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Best proximity point of generalized F-proximal non-self contractions

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Abstract. In this paper, we introduce the concept of generalized F-proximal contraction mappings and prove some best proximity point theorems for a non-self mapping in a complete metric space. Then some of the well-known results in the existing literature are generalized/extended using these newly obtained results. An example is being given to demonstrate the usefulness of our results.

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Keywords. Best proximity point, generalized F-proximal contraction, p-property.

1. Introduction

Following question was one of the major point of reseach in applied mathematics and nonlinear functional analysis during the last decade see [1,2,4,6, 9–11];

Is there a point u_0 in the metric space (U, d) such that $d(u_0, Pu_0) = d(E, G)$ where E, G are non-empty subsets of U, $P : E \to G$ is a non-self mapping and $d(E, G) = \inf\{d(e, g) : e \in E, g \in G\}$?

The point $u_0 \in U$ is called the *best proximity point*. In best proximity point theory we attempt to find minimum conditions on the non-self mapping P to ensure the existence and uniqueness of the best proximity point. This new setting is richer and more general than the metric fixed point theory. The mappings considered are not essentially self-mappings like fixed point theory. Fixed point theory incorporates different mathematical disciplines, such as topology, operator theory and geometry, to demonstrate the existence of $Pu = u$ solutions under certain conditions. On the other hand, if P is not a self-mapping, there may be no solutions to the $Pu = u$ equation and in this case, it is of fundamental importance to decide an element u that is closest to Pu in any way. Wardowski [12] introduced a new contraction definition and proved a fixed point theorem that generalizes the theory of

Banach contraction. Cosentino and Vetro [5] recently presented a new Hardy-Rogers-type definition of F -contraction and proved a fixed point theorem for self-mapping on complete metric spaces. A new idea, F-contractive non-self mappings, was introduced by Omidvari et al. [7] and proved the best proximity point theorems. In this paper, we introduce a new concept of generalized F -proximal contractions (of first and second kind) and then prove the best proximity point results on complete metric spaces. The paper is arranged as follows: In Sect. 2 we recall some basic notations and definitions from the existing literature for subsequent use. In Sect. 3, by using Hardy-Rogers-type F-contraction, we obtain sufficient conditions for the existence of the best proximity point. Also we define generalized F - proximal contractions of the first and second kind and prove the best proximity point results on complete metric spaces.

2. Preliminaries

In this article, U, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the non-void set, the positive real number set, the positive integer set, and the non-negative integer set.
First, we recall the concept of a control function which is introduced by Wardowski [12]. Let $\Im$ denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:

(F_1) F is strictly increasing;
(F_2) for each sequence $\{\alpha_n\}$ of positive numbers, we have

$$\lim_{n\to\infty} \alpha_n = 0 \iff \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F_3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We give some examples of functions belonging to $\Im$ as follows:

Example 2.1. Let functions F_1, F_2, F_3, F_4 : $\mathbb{R}^+ \to \mathbb{R}$ be defined by:

1. $F_1(\beta) = \ln \beta$ for $\beta > 0$;
2. $F_2(\beta) = \beta + \ln \beta$ for all $\beta > 0$;
3. $F_3(\beta) = -\frac{1}{\sqrt{\beta}}$ for all $\beta > 0$;
4. $F_4(\beta) = \ln(\beta^2 + \beta)$ for all $\beta > 0$.

Then $F_1, F_2, F_3, F_4 \in \Im$.

Definition 2.1. [12] A self-mapping P on a metric space U is called an F -contraction if there exist $F \in \Im$ and $\tau \in \mathbb{R}^+$ such that

$$\tau + F(d(Pu, Pv)) \leq F(d(u, v)),$$

for all $u, v \in U$ with $d(Pu, Pv) > 0$.

Next we state some notations for subsequent use. If E and G non-void subsets of U, then we denote by.

$$d(e, G) := \inf\{d(e, g) : g \in G\},\ e \in E,$$
$$E_0 := \{e \in E : d(e, g) = d(E, G) \text{ for some } g \in G\},$$
$$G_0 := \{g \in G : d(e, g) = d(E, G) \text{ for some } e \in E\}.$$

In case E and G are closed subsets of a normed space and $d(E,G) > 0$, then E_0 and G_0 are contained in the boundaries of E and G respectively [3].

Definition 2.2. [8] Let (U,d) be a metric space and (E,G) be a pair of non-void subsets of (U,d) with $E_0 \neq \phi$. If for every $u_1, u_2 \in E$ and every $v_1, v_2 \in G$

$$\left.\begin{aligned} d(u_1,v_1) = d(E,G) \\ d(u_2,v_2) = d(E,G) \end{aligned}\right\} \Rightarrow d(u_1,u_2) = d(v_1,v_2),$$

then the pair (E,G) is said to have the p-property.

Definition 2.3. [3] A set G is called approximately compact with respect to E if every sequence $\{g_n\}$ of G with $d(e,g_n) \to d(e,G)$ for some $e \in E$ has a convergent subsequence.

3. Main results

In this section, inspired by the notions of F-contraction of Hardy–Rogers-type, we introduce new generalized F- proximal contractions of the first and second kind and prove some best proximity point theorems for generalized F-proximal contractions of the first and second kind on complete metric space.

Definition 3.1. A mapping $P : E \to G$ is said to be a generalized F-proximal contraction of the first kind if there exist $F \in \Im$ and $a,b,c,h,\tau > 0$ with $a+b+c+2h = 1$, $c \neq 1$ such that the conditions

$$\begin{aligned} &\left.\begin{aligned} d(u_1,Pv_1) = d(A,B) \\ d(u_2,Pv_2) = d(A,B) \end{aligned}\right\} \Rightarrow \tau + F(d(u_1,u_2)) \\ &\quad \leq F\big(ad(v_1,v_2) + bd(u_1,v_1) \\ &\qquad + cd(u_2,v_2) + h(d(v_1,u_2) \\ &\qquad + d(v_2,u_1))\big) \end{aligned}$$

for all u_1, u_2, v_1, v_2 in E and $u_1 \neq u_2$.

Definition 3.2. A mapping $P : E \to G$ is said to be a generalized F-proximal contraction of the second kind if there exist $F \in \Im$ and $a,b,c,h,\tau > 0$ with $a+b+c+2h = 1$, $c \neq 1$ such that the conditions

$$\begin{aligned} &\left.\begin{aligned} d(u_1,Pv_1) = d(A,B) \\ d(u_2,Pv_2) = d(A,B) \end{aligned}\right\} \Rightarrow \tau + F(d(Pu_1,Pu_2)) \\ &\quad \leq F\big(ad(Pv_1,Pv_2) \\ &\qquad + bd(Pu_1,Pv_1) + cd(Pu_2,Pv_2) \\ &\qquad + h(d(Pv_1,Pu_2) + d(Pv_2,Pu_1))\big) \end{aligned}$$

for all u_1, u_2, v_1, v_2 in E and $Pu_1 \neq Pu_2$.

Theorem 3.1. *Let (U,d) be a complete metric space and (E,G) be a pair of non-void closed subsets of (U,d). If G is approximately compact with respect to E and $P : E \to G$ satisfy the following conditions:*

(i) $P(E_0) \subseteq G_0$ *and* (E,G) *satisfies the* p *-property;*
(ii) P *is a generalized* F*-proximal contraction of the first kind.*

Then, there exists a unique $u \in E$ *such that* $d(u,Pu) = d(E,G)$. *In addition, for any fixed element* $u_0 \in E_0$, *sequence* $\{u_n\}$ *defined by*

$$d(u_{n+1}, Pu_n) = d(E,G),$$

converges to the best proximity point u.

Proof. Choose an element $u_0 \in E_0$. As, $P(E_0) \subseteq G_0$, therefore there is an element $u_1 \in E_0$ satisfying

$$d(u_1, Pu_0) = d(E,G).$$

Since $Pu_1 \in P(E_0) \subseteq G_0$, it further implies that there is an element $u_2 \in E_0$ such that

$$d(u_2, Pu_1) = d(E,G).$$

Continuing in this way, we can choose an element $u_{n+1} \in E_0$ satisfying the condition that

$$d(u_{n+1}, Pu_n) = d(E,G), \tag{3.1}$$

for every non-negative integer n owing to the hypothesis that $P(E_0)$ is contained in G_0. From the p-property framework and (3.1) we get

$$d(u_n, u_{n+1}) = d(Pu_{n-1}, Pu_n), \quad \forall n \in \mathbb{N}.$$

If for some n_0, $d(u_{n_0}, u_{n_0+1}) = 0$, consequently $d(Pu_{n_0-1}, Pu_{n_0}) = 0$. So $Pu_{n_0-1} = Pu_{n_0}$, hence $d(E,G) = d(u_{n_0}, Pu_{n_0})$. Thus the conclusion is immediate. So let for any $n \geq 0$, $d(u_n, u_{n+1}) > 0$. In view of the fact P is a generalized F-proximal contraction of the first kind, we have that

$$\begin{aligned}
\tau + F(d(u_n, u_{n+1})) &\leq F\big(ad(u_{n-1}, u_n) + bd(u_{n-1}, u_n) + cd(u_n, u_{n+1}) \\
&\quad + hd(u_{n-1}, u_{n+1})\big) \\
&\leq F\big(ad(u_{n-1}, u_n) + bd(u_{n-1}, u_n) + cd(u_n, u_{n+1}) \\
&\quad + h[d(u_{n-1}, u_n) + d(u_n, u_{n+1})]\big) \\
&= F\big((a+b+h)d(u_{n-1}, u_n) + (c+h)d(u_n, u_{n+1})\big).
\end{aligned}$$

Since F is strictly increasing, we deduce

$$d(u_n, u_{n+1}) \leq (a+b+h)d(u_{n-1}, u_n) + (c+h)d(u_n, u_{n+1}).$$

Thus

$$d(u_n, u_{n+1}) \leq \left(\frac{a+b+h}{1-c-h}\right) d(u_n, u_{n-1}), \quad \forall n \in \mathbb{N}$$

From $a+b+c+2h = 1$ and $c \neq 1$, we have that $1-c-h > 0$ and so

$$d(u_n, u_{n+1}) \leq \left(\frac{a+b+h}{1-c-h}\right) d(u_n, u_{n-1}) = d(u_n, u_{n-1}), \quad \forall n \in \mathbb{N}.$$

Consequently,

$$\tau + F(d(u_n, u_{n+1})) \leq F(d(u_n, u_{n-1})), \quad \forall n \in \mathbb{N}.$$

It implies

$$F(d(u_n,u_{n+1})) \le F(d(u_n,u_{n-1})) - \tau \le \cdots$$
$$\le F(d(u_0,u_1)) - n\tau, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

Put $t_n := d(u_n,u_{n+1})$. From (3.2) $\lim_{n\to\infty} F(t_n) = -\infty$. By the properties (F_2), we get that

$$t_n \to 0 \text{ as } n \to \infty.$$

Now, let $k \in (0,1)$ such that $\lim_{n\to\infty} t_n^k F(t_n) = 0$. By (3.2), the following holds for all $n \in \mathbb{N}$:

$$t_n^k F(t_n) - t_n^k F(t_0) \le -nt_n^k\tau \le 0. \tag{3.3}$$

Letting $k \to \infty$ in (3.3) , we have

$$\lim_{n\to\infty} nt_n^k = 0.$$

Therefore $\lim_{n\to\infty} n^{\frac{1}{k}} t_n = 0$. Now, $\lim_{n\to\infty} n^{\frac{1}{k}} t_n = 0$ implies that the series $\sum_{n=1}^{\infty} t_n$ is convergent.It further implies that $\{u_n\}$ is a Cauchy sequence. Because the space is complete, the sequence $\{u_n\}$ converges to some element u in E.

Also,

$$\begin{aligned} d(u,G) \le d(u,Pu_n) &\le d(u,u_{n+1}) + d(u_{n+1},Pu_n) \\ &= d(u,u_{n+1}) + d(E,G) \\ &\le d(u,u_{n+1}) + d(u,G). \end{aligned}$$

Therefore, $d(u,Pu_n) \to d(u,G)$. In spite of the fact that G is approximately compact with respect to E, the sequence $\{Pu_n\}$ has a subsequence $\{Pu_{n_k}\}$ converging to some element v in G. So it turns out that

$$d(u,v) = \lim_{n\to\infty} d(u_{n_k+1}, Pu_{n_k}) = d(E,G). \tag{3.4}$$

Thus u must be an element of E_0. Owing to the fact that $P(E_0) \subseteq G_0$,

$$d(t,Pu) = d(E,G)$$

for some element t in E. Using the p-property and (3.4) we have

$$d(u_{n_k+1},t) = d(Pu_{n_k},Pu), \quad \forall n_k \in \mathbb{N}.$$

If for some n_0, $d(t,u_{n_0+1}) = 0$, consequently $d(Pu_{n_0},Pu) = 0$. So $Pu_{n_0} = Pu$, hence $d(E,G) = d(u,Pu)$. Thus the conclusion is immediate. So let for any $n \ge 0$, $d(t,u_{n+1}) > 0$. Since P is a generalized F-proximal contraction of the first kind, it follows from this that

$$\begin{aligned} \tau + F(d(t,u_{n+1})) \le F\big(&ad(u,u_n) + bd(t,u) + cd(u_n,u_{n+1}) \\ &+ h[d(u,u_{n+1}) + d(u_n,t)]\big). \end{aligned}$$

Since F is strictly increasing, we have

$$\begin{aligned} d(t,u_{n+1}) \le\ &ad(u,u_n) + bd(t,u) + cd(u_n,u_{n+1}) \\ &+ h[d(u,u_{n+1}) + d(u_n,t)]. \end{aligned}$$

As $n \to \infty$,

$$d(t,u) \le (b+h)d(t,u),$$

which further implies that u and t must be identical. It follows, that

$$d(u,Pu) = d(t,Pu) = d(E,G).$$

Uniqueness; Suppose that there is another best proximity point u^* of the mapping P such that

$$d(u^*,Pu^*) = d(E,G).$$

Since P is a generalized F-proximal contraction of the first kind, thereore

$$\tau + F(d(u,u^*)) \le F\big((a+2h)d(u,u^*)\big).$$

Since F strictly increasing,

$$d(u,u^*) \le (a+2h)d(u,u^*).$$

Therefore, u and u^* must be identical. Hence, P has a unique best proximity point. □

Next, we state and prove the best proximity point theorem for non-self generalized F-proximal contraction of the second kind.

Theorem 3.2. *Let (U,d) be a complete metric space and (E,G) be a pair of non-void closed subsets of (U,d) such that E is approximately compact with respect to G . Let $P : E \to G$ satisfy the following conditions:*

(i) *$P(E_0) \subseteq G_0$ and (E,G) satisfies the p -property;*
(ii) *P is a continuous generalized F-proximal contraction of the second kind.*

Then, there exists a $u \in E$ such that

$$d(u,Pu) = d(E,G),$$

and $u_n \to u$, where u_0 is any fixed point in E_0 and $d(u_{n+1},Pu_n) = d(E,G)$ for $n \ge 0$. Further, if u^* is another best proximity point of P, then $Pu = Pu^*$.

Proof. Similar to Theorem 3.1, we can find a sequence $\{u_n\}$ in E_0 such that

$$d(u_{n+1},Pu_n) = d(E,G), \tag{3.5}$$

for all non-negative integral values of n. From the p-property and (3.5) we get

$$d(u_n,u_{n+1}) = d(Pu_{n-1},Pu_n), \quad \forall n \in \mathbb{N}.$$

If for some n_0, $d(u_{n_0+1},u_{n_0+2}) = 0$, consequently $d(Pu_{n_0},Pu_{n_0+1}) = 0$. So $Pu_{n_0} = Pu_{n_0+1}$, hence $d(E,G) = d(u_{n_0+1},Pu_{n_0+1})$. Thus the conclusion is immediate. So let for any $n \geq 0$, $d(Pu_n, Pu_{n+1}) > 0$. In view of the reality that P is a generalized second-kind F-proximal contraction,

$$\begin{aligned}\tau + F\big(d(Pu_n, Pu_{n+1})\big) &\leq F\big(ad(Pu_{n-1}, Pu_n) + bd(Pu_{n-1}, Pu_n) \\ &\quad + cd(Pu_n, Pu_{n+1}) + hd(Pu_{n-1}, Pu_{n+1})\big) \\ &\leq F\big(ad(Pu_{n-1}, Pu_n) + bd(Pu_{n-1}, Pu_n) \\ &\quad + cd(Pu_n, Pu_{n+1}) + h[d(Pu_{n-1}, Pu_n) \\ &\quad + d(Pu_n, Pu_{n+1})]\big) \\ &\leq F\big((a+b+h)d(Pu_{n-1}, Pu_n) \\ &\quad + (c+h)d(Pu_n, Pu_{n+1})\big).\end{aligned}$$

We deduce that since F strictly increasing,

$$d(Pu_n, Pu_{n+1}) \leq (a+b+h)d(Pu_{n-1}, Pu_n) + (c+h)d(Pu_n, Pu_{n+1})$$

and thus

$$d(Pu_n, Pu_{n+1}) \leq \left(\frac{a+b+h}{1-c-h}\right)d(Pu_n, Pu_{n-1}), \quad \forall n \in \mathbb{N}$$

From $a+b+c+2h = 1$ and $c \neq 1$, we deduce that $1-c-h > 0$ and so

$$d(Pu_n, Pu_{n+1}) \leq \left(\frac{a+b+h}{1-c-h}\right)d(Pu_n, Pu_{n-1}) = d(Pu_n, Pu_{n-1}), \quad \forall n \in \mathbb{N}.$$

Therefore,

$$\tau + F(d(Pu_n, Pu_{n+1})) \leq F(d(Pu_n, Pu_{n-1})), \quad \forall n \in \mathbb{N}.$$

It further implies that

$$\begin{aligned}F(d(Pu_n, Pu_{n+1})) &\leq F(d(Pu_n, Pu_{n-1})) - \tau \\ &\vdots \\ &\leq F(d(Pu_0, Pu_1)) - n\tau, \quad \forall n \in \mathbb{N}.\end{aligned} \tag{3.6}$$

Put $s_n := d(Pu_n, Pu_{n+1})$. From (3.6) $\lim_{n\to\infty} F(s_n) = -\infty$. By the properties (F_2), we get that

$$s_n \to 0 \text{ as } n \to \infty.$$

Now, let $k \in (0,1)$ such that $\lim_{n\to\infty} s_n^k F(s_n) = 0$. By (3.6), the following holds for all $n \in \mathbb{N}$:

$$s_n^k F(s_n) - s_n^k F(s_0) \leq -ns_n^k \tau \leq 0. \tag{3.7}$$

As $k \to \infty$ in (3.7) , we deduce

$$\lim_{n\to\infty} ns_n^k = 0.$$

Thus $\lim_{n\to\infty} n^{\frac{1}{k}} s_n = 0$. Now, $\lim_{n\to\infty} n^{\frac{1}{k}} s_n = 0$ ensures that the series $\sum_{n=1}^{\infty} s_n$ is convergent. This implies that $\{Pu_n\}$ is a Cauchy sequence. Because the space is complete, the sequence $\{Pu_n\}$ converges to some element

v in G.
Moreover,

$$\begin{aligned} d(v,E) \le d(v,u_{n+1}) &\le d(v,Pu_n) + d(Pu_n,u_{n+1}) \\ &= d(v,Pu_n) + d(E,G) \\ &\le d(v,Pu_n) + d(v,E). \end{aligned}$$

Therefore, $d(v,u_n) \to d(v,E)$. Since E is approximately compact with respect to G, the sequence $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ converging to some $u \in E$. Because P is a continuous mapping,

$$d(u,Pu) = \lim_{n\to\infty} d(u_{n+1},Pu_n) = d(E,G).$$

Uniqueness; Assume that in E, there exist another best point of proximity, u^*, so that

$$d(u^*,Pu^*) = d(E,G).$$

Because P is a generalized proximal contraction of the second kind,

$$\tau + F(d(Pu,Pu^*)) \le F\big((a+2h)d(Pu,Pu^*)\big).$$

We deduce that since F strictly increasing,

$$d(Pu,Pu^*) \le (a+2h)d(Pu,Pu^*).$$

Thus $Pu = Pu^*$. Hence, P has a unique best proximity point. □

Our next result is for non-self generalized proximal contractions of the first kind as well as generalized F-proximal contractions of the second kind without the assumption of approximately compactness of the domains or the co-domain of the mappings..

Theorem 3.3. *Let (U,d) be a complete metric space and (E,G) be a pair of non-void closed subsets of (U,d). Let $P : E \to G$ satisfy the following conditions:*

(i) *$P(E_0) \subseteq G_0$ and (E,G) satisfies the p -property;*
(ii) *P is a generalized F-proximal contraction of the first kind as well as a generalized F-proximal contraction of the second kind.*

Then, there exists a unique element $u \in E$ such that

$$d(u,Pu) = d(E,G),$$

and $u_n \to u$, where u_0 is any fixed element in E_0 and $d(u_{n+1},Pu_n) = d(E,G)$ for $n \ge 0$.

Proof. Similar to Theorem 3.1, we find a sequence $\{u_n\}$ in E_0 such that

$$d(u_{n+1},Pu_n) = d(E,G)$$

for all non-negative integral values of n. Similar to Theorem 3.1, we can show that sequence $\{u_n\}$ is a Cauchy sequence. Thus converges to some element u in E. As in Theorem 3.2, it can be shown that the sequence $\{Pu_n\}$ is a Cauchy sequence and converges to some element v in G. Therefore,

$$d(u,v) = \lim_{n\to\infty} d(u_{n+1},Pu_n) = d(E,G). \tag{3.8}$$

Eventually, u becomes an element of E_0. In light of the fact that $P(E_0)$ is contained in G_0,

$$d(t, Pu) = d(E, G)$$

for some element t in E. From the p-property framework and (3.8) we get

$$d(u_{n+1}, t) = d(Pu_n, Pu), \quad \forall n \in \mathbb{N}.$$

If for some n_0, $d(t, u_{n_0+1}) = 0$, consequently $d(Pu_{n_0}, Pu) = 0$. So $Pu_{n_0} = Pu$, hence $d(E, G) = d(u, Pu)$. Thus the conclusion is immediate. So let for any $n \geq 0$, $d(t, u_{n+1}) > 0$. Since P is a generalized F-proximal contraction of the first kind, it can be seen that

$$\tau + F(d(t, u_{n+1})) \leq F\big(ad(u, u_n) + bd(t, u) + cd(u_n, u_{n+1}) \\ + h[d(u, u_{n+1}) + d(u_n, t)]\big).$$

We deduce that since F strictly increasing,

$$d(t, u_{n+1}) \leq ad(u, u_n) + bd(t, u) + cd(u_n, u_{n+1}) \\ + h[d(u, u_{n+1}) + d(u_n, t)].$$

As $n \to \infty$, $d(t, u) \leq (b + h)d(t, u)$, which means that u and t must be identical. It follows, thus, that

$$d(u, Pu) = d(t, Pu) = d(E, G).$$

Also, as in the theorem 3.1, the uniqueness of the best proximity point of mapping P follows. □

Example 3.4. Consider $U = \mathbb{R}^2$ and define the metric d on U by

$d((u_1, u_2), (v_1, v_2)) = |u_1 - v_1| + |u_2 - v_2|, \forall\, (u_1, u_2), (v_1, v_2) \in \mathbb{R}^2$

We know, (U, d) is a complete metric space. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be denoted by $F(a) = \ln a$. It is obvious that, for any $k \in (0, 1)$, the function F satisfies the conditions $(F_1) - (F_3)$. Let $E = \{(e, 0) : e \geq 0\}$ and $G = \{(g, 1) : g \geq 0\}$. We have $E = E_0$ and $G = G_0$. Let $P : E \to G$ be a mapping defined by, for each $(e, 0) \in E$,

$$P(e, 0) = (T(e), 1),$$

where

$$T(e) = \frac{1}{2} - \frac{1}{e + 2}.$$

It is clear that, for each $c, d \geq 0$,

$$\big|T(c) - T(d)\big| \leq \big|c - d\big|.$$

It is clear that E is approximately compact with respect to G, (E, G) satisfies the p-property, P is continuous and $P(E_0) \subseteq G_0$. We show that In fact, r, s, i, j be elements in E such that $d(r, Pi) = d(s, Pj) = d(E, G)$. We write $i = (e_1, 0)$ and $j = (e_2, 0)$ for some $e_1, e_2 \geq 0$. So $r = (T(e_1), 1)$ and $s = (T(e_2), 1)$. We obtain

$$d(Pr, Ps) = \big|T^2(e_1) - T^2(e_2)\big| \leq e^{-\tau} d(Pi, Pj).$$

Consequently, P is a generalized F-proximal contraction of the second kind with $e^{-\tau} = \frac{16}{7}$ or $\tau = \ln \frac{7}{16}$, $b = c = h = 0$. Thus, all the conditions of Theorem 3.2 are satisfied. Hence, P has a unique best proximity point $(0, 0)$.

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