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Fixed point theorems for asymptotically regular semigroups equipped with generalized Lipschitzian conditions in metric spaces

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Abstract. In this paper, we prove some common fixed point theorems for one-parameter semigroups of asymptotically regular mappings which satisfy certain generalized Lipschitzian conditions in metric spaces. Our results do not assume the continuity of the mappings in the semigroups. The results extend some relevant common fixed point theorems in Górnicki (Colloq Math 64:55–57, 1993), Imdad and Soliman (Fixed Point Theory Appl 2010:692401, 2010), Imdad and Soliman (Bull Malays Math Sci Soc 2(35):687–694, 2012), Yao and Zeng (J Nonlinear Convex Anal 8:153–163, 2007).

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1. Introduction

Let (X, d) be a metric space. A self-mapping T on X is said to be Lipschitzian if for any $n \geq 1$, there exists $k_n \geq 0$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y) \tag{1.1}$$

for all $x, y \in X$. A Lipschitzian mapping T is called a k-uniformly Lipschitzian (or, uniformly Lipschitzian) mapping (see, [17] and Section 8 of [18]) if $k_n \leq k$ for all $n \geq 1$. In particular, T is said to be nonexpansive if $k \leq 1$ and contraction if $k < 1$. On the other hand, a self-mapping T on X is said to be asymptotically regular (see, [5]) if

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0$$

for all $x \in X$.

In the 1970s, a relation between the concepts of nonexpansive mappings and asymptotically regular mappings (i.e., the concept of average mappings) has been used to approximate fixed points of nonexpansive mappings in normed linear spaces (see, [3,15,26]). Since then, many researchers have studied the existence of common fixed points of semigroups of asymptotically regular mappings equipped with some Lipschitzian conditions in Banach spaces and metric spaces (see, [7,11–13,20,21,44,45] and the references therein). In particular, Górnicki [20] and Yao and Zeng [45] utilized geometrical constants of metric spaces, namely the Lifšic constant $\kappa(X)$ and the normal structure constant $\widetilde{N}(X)$.

Clearly, any Lipschitzian mapping T on X is continuous. Also, any contraction mapping T is asymptotically regular. But, in a more general situation, the class of nonexpansive mappings (or, more generally, the class of continuous mappings) and the class of asymptotically regular mappings are independent (see, e.g., Example 2.2 and Example 2.3 of [23], and Example 5.3 of [42]). Moreover, any continuous asymptotically regular mapping T on X always has a fixed point, provided the Picard iteration $\{T^n x_0\}$ is convergent (see, Proposition 1 of [15]).

Motivated by this fact, we investigate the existence of fixed points of an asymptotically regular mapping, of which the iterations are not necessarily continuous and it satisfies a certain condition that generalizes condition (1.1). Precisely, in Sect. 3, we prove some common fixed point theorems for (one-parameter) semigroups of asymptotically regular mappings which satisfy certain generalized Lipschitzian conditions (see, Definition 2.1 and Definition 2.2) in metric spaces. We utilize the normal structure constant $\widetilde{N}(X)$ and the Lifšic constant $\kappa(X)$ in Theorems 3.1 and 3.6, respectively. Those theorems extend some common fixed point theorems for semigroups of asymptotically regular mappings under condition (1.1), i.e., Theorem 3.2 of [45] and the theorem on page 55 of [20]. Using a technique that is different from Theorem 3.2 of [24] and Theorem 3.1 and Theorem 3.2 of [25], Theorem 3.1 extends those theorems to a more general class of semigroups, of which the mappings are not necessarily continuous.

2. Preliminaries

In this section, we discuss some preliminary concepts which are needed in the next section.

Let (X, d) be a metric space. The following concepts related to the normal structure constant $\widetilde{N}(X)$. Let M be a nonempty bounded subset of X. We shall use the following notations.

$$\begin{aligned} r(x, M) &= \sup\{d(x, y) : y \in M\}, \quad x \in X, \\ \delta(M) &= \sup\{r(x, M) : x \in M\}, \\ R(M) &= \inf\{r(x, M) : x \in M\}. \end{aligned}$$

We note that $\delta(M) = diam(M)$, i.e. the diameter of M.

The following concept of a convexity structure is contained in [39] and it has been developed by Khamsi [27]. Let $\mathcal{F}$ be a nonempty family of subsets of X. We say that $\mathcal{F}$ defines a convexity structure on X if for each nonempty family $\{M_\alpha : \alpha \in I\} \subseteq \mathcal{F}$, one has

$$\bigcap_{\alpha \in I} M_\alpha \in \mathcal{F}.$$

Moreover, a subset of X is said to be admissible (see, page 459 of [14]) if it can be expressed as an intersection of closed balls. We denote $\mathcal{A}(X)$ as the family of all admissible subsets of X. Obviously, $\mathcal{A}(X)$ defines a convexity structure on X. In this work, we assume that any other convexity structure defined by $\mathcal{F}$ on X always contains $\mathcal{A}(X)$.

For any bounded subset M of X, we denote $cov(M)$ as the intersection of all closed balls in X which contain M. Thus,

$$M \in \mathcal{A}(X) \Longleftrightarrow M = cov(M).$$

The notation $ad(M)$ is usually used to express the admissible hull of M. It is defined as the intersection of all those admissible subsets of X which contain M. Actually, it is easy to see that $ad(M) = cov(M)$.

The following concept of a uniform normal structure is contained in [27] and it is the metric space version of a concept due to Gillespie and Williams [16]. A metric space (X, d) is said to have normal (resp. uniform normal) structure (see, Definition 6 of [27]) if there exists a convexity structure defined by $\mathcal{F}$ on X such that $R(M) < \delta(M)$ (resp. $R(M) \le c \cdot \delta(M)$ for some $c \in (0, 1)$) for all $M \in \mathcal{F}$ which is bounded and $\delta(M) > 0$. In this case, $\mathcal{F}$ is said to be normal (resp. uniformly normal).

The following notion is contained in [35] and it is the metric space version of a notion due to Bynum [8]. The normal structure constant $\widetilde{N}(X)$ of X (with respect to a given convexity structure by $\mathcal{F}$) (see, page 1232 of [35]) is defined as the real number

$$\sup\left\{\frac{R(M)}{\delta(M)} : M \in \mathcal{F} \text{ and } M \text{ is bounded with } \delta(M) > 0\right\}.$$

We note that $\delta(M) \le 2R(M)$ for any nonempty bounded subset M of X. Thus, $\frac{1}{2} \le \widetilde{N}(X) \le 1$ for any metric space X. In particular, X has uniform normal structure if and only if $\frac{1}{2} \le \widetilde{N}(X) < 1$. It is well known that for a hyperconvex metric space X, $\widetilde{N}(X) = \frac{1}{2}$ (with respect to a convexity structure defined by $\mathcal{A}(X)$) because $\delta(M) = 2R(M)$ for all $M \in \mathcal{A}(X)$ with $\delta(A) > 0$. These values also apply in any $\mathbb{R}$-tree (see, Proposition 3.1 of [30]). For more detailed discussions related to the concept of normal structure constants and their values, see [8,29,31,32,36] and the references therein.

For a metric space (X, d) we shall denote the closed ball of center $x \in X$ and radius $r > 0$ by $B(x, r)$. The Lifšic constant $\kappa(X)$ of X (see, [34]) is defined as the real number

$$\sup\left\{b>0: \begin{array}{c}\exists a>1 \text{ such that } \forall x,y\in X \text{ and } \forall r>0, d(x,y)>r\\ \Rightarrow \exists z\in X \text{ such that } B(x,br)\cap B(y,ar)\subseteq B(z,r)\end{array}\right\}. \quad (2.1)$$

It is clear that $\kappa(X)\geq 1$ for any metric space X.

We note that the exact value of $\kappa(X)$ when X is an infinite-dimensional Hilbert space, is given by $\kappa(X)=\sqrt{2}$ (see, Remark 1 of [43]), when X is a nonreflexive Banach space, is given by $\kappa(X)=1$ (see, Section 37.5 of [33]), and when X is an $\mathbb{R}$-tree, is given by $\kappa(X)=2$ (see, Theorem 5 of [10] and Theorem 3.16 of [1]). Moreover, if X is a Hilbert space (or, more generally, a complete CAT(0) space), then $\kappa(X)\geq\sqrt{2}$ (see, e.g., Remark 1 of [43] and Theorem 5 of [10]). For a more detailed discussion about the concept of the Lifšic constant $\kappa(X)$ and related constants in Banach spaces, see [2,13,43] and the references therein.

From Lemma 1 of [43] we see that

$$R(M)\kappa(M)\leq\delta(M) \quad (2.2)$$

for any bounded subspace M of X. If X is a singleton, then the set in (2.1) is not bounded above, and also any mapping on X is the identity mapping. By considering these facts, for the rest of this work we should assume that X is not a singleton. Now, from (2.2) we can state that $1\leq\kappa(X)\leq 2$ for any bounded metric space X.

Let us denote $N(X)=\frac{1}{\widetilde{N}(X)}$, and let $\bar{N}(X)$ be a real number defined by

$$\overline{N}(X)=\inf\left\{\frac{\delta(M)}{r(M,X)}: M\in\mathcal{F} \text{ and } M \text{ is bounded with } \delta(M)>0\right\},$$

where $\mathcal{F}$ defines the same convexity structure as for $\widetilde{N}(X)$ and $r(M,X)=\inf\{r(x,M): x\in X\}$. Using Lemma 1 of [43], we see that

$$\max\{\kappa(X),N(X)\}\leq\overline{N}(X)$$

for any bounded metric space X.

We now turn to the discussion related to a semigroup of mappings for which, in the next section, we investigate the existence of the common fixed points. Let (X,d) be a metric space. Let G be an unbounded subset of $[0,\infty)$ such that $t+s, t-s\in G$ for all $s,t\in G$ with $t\geq s$ (often, $G=[0,\infty)$ or $G=\mathbb{N}\cup\{0\}$). A family of mappings $\mathcal{T}=\{T_t: t\in G\}$ from X into itself is said to be a (one-parameter) semigroup on X if for any $s,t\in G$ and $x\in X$ we have $T_{s+t}x=T_sT_tx$ and $T_0x=x$. A semigroup $\mathcal{T}=\{T_t: t\in G\}$ on X is said to be asymptotically regular at a point $x\in X$ if

$$\lim_{t\to\infty} d(T_{t+h}x, T_tx)=0$$

for all $h\in G$. If $\mathcal{T}$ is asymptotically regular at each $x\in X$, then $\mathcal{T}$ is said asymptotically regular on X.

Let $\mathcal{T}_1=\{T_t: t\in G\}$ be a semigroup on X. Assume that any T_t is a Zamfirescu mapping (see, [46]). Then for any $x,y\in X$, at least one of the following conditions is satisfied.

(i) $d(T_tx,T_ty)\leq a_td(x,y)$,

(ii) $d(T_tx, T_ty) \le \frac{b_t}{2}(d(x, T_tx) + d(y, T_ty))$,
(iii) $d(T_tx, T_ty) \le \frac{c_t}{2}(d(x, T_ty) + d(y, T_tx))$,

where $0 \le a_t, b_t, c_t < 1$. We now consider a semigroup $\mathcal{T}_2 = \{T_t : t \in G\}$ on X such that any mapping T_t satisfies at least one of conditions (i)–(iii) by relaxing the assumption of the constants a_t, b_t, and c_t to be $0 \le a_t, b_t, c_t < \infty$. Then T_t is not necessarily a Lipschitzian mapping.

In the next section, we investigate the existence of common fixed points of an asymptotically regular semigroup on X together with a semigroup $\mathcal{T}_3 = \{T_t : t \in G\} \subseteq \mathcal{T}_2$, i.e., for any $t \in G$, there exists at least one of the constants $a_t, b_t, c_t \ge 0$ such that

(i′) $d(T_tx, T_ty) \le a_t d(x, y)$,
(ii′) $d(T_tx, T_ty) \le \frac{b_t}{2}(d(x, T_tx) + d(y, T_ty))$,
(iii′) $d(T_tx, T_ty) \le \frac{c_t}{2}\max\{d(x, T_ty), d(y, T_tx)\}$,

where $x, y \in X$. In Example 2.5, we give a simple example of a semigroup of mappings which satisfies conditions (i′)–(iii′), but the mappings are not continuous.

Motivated by the work of Rhoades [41], we may restate the semigroup $\mathcal{T}_3$ as follows.

Definition 2.1. Let (X, d) be a metric space. A semigroup $\mathcal{T} = \{T_t : t \in G\}$ on X is said to be generalized Lipschitzian type-1 if for any $t \in G$, there exists $k_t \ge 0$ such that

$$d(T_tx, T_ty) \le k_t \max\left\{d(x, y), \frac{1}{2}(d(x, T_tx) + d(y, T_ty)), \frac{1}{2}d(x, T_ty), \frac{1}{2}d(y, T_tx)\right\} \tag{2.3}$$

for all $x, y \in X$.

From conditions (i′) and (iii′) on the definition of the semigroup $\mathcal{T}_3$, we also consider the following generalized Lipschitzian type semigroup.

Definition 2.2. Let (X, d) be a metric space. A semigroup $\mathcal{T} = \{T_t : t \in G\}$ on X is said to be generalized Lipschitzian type-2 if for any $t \in G$, there exists $k_t \ge 0$ such that

$$d(T_tx, T_ty) \le k_t \max\left\{d(x, y), \frac{1}{2}d(x, T_ty), \frac{1}{2}d(y, T_tx)\right\} \tag{2.4}$$

for all $x, y \in X$.

Remark 2.3. We see that every generalized Lipschitzian semigroup type-1 gives relatively sharper constants than generalized Lipschitzian semigroup type-2. To see this, let (X, d) be a metric space and $\mathcal{T} = \{T_t : t \in G\}$ be a generalized Lipschitzian semigroup type-2. We use notations ${k_t}'$ and ${k_t}''$ to denote the infimum of constants k_t in inequalities (2.3) and (2.4), respectively.

We have

$$\begin{aligned} &d(T_tx, T_ty) \\ &\quad \le {k_t}'' \max\left\{d(x,y), \frac{1}{2}d(x,T_ty), \frac{1}{2}d(y,T_tx)\right\} \\ &\quad \le {k_t}'' \max\left\{d(x,y), \frac{1}{2}(d(x,T_tx)+d(y,T_ty)), \frac{1}{2}d(x,T_ty), \frac{1}{2}d(y,T_tx)\right\} \end{aligned}$$

for any $x, y \in X$. From the definition of the constant ${k_t}'$, we obtain ${k_t}' \le {k_t}''$. Moreover, for a certain case we have ${k_{t_0}}' < {k_{t_0}}''$ for some $t_0 \in G$. Indeed, let $X = [0,1]$ be a subspace of the usual metric space $\mathbb{R}$. Let $G = \mathbb{N} \cup \{0\}$ and $\mathcal{T} = \{T_n : n \in G\}$ be a semigroup of mappings on X, where $T_0 = I_X$ is the identity mapping and $T_n = T^n$ is the n$^{\text{th}}$ iteration of a mapping given by

$$Tx = \begin{cases} 1 & \text{if } x = 0 \\ \frac{2}{3} & \text{if } x \in (0,1]. \end{cases}$$

We note that $T_nx = \frac{2}{3}$ for all $x \in X$ and $n > 1$, and $|Tx - Ty| = 0$ for all $x, y \in (0,1]$. It is easy to see that

$$\begin{aligned} &{k_1}'' \\ &\quad = \sup\left\{\frac{|Tx-Ty|}{\max\left\{|x-y|, \frac{1}{2}|x-Ty|, \frac{1}{2}|y-Tx|\right\}} : x, y \in [0,1], x \ne y\right\}. \end{aligned} \tag{2.5}$$

From (2.5) and the definition of T we get

$$\begin{aligned} {k_1}'' &= \sup_{y\in(0,1]} \frac{|T0-Ty|}{\max\left\{|0-y|, \frac{1}{2}|T0-Ty|, \frac{1}{2}|y-T0|\right\}} \\ &\le \frac{\frac{1}{3}}{\max\left\{\inf_{y\in(0,1]} y, \frac{1}{3}, \frac{1}{2}\inf_{y\in(0,1]}(1-y)\right\}} = 1. \end{aligned}$$

There are also $x, y \in [0,1]$, i.e., $x = 0$ and $y = \frac{1}{3}$ which satisfy

$$\frac{|Tx-Ty|}{\max\left\{|x-y|, \frac{1}{2}|x-Ty|, \frac{1}{2}|y-Tx|\right\}} = 1.$$

Thus, $\mathcal{T}$ is a generalized Lipschitzian semigroup type-2 with the constants ${k_1}'' = 1$ and ${k_n}'' = 0$ for all $n > 1$. On the other hand, similar to the above case, by the definition of T we have

$$
\begin{aligned}
&{k_1}' \\
&= \sup\left\{\frac{|Tx-Ty|}{\max\left\{|x-y|, \frac{1}{2}(|x-Tx|+|y-Ty|), \frac{1}{2}|x-Ty|, \frac{1}{2}|y-Tx|\right\}}\right. \\
&\qquad \left. : x, y \in [0,1], x \neq y\right\} \\
&= \sup_{y\in(0,1]} \frac{|T0-Ty|}{\max\left\{|0-y|, \frac{1}{2}(|0-T0|+|y-Ty|), \frac{1}{2}|0-Ty|, \frac{1}{2}|y-T0|\right\}} \\
&\leq \frac{\frac{1}{3}}{\max\left\{\inf_{y\in(0,1]} y, \frac{1}{2}\inf_{y\in(0,1]}\left(1+\left|y-\frac{2}{3}\right|\right), \frac{1}{3}, \frac{1}{2}\inf_{y\in(0,1]}(1-y)\right\}} = \frac{2}{3}.
\end{aligned}
$$

Since we can find $x, y \in X$, i.e., $x = 0$ and $y = \frac{2}{3}$ such that

$$
\frac{|Tx-Ty|}{\max\left\{|x-y|, \frac{1}{2}(|x-Tx|+|y-Ty|), \frac{1}{2}|x-Ty|, \frac{1}{2}|y-Tx|\right\}} = \frac{2}{3},
$$

then $\mathcal{T}$ is also a generalized Lipschitzian semigroup type-1 with the constants ${k_1}' = \frac{2}{3}$ and ${k_n}' = 0$ for all $n > 1$. From this observation, we conclude that ${k_1}' < {k_1}''$.

Remark 2.4. In the framework of Banach spaces, the study related to the existence of common fixed points of a certain generalized Lipschitzian semigroups which is slightly different from Definition 2.1 has been discussed in [37,38]. Razani and Goodarzi [40] have also investigated the problem of finding a common fixed point of a certain generalized Lipschitzian semigroup, i.e., the semigroup $\mathcal{T} = \{T^n : n \in \mathbb{N} \cup \{0\}\}$ on a nonempty closed convex subset of a Banach space, where T^n is n^{th} iteration of a quasi-contraction mapping T (see, [9]) with the constants $k_n = k < \frac{1}{2}$.

Example 2.5. Let $X = [0,1]$ be a subspace of the usual metric space $\mathbb{R}$ and $G = \mathbb{N} \cup \{0\}$. We define a semigroup $\mathcal{T} = \{T_n : n \in G\}$ of mappings on X as the semigroup of iterations of a mapping given by

$$
Tx = \begin{cases} \gamma x & \text{if } x \in \left[0, \frac{2}{3}\right] \\ 1-x & \text{if } x \in \left(\frac{2}{3}, 1\right], \end{cases}
$$

where $\gamma \in \left(0, \frac{1}{2}\right)$, and $T_0 = I_X$ is the identity mapping. It is clear that for any $n \geq 1$, T_n is not continuous at $\frac{2}{3}$. Let $n \geq 1$ be fixed. Then,

$$
|T_n x - T_n y| = \gamma^n |x-y| \quad \forall x, y \in \left[0, \frac{2}{3}\right]
$$

and

$$
|T_n x - T_n y| = \gamma^{n-1} |x-y| \quad \forall x, y \in \left(\frac{2}{3}, 1\right].
$$

If $x \in \left[0, \frac{2}{3}\right]$ and $y \in \left(\frac{2}{3}, 1\right]$, then

$$
|T_n x - T_n y| = \gamma^{n-1} |\gamma x - (1-y)|. \tag{2.6}
$$

Assume that $\gamma x \leq 1 - y$. From (2.6), we have

$$|T_n x - T_n y| = \gamma^{n-1}(1 - y - \gamma x) < \gamma^{n-1}\left(\frac{1}{2}y - \gamma x\right)$$
$$\leq \gamma^{n-1}\frac{1}{2}(y - \gamma^n x) = \gamma^{n-1}\frac{1}{2}(y - T_n x).$$

For the case $\gamma x > 1 - y$, from (2.6), we have

$$|T_n x - T_n y| = \gamma^{n-1}(\gamma x - (1 - y)) < \gamma^{n-1}\left(\frac{1}{2}x - (1 - y)\right)$$
$$\leq \gamma^{n-1}\frac{1}{2}(x - \gamma^{n-1}(1 - y)) = \gamma^{n-1}\frac{1}{2}(x - T_n y).$$

From this observation, we obtain

$$|T_n x - T_n y| \leq \gamma^{n-1}\max\left\{|x - y|, \frac{1}{2}|x - T_n y|, \frac{1}{2}|y - T_n x|\right\}.$$

Thus, $\mathcal{T}$ is a generalized Lipschitzian semigroup type-2 on X. Moreover, we claim that $\mathcal{T}$ is asymptotically regular on X. Indeed, for any $h \geq 0$, we have

$$\lim_{n\to\infty} |T_{h+n}x - T_n x| = \lim_{n\to\infty} \gamma^n|\gamma^h x - x| = 0$$

for all $x \in \left[0, \frac{2}{3}\right]$, and

$$\lim_{n\to\infty} |T_{h+n}x - T_n x| = \lim_{n\to\infty} \gamma^{n-1}|\gamma^h(1 - x) - (1 - x)| = 0$$

for all $x \in \left(\frac{2}{3}, 1\right]$.

Let (X, d) be a metric space and $\mathcal{T} = \{T_t : t \in G\}$ be a semigroup on X. For the rest of this work, we denote $\omega(\infty)$ as the set:

$$\omega(\infty) = \{\{t_n\} : \{t_n\} \text{ in } G \text{ such that } \{t_n\} \text{ increases monotonically to } \infty\}.$$

We give an important property of asymptotically regular and generalized Lipschitzian semigroups in the following lemma. This lemma is a slight modification of Lemma 3.1 of [37] and thus, we do not write the proof as it is an analogue.

Lemma 2.6. *Let (X, d) be a metric space and $\mathcal{T} = \{T_t : t \in G\}$ be an asymptotically regular semigroup on X. Suppose that $\mathcal{T}$ is a generalized Lipschitzian semigroup type-1 on X such that*

$$\lim_{n\to\infty} k_{t_n} < 2,$$

where $T_{t_n}u \to v$ as $n \to \infty$ for some $u, v \in X$ and $\{t_n\} \in \omega(\infty)$. Then, $T_t v = v$ for all $t \in G$.

We next discuss some concepts from the works of Lim and Xu [35] and Yao and Zeng [45].

Definition 2.7. [35, Definition 5] A metric space (X, d) is said to have property (P) if for any two bounded sequences $\{x_n\}$ and $\{z_n\}$ in X, one can find $z \in \cap_{n\geq 1} ad(\{z_j : j \geq n\})$ such that

$$\limsup_{n\to\infty} d(z, x_n) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} d(z_j, x_n).$$

Remark 2.8. Lim and Xu noted that if X is a weakly compact convex subset of a normed linear space, then X has property (P) (see, the remark on page 1233 of [35]). We know that any compact metric space also has property (P). Another case of a metric space which has property (P) is a complete CAT(0) space (see, Theorem 8 of [10]).

The following concept of a property $(*)$ for semigroups was introduced by Yao and Zeng [45]. The trivial case is, if a bounded metric space has property (P) then any semigroup on it possess this property.

Definition 2.9. [45, Definition 2.4] Let (X, d) be a metric space and $\mathcal{T} = \{T_t : t \in G\}$ be a semigroup on X. A semigroup $\mathcal{T}$ is said to have property $(*)$ if for each $x \in X$ and $\{t_n\} \in \omega(\infty)$, the following conditions are satisfied:

(i) the sequence $\{T_{t_n}x\}$ is bounded,
(ii) for any sequence $\{z_n\}$ in $ad(\{T_{t_n}x : n \geq 1\})$, there exists $z \in \cap_{n\geq 1} ad(\{z_j : j \geq n\})$ such that

$$\limsup_{n\to\infty} d(z, T_{t_n}x) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} d(z_j, T_{t_n}x).$$

From the proof of Theorem 3.2 of [45] we formulate the following lemma.

Lemma 2.10. [45, page 159] *Let $\{a_t\}_{t\in G}$ be a net of real numbers. Suppose that $\liminf_{t\to\infty} a_t = a \in \mathbb{R}$. Then there exists a sequence $\{t_n\} \in \omega(\infty)$ such that $\{a_{t_n}\}$ converges to a and $\{t_{n+1} - t_n\} \in \omega(\infty)$. Moreover,*

$$\{a_{t_i - t_j} : i > j \geq n\} \subseteq \{a_t : t_{n+1} - t_n \leq t \in G\}$$

for all $n \geq 1$.

Lastly, we mention a lemma, which is important to the proof of Theorem 3.1.

Lemma 2.11. [45, Lemma 3.1] *Let (X, d) be a complete metric space with uniform normal structure and $\mathcal{T} = \{T_t : t \in G\}$ be a semigroup on X with property $(*)$. Then, for each $x \in X$, each $\{t_n\} \in \omega(\infty)$, and for any $\bar{c} > \widetilde{N}(X)$ ($\widetilde{N}(X)$ is the normal structure constant with respect to a given convexity structure by $\mathcal{F}$), there exists $z \in X$ satisfying the following properties:*

(i) $\limsup_{n\to\infty} d(z, T_{t_n}x) \leq \bar{c} \lim_{n\to\infty} diam(\{T_{t_j}x : j \geq n\})$,
(ii) $d(z, y) \leq \limsup_{n\to\infty} d(T_{t_n}x, y)$ *for all $y \in X$.*

Remark 2.12. Yao and Zeng [45] actually stated Lemma 2.11 for the semigroup $\mathcal{T} = \{T_t : t \in G\}$, where T_t is continuous for all $t \in G$. But we see from their proof that the continuity condition can be removed.

3. Main results

We now prove our first common fixed point result, which uses the normal structure constant $\widetilde{N}(X)$. Our result is more general than Theorem 3.2 of [24] and Theorem 3.1 and Theorem 3.2 of [25]. Moreover, our technique of proof is different and is not derived from the proofs of those theorems.

Theorem 3.1. *Let (X, d) be a complete bounded metric space with uniform normal structure and $\mathcal{T} = \{T_t : t \in G\}$ be an asymptotically regular semigroup on X with property $(*)$. Suppose that $\mathcal{T}$ is a generalized Lipschitzian semigroup type-1 on X such that*

$$\left(\liminf_{t\to\infty} k_t\right)\left(\limsup_{t\to\infty} k_t\right) < \left[\widetilde{N}(X)\right]^{-1}.$$

Then there exists $z \in X$ such that $T_t z = z$ for all $t \in G$.

Proof. Let us write

$$k = \liminf_{t\to\infty} k_t \quad \text{and} \quad \hat{k} = \limsup_{t\to\infty} k_t.$$

We first choose a sequence $\{t_n\} \in \omega(\infty)$ such that $\lim_{n\to\infty} k_{t_n} = k$. If $k < 1$, then from the assumption, there exists $t_{n^*} \in G$ such that $T_{t_{n^*}}$ is a Zamfirescu mapping. Theorem 1 of [46] ensures that there exists a unique $z \in X$ such that $T_{t_{n^*}} z = z$. Following the idea of Bryant [6], we see that $T_t z = z$ for all $t \in G$. For the rest of the proof, we should consider $k \geq 1$.

Note that, since $\widetilde{N}(X) \geq \frac{1}{2}$ then

$$\hat{k} < \frac{1}{\widetilde{N}(X)k} \leq \frac{2}{k} \leq 2.$$

Using Lemma 2.10, we may assume that the sequence $\{t_n\}$ has properties:

$$\{t_{n+1} - t_n\} \in \omega(\infty)$$

and

$$\{k_{t_i - t_j} : i > j \geq n\} \subseteq \{k_t : t_{n+1} - t_n \leq t \in G\}$$

for all $n \geq 1$. Let $x \in X$ be fixed and $\bar{c}$ be a positive number satisfying

$$k\hat{k} < \frac{1}{\bar{c}} < \left[\widetilde{N}(X)\right]^{-1}.$$

Using Lemma 2.11, it is possible to construct a sequence $\{x_m\}$ in X inductively with properties: $x_0 = x$ and for any $m \geq 0$,

(a) $\limsup_{n\to\infty} d(x_{m+1}, T_{t_n} x_m) \leq \bar{c} \lim_{n\to\infty} diam(\{T_{t_j} x_m : j \geq n\})$,
(b) $d(x_{m+1}, y) \leq \limsup_{n\to\infty} d(T_{t_n} x_m, y)$ for all $y \in X$.

Let us write $\eta = \bar{c}k\hat{k} < 1$ and

$$d_m = \limsup_{n\to\infty} d(T_{t_n} x_m, x_{m+1})$$

for all $m \geq 0$.

We shall show that $d_m \leq \eta d_{m-1}$ for all $m \geq 1$. Let $m \geq 1$ be fixed. Observe that for each $i > j \geq 1$,

$$\begin{aligned} & d(T_{t_i}x_m, T_{t_j}x_m) \\ & = d(T_{t_j}x_m, T_{t_j}T_{t_i-t_j}x_m) \\ & \leq k_{t_j} \max\Big\{ d(x_m, T_{t_i-t_j}x_m), \frac{1}{2}\big(d(x_m, T_{t_j}x_m) + d(T_{t_i-t_j}x_m, T_{t_j}T_{t_i-t_j}x_m)\big), \\ & \quad \frac{1}{2}d(x_m, T_{t_j}T_{t_i-t_j}x_m), \frac{1}{2}d(T_{t_i-t_j}x_m, T_{t_j}x_m)\Big\} \\ & = k_{t_j} \max\Big\{ d(x_m, T_{t_i-t_j}x_m), \frac{1}{2}\big(d(x_m, T_{t_j}x_m) + d(x_m, T_{t_i}x_m)\big), \\ & \quad \frac{1}{2}d(T_{t_i-t_j}x_m, T_{t_j}x_m), \frac{1}{2}d(T_{t_i-t_j}x_m, T_{t_i}x_m)\Big\}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \lim_{n\to\infty} \sup_{i>j\geq n} d(T_{t_i}x_m, T_{t_j}x_m) \\ & \quad \leq \left(\lim_{n\to\infty} \sup_{j\geq n} k_{t_j}\right) \max\Big\{ \lim_{n\to\infty} \sup_{i>j\geq n} d(x_m, T_{t_i-t_j}x_m), \\ & \quad \frac{1}{2} \lim_{n\to\infty} \sup_{i>j\geq n} \Big(d(x_m, T_{t_j}x_m) + d(x_m, T_{t_i}x_m)\Big), \\ & \quad \frac{1}{2} \lim_{n\to\infty} \sup_{i>j\geq n} d(T_{t_i-t_j}x_m, T_{t_i}x_m), \frac{1}{2} \lim_{n\to\infty} \sup_{i>j\geq n} d(T_{t_i-t_j}x_m, T_{t_j}x_m)\Big\} \\ & \quad \leq k \max\Big\{ \lim_{n\to\infty} \sup_{i>j\geq n} d(x_m, T_{t_i-t_j}x_m), \qquad (3.1) \\ & \quad \frac{1}{2}\left(\lim_{n\to\infty} \sup_{j\geq n} d(x_m, T_{t_j}x_m) + \lim_{n\to\infty} \sup_{i\geq n} d(x_m, T_{t_i}x_m)\right), \\ & \quad \frac{1}{2} \lim_{n\to\infty} \sup_{i>j\geq n} d(T_{t_i-t_j}x_m, T_{t_i}x_m), \frac{1}{2} \lim_{n\to\infty} \sup_{i>j\geq n} d(T_{t_i-t_j}x_m, T_{t_j}x_m)\Big\} \\ & \quad = k \max\Big\{ \lim_{n\to\infty} \sup_{i>j\geq n} d(x_m, T_{t_i-t_j}x_m), \limsup_{j\to\infty} d(x_m, T_{t_j}x_m), \\ & \quad \frac{1}{2} \lim_{n\to\infty} \sup_{i>j\geq n} d(T_{t_i-t_j}x_m, T_{t_i}x_m), \frac{1}{2} \lim_{n\to\infty} \sup_{i>j\geq n} d(T_{t_i-t_j}x_m, T_{t_j}x_m)\Big\}. \end{aligned}$$

To derive inequality (3.1), we need to calculate each element of the set on the right side of this inequality.

Claim 1

$$\lim_{n\to\infty} \sup_{i>j\geq n} d(x_m, T_{t_i-t_j}x_m) \leq \hat{k} d_{m-1}.$$

Proof of Claim 1 Using (b) we have

$$d(x_m, T_{t_i-t_j}x_m) \leq \limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_i-t_j}x_m). \qquad (3.2)$$

Using the asymptotic regularity of $\mathcal{T}$, observe that for each $i > j \geq 1$,

$$\begin{aligned}&\limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_i-t_j}x_m)\\ &\quad\leq \limsup_{p\to\infty}\big(d(T_{t_p}x_{m-1}, T_{t_p}T_{t_i-t_j}x_{m-1}) + d(T_{t_i-t_j}T_{t_p}x_{m-1}, T_{t_i-t_j}x_m)\big)\\ &\quad\leq k_{t_i-t_j}\max\left\{\limsup_{p\to\infty} d(T_{t_p}x_{m-1}, x_m), \frac{1}{2}d(x_m, T_{t_i-t_j}x_m),\right.\\ &\qquad\left.\frac{1}{2}\limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_i-t_j}x_m), \frac{1}{2}\limsup_{p\to\infty} d(x_m, T_{t_i-t_j}T_{t_p}x_{m-1})\right\}.\end{aligned}$$

Next, using (b) and the asymptotic regularity of $\mathcal{T}$, from the last inequality, we have

$$\begin{aligned}&\limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_i-t_j}x_m)\\ &\quad\leq k_{t_i-t_j}\max\left\{d_{m-1}, \frac{1}{2}\limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_i-t_j}x_m),\right.\\ &\qquad\left.\frac{1}{2}\left(\limsup_{p\to\infty} d(x_m, T_{t_p}x_{m-1}) + \limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_p}T_{t_i-t_j}x_{m-1})\right)\right\}\\ &\quad= k_{t_i-t_j}\max\left\{d_{m-1}, \frac{1}{2}\limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_i-t_j}x_m)\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{n\to\infty}\sup_{i>j\geq n}\limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_i-t_j}x_m)\\ &\quad\leq\left(\lim_{n\to\infty}\sup_{t_{n+1}-t_n\leq t\in G} k_t\right)\\ &\qquad\cdot\max\left\{d_{m-1}, \frac{1}{2}\lim_{n\to\infty}\sup_{i>j\geq n}\limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_i-t_j}x_m)\right\}\\ &\quad= \hat{k}\max\left\{d_{m-1}, \frac{1}{2}\lim_{n\to\infty}\sup_{i>j\geq n}\limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_i-t_j}x_m)\right\}.\end{aligned} \tag{3.3}$$

Let

$$\frac{1}{2}\lim_{n\to\infty}\sup_{i>j\geq n}\limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_i-t_j}x_m)$$

be the maximum. Using the fact that $\hat{k} < 2$, from (3.2), we obtain

$$\lim_{n\to\infty}\sup_{i>j\geq n}\limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_i-t_j}x_m) = 0.$$

Thus, from the assumption we get $d_{m-1} = 0$. Lemma 2.6 ensures x_m as the common fixed point of $\mathcal{T}$. Thus, the only case that has to be verified is the maximum of the set in (3.3) is d_{m-1}. In this case, from (3.2) and (3.3), we obtain

$$\begin{aligned}&\lim_{n\to\infty}\sup_{i>j\geq n} d(x_m, T_{t_i-t_j}x_m)\\&\quad\leq \lim_{n\to\infty}\sup_{i>j\geq n}\limsup_{p\to\infty} d(T_{t_p}x_{m-1}, T_{t_i-t_j}x_m)\\&\quad\leq \hat{k}d_{m-1}.\end{aligned}\tag{3.4}$$

For the calculation of the second element of the set in (3.1), we see that by replacing $T_{t_i-t_j}$ in Claim 1 by T_{t_j} and using a similar argument as in the proof of Claim 1,

$$\limsup_{j\to\infty} d(x_m, T_{t_j}x_m) \leq kd_{m-1} \leq \hat{k}d_{m-1}.\tag{3.5}$$

For the third one, from (3.4) and (3.5), we have

$$\begin{aligned}&\lim_{n\to\infty}\sup_{i>j\geq n} d(T_{t_i-t_j}x_m, T_{t_i}x_m)\\&\quad\leq \lim_{n\to\infty}\sup_{i>j\geq n} d(x_m, T_{t_i-t_j}x_m) + \limsup_{i\to\infty} d(x_m, T_{t_i}x_m)\\&\quad= \lim_{n\to\infty}\sup_{i>j\geq n} d(x_m, T_{t_i-t_j}x_m) + \limsup_{j\to\infty} d(x_m, T_{t_j}x_m)\\&\quad\leq 2\hat{k}d_{m-1}.\end{aligned}\tag{3.6}$$

For the case of the last element, we calculate from (3.4) and (3.5),

$$\begin{aligned}&\lim_{n\to\infty}\sup_{i>j\geq n} d(T_{t_i-t_j}x_m, T_{t_j}x_m)\\&\quad\leq \lim_{n\to\infty}\sup_{i>j\geq n} d(x_m, T_{t_i-t_j}x_m) + \limsup_{j\to\infty} d(x_m, T_{t_j}x_m)\\&\quad\leq 2\hat{k}d_{m-1}.\end{aligned}\tag{3.7}$$

It follows from (3.1), (3.4), (3.5), (3.6), and (3.7) that

$$\lim_{n\to\infty}\sup_{i>j\geq n} d(T_{t_i}x_m, T_{t_j}x_m) \leq k\hat{k}d_{m-1}.\tag{3.8}$$

Using inequality (3.8), condition (a) implies

$$\begin{aligned}d_m &\leq \bar{c}\lim_{n\to\infty} diam(\{T_{t_j}x_m : j\geq n\})\\&= \bar{c}\lim_{n\to\infty}\sup\left\{\{d(T_{t_i}x_m, T_{t_j}x_m) : i>j\geq n\}\cup\{0\}\right\}\\&= \bar{c}\lim_{n\to\infty}\sup_{i>j\geq n} d(T_{t_i}x_m, T_{t_j}x_m)\\&\leq \bar{c}k\hat{k}d_{m-1}.\end{aligned}$$

Thus,

$$d_m \leq \eta d_{m-1}.\tag{3.9}$$

Moreover, since (3.9) holds for all $m \geq 1$, then by an induction one can easily see that

$$d_m \leq \eta^m d_0 \to 0 \quad \text{as } m\to\infty.\tag{3.10}$$

By replacing T_{t_j} in (3.5) by T_{t_q}, we have

$$\begin{aligned} d(x_m, x_{m+1}) &\leq \limsup_{q\to\infty} d(x_m, T_{t_q} x_m) + \limsup_{q\to\infty} d(T_{t_q} x_m, x_{m+1}) \\ &\leq \hat{k} d_{m-1} + d_m \\ &\leq (\hat{k}\eta^{m-1} + \eta^m) d_0 \end{aligned} \tag{3.11}$$

for all $m \geq 1$. From (3.10) and (3.11), we see that $\{x_m\}$ is a Cauchy sequence and thus, it is convergent. Let $z = \lim_{m\to\infty} x_m$. For any $q, m \geq 1$, we have

$$\begin{aligned} &d(z, T_{t_q} z) \\ &\leq d(z, x_m) + d(x_m, T_{t_q} x_m) + d(T_{t_q} x_m, T_{t_q} z) \\ &\leq d(z, x_m) + d(x_m, T_{t_q} x_m) \\ &\quad + k_{t_q} \max\left\{ d(x_m, z), \frac{1}{2}\left(d(x_m, T_{t_q} x_m) + d(z, T_{t_q} z)\right), \right. \\ &\quad \left. \frac{1}{2}\left(d(x_m, z) + d(z, T_{t_q} z)\right), \frac{1}{2}\left(d(x_m, z) + d(x_m, T_{t_q} x_m)\right) \right\}. \end{aligned} \tag{3.12}$$

Using the same argument as in (3.11), we take the limit superior as $q \to \infty$ into (3.12) to obtain

$$\begin{aligned} &\limsup_{q\to\infty} d(z, T_{t_q} z) \\ &\leq d(x_m, z) + \hat{k} d_{m-1} \\ &\quad + \hat{k} \max\left\{ d(x_m, z), \frac{1}{2}\left(\hat{k} d_{m-1} + \limsup_{q\to\infty} d(z, T_{t_q} z)\right), \right. \\ &\quad \left. \frac{1}{2}\left(d(x_m, z) + \limsup_{q\to\infty} d(z, T_{t_q} z)\right), \frac{1}{2}\left(d(x_m, z) + \hat{k} d_{m-1}\right) \right\}. \end{aligned} \tag{3.13}$$

Moreover, by taking the limit as $m \to \infty$ into (3.13) we get

$$\limsup_{q\to\infty} d(z, T_{t_q} z) \leq \frac{\hat{k}}{2} \limsup_{q\to\infty} d(z, T_{t_q} z).$$

Since $\hat{k} < 2$, we conclude that $\lim_{q\to\infty} d(T_{t_q} z, z) = 0$. Finally, Lemma 2.6 ensures $T_t z = z$ for all $t \in G$. □

Remark 3.2. Theorem 3.1 extends Theorem 3.2 of [45], Theorem 3.2 of [24], and Theorem 3.1 and Theorem 3.2 of [25] to a more general class of semigroups, of which the mappings are not necessarily continuous.

From Remark 2.8 and Theorem 3.1, we have the following corollaries.

Corollary 3.3. *Let C be a nonempty weakly compact convex subset of a Banach space X with uniform normal structure and $\mathcal{T} = \{T_t : t \in G\}$ be an asymptotically regular semigroup on C. Suppose that $\mathcal{T}$ is a generalized Lipschitzian semigroup type-1 such that*

$$\left(\liminf_{t\to\infty} k_t\right)\left(\limsup_{t\to\infty} k_t\right) < \left[\widetilde{N}(X)\right]^{-1}.$$

Then there exists $z \in X$ such that $T_t z = z$ for all $t \in G$.

Corollary 3.4. *Let* (X, d) *be a complete bounded* CAT(0) *space and* $\mathcal{T} = \{T_t : t \in G\}$ *be an asymptotically regular semigroup on* X*. Suppose that* $\mathcal{T}$ *is a generalized Lipschitzian semigroup type-1 such that*

$$\left(\liminf_{t\to\infty} k_t\right)\left(\limsup_{t\to\infty} k_t\right) < \left[\widetilde{N}(X)\right]^{-1}.$$

Then there exists $z \in X$ *such that* $T_t z = z$ *for all* $t \in G$*.*

Remark 3.5. Some values of the constant $\widetilde{N}(X)$ in Corollary 3.3 can be found in [8,32,36] and the references therein. Furthermore, there are some common fixed point theorems for asymptotically regular semigroups in Banach spaces by utilizing the constant $\widetilde{N}(X)$, for example, see [19,47]. Unfortunately, the proofs of Theorem 3 of [19] and Theorem 2.1 and Theorem 3.1 of [47] seem not correct, because the inequality

$$\lim_{n\to\infty} \sup\{\|x_i - x_j\| : i, j \geq n\} \leq \limsup_{i\to\infty}\limsup_{j\to\infty} \|x_i - x_j\|$$

is false (see, Remark 12 of [22]). Regarding Corollary 3.4, the metric space X has uniform normal structure with $\widetilde{N}(X) \leq 1/\sqrt{2}$ (see, page 765 of [10]). In particular, $\widetilde{N}(X) = \frac{1}{2}$ when X is an $\mathbb{R}$-tree (see, Proposition 3.1 of [30]). Recently, Khamsi and Shukri [29] generalized the notion of CAT(0) spaces and proved a fixed point theorem for uniformly Lipschitzian mappings. By considering some results in [29] and by modifying the proof of Theorem 3.1, one can investigate the structure of fixed point sets of asymptotically regular semigroups in generalized CAT(0) spaces.

In the following result, we use some properties of the generalized Lipschitzian condition type-2 and the Lifšic constant $\kappa(X)$ to remove the assumptions of uniform normal structure of the space and property $(*)$ of the semigroup in Theorem 3.1.

Theorem 3.6. *Let* (X, d) *be a complete bounded metric space and* $\mathcal{T} = \{T_t : t \in G\}$ *be an asymptotically regular semigroup on* X*. Suppose that* $\mathcal{T}$ *is a generalized Lipschitzian semigroup type-2 on* X *such that*

$$\liminf_{t\to\infty} k_t < \kappa(X).$$

Then there exists $z \in X$ *such that* $T_t z = z$ *for all* $t \in G$*.*

Proof. We first choose $\{t_n\} \in \omega(\infty)$ such that

$$\liminf_{t\to\infty} k_t = \lim_{n\to\infty} k_{t_n} = k < \kappa(X).$$

If $\kappa(X) = 1$, then we use a similar argument to that of the proof of Theorem 3.1 to obtain the existence of a common fixed point of $\mathcal{T}$. For the rest of the proof, we consider $\kappa(X) > 1$ and $k \geq 1$. Since X is bounded, then for each $y \in X$ we can define a nonnegative real number $r(y)$ by

$$r(y) = \inf\left\{R > 0 : \exists x \in X, \limsup_{n\to\infty} d(T_{t_n}x, y) \leq R\right\}.$$

Claim 1 If $r(y) = 0$, then y is a common fixed point of $\mathcal{T}$.

Proof of Claim 1 Let $\{s_r\}$ be a subsequence of $\{t_n\}$ such that $\sup_{r\geq 1} k_{s_r} = \delta < 2$. For any $\varepsilon > 0$ we choose $x \in X$ such that $\limsup_{n\to\infty} d(T_{t_n}x, y) < \varepsilon$. Then using the asymptotic regularity of $\mathcal{T}$, for any $r \geq 1$, we have

$$\begin{aligned}
&d(T_{s_r}y, y)\\
&\quad\leq \limsup_{n\to\infty} \left(d(T_{s_r}y, T_{s_r+t_n}x) + d(T_{s_r+t_n}x, T_{t_n}x) + d(T_{t_n}x, y)\right)\\
&\quad\leq \limsup_{n\to\infty} d(T_{s_r}y, T_{s_r+t_n}x) + \limsup_{n\to\infty} d(T_{t_n}x, y)\\
&\quad< \varepsilon + k_{s_r} \max\Bigg\{\limsup_{n\to\infty} d(T_{t_n}x, y),\\
&\qquad \frac{1}{2}\limsup_{n\to\infty}\left(d(y, T_{t_n}x) + d(T_{t_n}x, T_{s_r+t_n}y)\right), \frac{1}{2}\limsup_{n\to\infty}\left(d(T_{t_n}x, y) + d(y, T_{s_r}y)\right)\Bigg\}\\
&\quad< \varepsilon + \delta \max\left\{\varepsilon, \frac{1}{2}(\varepsilon + d(T_{s_r}y, y))\right\}.
\end{aligned}$$

It follows that

$$d(T_{s_r}y, y) < \varepsilon + \delta\left(\frac{3}{2}\varepsilon + \frac{1}{2}d(T_{s_r}y, y)\right). \tag{3.14}$$

Since $\delta < 2$, from (3.14) we obtain

$$d(T_{s_r}y, y) < \left(\frac{2}{2-\delta}\right)\left(1 + \frac{3\delta}{2}\right)\varepsilon \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

Thus, $T_{s_r}y = y$ for all $r \geq 1$. The asymptotic regularity of $\mathcal{T}$ ensures that

$$d(T_t y, y) = \lim_{r\to\infty} d(T_t T_{s_r}y, T_{s_r}y) = 0$$

for all $t \in G$. It is evidence that y is the common fixed point of $\mathcal{T}$.

Assume that $r(y) > 0$. Let a and b be constants associated with the definition of the Lifšic constant $\kappa(X)$, where $b \in (k, \kappa(X))$. Choose $\lambda \in (0,1)$ such that

$$\gamma = \min\left\{a\lambda, \sqrt{\frac{b\lambda}{k}}\right\} > 1.$$

Then denote $R_y = \limsup_{n\to\infty} d(T_{t_n}y, y)$. We may assume that $R_y > 0$. Otherwise, y is the common fixed point of $\mathcal{T}$ according to Lemma 2.6. From the definition of $r(y)$, it is clear that $\lambda r(y) < R_y$. Therefore, using the fact that $k < 2$ we can choose $n_0 \geq 1$ such that

$$\lambda r(y) < d(y, T_{t_{n_0}}y) \tag{3.15}$$

and

$$k_{t_{n_0}} < \min\{k\gamma, 2\}. \tag{3.16}$$

From the definition of $r(y)$, we can also choose $R' \in (r(y), \gamma r(y))$ such that

$$\limsup_{n\to\infty} d(T_{t_n}x, y) \leq R' < \gamma r(y) \tag{3.17}$$

for some $x \in X$. Then for any $n \geq 1$, we have

$$\begin{aligned} & d(T_{t_n}x, T_{t_{n_0}}y) \\ & \leq d(T_{t_n}x, T_{t_n+t_{n_0}}x) + d(T_{t_n+t_{n_0}}x, T_{t_{n_0}}y) \\ & \leq d(T_{t_n}x, T_{t_n+t_{n_0}}x) + k_{t_{n_0}} \max\Big\{ d(T_{t_n}x, y), \\ & \frac{1}{2}d(T_{t_n}x, T_{t_{n_0}}y), \frac{1}{2}\Big(d(y, T_{t_n}x) + d(T_{t_n}x, T_{t_n+t_{n_0}}x)\Big)\Big\}. \end{aligned} \tag{3.18}$$

By taking the limit superior as $n \to \infty$ into (3.18), and then, using the asymptotic regularity of $\mathcal{T}$ and (3.17) we see that

$$\limsup_{n\to\infty} d(T_{t_n}x, T_{t_{n_0}}y) < k_{t_{n_0}} \max\left\{\gamma r(y), \frac{1}{2}\limsup_{n\to\infty} d(T_{t_n}x, T_{t_{n_0}}y)\right\} \tag{3.19}$$

Let

$$\frac{1}{2}\limsup_{n\to\infty} d(T_{t_n}x, T_{t_{n_0}}y)$$

be the maximum. Using (3.16), from (3.19) we obtain

$$\limsup_{n\to\infty} d(T_{t_n}x, T_{t_{n_0}}y) < \frac{k_{t_{n_0}}}{2}\limsup_{n\to\infty} d(T_{t_n}x, T_{t_{n_0}}y),$$

a contradiction. Thus, inequality (3.18) can be derived into the following inequality:

$$\limsup_{n\to\infty} d(T_{t_n}x, T_{t_{n_0}}y) < k\gamma^2 r(y). \tag{3.20}$$

Now, from (3.17) and (3.20) we choose $n_1 \geq 1$ such that for any $n \geq n_1$,

$$\begin{aligned} T_{t_n}x &\in B(y, \gamma r(y)) \cap B\big(T_{t_{n_0}}y, k\gamma^2 r(y)\big) \\ &\subseteq B(y, a\lambda r(y)) \cap B\big(T_{t_{n_0}}y, b\lambda r(y)\big). \end{aligned}$$

According to the definition of constants a and b above, we ensure by (3.15) that there exists $w = w(y) \in X$ such that

$$d(T_{t_n}x, w) \leq \lambda r(y) \quad \forall n \geq n_1. \tag{3.21}$$

It follows from (3.21) and (3.17) that

$$r(w) \leq \lambda r(y)$$

and

$$\begin{aligned} d(w, y) &\leq \limsup_{n\to\infty} d(T_{t_n}x, w) + \limsup_{n\to\infty} d(T_{t_n}x, y) \\ &\leq \lambda r(y) + \gamma r(y) = \mu r(y), \end{aligned}$$

where $\mu = \lambda + \gamma$.

We process the above procedure to obtain a sequence $\{w_m\}$ in X with $w_0 = y$ and $w_m = w(w_{m-1})$ such that

$$r(w_m) \leq \lambda^m r(w_0) \to 0 \quad \text{as } m \to \infty$$

and

$$d(w_{m+1}, w_m) \leq \mu r(w_m) \leq \lambda^m \mu r(w_0).$$

This, in turn, implies that $\{w_m\}$ is a Cauchy sequence and thus, it is convergent. Let z be the limit of $\{w_m\}$. Let $\varepsilon > 0$ be fixed. For sufficiently large m we choose $x_m, w_m \in X$ such that $\limsup_{n\to\infty} d(T_{t_n} x_m, w_m) < \frac{\varepsilon}{2}$ and $d(w_m, z) < \frac{\varepsilon}{2}$. We may assume $\limsup_{n\to\infty} d(T_{t_n} x_m, w_m) > 0$. Otherwise, w_m is the common fixed point of $\mathcal{T}$ according to Lemma 2.6. Then, we have

$$\limsup_{n\to\infty} d(T_{t_n} x_m, z) \leq \limsup_{n\to\infty} d(T_{t_n} x_m, w_m) + d(w_m, z) < \varepsilon,$$

which implies $r(z) = 0$. As in Claim 1, we get $T_t z = z$ for all $t \in G$. □

Remark 3.7. Comparing to the theorem on page 55 of [20], Theorem 3.6 extends it in two aspects: (1) the concept of a family of iterations of mapping is replaced by the concept of a one-parameter semigroup; (2) the exact Lipschitz constant assumption is replaced by the corresponding generalized Lipschitz constant.

Remark 3.8. Let $X = [0, 1]$ be a subspace of the usual metric space $\mathbb{R}$ and $\mathcal{T}$ be the semigroup defined in Example 2.5. By Proposition 4.1 of [28] we see that $\mathbb{R}$ is a hyperconvex metric space. Since X is admissible, then by Proposition 4.5 of [28] we deduce that X is hyperconvex. Thus, $\widetilde{N}(X) = \frac{1}{2}$. Moreover, Remark 2.8 shows that X has property (P). It follows immediately that $\mathcal{T}$ has property $(*)$. Thus, all the assumptions of Theorem 3.1 are satisfied with $\limsup_{n\to\infty} k_n = \limsup_{n\to\infty} \gamma^{n-1} = 0$. On the other hand, since the metric space X is hyperconvex, then we see from page 5 of [4] and Theorem 3.2 of [30] that X is an $\mathbb{R}$-tree. Therefore, $\kappa(X) = 2$. Thus, all the assumptions of Theorem 3.6 are also satisfied with $\liminf_{n\to\infty} k_n = 0$. Note that, the semigroup $\mathcal{T}$ in Example 2.5 cannot be employed by Theorem 3.2 of [45], Theorem 3.2 of [24], and Theorem 3.1 and Theorem 3.2 of [25]. Also, the mapping T in Example 2.5 cannot be employed by the theorem on page 55 of [20].

Remark 3.9. Lastly, we note here that conditions (a) and (c) in the proof of the main result of Yao and Zeng, i.e., Theorem 3.2 of [45] actually are equivalent. In a more general situation, we have the following claim. Let (X, d) be a metric space and $\{x_n\}$ be a bounded sequence in X. Let us denote

$$A = \bigcap_{n\geq 1} ad(\{x_i : i \geq n\}).$$

We claim that $z \in A$ if and only if

$$d(z, y) \leq \limsup_{n\to\infty} d(x_n, y)$$

for all $y \in X$. Since the implication has been proved on page 1233 of [35], then we just need to prove the converse. Let us denote $A_n = \{x_i : i \geq n\}$ for all $n \geq 1$. There are two possible cases.

Case 1 There exists $n' \geq 1$ with $A_{n'}$ is a singleton. We write $A_{n'} = \{x^*\}$. It is easy to see that $x^* \in A$. Moreover, from the assumption we have

$$d(z, y) \leq \limsup_{n\to\infty} d(x_n, y) = d(x^*, y)$$

for all $y \in X$. In particular, $d(z, x^*) \leq d(x^*, x^*) = 0$. It is evidence that $z \in A$.

Case 2 A_n consists of more than one point for any $n \geq 1$. From the assumption, for any $n \geq 1$ and $y \in X$ we have

$$d(z, y) \leq \sup\{d(x_i, y) : i \geq n\} = r(y, A_n).$$

Therefore, $z \in B(y, r(y, A_n))$. It follows that

$$z \in \bigcap_{y\in X} B(y, r(y, A_n)).$$

Using Proposition 5.3.(1) of [28] we immediately obtain $z \in cov(A_n)$ and thus, $z \in ad(A_n)$ for all $n \geq 1$. Hence,

$$z \in \bigcap_{n\geq 1} ad(A_n) = \bigcap_{n\geq 1} ad(\{x_i : i \geq n\}).$$

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References

[1] Aksoy, A.G., Borman, M.S., Westfahl, A.L.: Compactness and measures of noncompactness in metric trees. In: Kato, M., Maligranda, L. (eds.) Banach and Function Spaces II, pp. 277–292. Yokohama Publishers, Yokohama (2008)

[2] Ayerbe Toledano, J.M., Domínguez Benavides, T., López Acedo, G.: Measures of Noncompactness in Metric Fixed Point Theory. Birkhäuser, Basel (1997)

[3] Baillon, J.B., Bruck, R.E., Reich, S.: On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces. Houst. J. Math. **4**, 1–9 (1978)

[4] Bridson, M.R., Haefliger, A.: Metric Spaces of Non-Positive Curvature. Springer, New York (1999)

[5] Browder, F.E., Petryshyn, W.V.: The solution by iteration of nonlinear functional equations in Banach spaces. Bull. Am. Math. Soc. **72**, 571–575 (1966)

[6] Bryant, V.W.: A remark on fixed-point theorem for iterated mappings. Am. Math. Mon. **75**, 399–400 (1968)

[7] Budzyńska, M., Kuczumow, T., Reich, S.: Uniform asymptotic normal structure, the uniform semi-Opial property, and fixed points of asymptotically regular uniformly Lipschitzian semigroups: Part II. Abstr. Appl. Anal. **3**, 247–263 (1998)

[8] Bynum, W.L.: Normal structure coefficients for Banach spaces. Pac. J. Math. **86**, 427–436 (1980)

[9] Ćirić, L.B.: A generalization of Banach's contraction principle. Proc. Am. Math. Soc. **45**, 267–273 (1974)

[10] Dhompongsa, S., Kirk, W.A., Sims, B.: Fixed points of uniformly lipschitzian mappings. Nonlinear Anal. **65**, 762–772 (2006)

[11] Domínguez Benavides, T., Japón, M.A.: Opial modulus, moduli of noncompact convexity and fixed points for asymptotically regular mappings. Nonlinear Anal. **41**, 617–630 (2000)

[12] Domínguez Benavides, T., Japón, M.A., Sadeghi Hafshejani, A.: Fixed point theorems for asymptotically regular mappings in modular and metric spaces. J. Fixed Point Theory Appl. **22**, 12 (2020)

[13] Domínguez Benavides, T., Xu, H.K.: A new geometrical coefficient for Banach spaces and its applications in fixed point theory. Nonlinear Anal. **25**, 311–325 (1995)

[14] Dunford, N., Schwartz, J.T.: Linear Operator. Part I: General Theory. Interscience, New York (1958)

[15] Edelstein, M., O'Brein, R.C.: Nonexpansive mappings, asymptotically regularity, and successive approximations. J. Lond. Math. Soc. **2**(17), 547–554 (1978)

[16] Gillespie, A.A., Williams, B.B.: Fixed point theorem for nonexpansive mappings on Banach spaces with uniformly normal structure. Appl. Anal. **9**, 121–124 (1979)

[17] Goebel, K., Kirk, W.A.: A fixed point theorem for transformations whose iterates have uniform Lipschitz constant. Studia Math. **47**, 135–140 (1973)

[18] Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker, New York (1984)

[19] Górnicki, J.: Fixed points of asymptotically regular mappings in spaces with uniformly normal structure. Comment. Math. Univ. Carolin. **32**, 639–643 (1991)

[20] Górnicki, J.: A fixed point theorem for asymptotically regular mappings. Colloq. Math. **64**, 55–57 (1993)

[21] Górnicki, J.: Fixed points of asymptotically regular semigroups in Banach spaces. Rend. Circ. Mat. Palermo **2**(46), 89–118 (1997)

[22] Górnicki, J.: Structure of the fixed point set of asymptotically regular mappings in uniformly convex Banach spaces. Taiwan. J. Math. **15**, 1007–1020 (2011)

[23] Górnicki, J.: Remarks on asymptotic regularity and fixed points. J. Fixed Point Theory Appl. **21**, 29 (2019)

[24] Imdad, M., Soliman, A.H.: On uniformly generalized Lipschitzian mappings. Fixed Point Theory Appl. **2010**, 692401 (2010)

[25] Imdad, M., Soliman, A.H.: On one parameter semigroup of self mappings uniformly satisfying expansive Kannan condition. Bull. Malays. Math. Sci. Soc. **2**(35), 687–694 (2012)

[26] Ishikawa, S.: Fixed points and iteration of a nonexpansive mapping in a Banach space. Proc. Am. Math. Soc. **59**, 65–71 (1976)

[27] Khamsi, M.A.: On metric spaces with uniform normal structure. Proc. Am. Math. Soc. **106**, 723–726 (1989)

[28] Khamsi, M.A., Kirk, W.A.: An Introduction to Metric Spaces and Fixed Point Theory. Wiley, New York (2001)

[29] Khamsi, M.A., Shukri, S.A.: Generalized CAT(0) spaces. Bull. Belg. Math. Soc. Simon Stevin **24**, 1–10 (2017)

[30] Kirk, W.A.: Hyperconvexity of $\mathbb{R}$-trees. Fundam. Math. **156**, 67–72 (1998)

[31] Kirk, W.A.: Geodesic geometry and fixed point theory II. In: García Falset, J., Llorens Fuster, E., Sims, B. (eds.) Fixed Point Theory and Applications, pp. 113–142. Yokohama Publishers, Yokohama (2004)

[32] Kirk, W.A., Sims, B.: Uniform normal structure and related notions. J. Nonlinear Convex. Anal. **2**, 129–138 (2001)

[33] Krasnosel'skii, M.A., Zabreiko, P.P.: Geometrical Methods of Nonlinear Analysis. Springer, Berlin (1984)

[34] Lifšic, E.A.: Fixed point theorems for operators in strongly convex spaces. Voronež Gos. Univ. Trudy Mat. Fak. **16**, 23–28 (1975). (in Russian)

[35] Lim, T.C., Xu, H.K.: Uniformly Lipschitzian mappings in metric spaces with uniform normal structure. Nonlinear Anal. **25**, 1231–1235 (1995)

[36] Llorens Fuster, E.: Some moduli and constants related to metric fixed point theory. In: Kirk, W.A., Sims, B. (eds.) Handbook of Metric Fixed Point Theory, pp. 133–175. Kluwer Acad. Publishers, Dordrecht (2001)

[37] Najibufahmi, M., Zulijanto, A.: Common fixed points of asymptotically regular semigroups equipped with generalized Lipschitzian conditions. Fixed Point Theory **19**, 681–706 (2018)

[38] Najibufahmi, M., Zulijanto, A.: A fixed point theorem for generalized Lipschitzian semigroups in Hilbert spaces. Thai J. Math. **17**, 639–648 (2019)

[39] Penot, J.P.: Fixed point theorems without convexity. Bull. Soc. Math. Fr. **60**, 129–152 (1979)

[40] Razani, A., Goodarzi, Z.: Iteration by Cesàro means for quasi-contractive mappings. Filomat **28**, 1575–1584 (2014)

[41] Rhoades, B.E.: A comparison of various definitions of contractive mappings. Trans. Am. Math. Soc. **226**, 257–290 (1977)

[42] Sahu, D.R., Agarwal, R.P., O'Regan, D.: The structure of fixed-point sets of Lipschitzian type semigroups. Fixed Point Theory Appl. **2012**, 163 (2012)

[43] Webb, J.R.L., Zhao, W.: On connections between set and ball measures of noncompactness. Bull. Lond. Math. Soc. **22**, 471–477 (1990)

[44] Wiśnicki, A.: On the structure of fixed-point sets of asymptotically regular semigroups. J. Math. Anal. Appl. **393**, 177–184 (2012)

[45] Yao, J.C., Zeng, L.C.: Fixed point theorem for asymptotically regular semigroups in metric spaces with uniform normal structure. J. Nonlinear Convex Anal. **8**, 153–163 (2007)

[46] Zamfirescu, T.: Fix point theorems in metric spaces. Arch. Math. (Basel) **23**, 292–298 (1972)

[47] Zeng, L.C., Yang, Y.L.: On the existence of fixed points for Lipschitzian semigroups in Banach spaces. Chin. Ann. Math. **22B**, 397–404 (2001)

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