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Topological analysis of doubly nonlocal boundary value problems

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This paper is dedicated to the memory of Maddie Goodrich on the occasion of what would have been her 19th birthday.

Abstract. We consider both Hammerstein integral equations and nonlocal boundary value problems in possession of two different nonlocal elements. The first occurs in the differential equation itself and takes the form $\|u\|_q^q$. The second occurs in the boundary condition and takes the form of a Stieltjes integral. Because the nonlocal elements are not necessarily related, a careful analysis is required to control each nonlocal element simultaneously. Topological fixed point theory is used to deduce existence of at least one positive solution to the boundary value problem. And we illustrate the application of the results with an example.

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1. Introduction

In this paper we consider the perturbed Hammerstein integral equation

$$u(t) = \gamma(t)H\big(\varphi(u)\big) + \lambda \int_0^1 \left(A\left(\int_0^1 \big|u(\xi)\big|^q \,\mathrm{d}\xi\right)\right)^{-1} G(t,s)f\big(s,u(s)\big)\,\mathrm{d}s \tag{1.1}$$

where $q \geq 1$ is a constant, each of $A : [0,+\infty) \to \mathbb{R}$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$, $G : [0,1] \times [0,1] \to [0,+\infty)$, $H : [0,+\infty) \to [0,+\infty)$, and $\gamma : [0,1] \to [0,1]$ is continuous, and where

$$\varphi(u) := \int_0^1 u(s)\,\mathrm{d}\alpha(s); \tag{1.2}$$

the integrator α in (1.2) is of bounded variation on $[0,1]$ and monotone increasing, the latter assumption so that $\varphi(u) \leq \varphi(w)$ whenever $u \leq w$, which will be important in what follows. Solutions of (1.1) can then be associated to solutions of a boundary value problem, whose boundary conditions will depend on the choice of γ and G. For example, if $\gamma(t) = 1 - t$ and

$$G(t,s) := \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1 \end{cases},$$

then a solution of (1.1) is a solution of the following nonlocal boundary value problem.

$$\begin{aligned} -A\left(\int_0^1 \big|u(s)\big|^q \,\mathrm{d}s\right) u''(t) &= \lambda f\big(t, u(t)\big),\ 0 < t < 1 \\ u(0) &= H\big(\varphi(u)\big) \\ u(1) &= 0 \end{aligned} \tag{1.3}$$

Problem (1.3) is an example of a "doubly nonlocal" differential equation in the sense that (1.3) contains two different nonlocal elements.

1. The first nonlocal element is $A\left(\int_0^1 \big|u(s)\big|^q \,\mathrm{d}s\right)$, and it occurs in the differential equation itself. Note that this nonlocal element can be written as $A\left(\|u\|_q^q\right)$ with $\|u\|_q$ the L^q norm of u on $[0,1]$.
2. The second nonlocal element is $\varphi(u)$, and it occurs in the boundary condition at $t = 0$. This nonlocal element is a Stieltjes integral, which, therefore, can accommodate many different types of nonlocal boundary conditions by suitably choosing the integrator α. For example, we can accommodate both multipoint-type and integral-type boundary conditions.

Note that the boundary condition at $t = 0$ is not only nonlocal (due to $\varphi(u)$) but is also (possibly) nonlinear (due to H).

On the one hand perturbed Hammerstein equations with nonlocal elements of the type $\varphi(u)$ in (1.2) have been studied extensively in recent years since, as noted, their solutions can be associated to solutions of boundary value problem with nonlocal boundary conditions. For example, one may consult the papers by Anderson [3], Cabada et al. [15], Goodrich [23–25], Graef and Webb [37], Infante and Pietramala [41,42,44–46], Infante et al. [47], Jankowski [48], Karakostas and Tsamatos [49,50], Karakostas [51], Webb and Infante [55,56], and Yang [59–63]. Boundary nonlocal elements can arise naturally in mathematical modeling (e.g., beam deflection, chemical reactor theory, and thermodynamics)—see, for example, Cabada et al. [15], Infante and Pietramala [43], and Infante, Pietramala, and Tenuta [47].

On the other hand differential equations with a nonlocal element in the differential equation itself also have been studied extensively. These arise naturally in fractional differential equations since fractional operators are finite convolution operators and thus nonlocal—see, for example, [34, §2], [35, Examples 3, 4, and 5], and [36, Example 5.12]. Nonetheless, more often than not these have fallen into one of two types. The first type has as a model case the equation

$$-M\left(\int_0^1 \big(u(s)\big)^q \,\mathrm{d}s\right)u''(t)=\lambda f\big(t,u(t)\big),\ t\in(0,1), \tag{1.4}$$

where M is some continuous function. This type of equation encompasses as a special case the mean field equation (see Infante [39, (1.2)]), which in its elliptic PDE form is

$$-\left(\int_\Omega e^u \,\mathrm{d}\boldsymbol{x}\right)\Delta u=\lambda e^u.$$

The second type has as a model case the equation

$$-M\left(\int_0^1 \big(u'(s)\big)^q \,\mathrm{d}s\right)u''(t)=\lambda f\big(t,u(t)\big),\ t\in(0,1). \tag{1.5}$$

This equation is a particular example of a one-dimensional Kirchhoff equation; higher dimensional Kirchhoff-type equations, which lead to elliptic- and parabolic-type PDEs, have been extensively studied, too. Examples of papers studying equation (1.4) are those by Alves and Covei [4], Aly [2], Bavaud [6], Biler et al. [7], Biler and Nadzieja [8,9], Caglioti et al. [16], Corrêa [19], Corrêa et al. [20], do Ó et al. [21], Esposito et al. [22], Goodrich [32], Stańczy [52], Wang et al. [53], Yan and Ma [57], and Yan and Wang [58]. And some examples in the case of equation (1.5) are papers by Afrouzi et al. [1], Azzouz and Bensedik [5], Bouizem et al. [10], Boulaaras [11], Boulaaras et al. [12,14], Boulaaras and Guefaifia [13], Chung [17], Goodrich [33], and Infante [39,40].

Recently the author has introduced separately a methodology for these types of nonlocal DEs—i.e., one methodology for nonlocal boundary conditions and another for nonlocal equations such as (1.4). The idea in each case was to consider functions that make the nonlocal element coercive—namely,

$$\int_0^1 u(s)\,\mathrm{d}\alpha(s)\ge C_0\|u\| \text{ or } \int_0^1 u(s)\,\mathrm{d}s\ge C_0\|u\|,$$

for some suitably chosen constant $C_0\in(0,1]$; for example, see [28, (1.9)] and [32, (1.7)]. In addition and respectively, sets of the form

$$\left\{u\in\mathscr{K}_0\ :\ \int_0^1 u(s)\,\mathrm{d}\alpha(s)<\rho\right\} \text{ or } \left\{u\in\mathscr{K}_0\ :\ \int_0^1 \big(u(s)\big)^q\,\mathrm{d}s<\rho\right\},$$

where $\mathscr{K}_0$ is some suitable order cone and $\rho>0$ is some given number, were used in order to provide direct control over the nonlocal elements. These two ideas used in tandem then allowed for weaker hypotheses on the nonlocal elements in the problem. One of the key ideas is that the boundaries of the above sets are, respectively,

$$\left\{u\in\mathscr{K}_0\ :\ \int_0^1 u(s)\,\mathrm{d}\alpha(s)=\rho\right\} \text{ or } \left\{u\in\mathscr{K}_0\ :\ \int_0^1 \big(u(s)\big)^q\,\mathrm{d}s=\rho\right\}.$$

Thus, for elements of the boundary we have exact control over the nonlocal elements. This fact turns out to be very important in the application of the topological fixed point theory—cf., Lemma 2.12.

Our goal in this paper is to provide a methodology to combine these two types of nonlocal problems. This might seem at first glance to be trivial; after all, it literally simply appears to be combining two already established methodologies. However, it is not so simple. The problem is one of elementary topology. Given two sets A and B, in general, it can***not*** be expected that $\partial(A \cap B) = \partial A \cap \partial B$. This leads to a fundamental problem since the natural (and, indeed, trivial) way to study jointly these two types of nonlocal problems would be simply to consider

$$\left\{u \in \mathscr{K}_0 \ : \ \int_0^1 u(s)\ \mathrm{d}\alpha(s) < \rho\right\} \cap \left\{u \in \mathscr{K}_0 \ : \ \int_0^1 \big(u(s)\big)^q\ \mathrm{d}s < \rho\right\}.$$

But then, generally speaking,

$$\begin{aligned}&\partial\left(\left\{u \in \mathscr{K}_0 \ : \ \int_0^1 u(s)\ \mathrm{d}\alpha(s) < \rho\right\} \cap \left\{u \in \mathscr{K}_0 \ : \ \int_0^1 \big(u(s)\big)^q\ \mathrm{d}s < \rho\right\}\right)\\ &\quad \neq \left\{u \in \mathscr{K}_0 \ : \ \int_0^1 u(s)\ \mathrm{d}\alpha(s),\ \int_0^1 \big(u(s)\big)^q\ \mathrm{d}s = \rho\right\}.\end{aligned}$$

Due to this problem we must instead study problem (1.1) more carefully. Our methodology consists of using the cone

$$\begin{aligned}\mathscr{K} := \Bigg\{&u \in \mathscr{C}\big([0,1]\big) \ : \\ &\int_0^1 u(s)\ \mathrm{d}\alpha(s) \geq C_0\|u\|,\ \int_0^1 u(s)\ \mathrm{d}s \geq C_0\|u\|,\ \min_{t\in[a,b]} u(t) \geq \eta_0\|u\|,\ u \geq 0\Bigg\},\end{aligned} \tag{1.6}$$

where C_0, $\eta_0 \in (0,1]$ are constants introduced in Sect. 2. The cone is an amalgamation of the cones used separately to study each type of nonlocal problem. Then using topological fixed point theory we make a dual use of the sets identified above—though *individually* rather than in intersection. This requires studying carefully the connections, albeit indirect, between the quantities $\int_0^1 u(s)\ \mathrm{d}\alpha(s)$ and $\int_0^1 \big(u(s)\big)^q\ \mathrm{d}s$. Coordinating this is somewhat of a delicate balancing act as the proofs in the next section will reveal.

In the end, as our main results, Theorems 2.13 and 2.16 together with Corollary 2.14, demonstrate, we are able to achieve the same sorts of good features obtainable when studying the two types of problems separately—namely,

1. A need be neither monotone nor strictly positive nor satisfy any global growth condition; and
2. H, likewise, need be neither monotone nor satisfy any either asymptotic or global growth condition.

In particular, Example 2.17, which concludes this paper, demonstrates each of these points. Since condition (1), in particular, is nearly universal among the existing literature, e.g., [4, Condition (2), p. 1], [17, Conditions (M0), (3), (4)], [21, Condition (H1), p. 299], [39, Theorem 2.3], [52, Theorem 2.2], [53, Condition (H1), p. 2], [57, p. 1], and [58, Theorem 4.1, p. 84], it is important to note that we still recover this improvement in spite of the additional complexity created by mixing the two types of nonlocal elements.

2. Main results and an example

As mentioned in Sect. 1 our approach is to use a coordinated pair of open sets in order to apply topological fixed point theory to problem (1.1). In particular, we will consider the open sets $\widehat{V}_{\rho}^{q}$, $\widehat{W}_{\rho} \subseteq \mathscr{K}$ defined as follows; here and throughout $\mathscr{K}$ is as in (1.6).

$$\widehat{V}_{\rho}^{q} := \left\{u \in \mathscr{K} \ : \ \int_0^1 \big(u(s)\big)^q \ \mathrm{d}s < \rho\right\}$$
$$\widehat{W}_{\rho} := \left\{u \in \mathscr{K} \ : \ \int_0^1 u(s) \ \mathrm{d}\alpha(s) < \rho\right\}$$

We note that the open set $\widehat{V}_{\rho}^{q}$ has been previously introduced in [32], whereas the open set $\widehat{W}_{\rho}$ has been previously used in [26–31], for example. However, the *dual use* of these sets has not been used. In fact, their dual use is nontrivial because as mentioned in Sect. 1 we must carefully analyze the interaction between these two sets.

Going forward it will be useful to make use of some notation. First of all, by **1** we denote the function that is identically the constant polynomial 1 on all of $\mathbb{R}$—that is,

$$\mathbf{1} := \mathbf{1}(x) \equiv 1,\ x \in \mathbb{R}.$$

Second of all, for a continuous function $f \ : \ [0,1] \times [0,+\infty) \to \mathbb{R}$ and for numbers $0 \le a < b \le 1$ and $0 \le c < d < +\infty$ we denote by $f_{[a,b]\times[c,d]}^{m}$ and $f_{[a,b]\times[c,d]}^{M}$, respectively, the numbers

$$f_{[a,b]\times[c,d]}^{m} := \min_{(t,y)\in[a,b]\times[c,d]} f(t,y)$$

and

$$f_{[a,b]\times[c,d]}^{M} := \max_{(t,y)\in[a,b]\times[c,d]} f(t,y).$$

For a continuous function $H \ : \ \mathbb{R} \to \mathbb{R}$ we will write similarly

$$H_{[a,b]}^{m} := \min_{y\in[a,b]} H(y) \text{ and } H_{[a,b]}^{M} := \max_{y\in[a,b]} H(y),$$

for any numbers $-\infty < a < b < +\infty$.

We assume throughout that $\mathscr{C}\big([0,1]\big)$ is equipped with the usual maximum norm denoted by $\|\cdot\|$. The coercivity constant, C_0, in the definition of $\mathscr{K}$ is defined by

$$C_0 := \min\left\{\varphi(\mathbf{1}), \varphi(\gamma), \int_0^1 \gamma(t)\ \mathrm{d}t, \inf_{s\in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\ \mathrm{d}t,\right.$$
$$\left.\inf_{s\in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\ \mathrm{d}\alpha(t)\right\},$$

where

$$\mathscr{G}(s) := \max_{t\in[0,1]} G(t,s) \tag{2.1}$$

for $s \in [0,1]$, and $S_0 \subseteq [0,1]$ is a set of full measure on which $\mathscr{G}(s) \neq 0$. We also will make the following general assumptions on the functions appearing in integral equation (1.1).

H1: The functions $\gamma : [0,1] \to [0,\infty)$, $A : [0,\infty) \to \mathbb{R}$, $f : [0,1]\times[0,\infty) \to [0,\infty)$, and $H : [0,\infty) \to [0,\infty)$ are continuous. In addition, the function $\alpha : [0,1] \to \mathbb{R}$ is of bounded variation on $[0,1]$ and is monotone increasing.

H2: The function $G : [0,1] \times [0,1] \to [0,\infty)$ is continuous, and there exist numbers $0 \leq a < b \leq 1$ and number $\eta_0 \in (0,1]$ such that $\min\limits_{t\in[a,b]} G(t,s) \geq \eta_0\mathscr{G}(s)$, for each $s \in [0,1]$, where $\mathscr{G}$ is defined as in (2.1). Moreover, the set S_0 as described above satisfies $\left|S_0\right| = 1$, where by $|\cdot|$ we mean the usual Lebesgue measure.

H3: The function γ satisfies the following three conditions:
1. $\min\limits_{t\in[a,b]} \gamma(t) \geq \eta_0\|\gamma\|$, where a, b, and η_0 are the same numbers as in condition (H2);
2. $0 < \|\gamma\| \leq 1$; and
3. $\varphi(\gamma) \geq C_0\|\gamma\|$ and $\displaystyle\int_0^1 \gamma(s)\ \mathrm{d}s \geq C_0\|\gamma\|$.

Finally, we will define the operator $T : \mathscr{C}\big([0,1]\big) \to \mathscr{C}\big([0,1]\big)$ by

$$(Tu)(t):=\gamma(t)H\big(\varphi(u)\big)+\lambda\int_0^1\left(A\left(\int_0^1\big(u(\xi)\big)^q\ \mathrm{d}\xi\right)\right)^{-1}G(t,s)f\big(s,u(s)\big)\ \mathrm{d}s.$$

A fixed point of the operator T will correspond to a solution of integral equation (1.1).

Remark 2.1. Note that conditions (H1) and (H3) ensure that $\gamma \in \mathscr{K}$ since $\gamma(t) \geq 0$, $\varphi(\gamma) \geq C_0\|\gamma\|$, $\min\limits_{t\in[a,b]}\gamma(t) \geq \eta_0\|\gamma\|$, and $\displaystyle\int_0^1 \gamma(s)\ \mathrm{d}s \geq C_0\|\gamma\|$. Therefore, $\mathscr{K} \neq \varnothing$.

Remark 2.2. The function $\gamma(t) := 1 - t$ satisfies conditions (H3.1), (H3.2), and (H3.3). For this choice of γ solutions of (1.1) correspond to solutions of boundary value problem (1.3).

Remark 2.3. Because we will work within the cone $\mathscr{K}$ we will henceforth write $\big(u(\xi)\big)^q$ instead of $\big|u(\xi)\big|^q$ when studying problem (1.1)—just as we did in the definition of T above.

We next present a collection of preliminary lemmata. These will be important in the existence theorems for integral equation (1.1). Our first lemma demonstrates that T is a reflexive map on a particular annular subregion of $\mathscr{K}$.

Lemma 2.4. *Suppose that conditions (H1)–(H3) are satisfied. In addition, assume that* $A(t) > 0$ *whenever* $t \in [\rho_1, \rho_2]$. *Then* $T\left(\overline{\widehat{V}_{\rho_2}^q} \setminus \widehat{V}_{\rho_1}^q\right) \subseteq \mathscr{K}$.

Proof. The proof is similar to a combination of part of the proofs of [28, Theorem 3.1] and [32, Lemma 2.3]. We include the details for completeness.

First note that

$$A\left(\int_0^1 \big(u(\xi)\big)^q \, \mathrm{d}\xi\right) > 0$$

for each $u \in \overline{\widehat{V}_{\rho_2}^q} \setminus \widehat{V}_{\rho_1}^q$ by virtue of the fact that for any such u it follows that

$$\rho_1 \le \int_0^1 \big(u(\xi)\big)^q \, \mathrm{d}\xi \le \rho_2.$$

Then the assumption in the statement of the lemma establishes the desired claim. So, in particular, the operator T and thus integral equation (1.1) are well defined on the annular region $\overline{\widehat{V}_{\rho_2}^q} \setminus \widehat{V}_{\rho_1}^q$.

We next show that T satisfies the coercivity condition for the functional $u \mapsto \displaystyle\int_0^1 u(s)\, \mathrm{d}s$, namely that

$$\int_0^1 (Tu)(s)\, \mathrm{d}s \ge C_0 \|Tu\|.$$

To see that this is true we calculate

$$\begin{aligned}
\int_0^1 (Tu)(t)\, \mathrm{d}t &= H(\varphi(u)) \int_0^1 \gamma(t)\, \mathrm{d}t \\
&\quad + \lambda \int_0^1 \int_0^1 \left(A\left(\int_0^1 \big(u(\xi)\big)^q \, \mathrm{d}\xi\right)\right)^{-1} G(t,s) f\big(s, u(s)\big)\, \mathrm{d}s\, \mathrm{d}t \\
&\ge H(\varphi(u)) \int_0^1 \gamma(t)\, \mathrm{d}t \\
&\quad + \lambda \int_0^1 \left[\inf_{s \in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, \mathrm{d}t\right] \left(A\left(\int_0^1 \big(u(\xi)\big)^q \, \mathrm{d}\xi\right)\right)^{-1} \mathscr{G}(s) f\big(s, u(s)\big)\, \mathrm{d}s \\
&\ge H(\varphi(u)) \int_0^1 \gamma(t)\, \mathrm{d}t + C_0 \lambda \int_0^1 \left(A\left(\int_0^1 \big(u(\xi)\big)^q \, \mathrm{d}\xi\right)\right)^{-1} \mathscr{G}(s) f\big(s, u(s)\big)\, \mathrm{d}s \\
&\ge C_0 \left[H(\varphi(u)) \|\gamma\| + \lambda \int_0^1 \left(A\left(\int_0^1 \big(u(\xi)\big)^q \, \mathrm{d}\xi\right)\right)^{-1} \mathscr{G}(s) f\big(s, u(s)\big)\, \mathrm{d}s\right] \\
&\ge C_0 \|Tu\|,
\end{aligned}$$

using that

$$C_0 := \min\left\{\int_0^1 \gamma(t)\, \mathrm{d}t,\ \inf_{s \in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, \mathrm{d}t,\ \int_0^1 \gamma(t)\, \mathrm{d}\alpha(t),\ \inf_{s \in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\, \mathrm{d}\alpha(t)\right\}.$$

In a similar manner we can show that T satisfies the coercivity condition for the functional $u \mapsto \int_0^1 u(s)\ \mathrm{d}\alpha(s)$. In particular,

$$\begin{aligned}
\int_0^1 (Tu)(s)\ \mathrm{d}\alpha(s) &= H(\varphi(u))\varphi(\gamma)\\
&\quad + \lambda \int_0^1 \int_0^1 \left(A\left(\int_0^1 (u(\xi))^q\ \mathrm{d}\xi\right)\right)^{-1} G(t,s)f(s,u(s))\ \mathrm{d}s\ \mathrm{d}\alpha(t)\\
&\geq H(\varphi(u))\varphi(\gamma) + \lambda \int_0^1 \left[\inf_{s\in S_0} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\ \mathrm{d}\alpha(t)\right]\\
&\quad \times \left(A\left(\int_0^1 (u(\xi))^q\ \mathrm{d}\xi\right)\right)^{-1} \mathscr{G}(s)f(s,u(s))\ \mathrm{d}s\\
&\geq C_0\left[H(\varphi(u))\|\gamma\| + \lambda \int_0^1 \left(A\left(\int_0^1 (u(\xi))^q\ \mathrm{d}\xi\right)\right)^{-1} \mathscr{G}(s)f(s,u(s))\ \mathrm{d}s\right]\\
&\geq C_0\|Tu\|,
\end{aligned}$$

once again using the definition of C_0.

On the other hand, since both G and γ satisfy, via conditions (H2)–(H3), Harnack-like inequalities, it is straightforward to demonstrate that $\min_{t\in[a,b]}(Tu)(t) \geq \eta_0\|Tu\|$. Finally, that $(Tu)(t) \geq 0$ for each $t \in [0,1]$ follows directly from the definition of T and the nonnegativity of f, G, A, and γ. And this completes the proof. □

Remark 2.5. If we put $\gamma(t) := 1 - t$ and

$$G(t,s) := \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1 \end{cases},$$

then we recover problem (1.3)—i.e., inhomogeneous Dirichlet boundary conditions. In this case we see that

$$\min\left\{\int_0^1 1 - t\ \mathrm{d}t,\ \inf_{s\in(0,1)} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\ \mathrm{d}t\right\} = \frac{1}{2},$$

which matches what was obtained in [32]. Moreover, in this case it is well known that $\eta_0 := \min\{a, 1-b\}$.

We next demonstrate a relationship between the $\widehat{V}_\rho^q$ and $\widehat{W}_\rho$ sets. This is crucial for the correct application of the fixed point theorem, Lemma 2.12, later.

Lemma 2.6. *For any numbers $q \geq 1$ and $0 < \rho_1 < C_0\rho_2^{\frac{1}{q}}$ it holds that $\widehat{W}_{\rho_1} \subset \widehat{V}^q_{\left(\frac{\rho_1}{C_0}\right)^q} \subset \widehat{V}^q_{\rho_2}$.*

Proof. Let $u \in \widehat{W}_{\rho_1}$. Then

$$C_0\|u\| \leq \int_0^1 u(s)\ \mathrm{d}\alpha(s) < \rho_1$$

so that

$$\|u\| < \frac{\rho_1}{C_0}.$$

At the same time

$$\int_0^1 \big(y(s)\big)^q \, \mathrm{d}s \le \|u\|^q < \left(\frac{\rho_1}{C_0}\right)^q.$$

Since, by assumption, we have that $\rho_1 < C_0\rho_2^{\frac{1}{q}}$, it follows that $\widehat{W}_{\rho_1} \subset \widehat{V}^q_{\left(\frac{\rho_1}{C_0}\right)^q} \subset \widehat{V}^q_{\rho_2}$, as was claimed. $\square$

Remark 2.7. Notice that the condition $\rho_1 < C_0\rho_2^{\frac{1}{q}}$ depends on initial data only—namely, ρ_1, ρ_2, q, and C_0.

Another relationship between the sets $\widehat{W}_{\rho_1}$ and $\widehat{V}^q_{\rho_2}$ is stated in the next corollary. As with Lemma 2.6 that Corollary 2.8 holds is essential for the correct application of the fixed point theory in the sequel.

Corollary 2.8. *For any numbers $q \ge 1$ and $0 < \rho_1 < C_0\rho_2^{\frac{1}{q}}$ it holds that $\widehat{V}^q_{\rho_2} \setminus \overline{\widehat{W}_{\rho_1}} \neq \varnothing$.*

Proof. For any $\rho > 0$ define the set Ω_ρ by

$$\Omega_\rho := \big\{u \in \mathscr{K} \ : \ \|u\| < \rho\big\},$$

and consider the collection

$$\mathscr{K} \supseteq \Omega_{\rho_2^{\frac{1}{q}}} \setminus \overline{\Omega_{\frac{\rho_1}{C_0}}}.$$

Then given any $u \in \Omega_{\rho_2^{\frac{1}{q}}} \setminus \overline{\Omega_{\frac{\rho_1}{C_0}}}$ we have that

$$\frac{\rho_1}{C_0} < \|u\| < \rho_2^{\frac{1}{q}}.$$

Consequently,

$$\int_0^1 u(s) \, \mathrm{d}\alpha(s) \ge C_0\|u\| > \rho_1,$$

where we have used the coercivity of the functional $u \mapsto \displaystyle\int_0^1 u(s) \, \mathrm{d}\alpha(s)$. Similarly,

$$\int_0^1 \big(u(s)\big)^q \, \mathrm{d}s \le \|u\|^q < \rho_2.$$

Thus, we conclude that

$$u \in \Omega_{\rho_2^{\frac{1}{q}}} \setminus \overline{\Omega_{\frac{\rho_1}{C_0}}} \Longrightarrow u \in \widehat{V}^q_{\rho_2} \setminus \overline{\widehat{W}_{\rho_1}}.$$

But since

$$\frac{\rho_1}{C_0} < \rho_2^{\frac{1}{q}},$$

by assumption, it follows that

$$\Omega_{\rho_2^{\frac{1}{q}}} \setminus \overline{\Omega_{\frac{\rho_1}{C_0}}} \neq \varnothing$$

so that

$$\widehat{V}^q_{\rho_2} \setminus \overline{\widehat{W}_{\rho_1}} \neq \varnothing.$$

And this completes the proof of the corollary. □

Our next preliminary lemma demonstrates an important topological condition of the sets $\widehat{V}^q_\rho$ and $\widehat{W}_\rho$. This lemma is essential for the correct application of the fixed point result Lemma 2.12.

Lemma 2.9. *For each $\rho > 0$ each of the sets $\widehat{V}^q_\rho$ and $\widehat{W}_\rho$ is bounded. Moreover, each set is relatively open in $\mathscr{K}$.*

Proof. Suppose that $u \in \widehat{V}^q_\rho$. Then by Jensen's inequality

$$C_0^q \|u\|^q \leq \int_0^1 \big(u(s)\big)^q \, \mathrm{d}s < \rho$$

so that

$$\|u\| < \frac{\rho^{\frac{1}{q}}}{C_0}. \tag{2.2}$$

So, inequality (2.2) implies that $\widehat{V}^q_\rho$ is bounded. In a similar way, we see that if $u \in \widehat{W}_\rho$, then

$$C_0 \|u\| \leq \int_0^1 u(s) \, \mathrm{d}\alpha(s) < \rho. \tag{2.3}$$

So, inequality (2.3) implies that $\widehat{W}_\rho$ is bounded. Finally, that each of these sets is relatively open in $\mathscr{K}$ is a simple consequence of the definition of the functionals as well as the fact that $u \in \mathscr{C}\big([0,1]\big)$. □

The next lemma will be used in the existence theorems. It establishes that a certain interval of interest is nonempty.

Lemma 2.10. *For each $\rho_2 > 0$, $\eta_0 \in (0,1)$, $0 \leq a < b \leq 1$, $C_0 \in (0,1]$, and $q \geq 1$, it holds that*

$$\left[\left(\frac{\eta_0 \rho_2}{\varphi(\mathbf{1})}\right)^q (b-a), \left(\frac{\rho_2}{C_0}\right)^q\right] \neq \varnothing.$$

Proof. Note that the interval is nonempty if and only if

$$\frac{\eta_0^q}{\big(\varphi(\mathbf{1})\big)^q}(b-a) < C_0^{-q}$$

or, equivalently,

$$\frac{\eta_0}{\varphi(\mathbf{1})}(b-a)^{\frac{1}{q}} < C_0^{-1}. \tag{2.4}$$

Since $\varphi(\mathbf{1}) \geq C_0$, note that

$$\frac{\eta_0}{\varphi(\mathbf{1})}(b-a)^{\frac{1}{q}} < \frac{\eta_0}{C_0}(b-a)^{\frac{1}{q}}.$$

Therefore, if

$$C_0^{-1} > \frac{\eta_0}{C_0}(b-a)^{\frac{1}{q}}, \tag{2.5}$$

then inequality (2.4) will be satisfied. But inequality (2.5) reduces to

$$1 > \eta_0(b-a)^{\frac{1}{q}},$$

which is always satisfied since $0 < b-a \leq 1$. Therefore, inequality (2.4) holds, and so, we conclude that the interval is nonempty, as claimed. $\square$

As a consequence of Lemma 2.10 we can prove the following lemma, which concerns a certain inequality involving the coefficient function A.

Lemma 2.11. *Suppose that $u \in \partial\widehat{W}_{\rho_2}$ for some number $\rho_2 > 0$. If the function A is monotone increasing on the set*

$$\left[\left(\frac{\eta_0\rho_2}{\varphi(\mathbf{1})}\right)^q(b-a), \left(\frac{\rho_2}{C_0}\right)^q\right],$$

then

$$\left(A\left(\left(\frac{\rho_2}{C_0}\right)^q\right)\right)^{-1} \leq \left(A\left(\int_0^1 (u(\xi))^q \ \mathrm{d}\xi\right)\right)^{-1} \leq \left(A\left(\left(\frac{\eta_0\rho_2}{\varphi(\mathbf{1})}\right)^q(b-a)\right)\right)^{-1}.$$

Proof. Due to Lemma 2.10 we already know that $\left[\left(\frac{\eta_0\rho_2}{\varphi(\mathbf{1})}\right)^q(b-a), \left(\frac{\rho_2}{C_0}\right)^q\right] \neq \varnothing$. Now, since $u \in \partial\widehat{W}_{\rho_2}$ we can write (using that α is a monotone increasing integrator)

$$\rho_2 = \int_0^1 u(s) \ \mathrm{d}\alpha(s) \leq \varphi(\mathbf{1})\|u\|. \tag{2.6}$$

Similarly, it holds that

$$C_0\|u\| \leq \int_0^1 u(s) \ \mathrm{d}\alpha(s) = \rho_2. \tag{2.7}$$

Therefore, inequalities (2.6)–(2.7) imply that

$$\frac{\rho_2}{\varphi(\mathbf{1})} \leq \|u\| \leq \frac{\rho_2}{C_0}. \tag{2.8}$$

At the same time, using the fact that u satisfies the Harnack inequality from $\mathscr{K}$ together with inequality (2.8) we calculate

$$\begin{aligned}\left(\frac{\rho_2}{C_0}\right)^q &\geq \|u\|^q \geq \int_0^1 (u(\xi))^q \ \mathrm{d}\xi \geq \int_a^b (u(\xi))^q \ \mathrm{d}\xi \geq \eta_0^q\|u\|^q(b-a)\\ &\geq \left(\frac{\eta_0\rho_2}{\varphi(\mathbf{1})}\right)^q(b-a).\end{aligned} \tag{2.9}$$

Thus, inequality (2.9) demonstrates that

$$\int_0^1 (u(\xi))^q \ \mathrm{d}\xi \in \left[\left(\frac{\eta_0\rho_2}{\varphi(\mathbf{1})}\right)^q(b-a), \left(\frac{\rho_2}{C_0}\right)^q\right],$$

which is precisely the interval on which A is monotone increasing. Therefore, we conclude that

$$A\left(\left(\frac{\rho_2}{C_0}\right)^q\right) \geq A\left(\int_0^1 \left(u(\xi)\right)^q \, \mathrm{d}\xi\right) \geq A\left(\left(\frac{\eta_0\rho_2}{\varphi(\mathbf{1})}\right)^q (b-a)\right)$$

so that

$$\left(A\left(\left(\frac{\rho_2}{C_0}\right)^q\right)\right)^{-1} \leq \left(A\left(\int_0^1 \left(u(\xi)\right)^q \, \mathrm{d}\xi\right)\right)^{-1} \leq \left(A\left(\left(\frac{\eta_0\rho_2}{\varphi(\mathbf{1})}\right)^q (b-a)\right)\right)^{-1},$$

as desired. □

We conclude our preliminary lemmata with a lemma regarding a fixed point result. For further details on this and related results one may consult, for example, Cianciaruso, Infante, and Pietramala [18, Lemma 2.3], Guo and Lakshmikantham [38], Infante, Pietramala, and Tenuta [47], or Zeidler [64].

Lemma 2.12. *Let U be a bounded open set and, with $\mathcal{K}$ a cone in a real Banach space $\mathscr{X}$, suppose both that $U_{\mathscr{K}} := U \cap \mathcal{K} \supseteq \{0\}$ and that $\overline{U_{\mathscr{K}}} \neq \mathscr{K}$. Assume that $T : \overline{U_{\mathscr{K}}} \to \mathscr{K}$ is a compact map such that $x \neq Tx$ for each $x \in \partial U_{\mathscr{K}}$. Then the fixed point index $i_{\mathscr{K}}(T, U_{\mathscr{K}})$ has the following properties.*

1. *If there exists $e \in \mathscr{K} \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for each $x \in \partial U_{\mathscr{K}}$ and each $\lambda > 0$, then $i_{\mathscr{K}}(T, U_{\mathscr{K}}) = 0$.*
2. *If $\mu x \neq Tx$ for each $x \in \partial U_{\mathscr{K}}$ and for each $\mu \geq 1$, then $i_{\mathscr{K}}(T, U_{\mathscr{K}}) = 1$.*
3. *If $i_{\mathscr{K}}(T, U_{\mathscr{K}}) \neq 0$, then T has a fixed point in $U_{\mathscr{K}}$.*
4. *Let U^1 be open in X with $\overline{U^1_{\mathscr{K}}} \subseteq U_{\mathscr{K}}$. If $i_{\mathscr{K}}(T, U_{\mathscr{K}}) = 1$ and $i_{\mathscr{K}}\left(T, U^1_{\mathscr{K}}\right) = 0$, then T has a fixed point in $U_{\mathscr{K}} \setminus \overline{U^1_{\mathscr{K}}}$. The same result holds if $i_{\mathscr{K}}(T, U_{\mathscr{K}}) = 0$ and $i_{\mathscr{K}}\left(T, U^1_{\mathscr{K}}\right) = 1$.*

We now present three representative existence results for problem (1.1). The first of these, Theorem 2.13, uses a $\widehat{W}_\rho$-type set on the "inner" boundary and a $\widehat{V}^q_\rho$-type set on the "outer" boundary.

Theorem 2.13. *Suppose that conditions (H1)–(H3) are satisfied. In addition, suppose that there exists numbers ρ_1 and ρ_2, where $0 < \rho_1 < C_0\rho_2^{\frac{1}{q}}$, such that*

1. *A is monotone increasing on $\left[\left(\frac{\eta_0\rho_1}{\varphi(\mathbf{1})}\right)^q (b-a), \left(\frac{\rho_1}{C_0}\right)^q\right]$;*
2. *$A(t) > 0$ for $t \in \left[\left(\frac{\rho_1 C_0}{\varphi(\mathbf{1})}\right)^q, \rho_2\right]$;*
3. *$H(\rho_1)\varphi(\gamma) + \lambda\left(A\left(\left(\frac{\rho_1}{C_0}\right)^q\right)\right)^{-1} f^m_{[a,b]\times\left[\frac{\eta_0\rho_1}{\varphi(\mathbf{1})}, \frac{\rho_1}{C_0}\right]} \int_0^1 \int_a^b G(t,s) \, \mathrm{d}s \, \mathrm{d}\alpha(t) > \rho_1$; and*
4. *$\displaystyle\int_0^1 \left[\gamma(t) H^M_{\left[C_0\rho_2^{\frac{1}{q}}, \frac{\rho_2^{\frac{1}{q}}\varphi(\mathbf{1})}{C_0}\right]} + \frac{\lambda}{A(\rho_2)} f^M_{[0,1]\times\left[0, \frac{\rho_2^{\frac{1}{q}}}{C_0}\right]} \int_0^1 G(t,s) \, \mathrm{d}s\right]^q \mathrm{d}t < \rho_2.$*

Then problem (1.1) *has at least one positive solution,* u_0*, satisfying the localization*

$$u_0 \in \widehat{V}^q_{\rho_2} \setminus \overline{\widehat{W}_{\rho_1}}.$$

Proof. As a preliminary observation let us first notice that

$$\left(A\left(\int_0^1 \big(u(\xi)\big)^q \ \mathrm{d}\xi\right)\right)^{-1} > 0$$

whenever $u \in \widehat{V}^q_{\rho_2} \setminus \overline{\widehat{W}_{\rho_1}}$. Indeed, we see that

$$\|u\|\varphi(\mathbf{1}) > \int_0^1 u(s) \ \mathrm{d}\alpha(s) > \rho_1 \tag{2.10}$$

since $u \in \mathscr{K} \setminus \overline{\widehat{W}_{\rho_1}}$, and that

$$\rho_2 > \int_0^1 \big(u(s)\big)^q \ \mathrm{d}s \geq C_0^q \|u\|^q \tag{2.11}$$

since $u \in \widehat{V}^q_{\rho_2}$. Putting (2.10) and (2.11) together we see that

$$\left(\frac{\rho_1 C_0}{\varphi(\mathbf{1})}\right)^q \leq \int_0^1 \big(u(s)\big)^q \ \mathrm{d}s < \rho_2,$$

which establishes the desired claim due to assumption (2).

We first assume for contradiction the existence of $u \in \partial\widehat{W}_{\rho_1}$ and $\mu > 0$ such that $u(t) = (Tu)(t) + \mu e(t)$, for $t \in [0, 1]$, with $e(t) \equiv \mathbf{1}$, thereby trying to invoke part (1) of Lemma 2.12. Note that $\mathbf{1} \in \mathscr{K}$ by the definition of C_0 and the fact that $\|\mathbf{1}\| = 1$. Then applying φ to both sides of $u = Tu + \mu e$ yields

$$\begin{aligned}\rho_1 &= \varphi(u) \geq \varphi(\gamma)H\big(\varphi(u)\big)\\ &\quad + \lambda \int_0^1 \int_0^1 \left(A\left(\int_0^1 \big(u(\xi)\big)^q \ \mathrm{d}\xi\right)\right)^{-1} G(t,s) f\big(s, u(s)\big) \ \mathrm{d}s \ \mathrm{d}\alpha(t)\\ &\geq H\left(\rho_1\right)\varphi(\gamma)\\ &\quad + \lambda \left(A\left(\left(\frac{\rho_1}{C_0}\right)^q\right)\right)^{-1} \int_0^1 \int_0^1 G(t,s) f\big(s, u(s)\big) \ \mathrm{d}s \ \mathrm{d}\alpha(t)\\ &\geq H\left(\rho_1\right)\varphi(\gamma)\\ &\quad + \lambda \left(A\left(\left(\frac{\rho_1}{C_0}\right)^q\right)\right)^{-1} f^m_{[a,b]\times\left[\frac{\eta_0\rho_1}{\varphi(\mathbf{1})}, \frac{\rho_1}{C_0}\right]} \int_0^1 \int_a^b G(t,s) \ \mathrm{d}s \ \mathrm{d}\alpha(t)\\ &> \rho_1,\end{aligned} \tag{2.12}$$

where we have used Lemma 2.11 to obtain the lower bound on $\big(A(\cdot)\big)^{-1}$. We have also used in inequality (2.12) the fact that by (2.8) it follows that

$$\frac{\rho_1}{C_0} \geq \|u\| \geq \min_{t\in[a,b]} u(t) \geq \eta_0\|u\| \geq \frac{\eta_0\rho_1}{\varphi(\mathbf{1})}$$

so that

$$f\big(s,u(s)\big) \geq f^m_{[a,b]\times\left[\frac{\eta_0\rho_1}{\varphi(\mathbf{1})},\frac{\rho_1}{C_0}\right]}.$$

Since inequality (2.12) is a contradiction, from Lemma 2.12 we conclude that

$$i_{\mathcal{K}}\left(T,\widehat{W}_{\rho_1}\right)=0. \tag{2.13}$$

On the other hand, suppose for contradiction the existence of $u \in \partial\widehat{V}^q_{\rho_2}$ and $\mu \geq 1$ such that $\mu u(t) = (Tu)(t)$ for each $t \in [0,1]$, thereby trying to invoke part (2) of Lemma 2.12. As a preliminary observation note that since $u \in \partial\widehat{V}^q_{\rho_2}$ we obtain from Jensen's inequality that

$$C_0^q\|u\|^q \leq \int_0^1 \big(u(s)\big)^q\ \mathrm{d}s = \rho_2 \Longrightarrow \int_0^1 u(s)\ \mathrm{d}\alpha(s) \leq \|u\|\varphi(\mathbf{1}) \leq \frac{\rho_2^{\frac{1}{q}}\varphi(\mathbf{1})}{C_0}.$$

In a similar manner we also deduce that

$$\|u\|^q \geq \int_0^1 \big(u(s)\big)^q\ \mathrm{d}s = \rho_2 \Longrightarrow \int_0^1 u(s)\ \mathrm{d}\alpha(s) \geq C_0\|u\| \geq C_0\rho_2^{\frac{1}{q}}.$$

Consequently,

$$H\big(\varphi(u)\big) = H\left(\int_0^1 u(s)\ \mathrm{d}\alpha(s)\right) \leq H^M_{\left[C_0\rho_2^{\frac{1}{q}},\frac{\rho_2^{\frac{1}{q}}\varphi(\mathbf{1})}{C_0}\right]}. \tag{2.14}$$

Then integrating from $t=0$ to $t=1$ both sides of $\big(\mu u(t)\big)^q = \big((Tu)(t)\big)^q$ yields

$$\begin{aligned}
\rho_2 &\leq \mu\int_0^1 \big(u(t)\big)^q\ \mathrm{d}t\\
&= \int_0^1 \underbrace{\left[\gamma(t)H\big(\varphi(u)\big) + \lambda\int_0^1 \left(A\left(\int_0^1 \big(u(\xi)\big)^q\ \mathrm{d}\xi\right)\right)^{-1} G(t,s)f\big(s,u(s)\big)\ \mathrm{d}s\right]^q}_{=\left((Tu)(t)\right)^q}\ \mathrm{d}t\\
&\leq \int_0^1 \left[\gamma(t)H^M_{\left[C_0\rho_2^{\frac{1}{q}},\frac{\rho_2^{\frac{1}{q}}\varphi(\mathbf{1})}{C_0}\right]} + \lambda\int_0^1 \big(A(\rho_2)\big)^{-1}G(t,s)f\big(s,u(s)\big)\ \mathrm{d}s\right]^q\ \mathrm{d}t\\
&\leq \int_0^1 \left[\gamma(t)H^M_{\left[C_0\rho_2^{\frac{1}{q}},\frac{\rho_2^{\frac{1}{q}}\varphi(\mathbf{1})}{C_0}\right]} + \frac{\lambda}{A(\rho_2)}f^M_{[0,1]\times\left[0,\frac{\rho_2^{\frac{1}{q}}}{C_0}\right]}\int_0^1 G(t,s)\ \mathrm{d}s\right]^q\ \mathrm{d}t\\
&< \rho_2,
\end{aligned} \tag{2.15}$$

where we have used inequality (2.14). And since inequality (2.15) is a contradiction, we conclude from Lemma 2.12 that

$$i_{\mathcal{K}}\left(T,\widehat{V}^q_{\rho_2}\right)=1. \tag{2.16}$$

Putting the index calculations (2.13) and (2.16) together we conclude from yet another application of Lemma 2.12 that T has a fixed point, say u_0, satisfying the localization $u_0 \in \widehat{V}^q_{\rho_2} \setminus \overline{\widehat{W}_{\rho_1}}$. Finally, we observe that

$$\widehat{V}^q_{\rho_2} \setminus \overline{\widehat{W}_{\rho_1}} \neq \varnothing$$

due to Corollary 2.8, which may be applied since we assumed in the statement of the theorem that $C_0^{\frac{1}{q}}\rho_2 > \rho_1 > 0$. Therefore, u_0 is a positive solution of integral equation (1.1). □

An immediate corollary of Theorem 2.13 is the following. The difference between the two results is twofold. First of all, Corollary 2.14 eliminates the local monotonicity assumption on A. Second of all, the corollary uses a simpler version of condition (3) in Theorem 2.13. Therefore, in practice Corollary 2.14 is likely to be much simpler to apply—a fact illustrated by Example 2.17.

Corollary 2.14. *Suppose that conditions (H1)–(H3) are satisfied. In addition, suppose that there exists numbers ρ_1 and ρ_2, where $0 < \rho_1 < C_0\rho_2^{\frac{1}{q}}$, such that*

1. $A(t) > 0$ *for* $t \in \left[\left(\frac{\rho_1 C_0}{\varphi(\mathbf{1})}\right)^q, \rho_2\right]$*;*
2. $\frac{H(\rho_1)}{\rho_1} > \frac{1}{\varphi(\gamma)}$*; and*
3. $\int_0^1 \left[\gamma(t)H^M_{\left[C_0\rho_2^{\frac{1}{q}}, \frac{\rho_2^{\frac{1}{q}}\varphi(\mathbf{1})}{C_0}\right]} + \frac{\lambda}{A(\rho_2)} f^M_{[0,1]\times\left[0, \frac{\rho_2^{\frac{1}{q}}}{C_0}\right]} \int_0^1 G(t,s)\, \mathrm{d}s\right]^q \mathrm{d}t < \rho_2.$

Then problem (1.1) *has at least one positive solution, u_0, satisfying the localization*

$$u_0 \in \widehat{V}^q_{\rho_2} \setminus \overline{\widehat{W}_{\rho_1}}.$$

Remark 2.15. It is certainly possible in Theorem 2.13 and, thus, in Corollary 2.14 to "reverse" the roles of ρ_1 and ρ_2 in the sense that the conditions (1)–(4) can be rewritten for the case $\rho_1 > \rho_2 > 0$. We omit the precise statement of this result.

Our third result is an alternative existence result, which complements both Theorem 2.13 and Corollary 2.14. The distinction here is that we use a $\widehat{W}_\rho$-type set on both boundaries. This results in a slight alteration of conditions (1), (2), and (4) from Theorem 2.13.

Theorem 2.16. *Suppose that conditions (H1)–(H3) are satisfied. In addition, suppose that there exists numbers ρ_1 and ρ_2, where $0 < \rho_1 < \rho_2$, such that*

1. *A is monotone increasing on* $\left[\left(\frac{\eta_0\rho_1}{\varphi(\mathbf{1})}\right)^q (b-a), \left(\frac{\rho_1}{C_0}\right)^q\right] \cup \left[\left(\frac{\eta_0\rho_2}{\varphi(\mathbf{1})}\right)^q (b-a), \left(\frac{\rho_2}{C_0}\right)^q\right]$*;*

2. $A(t) > 0$ *for* $t \in \left[\left(\frac{\rho_1 C_0}{\varphi(\mathbf{1})}\right)^q, \left(\frac{\rho_2}{C_0}\right)^q\right]$;
3. $H(\rho_1)\varphi(\gamma) + \lambda\left(A\left(\left(\frac{\rho_1}{C_0}\right)^q\right)\right)^{-1} f^m_{[a,b]\times\left[\frac{\eta_0\rho_1}{\varphi(\mathbf{1})}, \frac{\rho_1}{C_0}\right]} \int_0^1 \int_a^b G(t,s)\ \mathrm{d}s\ \mathrm{d}\alpha(t) > \rho_1$; *and*
4. $H(\rho_2)\varphi(\gamma) + \lambda\left(A\left(\left(\frac{\eta_0\rho_2}{\varphi(\mathbf{1})}\right)^q (b-a)\right)\right)^{-1} f^M_{[0,1]\times\left[0, \frac{\rho_2}{C_0}\right]} \int_0^1 \int_0^1 G(t,s)\ \mathrm{d}s\ \mathrm{d}\alpha(t) < \rho_2$.

Then problem (1.1) *has at least one positive solution,* u_0*, satisfying the localization*

$$u_0 \in \widehat{W}_{\rho_2} \setminus \overline{\widehat{W}_{\rho_1}}.$$

Proof. Since the proof of this theorem is very similar to the proof of Theorem 2.13, we will only sketch the relevant details. Indeed, really only the first part of the proof changes.

As a preliminary observation let us first notice that

$$\left(A\left(\int_0^1 \left(u(\xi)\right)^q\ \mathrm{d}\xi\right)\right)^{-1} > 0$$

whenever $u \in \widehat{W}_{\rho_2} \setminus \overline{\widehat{W}_{\rho_1}}$. To argue that this is true we first notice that

$$\|u\|\varphi(\mathbf{1}) > \int_0^1 u(s)\ \mathrm{d}\alpha(s) > \rho_1 \tag{2.17}$$

since $u \in \mathscr{K} \setminus \overline{\widehat{W}_{\rho_1}}$. At the same time since $u \in \widehat{W}_{\rho_2}$ it follows that

$$\rho_2 > \int_0^1 u(s)\ \mathrm{d}\alpha(s) \geq C_0\|u\|$$

so that

$$\frac{\rho_1}{\varphi(\mathbf{1})} \leq \|u\| \leq \frac{\rho_2}{C_0}. \tag{2.18}$$

Putting (2.17)–(2.18) together with Jensen's inequality we see that

$$\left(\frac{\rho_1 C_0}{\varphi(\mathbf{1})}\right)^q \leq C_0^q\|u\|^q \leq \int_0^1 \left(u(s)\right)^q\ \mathrm{d}s \leq \|u\|^q < \left(\frac{\rho_2}{C_0}\right)^q,$$

which establishes the desired claim due to assumption (2). Note that this assumption is only meaningful if

$$\frac{\rho_1 C_0}{\varphi(\mathbf{1})} < \frac{\rho_2}{C_0}.$$

In other words, it is meaningful only if

$$\rho_1 < \frac{\varphi(\mathbf{1})}{C_0^2}\rho_2.$$

But now recalling that $0 < C_0 \leq 1$ and $\varphi(\mathbf{1}) \geq C_0$, we deduce that

$$\rho_2 \leq \frac{1}{C_0}\rho_2 \leq \frac{\varphi(\mathbf{1})}{C_0^2}\rho_2.$$

Since the condition $\rho_1 < \rho_2$ was assumed in the statement of the theorem, we conclude that condition (2) is meaningful.

The first part of the proof is identical to the first part of the proof of Theorem 2.13. On the other hand, we next claim that for each $u \in \partial\widehat{W}_{\rho_2}$ it follows that $\mu u \neq Tu$ for each $\mu \geq 1$. So, for contradiction suppose not. Then there is $u \in \partial\widehat{W}_{\rho_2}$ and $\mu \geq 1$ such that $\mu u(t) = (Tu)(t)$ for each $t \in [0,1]$. Recall that since $u \in \partial\widehat{W}_{\rho_2}$ it follows that

$$\varphi(u) = \rho_2.$$

Therefore, integrating from $t = 0$ to $t = 1$ both sides of $\mu u(t) = (Tu)(t)$ against $\mathrm{d}\alpha(t)$ we deduce the following estimate:

$$\begin{aligned}
\rho_2 &= \varphi(u)\\
&\leq \mu \int_0^1 u(t)\ \mathrm{d}\alpha(t)\\
&= \int_0^1 \underbrace{\left[\gamma(t)H\big(\varphi(u)\big) + \lambda\int_0^1 \left(A\left(\int_0^1 \big(u(\xi)\big)^q\ \mathrm{d}\xi\right)\right)^{-1} G(t,s)f\big(s,u(s)\big)\ \mathrm{d}s\right]}_{=(Tu)(t)}\ \mathrm{d}\alpha(t)\\
&= H(\rho_2)\,\varphi(\gamma)\\
&\quad + \lambda\int_0^1\int_0^1 \left(A\left(\int_0^1 \big(u(\xi)\big)^q\ \mathrm{d}\xi\right)\right)^{-1} G(t,s)f\big(s,u(s)\big)\ \mathrm{d}s\ \mathrm{d}\alpha(t)\\
&\leq H(\rho_2)\,\varphi(\gamma)\\
&\quad + \lambda\int_0^1\int_0^1 \left(A\left(\left(\frac{\eta_0\rho_2}{\varphi(\mathbf{1})}\right)^q (b-a)\right)\right)^{-1} G(t,s)f\big(s,u(s)\big)\ \mathrm{d}s\ \mathrm{d}\alpha(t)\\
&\leq H(\rho_2)\,\varphi(\gamma)\\
&\quad + \lambda\left(A\left(\left(\frac{\eta_0\rho_2}{\varphi(\mathbf{1})}\right)^q (b-a)\right)\right)^{-1} f^M_{[0,1]\times\left[0,\frac{\rho_2}{C_0}\right]}\int_0^1\int_0^1 G(t,s)\ \mathrm{d}s\ \mathrm{d}\alpha(t)\\
&< \rho_2,
\end{aligned} \tag{2.19}$$

using that

$$u \in \partial\widehat{W}_{\rho_2} \Longrightarrow \|u\| \leq \frac{\rho_2}{C_0} \Longrightarrow f\big(s,u(s)\big) \leq f^M_{[0,1]\times\left[0,\frac{\rho_2}{C_0}\right]},\ s \in [0,1].$$

Note that to obtain the estimate $\left(A\left(\int_0^1 \big(u(\xi)\big)^q\ \mathrm{d}\xi\right)\right)^{-1} \leq \left(A\left(\left(\frac{\eta_0\rho_2}{\varphi(\mathbf{1})}\right)^q (b-a)\right)\right)^{-1}$, which is used in inequality (2.19), we have used Lemma 2.11. Since inequality (2.19) is a contradiction, we conclude from Lemma 2.12 that $i_{\mathscr{K}}\left(T,\widehat{W}_{\rho_2}\right) = 1$. Then just as in the proof of Theorem 2.13 we deduce from Lemma 2.12 the existence of at least one positive solution $u_0 \in \widehat{W}_{\rho_2}\setminus\overline{\widehat{W}_{\rho_1}} \neq \varnothing$ to integral equation (1.1). □

We conclude with an example and a remark.

Example 2.17. We will demonstrate the application of Corollary 2.14 to a problem of the form (1.3). In particular, suppose that we choose $A(t) := \sin t$, $\varphi(u) := \frac{1}{2}u\left(\frac{1}{3}\right) + \frac{1}{50}u\left(\frac{1}{10}\right)$, $\gamma(t) := 1-t$, $H(t) := \frac{9}{10}\sqrt{t}$, and $q := 2$ so that (1.3) becomes

$$\begin{aligned}-\sin\left(\int_0^1 \big(u(s)\big)^2\ \mathrm{d}s\right)u''(t) &= \lambda f\big(t,u(t)\big),\ 0<t<1\\ u(0) &= \frac{9}{10}\sqrt{\frac{1}{2}u\left(\frac{1}{3}\right)+\frac{1}{50}u\left(\frac{1}{10}\right)}\\ u(1) &= 0.\end{aligned} \tag{2.20}$$

Note that since φ is a multipoint-type nonlocal element with positive coefficients, it follows that the Stieltjes integrator, α, associated to it will be monotone increasing.

For the Green's function associated to the Dirichlet problem it is know that $\eta_0 = \min\{a, 1-b\}$. If we choose here $a := \frac{1}{4}$ and $b := \frac{3}{4}$, then $\eta_0 = \frac{1}{4}$. In addition, we calculate the following.

$$\begin{aligned}\int_0^1 \gamma(t)\ \mathrm{d}t &= \frac{1}{2}\\ \varphi(\mathbf{1}) &= \frac{1}{2}+\frac{1}{50} = \frac{13}{25}\\ \varphi(\gamma) &= \frac{1}{2}\left[1-\frac{1}{2}\right]+\frac{1}{50}\left[1-\frac{1}{10}\right] = \frac{67}{250}.\end{aligned}$$

At the same time since $\mathscr{G}(s) = s(1-s)$ we also calculate

$$\begin{aligned}\inf_{s\in(0,1)} \frac{1}{s(1-s)}\int_0^1 G(t,s)\ \mathrm{d}t &= \inf_{s\in(0,1)} \frac{1}{s(1-s)}\underbrace{\left[\int_0^s t(1-s)\ \mathrm{d}t + \int_s^1 s(1-t)\ \mathrm{d}t\right]}_{\frac{1}{2}s(1-s)}\\ &= \frac{1}{2}\end{aligned}$$

and

$$\begin{aligned}&\inf_{s\in(0,1)} \frac{1}{s(1-s)}\int_0^1 G(t,s)\ \mathrm{d}\alpha(t)\\ &= \inf_{s\in(0,1)} \frac{1}{s(1-s)}\left[\frac{1}{2}G\left(\frac{1}{3},s\right)+\frac{1}{50}G\left(\frac{1}{10},s\right)\right]\\ &= \inf_{s\in(0,1)} \frac{1}{s(1-s)}\begin{cases}\frac{1}{3}s+\frac{9}{500}s, & 0<s<\frac{1}{10}\\ \frac{1}{3}s+\frac{1}{500}(1-s), & \frac{1}{10}\le s<\frac{1}{3}\\ \frac{1}{6}(1-s)+\frac{1}{500}(1-s), & \frac{1}{3}\le s<1\end{cases}\\ &= \inf_{s\in(0,1)} \begin{cases}\frac{527}{1500(1-s)}, & 0<s<\frac{1}{10}\\ \frac{497s+3}{1500s(1-s)}, & \frac{1}{10}\le s<\frac{1}{3}\\ \frac{253}{1500s}, & \frac{1}{3}\le s<1\end{cases}\\ &= \frac{253}{1500}.\end{aligned}$$

Therefore, we conclude that

$$C_0 := \min\left\{\frac{1}{2}, \frac{13}{25}, \frac{67}{250}, \frac{1}{2}, \frac{253}{1500}\right\} = \frac{253}{1500}.$$

With these preliminary calculations completed we now examine conditions (1)–(3) in the statement of Corollary 2.14. Note that

$$\left[\left(\frac{\rho_1 C_0}{\varphi(\mathbf{1})}\right)^q, \rho_2\right] = \left[\left(\frac{253}{780}\right)^2 \rho_1^2, \rho_2\right] \approx \left[0.105\rho_1^2, \rho_2\right].$$

Now choose

$$\rho_1 := \frac{1}{20} \quad \text{and} \quad \rho_2 := \frac{\pi}{2}.$$

Then $A(t) = \sin t > 0$ on $\left[\left(\frac{253}{780}\right)^2 \rho_1^2, \rho_2\right]$. So, condition (1) of Corollary 2.14 is satisfied. Moreover, since

$$\rho_1 = \frac{1}{20} < \frac{253}{1500}\sqrt{\frac{\pi}{2}} = C_0\rho_2^{\frac{1}{2}},$$

it follows that the condition $\rho_1 < C_0\rho_2^{\frac{1}{q}}$ is also satisfied. In addition, condition (2) of the corollary is satisfied since

$$\frac{H(\rho_1)}{\rho_1} = \frac{H\left(\frac{1}{20}\right)}{\frac{1}{20}} = 18\sqrt{\frac{1}{20}} > \frac{250}{67} = \frac{1}{\varphi(\gamma)}.$$

Now suppose that both f and λ satisfies the inequality

$$\int_0^1 \left[\frac{9}{10}\sqrt{\frac{780}{253}\sqrt{\frac{\pi}{2}}}(1-t) + \lambda f^M_{[0,1]\times\left[0,\frac{1500}{253}\sqrt{\frac{\pi}{2}}\right]} \int_0^1 G(t,s)\, ds\right]^2 dt < \sqrt{\frac{\pi}{2}}. \tag{2.21}$$

Then condition (3) of the corollary will be satisfied. Therefore, provided that f and λ are such that inequality (2.21) holds, then by Corollary 2.14 problem (2.20) has at least one positive solution, u_0, satisfying the localization $u_0 \in \widehat{V}^2_{\frac{\pi}{2}} \setminus \overline{\widehat{W}_{\frac{1}{20}}}$.

Remark 2.18. Note in Example 2.17 that the coefficient function A is not nonnegative on $\mathbb{R}$—nor is it strictly positive on $\mathbb{R}$. It is also not monotone on $\mathbb{R}$. As explained in Sect. 1 these are all typical conditions on nonlocal functions in the existing literature. By using the nonstandard cone $\mathscr{K}$ together with the nonstandard open sets $\widehat{V}^q_\rho$ and $\widehat{W}_\rho$ we are able to avoid those more restrictive conditions on the coefficient function A. Yet at the same time we are able to recover the pointwise-type conditions on the coefficient A as well as the function H in the nonlocal boundary condition. And in this way, as explained in Sect. 1, we are able to merge the good features of the different methodologies in [27–29,32,36].

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