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A proximal point algorithm for finding minimizers and fixed points of quasi-pseudo-contractive mappings in CAT(0) spaces

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Abstract. The purpose of this article is twofold. One is to establish a proximal point algorithm for finding a minimizer of a proper convex and lower semi-continuous function and fixed points of quasi-pseudo-contractive mappings in CAT(0) spaces. The other is to point out and correct a basic and conceptual error in a paper of Ugwunnadi et al. [Theorem 3.1, J. Fixed Point Theory Appl. (2018) 20: 82].

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1. Introduction

Recently, Ugwunnadi et al. [1] introduced a hybrid proximal point algorithm and established some strong convergence theorems to a common solution of proximal point for a proper convex and lower semi-continuous function and a fixed point of a k-demicontractive mapping in the framework of a CAT(0) space. Particular, the following main result is given:

Theorem UKA [1, Theorem 3.1]. *Let (X, d) be a complete CAT(0) space, $f : X \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function and $T : X \to X$ be an L-Lipschitzian k-demicontractive mapping such that T is Δ-demiclosed. If $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1) satisfying the following conditions:*

(c1) $lim_{n\to\infty}\alpha_n = 0$;
(c2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c3) $0 < \varepsilon \le \beta_n < 1 - 2k,\ \forall n \ge 1,$ *where ε and $k \in [0, 1)$ are some positive constants,*

and $\Omega := Fix(T) \bigcap argmin_{y\in X} f(y) \neq \emptyset$, *then the sequence* $\{x_n\}$ *generated by given* $x_1 \in X$,

$$\begin{cases} z_n = argmin_{y\in X}\left[f(y) + \dfrac{1}{2\lambda_n} d^2(y, x_n)\right], \\ y_n = (1-\alpha_n)z_n, \\ x_{n+1} = (1-\beta_n)z_n \oplus \beta_n T y_n \end{cases} \tag{1.1}$$

converges strongly to some point $p \in \Omega$.

During carefully reading Theorem UKA and its proof, we found that there exist some basic and conceptual errors in it. Since (X, d) is a CAT(0) space, it is not linear. Therefore it does not have a scalar multiplication and element 0. These show that the sequences $\{y_n\}$ and $\{x_n\}$ defined by (1.1) are ill-posed. And the proof of Theorem UKA is also lack of rationality.

The main purpose of this paper is to establish a proximal point algorithm for finding minimizers of a proper convex and lower semi-continuous function and fixed points of quasi-pseudo-contractive mappings in CAT(0) spaces and to point out and correct a basic and conceptual error in Ugwunnadi et al. [1, Theorem 3.1].

2. Preliminaries

Let (X, d) be a metric space and $x, y \in X$. A geodesic path joining x to y is an isometry $c : [0,\ d(x; y)] \to X$ such that $c(0) = x$ and $c(d(x; y)) = y$. The image of a geodesic path joining x to y is called a geodesic segment between x and y. A metric space X is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic segment joining x and y for each $x, y \in X$.

Let X be a uniquely geodesic space. We write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining x to y such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)d(x, y)$. We also denote by $[x, y]$ the geodesic segment joining x to y, that is, $[x, y] = \{(1 - t)x \oplus ty : 0 \leq t \leq 1\}$. A subset C of X is convex if $[x, y] \subset C$ for all $x, y \in C$.

A uniquely geodesic space $(X; d)$ is a CAT(0) space, if and only if

$$d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y), \tag{2.1}$$

for all $x, y, z \in X$ and all $t \in [0, 1]$.

It is well-known that any complete and simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples of CAT(0) spaces include pre-Hilbert spaces [2], R-trees, Euclidean buildings [3].

Let X be a metric space, $\{x_n\}$ be a bounded sequence in X, and $r(., \{x_n\}) : X \to [0, \infty)$ be a continuous functional defined by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) := \inf\{r(x, x_n) : x \in X\}$ while the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$. It is generally known that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$

in X is said to be Δ-convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$.

In 2008 Berg and Nikolaev [4] (see also, Reich and Salinas [5]) introduced the concept of quasilinearization in CAT(0) space X as follows:

Denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a *vector*. *Quasi-linearization* in CAT(0) space X is defined as a mapping $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ such that

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)) \tag{2.2}$$

for all $a, b, c, d \in X$. It can be easily verified that

$$\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b),\ \langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle,\ and$$
$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle\ \forall a, b, c, d, e \in X.$$

Remark 2.1. [6] It is well known that if X is a complete CAT(0) space, then $\{x_n\}$ Δ-converges to $x^* \in X$ if and only if

$$\lim sup_{n\to\infty} \langle \overrightarrow{x^* x_n}, \overrightarrow{x^* y} \rangle \leq 0,\ \forall y \in X.$$

Let X be a CAT(0) space. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d),\ \forall a, b, c, d \in X. \tag{2.3}$$

It is known ([4], Corollary 3) that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

Ahmadi Kakavandi and Amini [7] have introduced the concept of *dual space* of a complete CAT(0) space X, based on a work of Berg and Nikolaev [4], as follows.

Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ defined by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab},\ \overrightarrow{ax} \rangle,\ (t \in \mathbb{R}, a, b, x \in X),$$

where $C(X, \mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwartz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b),\ (t \in \mathbb{R},\ a, b \in X)$, where

$$L(\phi) = \sup \left\{ \frac{\phi(x) - \phi(y)}{d(x; y)} : x, y \in X,\ x \neq y \right\}$$

is the Lipschitz semi-norm for any function $\phi : X \to \mathbb{R}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)),\ (t, s \in \mathbb{R},\ a, b, c, d \in X).$$

For a complete CAT(0) space (X, d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $(Lip(X, \mathbb{R}), L)$. And $D((t, a, b), (s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab},\ \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd},\ \overrightarrow{xy} \rangle$, for all $x, y \in X$. Hence D imposes an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t, a, b) is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : D((t, a, b), (s, c, d)) = 0\}.$$

The set

$$X^* = \{[t\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X\}$$

is a metric space which is called the dual space of $(X; d)$ with metric

$$D([t\overrightarrow{ab}], [s\overrightarrow{cd}]) := D((t, a, b), (s, c, d)),$$

The following inequalities can be proved easily.

Lemma 2.1. *Let X be a CAT(0) space. For all $x, y, z, u, w \in X$ and $t \in [0, 1]$, the following inequalities hold:*

(i) $d(tx \oplus (1-t)y,\ z) \leq td(x, z) + (1-t)d(y, z)$;
(ii) $d^2((1-t)x \oplus ty, z) \leq (1-t)^2 d^2(x, z) + t^2 d^2(y, z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz}\rangle$;
(iii) $d(tx \oplus (1-t)y,\ tu \oplus (1-t)w) \leq td(x, u) + (1-t)d(y, w)$.

In the sequel, we always assume that X is a complete CAT(0) space, C is a nonempty and closed convex subset of X and $Fix(T)$ is the fixed point set of a mapping T.

Definition 2.2. A mapping $T : C \to C$ is said to be

(1) contractive if there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y), \quad \forall x, y \in C;$$

if $k = 1$, then T is said to be nonexpansive;

(2) quasinonexpansive if $Fix(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p), \forall p \in Fix(T),\ x \in C;$$

(3) firmly nonexpansive if

$$d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy}\rangle,\ \forall x, y \in C; \tag{2.4}$$

(4) k-demicontractive [8] if $Fix(T) \neq \emptyset$ and there exists a constant $k \in [0; 1)$ such that

$$d^2(Tx, p) \leq d^2(x, p) + kd^2(x, Tx),\ \forall x \in C,\ p \in Fix(T);$$

(5) quasi-pseudo-contractive if $Fix(T) \neq \emptyset$ and

$$d^2(Tx, p) \leq d^2(x, p) + d^2(x, Tx),\ \forall x \in C,\ p \in Fix(T); \tag{2.5}$$

Remark 2.3. From the definitions above, it is easy to see that if $Fix(T) \neq \emptyset$, then the following implications hold:

$$(3) \Longrightarrow (2) \Longrightarrow (4) \Longrightarrow (5).$$

But the converse is not true. These show that the class of quasi-pseudo-contractive mappings is more general than the classes of k-demicontractive mappings, quasinonexpansive mappings.

Definition 2.4. Let (X, d) be a complete CAT(0) space. A mapping $T : X \to X$ is said to be Δ-demiclosed, if for any bounded sequence $\{x_n\}$ in X such that $\Delta - \lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $Tp = p$.

Example of quasi-pseudo-contractive mappings Let H be the closed interval $[0,1]$ with the absolute value as norm. Let $T : H \to H$ be the mapping defined by:

$$Tx = \begin{cases} k, & x \in [0,k],\ k \in (0,1) \\ 0, & x \in (k,1]. \end{cases} \tag{2.6}$$

It is clear that $Fix(T) = \{k\}$. Hence for $x \in [0,k]$ we have

$$|Tx - k|^2 = 0 \le |x-k|^2 + |x - Tx|^2.$$

Also for $x \in (k,1]$ we have

$$|Tx - k|^2 = k^2 \le |x-k|^2 + |Tx - x|^2.$$

These show that for $x \in [0,1]$ we have

$$|Tx - k|^2 \le |x-k|^2 + |x - Tx|^2,$$

i.e., T is a quasi-pseudo-contractive mapping. Also it is easy to see that T is demiclosed.

Definition 2.5. A function $f : C \to (-\infty,\infty]$ is said to be convex if for all $x, y \in C$ and all $\lambda \in [0,1]$ the following inequality holds

$$f(\lambda x \oplus (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y).$$

Lemma 2.6 [9,10]. *Let $f : X \to (-\infty,\infty]$ be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in CAT(0) space X as*

$$J_\lambda^f(x) = argmin_{y\in X}\left[f(y) + \frac{1}{2\lambda} d^2(y,x)\right], \quad \forall x \in X. \tag{2.7}$$

Then

(i) *the set $Fix(J_\lambda^f)$ of fixed points of the resolvent of f coincides with the set $argmin_{y\in X} f(y)$ of minimizers of f, and for any $\lambda > 0$, the resolvent J_λ^f of f is a firmly nonexpansive mapping. Hence it is non-expansive;*

(ii) *Since J_λ^f is a firmly nonexpansive mapping, if $Fix(J_\lambda^f) \ne \emptyset$, then from (2.4) we have*

$$d^2(J_\lambda^f x, p) \le d^2(x,p) - d^2(J_\lambda^f x, x),\ \forall x \in X,\ p \in Fix(J_\lambda^f). \tag{2.8}$$

(iii) *For any $x \in X$, and $\lambda > \mu > 0$, the following identity holds:*

$$J_\lambda^f(x) = J_\mu^f\left(\frac{\lambda-\mu}{\lambda} J_\lambda^f(x) \oplus \frac{\mu}{\lambda} x\right).$$

Lemma 2.7 (see also [11]). *Let X be a complete CAT(0) space and $T : X \to X$ be a L-Lipschitzian mapping with $L \ge 1$. Let $G : X \to X$ and $K : X \to X$ be two mappings defined by*

$$K(x) := (1-\xi)x \oplus \xi T(Gx); \quad G(x) := (1-\eta)x \oplus \eta Tx,\ \ x \in X. \tag{2.9}$$

If $0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}}$, then the following conclusions hold:

(1) *$Fix(T) = Fix(T(G)) = Fix(K)$;*

(2) *If T is Δ-demiclosed, then K is also Δ-demiclosed;*
(3) *$K : X \to X$ is L^2-Lipschitzian;*
(4) *In addition, if $T : X \to X$ is quasi-pseudo-contractive, then $K : X \to X$ is a quasi-nonexpansive mapping, i.e., for any $x \in X$ and $p \in Fix(K)(= Fix(T))$*

$$d^2(Kx, p) \le d^2(x, p) - \xi\eta(1 - 2\eta - L^2\eta^2)d^2(x, Tx) \le d^2(x, p). \quad (2.10)$$

Proof. Now we prove the conclusion (1).

If $x^* \in Fix(T)$, then

$$\begin{aligned} d(x^*, TGx^*) &= d(x^*, T((1-\eta)x^* \oplus \eta Tx^*) \\ &= d(x^*, Tx^*) = 0, \quad i.e., x^* \in Fix(TG). \end{aligned}$$

If $x^* \in Fix(TG)$, then

$$\begin{aligned} d(x^*, Kx^*) &= d(TG(x^*), (1-\xi)x^* \oplus \xi TG(x^*)) \\ &= (1-\xi)d(TG(x^*), x^*) = 0, \quad i.e., x^* \in Fix(K). \end{aligned}$$

If $x^* \in Fix(K)$, then

$$\begin{aligned} d(x^*, Tx^*) &= d((1-\xi)x^* \oplus \xi TG(x^*), Tx^*) \\ &\le (1-\xi)d(x^*, Tx^*) + \xi d(TG(x^*), Tx^*) \\ &\le (1-\xi)d(x^*, Tx^*) + \xi Ld(G(x^*), x^*). \end{aligned}$$

Simplifying we have

$$d(x^*, Tx^*) \le Ld(x^*, Gx^*) = Ld(x^*, (1-\eta)x^* \oplus \eta Tx^*) \le L\eta d(x^*, Tx^*).$$

Since $L\eta < 1$, this implies that $x^* \in Fix(T)$. The conclusion (1) is proved.

Now we prove the conclusion (2).

For any sequence $\{x_n\} \subset X$ with $\Delta-\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} d(x_n, Kx_n) = 0$, we show that $x \in Fix(K)$. By conclusion (1), it is sufficient to prove that $x \in Fix(T)$. In fact, since T is L-Lipschizian, we have

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, Kx_n) + d(Kx_n, Tx_n) = d(x_n, Kx_n) \\ &\quad + d((1-\xi)x_n \oplus \xi T(Gx_n), Tx_n) \\ &\le d(x_n, Kx_n) + (1-\xi)d(x_n, Tx_n) + \xi d(T(Gx_n), Tx_n). \end{aligned}$$

Simplifying we have

$$\begin{aligned} d(x_n, Tx_n) &\le \frac{1}{\xi}d(x_n, Kx_n) + d(T(Gx_n), Tx_n) \\ &\le \frac{1}{\xi}d(x_n, Kx_n) + Ld((1-\eta)x_n \oplus \eta Tx_n, x_n) \\ &\le \frac{1}{\xi}d(x_n, Kx_n) + L\eta d(Tx_n, x_n). \end{aligned}$$

This implies that

$$(1 - L\eta)d(x_n, Tx_n) \le \frac{1}{\xi}d(x_n, Kx_n).$$

Since $(1 - L\eta) > 0$ and $d(x_n, Kx_n) \to 0$, this implies that $d(x_n, Tx_n) \to 0$. Since T is Δ-demiclosed, $x \in Fix(T)$. Hence $x \in Fix(K)$, i.e., K is Δ-demiclosed.

The conclusion (2) is proved.

The conclusion (3) is obvious, the proof is omitted.

Now we prove the conclusion (4).

For any $p \in Fix(T)$ and $x \in X$, it follows from (2.1) that

$$\begin{aligned} d^2(Kx, p) &= d^2((1-\xi)x \oplus \xi T((1-\eta)x \oplus \eta Tx), p) \\ &\le (1-\xi)d^2(x,p) + \xi d^2(T((1-\eta)x \oplus \eta Tx), p) \\ &\quad - \xi(1-\xi)d^2(x, T((1-\eta)x \oplus \eta Tx)). \end{aligned} \tag{2.11}$$

Since T is quasi-pseudo-contractive, we have

$$\begin{aligned} d^2(T((1-\eta)x \oplus \eta Tx), p) &\le d^2((1-\eta)x \oplus \eta Tx), p) \\ &+ d^2((1-\eta)x \oplus \eta Tx), T((1-\eta)x \oplus \eta Tx)). \end{aligned} \tag{2.12}$$

From (2.1), we have

$$\begin{aligned} d^2((1-\eta)x \oplus \eta Tx), p) &\le (1-\eta)d^2(x,p) + \eta d^2(Tx,p) - \eta(1-\eta)d^2(x,Tx) \\ &\le (1-\eta)d^2(x,p) \\ &\quad + \eta\{d^2(x,p) + d^2(x,Tx)\} - \eta(1-\eta)d^2(x,Tx) \\ &= d^2(x,p) + \eta^2 d^2(x,Tx), \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} d^2((1-\eta)x \oplus \eta Tx, T((1-\eta)x \oplus \eta Tx)) &\le (1-\eta)d^2(x, T((1-\eta)x \oplus \eta Tx)) \\ &\quad + \eta d^2(Tx, T((1-\eta)x \oplus \eta Tx)) \\ &\quad - \eta(1-\eta)d^2(x,Tx) \\ &\le (1-\eta)d^2(x, T((1-\eta)x \oplus \eta Tx)) + \eta L^2 d^2(x, (1-\eta)x \\ &\quad \oplus \eta Tx) - \eta(1-\eta)d^2(x,Tx) \\ &\le (1-\eta)d^2(x, T((1-\eta)x \oplus \eta Tx)) \\ &\quad + \eta^3 L^2 d^2(x,Tx) - \eta(1-\eta)d^2(x,Tx) \\ &\le (1-\xi)d^2(x, T((1-\eta)x \oplus \eta Tx)) \\ &\quad - \eta((1-\eta-L^2\eta^2)d^2(x,Tx) \ (since\ \xi < \eta). \end{aligned} \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.12), after simplifying we have

$$\begin{aligned} d^2(T((1-\eta)x \oplus \eta Tx), p) &\le d^2(x,p) + (1-\xi)d^2(x, T((1-\eta)x \oplus \eta Tx)) \\ &\quad - \eta(1-2\eta-L^2\eta^2)d^2(x,Tx). \end{aligned} \tag{2.15}$$

Substituting (2.15) into (2.11), and after simplifying we have

$$d^2(Kx,p) \le d^2(x,p) - \xi\eta(1-2\eta-L^2\eta^2)d^2(x,Tx) \le d^2(x,p).$$

This completes the proof of Lemma 2.7. □

Lemma 2.8 [12]. *If $\{a_n\}$ is a sequence of real numbers and there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in N$, then there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$ and the following properties are satisfied:*

$$a_{m_k} \le a_{m_k+1} \text{ and } a_k \le a_{m_k+1}.$$

for all sufficiently large numbers $k \in N$. In fact, $m_k = max\{j \le k : a_j < a_{j+1}\}$.

Lemma 2.9 ([13]). *If $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the following conditions:*

$$a_{n+1} \le (1-\delta_n)a_n + \delta_n\sigma_n + \gamma_n, n \ge 0.$$

where $\{\delta_n\} \subset [0,1]$, $\sum_{n=0}^{\infty}\delta_n = \infty$; $\limsup_{n\to\infty}\sigma_n \le 0$; $\gamma_n \ge 0$, and $\sum_{n=0}^{\infty}\gamma_n < \infty$. Then $a_n \to 0$, as $n \to \infty$.

3. Main results

Throughout this section we assume that

(1) (X,d) is a complete CAT(0) space;
(2) $f : X \to (-\infty, +\infty]$ is a proper convex and lower semi-continuous function, and $J_{\lambda_n}^f : X \to X$ is the Moreau-Yosida resolvent of f;
(3) $T : X \to X$ is an L-Lipschitzian quasi-pseudo contractive mapping with $L \ge 1$ and T is Δ-demiclosed;
(4) Define the mappings $G : X \to X$ and $K : X \to X$ by

$$K(x) := (1-\xi)x \oplus \xi T(Gx); \quad G(x) := (1-\eta)x \oplus \eta Tx, \quad x \in X, \tag{3.1}$$

where $0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}}$.

Theorem 3.1. *Let (X,d), f, $J_{\lambda_n}^f$, T, K, G satisfy the conditions (1)-(4) as above. Let $u \in X$ be a given point. For any given point $x_1 \in X$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} z_n = J_{\lambda_n}^f(x_n) := argmin_{y\in X}\left[f(y) + \frac{1}{2\lambda_n}d^2(y,x_n)\right], \\ y_n = \alpha_n u \oplus (1-\alpha_n)z_n, \\ x_{n+1} = (1-\beta_n)z_n \oplus \beta_n K y_n, \end{cases} \quad n \ge 1. \tag{3.2}$$

If $\Omega := Fix(T)\bigcap argmin_{y\in X} f(y) \ne \emptyset$ and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(c1) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(c2) $0 < \varepsilon \le \beta_n \le b < 1$, $\lambda_n > \lambda > 0$, $\forall n \ge 1$, *where ε, b and λ are some positive constants,*

then the sequence $\{x_n\}$ converges strongly to some point in Ω.

Proof. First we observe that by the assumptions of Theorem 3.1, Lemma 2.7 and Lemma 2.5 we know that

(a) the mapping $K : X \to X$ is quasi-nonexpansive, Δ-demiclosed, L^2-Lipschitzian and $Fix(T) = Fix(K)$;
(b) $J^f_{\lambda_n}$ is nonexpansive, so it is Δ-demiclosed, and $Fix(J^f_{\lambda_n}) = argmin_{y \in X} f(y)$.

(I) Now we prove that the sequence $\{x_n\}$ is bounded.

In fact, if $p \in \Omega$, then $p = J^f_{\lambda_n}(p), \forall n \geq 1$, and $p \in Fix(T) = Fix(K)$. Since $J^f_{\lambda_n}$ is a nonexpansive mapping, we have

$$d(z_n, p) = d(J^f_{\lambda_n}(x_n), J^f_{\lambda_n}(p)) \leq d(x_n, p). \tag{3.3}$$

It follows from (3.2), (2.1) and Lemma 2.1(ii) that

$$\begin{aligned} d^2(y_n, p) &= d^2(\alpha_n u \oplus (1-\alpha_n) z_n, p) \\ &\leq \alpha_n d^2(u, p) + (1-\alpha_n) d^2(z_n, p) - \alpha_n(1-\alpha_n) d^2(u, z_n) \qquad (3.4) \\ &= \alpha_n^2 d^2(u, p) + (1-\alpha_n)^2 d^2(z_n, p) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{up}, \overrightarrow{z_n p} \rangle. \end{aligned}$$

Also from (3.2), (3.3) and (3.4), we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2((1-\beta_n) z_n \oplus \beta_n K y_n, p) \\ &\leq (1-\beta_n) d^2(z_n, p) + \beta_n d^2(K y_n, p) - \beta_n(1-\beta_n) d^2(z_n, K y_n) \\ &\leq (1-\beta_n) d^2(z_n, p) + \beta_n d^2(y_n, p) - \beta_n(1-\beta_n) d^2(z_n, K y_n) \qquad (3.5) \\ &\leq (1-\beta_n) d^2(z_n, p) + \beta_n\{\alpha_n d^2(u, p) + (1-\alpha_n) d^2(z_n, p) \\ &\quad - \alpha_n(1-\alpha_n) d^2(u, z_n)\} - \beta_n(1-\beta_n) d^2(z_n, K y_n). \end{aligned}$$

This implies that

$$\begin{aligned} d^2(x_{n+1}, p) &\leq (1-\alpha_n\beta_n) d^2(x_n, p) + \alpha_n\beta_n d^2(u, p) \\ &\leq max\{d^2(x_n, p), d^2(u, p)\}. \end{aligned}$$

By induction, we can prove that

$$d^2(x_n, p) \leq max\{d^2(x_1, p), d^2(u, p)\}, \quad \forall n \geq 1.$$

This implies that $\{x_n\}$ is a bounded sequence. So are $\{z_n\}$, $\{y_n\}$ and $\{K y_n\}$. The conclusion (I) is proved. □

(II) Next we prove that $\{x_n\}$ converges strongly to some point in Ω.

In fact, from (3.2) and (3.4) we have

$$\begin{aligned} d^2(x_{n+1},p) &= d^2((1-\beta_n)z_n \oplus \beta_n K y_n, p) \\ &\le (1-\beta_n)d^2(z_n,p) + \beta_n d^2(Ky_n,p) - \beta_n(1-\beta_n)d^2(z_n,Ky_n) \\ &\le (1-\beta_n)d^2(z_n,p) + \beta_n d^2(y_n,p) - \beta_n(1-\beta_n)d^2(z_n,Ky_n) \\ &\le (1-\beta_n)d^2(z_n,p) - \beta_n(1-\beta_n)d^2(z_n,Ky_n) \\ &\quad + \beta_n\{\alpha_n^2 d^2(u,p) + (1-\alpha_n)^2 d^2(z_n,p) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{up}, \overrightarrow{z_n p}\rangle\} \\ &\le (1-\alpha_n\beta_n)d^2(z_n,p) + \alpha_n\beta_n\{\alpha_n d^2(u,p) + 2(1-\alpha_n)\langle \overrightarrow{up}, \overrightarrow{z_n p}\rangle\} \\ &\quad - \beta_n(1-\beta_n)d^2(z_n,Ky_n) \\ &\le (1-\alpha_n\beta_n)d^2(x_n,p) + \alpha_n\beta_n\{\alpha_n d^2(u,p) + 2(1-\alpha_n)\langle \overrightarrow{up}, \overrightarrow{z_n p}\rangle\} \\ &\quad - \beta_n(1-\beta_n)d^2(z_n,Ky_n). \\ &= d^2(x_n,p) + \alpha_n\beta_n\xi_n - \beta_n(1-\beta_n)d^2(z_n,Ky_n), \end{aligned} \tag{3.6}$$

where

$$\xi_n = -d^2(x_n,p) + \alpha_n d^2(u,p) + 2(1-\alpha_n)\langle \overrightarrow{up}, \overrightarrow{z_n p}\rangle.$$

After simplifying and putting $M = \sup_{n\ge 1}|\xi_n|$, then we have

$$\beta_n(1-\beta_n)d^2(z_n,Ky_n) \le d^2(x_n,p) - d^2(x_{n+1},p) + \alpha_n\beta_n M, \tag{3.7}$$

Now we consider the following two cases:

Case 1: Assume that $\{d(x_n,p)\}$ is eventually nonincreasing. Hence there exists a sufficiently large positive integer n_0 such that $d(x_{n+1},p) \le d(x_n,p)$, $\forall n \ge n_0$. Since $\{x_n\}$ is bounded, the limit $\lim_{n\to\infty} d(x_n,p)$ exists. Since $\alpha_n \to 0$ and $0 < \varepsilon \le \beta_n \le b < 1$ (by conditions (c_1) and (c_2)), from (3.7) we have that

$$d(z_n,Ky_n) \to 0, \ \ as \ n \to \infty. \tag{3.8}$$

Also from (3.2) and Lemma 2.1(i), we obtain

$$\begin{aligned} d(y_n,z_n) &= (\alpha_n u \oplus (1-\alpha_n)z_n, z_n) \\ &\le \alpha_n d(u,z_n) \to 0, \ \ as \ n \to \infty. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9) we have

$$d(y_n,Ky_n) \to 0, \ \ as \ n \to \infty. \tag{3.10}$$

On the other hand, from (2.8) we have

$$d^2(x_n,z_n) \le d^2(x_n,p) - d^2(z_n,p). \tag{3.11}$$

Also, it follows from (3.5) that

$$d^2(x_{n+1},p) \le (1-\beta_n)d^2(x_n,p) + \beta_n\{\alpha_n d^2(u,p) + (1-\alpha_n)d^2(z_n,p)\}.$$

This implies that

$$d^2(x_n,p) \le \frac{1}{\beta_n}\{d^2(x_n,p) - d^2(x_{n+1},p)\} + \alpha_n d^2(u,p) + (1-\alpha_n)d^2(z_n,p).$$

(3.12)

Substituting (3.12) into (3.11), after simplifying we have

$$d^2(x_n, z_n) \leq \frac{1}{\beta_n}(d^2(x_n, p) - d^2(x_{n+1}, p)) + \alpha_n(d^2(u, p) - d^2(z_n, p)).$$

Since $\{z_n\}$ is bounded, $\{d(x_n, p)\}$ is convergent and $\alpha_n \to 0$, these show that

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{3.13}$$

Hence from (3.9)–(3.13) and Lemma 2.7 (3) we have

$$\begin{aligned} d(x_n, Kx_n) &\leq d(x_n, z_n) + d(z_n, y_n) + d(y_n, Ky_n) + d(Ky_n, Kx_n) \\ &\leq d(x_n, z_n) + d(z_n, y_n) \\ &\quad + d(y_n, Ky_n) + L^2 d(y_n, x_n) \to 0,\ as\ n \to \infty. \end{aligned} \tag{3.14}$$

As $\lambda_n \geq \lambda > 0$, so by Lemma 2.5 (iii) and (3.13), we have

$$\begin{aligned} d(J^f_\lambda(x_n), z_n) &= d(J^f_\lambda(x_n), J^f_{\lambda_n}(x_n)) \\ &= d(J^f_\lambda(x_n), J^f_\lambda\left(\frac{\lambda_n - \lambda}{\lambda_n} J^f_{\lambda_n}(x_n) \oplus \frac{\lambda}{\lambda_n} x_n\right) \\ &\leq d\left(x_n, \frac{\lambda_n - \lambda}{\lambda_n} J^f_{\lambda_n}(x_n) \oplus \frac{\lambda}{\lambda_n} x_n\right) \\ &\leq \left(1 - \frac{\lambda}{\lambda_n}\right) d(x_n, J^f_{\lambda_n}(x_n)) \\ &= \left(1 - \frac{\lambda}{\lambda_n}\right) d(x_n, z_n) \to 0,\ as\ n \to \infty. \end{aligned}$$

This together with (3.13) shows that

$$d(x_n, J^f_\lambda(x_n)) \leq d(x_n, z_n) + d(z_n, J^f_\lambda(x_n)) \to 0,\ as\ n \to \infty. \tag{3.15}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\Delta - \lim_{i\to\infty} x_{n_i} = x^* \in X$ and

$$\limsup_{n\to\infty}\langle \overrightarrow{ux^*}, \overrightarrow{x_n x^*}\rangle = \limsup_{i\to\infty}\langle \overrightarrow{ux^*}, \overrightarrow{x_{n_i} x^*}\rangle. \tag{3.16}$$

Since $\limsup_{i\to\infty}\langle \overrightarrow{ux^*}, \overrightarrow{x_{n_i} x^*}\rangle \leq 0$ by Remark 2.1, which shows

$$\limsup_{n\to\infty}\langle \overrightarrow{ux^*}, \overrightarrow{x_n x^*}\rangle \leq 0. \tag{3.17}$$

By virtue of (3.13), (3.16) and Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \limsup_{n\to\infty}\langle \overrightarrow{ux^*}, \overrightarrow{z_n x^*}\rangle &\leq \limsup_{n\to\infty}\langle \overrightarrow{ux^*}, \overrightarrow{z_n x_n}\rangle + \limsup_{n\to\infty}\langle \overrightarrow{ux^*}, \overrightarrow{x_n x^*}\rangle \\ &\leq \limsup_{n\to\infty} d(u, x^*) d(z_n, x_n)\rangle + \limsup_{n\to\infty}\langle \overrightarrow{ux^*}, \overrightarrow{x_n x^*}\rangle \leq 0. \end{aligned} \tag{3.18}$$

On the other hand, since K is Δ-demiclosed, from (3.14), $x^* \in Fix(K)$. Also since J^f_λ is nonexpansive, it is also Δ-demiclosed. From (3.15) $x^* \in Fix(J^f_\lambda)$. Hence $x^* \in \Omega$.

Taking $p = x^*$ in (3.6), we obtain

$$d^2(x_{n+1}, x^*) \leq (1 - \alpha_n\beta_n) d^2(x_n, x^*) + \alpha_n\beta_n\{\alpha_n d^2(u, x^*)$$

$$+2(1-\alpha_n)\langle\overrightarrow{ux^*},\overrightarrow{z_nx^*}\rangle\}. \tag{3.19}$$

Putting $a_n = d^2(x_n, x^*)$, $\delta_n = \alpha_n\beta_n$, $\sigma_n = \alpha_n d^2(u, x^*)+2(1-\alpha_n)\langle\overrightarrow{ux^*},\overrightarrow{z_nx^*}\rangle$ and $\gamma_n = 0$ in Lemma 2.9, we obtain that $d(x_n, x^*) \to 0$, i.e., $x_n \to x^* \in \Omega$.

Case 2: Assume that $\{d(x_n, p)\}$ is not eventually nonincreasing. Hence there exists a subsequence $\{n_i\} \subset \{n\}$ such that

$$d(x_{n_i}, p) < d(x_{n_i+1}, p), \quad \forall i \in N.$$

Hence by Lemma 2.8, there exists an increasing sequence $\{m_j\}$, $j \geq 1$ $m_j \to \infty$, such that

$$d(x_{m_j}, p) \leq d(x_{m_j+1}, p), \; and \;\; d(x_j, p) \leq d(x_{m_j+1}, p), \; \forall j \geq 1. \tag{3.20}$$

Also from (3.7) and the fact that $\alpha_{m_j} \to 0$, as $m_j \to \infty$ we obtain $d(z_{m_j}, Ky_{m_j}) \to 0$, as $j \to \infty$. Following arguments similar to those in the proof of Case 1, we can get

$$\limsup_{j\to\infty}\langle\overrightarrow{ux^*},\overrightarrow{x_{m_j}x^*}\rangle \leq 0. \tag{3.21}$$

Also from the inequality (3.6) we obtain that

$$\begin{aligned} d^2(x_{m_j+1}, x^*) &\leq (1-\alpha_{m_j}\beta_{m_j})d^2(x_{m_j}, x^*) \\ &\quad + \alpha_{m_j}\beta_{m_j}\{\alpha_{m_j}d^2(u, x^*) + 2(1-\alpha_{m_j})\langle\overrightarrow{ux^*},\overrightarrow{z_{m_j}x^*}\rangle\}. \end{aligned} \tag{3.22}$$

After simplifying we have

$$\begin{aligned} \alpha_{m_j}\beta_{m_j}d^2(x_{m_j}, x^*) &\leq d^2(x_{m_j}, x^*) - d^2(x_{m_j+1}, x^*) + \alpha_{m_j}\beta_{m_j}\{\alpha_{m_j}d^2(u, x^*) \\ &\quad + 2(1-\alpha_{m_j})\langle\overrightarrow{ux^*},\overrightarrow{z_{m_j}x^*}\rangle\} \\ &\leq \alpha_{m_j}\beta_{m_j}\{\alpha_{m_j}d^2(u, x^*) + 2(1-\alpha_{m_j})\langle\overrightarrow{ux^*},\overrightarrow{z_{m_j}x^*}\rangle\}. \end{aligned}$$

This implies that $d^2(x_{m_j}, x^*) \to 0$, as $j \to \infty$. From (3.22) it follows that $d^2(x_{m_j+1}, x^*) \to 0$, as $j \to \infty$. Hence from (3.20) we have that

$$\lim_{j\to\infty} d(x_j, x^*) \leq \lim_{j\to\infty} d(x_{m_j+1}, x^*) = 0, \; i.e., \; \lim_{j\to\infty} x_j = x^* \in \Omega.$$

This completes the proof of Theorem 3.1. □

In Theorem 3.1, if the mapping $T : X \to X$ is replaced by a k-demicontractive mapping, then the following result can be obtained from Theorem 3.1 immediately.

Corollary 3.2. *Let (X, d), f, $J^f_{\lambda_n}$ be the same as in Theorem 3.1. Let $T : X \to X$ be a L-Lipschitzian, k-demicontractive and Δ-demiclosed mapping with $L \geq 1$ and $k \in (0, 1)$. Denote the mapping $S : X \to X$ by*

$$Sx := \delta x \oplus (1-\delta)Tx, \; x \in X, \; 0 < k \leq \delta < 1. \tag{3.23}$$

Let $u \in X$ be a given point. For any given point $x_1 \in X$, let $\{x_n\}$ be the sequence generated by

$$\begin{cases} z_n = J^f_{\lambda_n}(x_n) := argmin_{y\in X}\left[f(y) + \dfrac{1}{2\lambda_n}d^2(y, x_n)\right], \\ y_n = \alpha_n u \oplus (1-\alpha_n)z_n, \\ x_{n+1} = (1-\beta_n)z_n \oplus \beta_n Sy_n, \end{cases} \quad \forall \; n \geq 1. \tag{3.24}$$

If $\Omega := Fix(T) \bigcap argmin_{y\in X} f(y) \neq \emptyset$ *and the sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *satisfying the following conditions:*

(c1) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(c2) $0 < \varepsilon \leq \beta_n \leq b < 1$, $\lambda_n > \lambda > 0$, $\forall n \geq 1$, *where* λ, b *and* ε *are some positive constants,*

then the sequence $\{x_n\}$ *converges strongly to some point in* Ω*.*

Proof. In order to prove Corollary 3.2, it is sufficient to prove that the mapping $S : X \to X$ defined by (3.23) has the following properties:

(1) Fix(T)=Fix(S); (2) S is demiclosed; (3) S is L-lipschitzian; (4) S is a quasi-nonexpensive mappings.

It is easy to prove that S has the properties (1)-(3). Next we prove that S has the property (4). In fact, since $Fix(T) = Fix(S)$, hence for any $p \in Fix(T) = Fix(S)$ and $x \in X$ it follows from (3.23) that

$$\begin{aligned} d^2(Sx, p) &= d^2(\delta x \oplus (1-\delta)Tx,\ p) \\ &\leq \delta d^2(x,p) + (1-\delta)d^2(Tx,\ p) - \delta(1-\delta)d^2(x,Tx) \\ &\leq \delta d^2(x,p) + (1-\delta)\{d^2(x,\ p) + kd^2(x,Tx)\} - \delta(1-\delta)d^2(x,Tx) \\ &= d^2(x,p) + (1-\delta)(k-\delta)d^2(x,Tx) \leq d^2(x,p)\ (since\ k \leq \delta). \end{aligned}$$

(4) is proved. This completes the proof of Corollary 3.2. □

Remark 3.3. Theorem 3.1 not only corrects some basic errors in Ugwunnadi et al. [1], but also extends the main results in [1] from k-demi-contractive mappings to quasi-pseudo-contractive mappings in CAT(0) space. Theorem 3.1 extends the result of Bačák [14] from weak convergence to strong convergence and the result of Cholamjiak et al. [15] from nonexpanvive mapping to Lipschitzian quasi-pseudo mapping. Also Theorem 3.1 extended the result in [16] from strict pseudo-contractive mapping in a real Hilbert space to Lipschitzian quasi-pseudo mapping in a more general space than Hilbert space. We studied a new hybrid proximal point algorithm for solving convex minimization problem as well as fixed point problem of Lipschitzian quasi-pseudo mapping in CAT(0) spaces. Our method of proof is different from that of Cholamjiak et al. [15] and Chang et al. [17].

4. Applications

Throughout this section we assume that (X, d) is a complete CAT(0) space and C is a non-empty closed and convex subset of X.

4.1. Application to convex minimization problem and equilibrium problem in CAT(0) space

The "so called" *equilibrium problem* for a bifunction $F : C \times C \to \mathbb{R}$ is to find a $x^* \in C$ such that

$$F(x^*, y) \geq 0,\ \forall y \in C, \tag{4.1}$$

where $F : C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0,\ \forall x \in C$;

(A2) F is monotone, i.e. , $F(x,y)+F(y,x)\le 0,\ \forall x,y\in C$;
(A3) The function $y\mapsto F(x,y)$ is convex for all $x\in C$;

By using F we define a mapping $T_r : X\to C,\ r>0$ as follows:

$$T_r(x):=\{z\in C : F(z,y)-\frac{1}{r}\langle\overrightarrow{yz},\ \overrightarrow{xz}\rangle\}\ge 0,\ \forall y\in C\}. \tag{4.2}$$

We have the following result

Lemma 4.1. [18] *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $F : C\times C\to\mathbb{R}$ be a bifunction satisfying the conditions (A1)–(A3). If the following condition is satisfied*

(A4) For each $\bar{x}\in X$ and $r>0$, there exists a compact subset $D_{\bar{x}}\subset C$ containing a point $y_{\bar{x}}\in D_{\bar{x}}\subset C$ such that

$$F(x,y_{\bar{x}})-\frac{1}{r}\langle\overrightarrow{xy_{\bar{x}}},\ \overrightarrow{x\bar{x}}\rangle<0,\forall x\in C\setminus D_{\bar{x}},$$

then, the following conclusions hold:

(a) *T_r is well defined in X and T_r is a single-valued mapping;*
(b) *T_r is firmly nonexpansive restricted to C, i.e., $\forall x,y\in C$*

$$d^2(T_rx,\ T_ry)\le\langle\overrightarrow{T_rxT_ry},\ \overrightarrow{xy}\rangle;$$

Therefore T_r is a a nonexpansive (i.e., 1-Lipschitzian) and demiclosed mapping restricted to C. In addition, if $Fix(T_r)\ne\emptyset$, then T_r is quasi-nonexpansive.

(c) *$Fix(T_r)=\Omega_1$, where Ω_1 is the solution set of problem (4.1);*
(d) *If $Fix(T_r)\ne\emptyset$, we have*

$$d^2(T_rx,x)\le d^2(x,p)-d^2(T_rx,p),\ \forall x\in C,\ \text{and}\ \forall p\in Fix(T_r).$$

Taking $T=K=T_r$ in Theorem 3.1, then the following theorem can be obtained from Theorem 3.1 and Lemma 4.1 immediately.

Theorem 4.2. *Let X be a complete CAT(0) space, C be a nonempty closed and convex subset of X. Let $f : C\to\mathbb{R}$ be a proper convex and lower semi-continuous function, and $J^f_{\lambda_n} : C\to C$ be the Moreau-Yosida resolvent of f. Let $F : C\times C\to\mathbb{R}$ be a bifunction satisfying the conditions (A1)- (A4) and $T_r,\ r>0$ be the mapping defined by (4.2). Let $u\in X$ be a given point. For any given point $x_1\in X$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} z_n=J^f_{\lambda_n}(x_n):=argmin_{y\in X}\left[f(y)+\dfrac{1}{2\lambda_n}d^2(y,x_n)\right], \\ y_n=\alpha_n u\oplus(1-\alpha_n)z_n, \\ x_{n+1}=(1-\beta_n)z_n\oplus\beta_n T_r y_n, \end{cases} \quad n\ge 1. \tag{4.3}$$

If $\Omega_2:=Fix(T_r)\bigcap argmin_{y\in X}f(y)\ne\emptyset$ and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(c1) $\{\alpha_n\}\subset(0,1),\ \lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^\infty\alpha_n=\infty$, (c2) $0 < \varepsilon \leq \beta_n \leq b < 1,\ \lambda_n > \lambda > 0,\ \forall n \geq 1$, *where* $\varepsilon,\ b$ *and* λ *are some positive constants,*

then (1) the sequence $\{x_n\}$ *converges strongly to some point in* Ω_2.

(2) Especially, if $f \equiv 0$, *then* $J^f_{\lambda_n} = I$ *(identity mapping) for all* $n \geq 1$. *Hence the sequence* $\{x_n\}$ *defined by*

$$\begin{cases} y_n = \alpha_n u \oplus (1-\alpha_n)x_n, \\ x_{n+1} = (1-\beta_n)x_n \oplus \beta_n T_r y_n, \end{cases} \quad n \geq 1. \tag{4.4}$$

converges strongly to a solution of equilibrium problem (4.1).

4.2. Application to saddle point problem in CAT(0) spaces

Let X_1 and X_2 be complete CAT(0) spaces. Then the product space $X = X_1 \times X_2$ is also a complete CAT(0) space (see [19, Page 239]. A function $H : X_1 \times X_2 \to \mathbb{R}$ is called a *saddle function* if

(i) $y \mapsto H(x, y)$ is convex on X_2 for each $x \in X_1$ and

(ii) $x \mapsto H(x, y)$ is concave, i.e., $x \mapsto -H(x, y)$ is convex on X_1 for each $y \in X_2$.

A point $z^* = (x^*, y^*) \in X_1 \times X_2$ is said to be a saddle point of H if

$$H(x, y^*) \leq H(x^*, y^*) \leq H(x^*, y),\ \forall z = (x, y) \in X_1 \times X_2. \tag{4.5}$$

We denote by Ω_3 the set of saddle points of problem (4.5).

Let $V_H : X = X_1 \times X_2 \to 2^{X_1^*} \times 2^{X_2^*}$ be a multivalued mapping associated with saddle function H (where X_i^* is the dual space of $X_i,\ i = 1, 2$, (see, (2.4)) defined by

$$V_H(x, y) = \partial(-H(., y))(x) \times \partial(H(x, .))(y),\ \forall (x, y) \in X_1 \times X_2, \tag{4.6}$$

Let us define the resolvent $J^{V_H}_\lambda : X = X_1 \times X_2 \to 2^{X_1 \times X_2}$ of V_H of order $\lambda > 0$ by

$$J^{V_H}_\lambda(x) := \{z \in X : [\frac{1}{\lambda}\overrightarrow{zx}] \in V_H(z)\},\ x \in X = X_1 \times X_2. \tag{4.7}$$

The following results hold.

Lemma 4.3. [20] *Let* X_1 *and* X_2 *be complete CAT(0) spaces,* H *be a saddle function on* $X = X_1 \times X_2$ *and* V_H *be the multivalued mapping defined by (4.6). Then*

(1) $J^{V_H}_\lambda : X \to X,\ \lambda > 0$ *is a single-valued and firmly nonexpansive mapping;*

(2) *A point* $z^* = (x^*, y^*) \in X$ *is a saddle point of* H *if and only if* $z^* \in Fix(J^{V_H}_\lambda)$.

In Theorem 3.1, taking $f \equiv 0,\ T = K = J^{V_H}_\lambda$, then the following result can be obtained from Theorem 3.1 immediately.

Theorem 4.4. *Let* $X_1,\ X_2,\ X,\ H,\ V_H$ *and* $J^{V_H}_\lambda$ *be the same as in Lemma 4.3. Let* $u \in X$ *be a given point. For any given point* $x_1 \in X$, *let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} y_n = \alpha_n u \oplus (1-\alpha_n)x_n, \\ x_{n+1} = (1-\beta_n)x_n \oplus \beta_n J^{V_H}_\lambda(y_n), \end{cases} \quad n \geq 1. \tag{4.8}$$

If $Fix(J_\lambda^{V_H}) \neq \emptyset$ *and the sequences* $\{\alpha_n\}$, *and* $\{\beta_n\}$ *satisfy the following conditions:*

(c1) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(c2) $0 < \varepsilon \leq \beta_n \leq b < 1$, $\forall n \geq 1$, *where* ε, b *are some positive constants, then the sequence* $\{x_n\}$ *converges strongly to a saddle point of problem (4.5).*

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