x - x'\| \cdot \cos(5\alpha_0) \geq 2r \sin\frac{\tau}{2} \cdot \cos(5\alpha_0)$$

and

$$\|x - p_x\| \leq \|x - x'\| \cdot \sin(5\alpha_0) \leq \tau \cdot \sin(5\alpha_0).$$

Let p be the projection of p_x onto the set $H_x \cap H_{x'}$. Clearly, this set is closed and convex, so the projection is a single, i.e., is well defined. Since vectors $\overline{x\,T(x)}$ and $\overline{p_x\,x'}$ are parallel, we can calculate the measure of the angle $\angle_{x'}(p_x, T(x'))$ in the following way:

$$\angle_{x'}(p_x, T(x')) = \pi - \gamma,$$

where γ denotes the angle between vectors $\overline{x\,T(x)}$ and $\overline{x'\,T(x')}$ (Fig. 8).

$$\angle_{x'}(T(x), p) = \frac{\pi}{2},$$

we can estimate $\angle_{x'}(p_x, p)$ by

$$\angle_{x'}(p_x, p) = \frac{\pi}{2} - \gamma \geq \frac{\pi}{2} - \tau.$$

For all points $h \in H_x \cap H_{x'}$, we have

$$\|h - x\| \geq \|h - p_x\| - \|p_x - x\| \geq \|p - p_x\| - \|p_x - x\|.$$

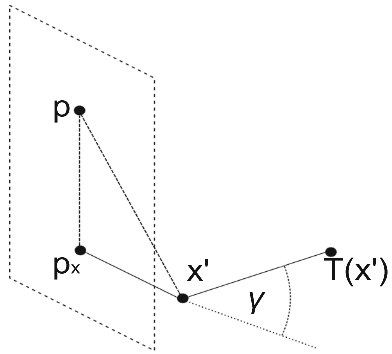

FIGURE 8. Measure of the angle $\angle_x y, z$

Simultaneously,

$$\|p - p_x\| = \|x' - p_x\| \cdot \tan \angle_{x'}(p, p_x) = \|x' - p_x\| \cdot \cot \gamma \geq \|x' - p_x\| \cdot \cot \tau.$$

Combining it with the earlier estimations, we obtain

$$\|h - x\| \geq 2r \sin \frac{\tau}{2} \cdot \cos(5\alpha_0) \cdot \cot \tau - \tau \cdot \sin(5\alpha_0).$$

The minuend is equal to

$$r \cdot \frac{2 \sin \frac{\tau}{2}}{\sin \tau} \cdot \cos(5\alpha_0) \cdot \cos \tau = r \cdot \frac{\cos(5\alpha_0) \ \cos \tau}{\cos \frac{\tau}{2}} \geq r \cdot \cos^2(9\alpha_0).$$

In a similar way, we can estimate the subtrahend by

$$\tau \cdot \sin(5\alpha_0) \leq 45(\alpha_0)^2.$$

Our considerations finally lead to

$$\|h - x\| \geq r \cdot \cos^2(9\alpha_0) - 45(\alpha_0)^2 \geq \sqrt{\frac{11 - 4\sqrt{3}}{9}} \cdot \cos^2(9 \cdot \pi/128)$$
$$-45(\pi/128)^2 \geq 0.6$$

for all points belonging to $H_x \cap H_{x'}$.

In this way, we have shown that the hyperplane H_x and $H_{x'}$ intersect outside the set U as long as $\tau \leq 9\alpha_0$. Furthermore, each point of the closed ball $\bar{B}(x, \alpha_m)$ can belong to at most one hyperplane H_y with $|\gamma^{-1}(y) - \gamma^{-1}(x)| \leq 9\alpha_0$.

Next, we will show that almost all points of the set U (more precisely, all points from V) satisfy the following condition:

$$x \text{ satisfies } (P) \text{ if } \qquad x \in U \wedge \left(\exists t \in [0, \infty) \colon \ \|x - \varphi(t)\| \leq 4\alpha_0 \wedge x \in H_{\varphi(t)}\right).$$

Let us consider the Cauchy sequence of points (x_n) such that $x_n \in U$ and all of them satisfy this property. As it was shown the limit point $x_0 = \lim x_n$ belongs to U and we will prove that x_0 also satisfies (P).

Since (x_n) is a Cauchy sequence, we may take a subsequence with $\|x_n - x_m\| \leq \alpha_0$. Then there must be

$$\|\varphi(s_n) - \varphi(s_m)\| \leq 9\alpha_0,$$

where $x_n \in H_{\varphi(s_n)}$.

So, without loss of generality we may assume that all s_n belong to $[m\alpha, (m+1)\alpha]$ and $x_0 \notin \varphi$. Otherwise, $x_0 \in H_{x_0}$ and the proof of the claim is complete. Then there is at least one accumulation point s_A. Let us assume that $s_{l(n)} \to s_A$. Since $x_n \not\to \varphi(s_A)$ and $s_{l(n)} \to s_A$, it must be

$$\cos_{\varphi(s_A)}(x_{l(n)}, T(\varphi(s_A))) \to 0.$$

This leads to

$$\cos_{\varphi(s_A)}(x_0, T(\varphi(s_A))) = 0$$

and completes the proof of our claim.

However, we will apply our considerations to prove an additional point. Namely, we will prove that the whole sequence (s_n) must tend to s_A. To see it, let us repeat our pattern for s_B—another accumulation point of (s_n). Let us assume that $s_{k(n)} \to s_B$. And again if $x_0 \notin \varphi$, there must be

$$\cos_{\varphi(s_B)}(x_{k(n)}, T(\varphi(s_B))) \to 0,$$

which yields

$$\cos_{\varphi(s_B)}(x_0, T(\varphi(s_B))) = 0,$$

but x_0 cannot belong to both hyperplanes $H_{\varphi(s_A)}$ and $H_{\varphi(s_B)}$, because it is too close to $\varphi(s_A)$. The proof of uniqueness of accumulation point for $x_0 \in \varphi$ is obvious.

Let us consider a subset V of U containing points u for which

$$\langle t_0 - T(t_0), u - T(t_0)\rangle \geq 0 \text{ as long as } \|u - t_0\| \leq 3\alpha_0.$$

Clearly, V is also closed. We will show that each point $u \in V$ satisfies the property (P).

Let us fix $u \in V$. From the inclusion $u \in U$ it follows that there is a positive number τ such that $\|u - \varphi(\tau)\| \leq \alpha_m$, when $\tau \in [m\alpha, (m+1)\alpha]$. Let us denote $x = \varphi(\tau)$. Let us assume that $\langle x-u, x-T(x)\rangle \neq 0$ is a negative number and we consider the line k containing u and parallel to the vector $\overline{x\,T(x)}$. Let $y_1 = k \cap H_x$.

Taking $x' = \varphi(\tau - 3\alpha_m)$, we obtain that u belongs to the metric segment $[y_0, y_1]$, where $y_0 = k \cap H_{x'}$. Otherwise, the hyperplanes H_x and $H_{x'}$ intersect too close to x. For each $t \in [\tau - 3\alpha_m, \tau]$ one can find a point $u_t = k \cap H_{\varphi(t)}$. Let $t_0 = \sup\{t \in [\tau - 3\alpha_m, \tau]\colon\ u \in [u_t, y_1]\}$. It is sufficient to prove that $u = u_{t_0}$. Indeed, for all $t > t_0$ there is $u_t \in [u, y_1]$, which means that $\langle \varphi(t) - u, \varphi(t) - T(\varphi(t))\rangle \leq 0$. And from the continuity of T it follows that $\langle \varphi(t_0) - u, \varphi(t_0) - T(\varphi(t_0))\rangle \leq$, i.e., $u_{t_0} \in [u, y_1]$. Therefore $u = u_{t_0}$, i.e., $u \in H_{\varphi(t_0)}$. Moreover,

$$\|u - \varphi(t_0)\| \leq \|u - x\| + \|x - \varphi(t_0)\| \leq \alpha_m + 3\alpha_m \leq 4\alpha_0$$

and finally u satisfies the property (P).

Step 5—definition of the map on U

Now, for each $y \in V$ there is precisely one point x on the curve such that $y \in H_x$, i.e.,

$$\langle x - y, x - T(x)\rangle = 0 \qquad \text{and} \qquad \|x - y\| \leq 4\alpha_0.$$

Hence, since T is INEA on the curve, we obtain

$$\|y - T(x)\| \geq \|x - T(x)\| \geq \|T(x) - T(T(x))\|.$$

So, one can set $T(y) = T(x)$. Then T is INEA on the whole set V.

Simultaneously, applying the same denotations as in the proof of the closedness of V, if the sequence (x_n) tends to x_0 (i.e., $x_n \in H_{\varphi(s_n)}$), then $x_0 \in H_{\varphi(s_0)}$, where $s_n \to s_0$. So, $T(x_n) = T(\varphi(s_n))$ tends to $T(\varphi(s_0)) = T(x_0)$ and T is also continuous on V. Now, we must only consider points from $\bar{B}(t_0, 3\alpha_0)$. Let u be such a point. If

$$\langle t_0 - u, t_0 - T(t_0)\rangle \geq 0,$$

then $T(u)$ has already been defined. Otherwise, we set $T(u) = T(t_0)$. Let us notice that T is still continuous and INEA also on the set U.

Step 6—definition of the map on the whole set B

In the previous step, we defined the mapping T on the whole set U. Since this is a closed subset of B and the curve φ is isomorphic to the set $[0, \infty)$, applying the Tietze extension theorem the mapping T can be extended to the whole set B. Therefore, we must show only that the continuous extension $\tilde{T}$ is also a fixed point free INEA mapping. The fact that $\tilde{T}$ is fixed point free is obvious. So let us take $x_0 \in B \setminus U$. Then $\tilde{T}(x_0) \in \varphi$. Since $\tilde{T}(x_0) = \varphi(t_0)$, one may assume that $t_0 \in [m\alpha, (m+1)\alpha]$. Since x_0 does not belong to U, the distance between x_0 and $\varphi(t_0)$ is bigger than α_m. Simultaneously, $\|T(\varphi(t_0)) - \varphi(t_0)\| < \alpha_m$. Hence,

$$\|x_0 - \tilde{T}(x_0)\| > \|\tilde{T}(x_0) - T(\tilde{T}(x_0))\| = \|\tilde{T}(x_0) - \tilde{T}(\tilde{T}(x_0))\|$$

and $\tilde{T}$ is INE. Moreover, since $\tilde{T}|_\varphi = T|_\varphi$, the extension is asymptotically regular and so is INEA. This fact completes the proof.

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