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Periodic boundary value problems involving Stieltjes derivatives

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Abstract. We are concerned with the study of a first-order nonlinear periodic boundary value problem

$$\begin{cases} u_g'(t) + b(t)u(t) = f(t, u(t)), \ t \in [0, T] \\ u(0) = u(T) \end{cases} \tag{1}$$

involving the Stieltjes derivative with respect to a left-continuous nondecreasing function. Based on Schaeffer's fixed point theorem and making use of a notion of partial Stieltjes derivative (along with its natural properties), we prove the existence of regulated solutions and provide a useful characterization in terms of Stieltjes integrals. The generality of our result is coming from the impressive number of particular cases of the described problem. Thus, first-order periodic differential equations, impulsive differential problems (including also the possibility to have Zeno points, i.e. accumulations of impulse moments), dynamic equations on time scales or generalized differential equations can all be studied through the theory of Stieltjes differential equations.

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1. Introduction

The theory of differential equations driven by measures has been continuously growing over the last decade (e.g. [2,4,7–9,21,27]) since it offers a tool to study in a unified way several classical problems: first-order differential equations (in the case when the driving measure is absolutely continuous with respect to the Lebesgue measure), impulsive differential problems (when we take into consideration a measure which can be written as a sum of Lebesgue measure with a discrete measure) with no limitations on the impulse moments, dynamic equations on time scales (see [4,8,9]) and generalized differential equations (e.g. [16,21,28,29]).

On the other hand, an equivalent formulation in terms of a notion of (Stieltjes) derivative with respect to a nondecreasing function is available (c.f. [22], see also [19]). This derivative, considered in [22] (even if the idea is not really new in literature, c.f. [31]) has found recent interesting applications in biology, population dynamics or chemistry (see [12,13] or [23]).
At the same time, it is well known that differential problems with periodic boundary conditions have wide applicability in various areas of science.
Relying on these considerations, we focus on first order nonlinear periodic boundary value problems of the form (1):

$$\begin{cases} u'_g(t) + b(t)u(t) = f(t, u(t)), \ t \in [0, T] \\ u(0) = u(T) \end{cases}$$

involving the Stieltjes derivative with respect to a function $g : [0, T] \to \mathbb{R}$ left-continuous and nondecreasing.

The maps $b : [0, T] \to \mathbb{R}$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ are supposed to be continuous at the continuity points of g.

In two steps (first, for the linear and then, applying Schaeffer's fixed point theorem, for the general case), we prove that the specified problem possesses solutions and provide a useful integral characterization via a Green function.
We essentially use an idea borrowed from [12] which consists in modifying in an appropriate manner the involved functions at the discontinuity points of g to adjust the g-derivative at these points.

This is, as far as the authors know, the first existence result for periodic boundary value problems in such a general framework (involving Stieltjes derivatives).

In the same spirit as described above for measure differential problems, in particular when g is the sum of an absolutely continuous function with a sum of Heaviside functions, we cover the framework of periodic impulsive differential problems, therefore we can deduce new results for impulsive equations (studied, under different assumptions, e.g. in [3,17,33]). Moreover, the number of impulses can be not only finite, but countable and it can contain accumulation points (known as Zeno points in the theory of hybrid systems, e.g. [18]). Our outcome is also related to some existence results for periodic dynamic equations on time scales (see [5,14] or [32]). We finally note that the same problem has also been studied in the framework of fractional differential equations (e.g. [1]) or in that of functional impulsive differential equations (as in [15]).

2. Notations and auxiliary results

A function $u : [0, T] \to \mathbb{R}$ is said to be *regulated* ([10]) if there exist the right and left limits $u(t+)$ and $u(s-)$ at every points $t \in [0, T)$ and $s \in (0, T]$. The set of discontinuity points of a regulated function is at most countable ([25]) and any function of bounded variation (and also any continuous function) is regulated. Regulated functions are bounded and the space of these functions is a Banach space when endowed with the norm $\|u\|_C = \sup_{t\in[0,T]} |u(t)|$.

A family $\mathcal{A}$ of regulated real-valued functions on $[0,T]$ is called *equiregulated* if for every $\bar{t}\in[0,T]$ and every $\varepsilon>0$ one can find $\delta>0$ such that for any $u\in\mathcal{A}$

$$|u(t)-u(\bar{t}-)|<\varepsilon,\ \text{for every } t\in(\bar{t}-\delta,\bar{t})$$

and

$$|u(t)-u(\bar{t}+)|<\varepsilon,\ \text{for every } t\in(\bar{t},\bar{t}+\delta).$$

The notion of equiregulatedness is related to compactness in the space of regulated functions.

Lemma 1. ([10, Corollary 2.4]) *A set of regulated functions is relatively compact if and only if it is equiregulated and pointwise bounded.*

This is a consequence of another interesting result.

Lemma 2. [10] *Let $(f_n)_n$ be an equiregulated sequence of functions which converges pointwise to a function f. Then $(f_n)_n$ converges uniformly to f.*

The following remark will be useful later:

Remark 3. Let $\mathcal{A}$ be a set of regulated functions. If there exists a regulated function $\chi:[0,T]\to\mathbb{R}$ such that for every $u\in\mathcal{A}$,

$$|u(t)-u(t')|\le|\chi(t)-\chi(t')|,\quad \forall\, 0\le t<t'\le T,$$

then $\mathcal{A}$ is equiregulated.

Let $g:[0,T]\to\mathbb{R}$ be a nondecreasing left-continuous function. In the whole paper, we deal with the Kurzweil–Stieltjes integral; to be allowed to use its properties, we recall below the basic facts concerning this type of integral.

Definition 4. [16,20,25,26,28] or [30] A function $f:[0,T]\to\mathbb{R}$ is said to be Kurzweil–Stieltjes integrable with respect to $g:[0,T]\to\mathbb{R}$ on $[0,T]$ (or KS-integrable) if there exists $\int_0^T f(s)\mathrm{d}g(s)\in\mathbb{R}$ such that, for every $\varepsilon>0$, there is a positive function δ_ε on $[0,T]$ with

$$\left|\sum_{i=1}^{p} f(\xi_i)(g(t_i)-g(t_{i-1}))-\int_0^T f(s)\mathrm{d}g(s)\right|<\varepsilon$$

for every δ_ε-fine partition $\{([t_{i-1},t_i],\xi_i):\ i=1,\dots,p\}$ of $[0,T]$.

A partition $\{([t_{i-1},t_i],\xi_i):\ i=1,\dots,p\}$ is δ_ε-fine if for all $i=1,\dots,p$, $[t_{i-1},t_i]\subset\,]\xi_i-\delta_\varepsilon(\xi_i),\xi_i+\delta_\varepsilon(\xi_i)[$. The KS-integrability is preserved on all sub-intervals of $[0,T]$.

In the particular case, where $g(t)=t$ for every $t\in[0,T]$, one finds the Henstock–Kurzweil integral (see [11]).
It is known that regulated functions are KS-integrable with respect to bounded variation functions and vice versa (see [30]). The properties of the primitive contained in the proposition below are important in what follows.

Proposition 5. ([30, Proposition 2.3.16]) *Let $f : [0, T] \to \mathbb{R}$ be KS-integrable w.r.t. $g : [0, T] \to \mathbb{R}$. If g is regulated, then so is the primitive $F : [0, T] \to \mathbb{R}$, $F(t) = \int_0^t f(s)\mathrm{d}g(s)$ and for every $t \in [0, T)$ and $s \in (0, T]$,*

$$F(t+) - F(t) = f(t)\left[g(t+) - g(t)\right] \quad \textit{and} \quad F(s) - F(s-) = f(s)\left[g(s) - g(s-)\right].$$

It follows that F is left-continuous, respectively, right-continuous at the points where g has the same property.
Moreover, when g is of bounded variation and f is bounded, F is of bounded variation as well.

Note that the Lebesgue–Stieltjes integrability of a function f (i.e., the abstract Lebesgue integrability w.r.t. the Stieltjes measure μ_g generated by g, see [24, Example 6.14]) implies the Kurzweil–Stieltjes integrability. In the framework of a left-continuous nondecreasing function g, as a consequence of [20, Theorem 6.11.3] (see also [25, Theorem 8.1]), for $t \in [0, T]$,

$$\int_0^t f(s)\mathrm{d}g(s) = \int_{[0,t]} f(s)\mathrm{d}\mu_g(s) - f(t)(g(t+) - g(t)) = \int_{[0,t)} f(s)\mathrm{d}\mu_g(s).$$

In [22], a notion of differentiability related to Stieltjes type integrals was introduced (following an idea in [31]).

Definition 6. Let $g : [0, T] \to \mathbb{R}$ be a nondecreasing left-continuous function. The derivative with respect to g (or the g-derivative) of a function $f : [0, T] \to \mathbb{R}$ at a point $\bar{t} \in [0, T]$ is given by

$$f'_g(\bar{t}) = \lim_{t \to \bar{t}} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } g \text{ is continuous at } \bar{t},$$

$$f'_g(\bar{t}) = \lim_{t \to \bar{t}+} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } g \text{ is discontinuous at } \bar{t},$$

provided the limit exists.

Define the following sets:

$$C_g = \{t \in [0, T] : g \text{ is constant on } (t - \varepsilon, t + \varepsilon) \text{ for some } \varepsilon > 0\}$$
$$D_g = \{t \in [0, T] : g(t+) - g(t) > 0\}.$$

It is worth mentioning that D_g is the set of atoms of the measure μ_g and if $t \in D_g$, the g-derivative $f'_g(t)$ exists if and only if the sided limit $f(t+)$ exists, and in this case

$$f'_g(t) = \frac{f(t+) - f(t)}{g(t+) - g(t)}.$$

Note that Definition 6 has no meaning in C_g. Anyway (see [22]), this set is not significant, in the sense that $\mu_g(C_g) = 0$.

What's more, if $C_g = \bigcup_{n \in \mathbb{N}} (u_n, v_n)$ is a disjoint decomposition of C_g and

$$N_g = \{u_n, v_n : n \in \mathbb{N}\} \setminus D_g,$$

then Definition 6 has no meaning in N_g neither, but again $\mu_g(N_g) = 0$.

This type of derivative has already been used in solving various problems where abrupt changes (corresponding to discontinuity points of g) and dead times (corresponding to intervals where g is constant) are present, such as [12,13,23] or [27].

Fundamental Theorems of Calculus are essential when taking into account the connection between integrals and derivatives. Such a result for Kurzweil–Stieltjes integrals can be found in [22, Theorem 6.5]:

Theorem 7. *Let $g : [0,T] \to \mathbb{R}$ be a nondecreasing left-continuous function. If $f : [0,T] \to \mathbb{R}$ is KS-integrable with respect to g and*

$$F(t) = \int_0^t f(s)\,\mathrm{d}g(s), \quad t \in [0,T],$$

then F is g-differentiable μ_g-a.e. on $[0,T]$ and $F'_g = f$.

It would be useful to know more precisely at which points the differentiability w.r.t. g is achieved.

Proposition 8. *Let $f : [0,T] \to \mathbb{R}$ be KS-integrable w.r.t. g and let $F(t) = \int_0^t f(s)\,\mathrm{d}g(s)$ be its primitive. Then F is g-differentiable (with the g-derivative equal to $f(t)$) at any point $t \in [0,T] \setminus (C_g \cup N_g)$ where f is continuous.*

Proof. Let $\bar{t} \in [0,T] \setminus (C_g \cup N_g)$ where f is continuous.

If $\bar{t} \notin D_g$, then

$$F'_g(\bar{t}) = \lim_{t \to \bar{t}} \frac{F(t) - F(\bar{t})}{g(t) - g(\bar{t})}.$$

We can write

$$\begin{aligned}\lim_{t \to \bar{t}, t > \bar{t}} \min_{\tau \in [\bar{t}, t)} f(\tau) &= \lim_{t \to \bar{t}, t > \bar{t}} \frac{\min_{\tau \in [\bar{t}, t)} f(\tau)(g(t) - g(\bar{t}))}{g(t) - g(\bar{t})} \\ &\le \lim_{t \to \bar{t}, t > \bar{t}} \frac{F(t) - F(\bar{t})}{g(t) - g(\bar{t})}\end{aligned}$$

and

$$\begin{aligned}\lim_{t \to \bar{t}, t > \bar{t}} \frac{F(t) - F(\bar{t})}{g(t) - g(\bar{t})} &\le \lim_{t \to \bar{t}, t > \bar{t}} \frac{\max_{\tau \in [\bar{t}, t)} f(\tau)(g(t) - g(\bar{t}))}{g(t) - g(\bar{t})} \\ &= \lim_{t \to \bar{t}, t > \bar{t}} \max_{\tau \in [\bar{t}, t)} f(\tau),\end{aligned}$$

whence, by the continuity of f at $\bar{t}$,

$$\lim_{t \to \bar{t}, t > \bar{t}} \frac{F(t) - F(\bar{t})}{g(t) - g(\bar{t})} = f(\bar{t}).$$

Similarly, it can be proved that

$$\lim_{t \to \bar{t}, t < \bar{t}} \frac{F(t) - F(\bar{t})}{g(t) - g(\bar{t})} = f(\bar{t}),$$

therefore

$$F'_g(\bar{t}) = f(\bar{t}).$$

On the other hand, if $\bar{t} \in D_g$, then

$$F_g'(\bar{t}) = \lim_{t \to \bar{t}, t > \bar{t}} \frac{F(t) - F(\bar{t})}{g(t) - g(\bar{t})}$$

and, as before, it equals $f(\bar{t})$. □

In the same line, we could define partial g-derivatives.

Definition 9. The partial g-derivative w.r.t. the variable t of a function $f : [0,T] \times [0,T] \to \mathbb{R}$ at a point $(\bar{t}, \bar{s}) \in [0,T] \times [0,T]$ is given by

$$\frac{\partial f}{\partial_g t}(\bar{t}, \bar{s}) = \lim_{t \to \bar{t}} \frac{f(t, \bar{s}) - f(\bar{t}, \bar{s})}{g(t) - g(\bar{t})} \quad \text{if } g \text{ is continuous at } \bar{t},$$
$$\frac{\partial f}{\partial_g t}(\bar{t}, \bar{s}) = \lim_{t \to \bar{t}+} \frac{f(t, \bar{s}) - f(\bar{t}, \bar{s})}{g(t) - g(\bar{t})} \quad \text{if } g \text{ is discontinuous at } \bar{t}$$

provided the limit exists.

Let us recall the chain rules for g-derivatives at the continuity points of g (borrowed from [22]):

Lemma 10. ([22, Theorem 2.3]) *Let f be a real function defined on a neighborhood of $t \in \mathbb{R} \backslash D_g$ and h be another function defined on a neighborhood of $f(t)$. Then*

1. *If $h'(f(t))$ and $f_g'(t)$ exist, then*
$$(h \circ f)_g'(t) = h'(f(t)) \cdot f_g'(t).$$
2. *If $h_g'(f(t))$, $g'(f(t))$ and $f_g'(t)$ exist, then*
$$(h \circ f)_g'(t) = h_g'(f(t)) \cdot g'(f(t)) \cdot f_g'(t).$$

We deduce several auxiliary (technical) results concerning the partial derivative of a function with respect to another function:

Lemma 11. i) *Let $h : \mathbb{R} \to \mathbb{R}$ be differentiable and $f : \mathbb{R}^2 \to \mathbb{R}$ be partially g-differentiable w.r.t. t at some point (t, s) with $t \in \mathbb{R} \backslash D_g$. Then $h \circ f$ is partially g-differentiable w.r.t. t and*

$$\frac{\partial (h \circ f)}{\partial_g t}(t, s) = h'(f(t, s)) \cdot \frac{\partial f}{\partial_g t}(t, s).$$

ii) *Let $h : \mathbb{R}^2 \to \mathbb{R}$ be partially g-differentiable w.r.t. both arguments u, v and $f, j : \mathbb{R} \to \mathbb{R}$ be g-differentiable at some point $t \in \mathbb{R} \backslash D_g$ such that g is differentiable at $f(t)$ and $j(t)$. Then $H(t) = h(f(t), j(t))$ is g-differentiable at t and*

$$H_g'(t) = \frac{\partial h}{\partial_g u}(f(t), j(t)) \cdot g'(f(t)) \cdot f_g'(t) + \frac{\partial h}{\partial_g v}(f(t), j(t)) \cdot g'(j(t)) \cdot j_g'(t).$$

Consequently, we can get the following version of Leibniz's rule for g-differentiation of Stieltjes integrals depending on a parameter:

Lemma 12. *Let* $g : [0, T] \to \mathbb{R}$ *be nondecreasing and left-continuous and* $\omega : [0, T] \times [0, T] \to \mathbb{R}$ *partially* g*-differentiable w.r.t.* t *be such that* $s \to \omega(t, s)$ *is Kurzweil–Stieltjes integrable w.r.t.* g*. Then*

$$J(t) = \int_{\phi(t)}^{\psi(t)} \omega(t, s)\mathrm{d}g(s),\ t \in [0, T]$$

is g*-differentiable at each point* $t \in [0, T]\backslash D_g$*, where*

- t, ϕ *and* ψ *are* g*-differentiable,*
- g *is differentiable at* t, $\phi(t)$ *and* $\psi(t)$*,*
- $s \to \omega(t, s)$ *is continuous at* $\phi(t)$ *and* $\psi(t)$*.*

Besides, $J'_g(t) =$

$$\int_{\phi(t)}^{\psi(t)} \frac{\partial \omega}{\partial_g t}(t, s)\mathrm{d}g(s) + \omega(t, \psi(t)) \cdot g'(\psi(t)) \cdot \psi'_g(t) - \omega(t, \phi(t)) \cdot g'(\phi(t)) \cdot \phi'_g(t).$$

Proof. Defining

$$\tilde{J}(u, v, t) = \int_u^v \omega(t, s)\mathrm{d}g(s)$$

one notices that

$$J(t) = \tilde{J}(\phi(t), \psi(t), t).$$

By Lemma 11,

$$J'_g(t) = \frac{\partial \tilde{J}}{\partial_g u} \cdot g'(u) \cdot u'_g + \frac{\partial \tilde{J}}{\partial_g v} \cdot g'(v) \cdot v'_g + \frac{\partial \tilde{J}}{\partial_g t} \cdot g'(t) \cdot t'_g.$$

Applying Proposition 8, whenever $s \to \omega(t, s)$ is continuous at u (resp. at v), one gets

$$\frac{\partial \tilde{J}}{\partial_g u} = -\omega(t, u), \quad \frac{\partial \tilde{J}}{\partial_g v} = \omega(t, v),$$

whence

$$J'_g(t) = -\omega(t, \phi(t)) \cdot g'(\phi(t)) \cdot \phi'_g(t) + \omega(t, \psi(t)) \cdot g'(\psi(t)) \cdot \psi'_g(t) + \int_{\phi(t)}^{\psi(t)} \frac{\partial \omega}{\partial_g t}(t, s)\mathrm{d}g(s) \cdot g'(t) \cdot t'_g.$$

At the same time, by definition, at a continuity point of g,

$$t'_g = \lim_{t' \to t} \frac{t' - t}{g(t') - g(t)},$$

while

$$g'(t) = \lim_{t' \to t} \frac{g(t') - g(t)}{t' - t},$$

and so,

$$g'(t) \cdot t'_g = 1.$$

□

A formula for the g-derivative of a product of two functions will also be necessary:

Lemma 13. *Let $f, h : [0, T] \to \mathbb{R}$ be g-differentiable at $\bar{t} \in D_g$. Then $f \cdot h$ is also g-differentiable at $\bar{t}$ and*

$$(f \cdot h)'_g(\bar{t}) = f'_g(\bar{t})h(\bar{t}+) + f(\bar{t})h'_g(\bar{t}).$$

Proof. Since $\bar{t} \in D_g$,

$$\begin{aligned}(f \cdot h)'_g(\bar{t}) &= \lim_{t \to \bar{t}, t > \bar{t}} \frac{(f \cdot h)(t) - (f \cdot h)(\bar{t})}{g(t) - g(\bar{t})} \\ &= \lim_{t \to \bar{t}, t > \bar{t}} \frac{(f(t) - f(\bar{t}))h(t) + f(\bar{t})(h(t) - h(\bar{t}))}{g(t) - g(\bar{t})} \\ &= f'_g(\bar{t})h(\bar{t}+) + f(\bar{t})h'_g(\bar{t}).\end{aligned}$$

□

3. Main results

Our goal is to provide existence of solutions for the boundary value problem with nonlinear right-hand side:

$$\begin{cases} u'_g(t) + b(t)u(t) = f(t, u(t)), \ t \in [0, T] \\ u(0) = u(T). \end{cases}$$

involving the g-derivative.

We impose the nonresonance condition

$$1 - b(t)\mu_g(\{t\}) \neq 0, \quad \text{for every } t \in [0, T].$$

Definition 14. A left-continuous, regulated function $u : [0, T] \to \mathbb{R}$ is a solution of our problem if it is constant on any interval where g is constant, g-differentiable μ_g-a.e. and it verifies the equality

$$u'_g(t) + b(t)u(t) = f(t, u(t)), \ \mu_g - a.e. \ on \ [0, T]$$

and the condition

$$u(0) = u(T).$$

3.1. Existence result for the linear problem

First, we study the linear periodic Stieltjes differential equation

$$\begin{cases} u'_g(t) + b(t)u(t) = f(t), \ t \in [0, T] \\ u(0) = u(T) \end{cases} \tag{2}$$

under the assumptions that $b : [0, T] \to \mathbb{R}$ and $f : [0, T] \to \mathbb{R}$ are continuous on $[0, T] \setminus D_g$.

To solve the problem (2), we must take into account the sign of $1 - b(t)\mu_g(\{t\})$. As in [12], if $b \in L^1_g([0, T]$, the set

$$D_g^- = \{t \in D_g : 1 - b(t)\mu_g(\{t\}) < 0\}$$

is finite since

$$\infty > \|b\|_{L^1_g} > \sum_{t\in D_g^-} b(t)\mu_g(\{t\}) > \sum_{t\in D_g^-} 1.$$

Denote by $t_1 < \cdots < t_k$ its elements and, for simplicity, let $t_0 = 0$ and $t_{k+1} = T$.
Let

$$\alpha(t) = \begin{cases} 1, \ if\ 0 \le t \le t_1 \\ (-1)^i, \ if\ t_i < t \le t_{i+1}, \ i = 1, \dots, k. \end{cases}$$

To simplify the proof of the existence theorem, we will use the following lemma.

Lemma 15. *Let $\bar{t} \in D_g$ and $c, h : [0, T] \to \mathbb{R}$ be KS-integrable w.r.t. g. Then*

i) *The function $t \to e^{\int_0^t c(r)\mathrm{d}g(r)}$ is g-differentiable at $\bar{t}$ and*

$$\left(e^{\int_0^t c(r)\mathrm{d}g(r)}\right)'_g(\bar{t}) = e^{\int_0^{\bar{t}} c(r)\mathrm{d}g(r)} \cdot \frac{e^{c(\bar{t})\mu_g(\{\bar{t}\})} - 1}{\mu_g(\{\bar{t}\})}.$$

ii)

1. *The function*

$$F(t) = \frac{1}{\alpha(t)} \int_0^t \alpha(s) e^{-\int_s^t c(r)\mathrm{d}g(r)} h(s)\mathrm{d}g(s)$$

is g-differentiable at $\bar{t}$ and

$$F'_g(\bar{t}) = \frac{e^{-c(\bar{t})\mu_g(\{\bar{t}\})} - 1}{\mu_g(\{\bar{t}\})} \cdot F(\bar{t}) + e^{-c(\bar{t})\mu_g(\{\bar{t}\})} h(\bar{t})$$

if $\bar{t} \in D_g \backslash \{t_1, \dots, t_k\}$ and

$$F'_g(\bar{t}) = \frac{-e^{-c(\bar{t})\mu_g(\{\bar{t}\})} - 1}{\mu_g(\{\bar{t}\})} \cdot F(\bar{t}) - e^{-c(\bar{t})\mu_g(\{\bar{t}\})} h(\bar{t})$$

if $\bar{t} = t_i, i = 1, \dots, k$.

2. *The function*

$$G(t) = \frac{1}{\alpha(t)} \int_t^T \alpha(s) e^{-\int_s^t c(r)\mathrm{d}g(r)} h(s)\mathrm{d}g(s)$$

is g-differentiable at $\bar{t}$ and

$$G'_g(\bar{t}) = \frac{e^{-c(\bar{t})\mu_g(\{\bar{t}\})} - 1}{\mu_g(\{\bar{t}\})} \cdot G(\bar{t}) - e^{-c(\bar{t})\mu_g(\{\bar{t}\})} h(\bar{t})$$

if $\bar{t} \in D_g \backslash \{t_1, \dots, t_k\}$ and

$$G'_g(\bar{t}) = \frac{-e^{-c(\bar{t})\mu_g(\{\bar{t}\})} - 1}{\mu_g(\{\bar{t}\})} \cdot G(\bar{t}) + e^{-c(\bar{t})\mu_g(\{\bar{t}\})} h(\bar{t})$$

if $\bar{t} = t_i, i = 1, \dots, k$.

Proof. i) By the definition of the g-derivative at a discontinuity point and using Proposition 5,

$$\left(e^{\int_0^t c(r)\mathrm{d}g(r)}\right)'_g(\bar{t}) = \frac{e^{\int_0^{\bar{t}+} c(r)\mathrm{d}g(r)} - e^{\int_0^{\bar{t}} c(r)\mathrm{d}g(r)}}{\mu_g(\{\bar{t}\})}$$
$$= \frac{e^{\int_0^{\bar{t}} c(r)\mathrm{d}g(r)}\left(e^{\int_{\bar{t}}^{\bar{t}+} c(r)\mathrm{d}g(r)} - 1\right)}{\mu_g(\{\bar{t}\})}$$
$$= e^{\int_0^{\bar{t}} c(r)\mathrm{d}g(r)} \cdot \frac{e^{c(\bar{t})\mu_g(\{\bar{t}\})} - 1}{\mu_g(\{\bar{t}\})}.$$

ii). 1. By definition, at any point $\bar{t} \in D_g \setminus \{t_1, \dots, t_k\}$,

$$F'_g(\bar{t}) = \frac{F(\bar{t}+) - F(\bar{t})}{\mu_g(\{\bar{t}\})}$$
$$= \frac{1}{\mu_g(\{\bar{t}\})}\left[\frac{1}{\alpha(\bar{t}+)}\int_0^{\bar{t}+} \alpha(s)e^{-\int_s^{\bar{t}+} c(r)\mathrm{d}g(r)}h(s)\mathrm{d}g(s)\right.$$
$$\left. - \frac{1}{\alpha(\bar{t})}\int_0^{\bar{t}} \alpha(s)e^{-\int_s^{\bar{t}} c(r)\mathrm{d}g(r)}h(s)\mathrm{d}g(s)\right]$$
$$= \frac{1}{\mu_g(\{\bar{t}\})}\left[\frac{1}{\alpha(\bar{t})}\int_0^{\bar{t}} \alpha(s)e^{-\int_s^{\bar{t}} c(r)\mathrm{d}g(r)}e^{-\int_{\bar{t}}^{\bar{t}+} c(r)\mathrm{d}g(r)}h(s)\mathrm{d}g(s)\right.$$
$$+\frac{1}{\alpha(\bar{t})}\int_{\bar{t}}^{\bar{t}+} \alpha(s)e^{-\int_s^{\bar{t}+} c(r)\mathrm{d}g(r)}h(s)\mathrm{d}g(s)$$
$$\left. - \frac{1}{\alpha(\bar{t})}\int_0^{\bar{t}} \alpha(s)e^{-\int_s^{\bar{t}} c(r)\mathrm{d}g(r)}h(s)\mathrm{d}g(s)\right].$$

Using Proposition 5,

$$F'_g(\bar{t}) = \frac{1}{\alpha(\bar{t})}\frac{e^{-c(\bar{t})\mu_g(\{\bar{t}\})} - 1}{\mu_g(\{\bar{t}\})} \cdot \int_0^{\bar{t}} \alpha(s)e^{-\int_s^{\bar{t}} c(r)\mathrm{d}g(r)}h(s)\mathrm{d}g(s)$$
$$+\frac{1}{\alpha(\bar{t})}\frac{\alpha(\bar{t})e^{-c(\bar{t})\mu_g(\{\bar{t}\})}h(\bar{t})\mu_g(\{\bar{t}\})}{\mu_g(\{\bar{t}\})}$$
$$= \frac{e^{-c(\bar{t})\mu_g(\{\bar{t}\})} - 1}{\mu_g(\{\bar{t}\})} \cdot F(\bar{t}) + e^{-c(\bar{t})\mu_g(\{\bar{t}\})}h(\bar{t}).$$

On the other hand, if $\bar{t} = t_i, i \in \{1, \dots, k\}$, then $\alpha(\bar{t}+) = -\alpha(\bar{t})$, so

$$F'_g(\bar{t}) = \frac{F(\bar{t}+) - F(\bar{t})}{\mu_g(\{\bar{t}\})}$$
$$= \frac{-e^{-c(\bar{t})\mu_g(\{\bar{t}\})} - 1}{\mu_g(\{\bar{t}\})} \cdot F(\bar{t}) - e^{-c(\bar{t})\mu_g(\{\bar{t}\})}h(\bar{t}).$$

The g-derivative of G can be computed in a similar way. □

Consider, following an idea used in [12] for initial value problems driven by measures,

$$\tilde{b}(t) = \begin{cases} b(t), \ if \ t \in [0,T] \backslash D_g \\ \frac{-\log|1-b(t)\mu_g(\{t\})|}{\mu_g(\{t\})}, \text{ if } t \in D_g \end{cases}$$

and

$$\tilde{f}(t) = \frac{f(t)}{1-b(t)\mu_g(\{t\})}.$$

Notice that $\tilde{f}(t) = f(t)$ whenever $t \notin D_g$.

Suppose that

$$\sum_{t \in D_g} |\log|1-b(t)\mu_g(\{t\})|| < \infty. \tag{3}$$

It implies that there exists a positive constant δ such that

$$|1-b(t)\mu_g(\{t\})| > \delta, \ \forall t \in D_g.$$

Indeed, if we consider the countable set D_g written as a sequence $(\tilde{t}_n)_n$, then

$$\big|\log|1-b(\tilde{t}_n)\mu_g(\{\tilde{t}_n\})|\big| \to 0 \quad \text{as } n \to \infty$$

which means that

$$|1-b(\tilde{t}_n)\mu_g(\{\tilde{t}_n\})| \to 1 \quad \text{as } n \to \infty$$

therefore there exists a positive constant δ such that

$$|1-b(t)\mu_g(\{t\})| > \delta, \ \forall t \in D_g.$$

What's more, as D_g^- is finite, from one place onwards $\tilde{t}_n \in D_g \backslash D_g^-$, whence

$$1-b(\tilde{t}_n)\mu_g(\{\tilde{t}_n\}) = |1-b(\tilde{t}_n)\mu_g(\{\tilde{t}_n\})| \to 1,$$

thus $b(\tilde{t}_n)\mu_g(\{\tilde{t}_n\}) \to 0$ as $n \to \infty$.

Lemma 16. *If $b, f : [0,T] \to \mathbb{R}$ are continuous on $[0,T] \backslash D_g$, then so are $\tilde{b}$ and $\tilde{f}$ under assumption (3).*

Proof. Let $\bar{t} \in [0,T] \backslash D_g$ and let $t_n \to \bar{t}$, $(t_n)_n \subset [0,T]$.
If $(t_n)_n \subset [0,T] \backslash D_g$, then $\tilde{b}(t_n) = b(t_n) \to b(\bar{t}) = \tilde{b}(\bar{t})$ by hypothesis.
If (at least on a subsequence) $t_n \in D_g$, then

$$\begin{aligned} \tilde{b}(t_n) &= \frac{-\log|1-b(t_n)\mu_g(\{t_n\})|}{\mu_g(\{t_n\})} \\ &= \frac{-\log|1-b(t_n)\mu_g(\{t_n\})|}{b(t_n)\mu_g(\{t_n\})} \cdot b(t_n) \to b(\bar{t}) = \tilde{b}(\bar{t}) \end{aligned}$$

since, by the previous discussion, $b(t_n)\mu_g(\{t_n\}) \to 0$ as $n \to \infty$.
So, the continuity of $\tilde{b}$ on $[0,T] \backslash D_g$ is proved. Obviously, the continuity of $\tilde{f}$ on $[0,T] \backslash D_g$ can be proved similarly. ☐

Theorem 17. *Let $g : [0, T] \to \mathbb{R}$ be a nondecreasing left-continuous function which, μ_g-a.e. on the set of continuity points, has the property that g is differentiable and the identical function is g-differentiable.*
Let $b : [0, T] \to \mathbb{R}$ be LS-integrable w.r.t. g, continuous on $[0, T]\backslash D_g$ and $f : [0, T] \to \mathbb{R}$ be continuous on $[0, T]\backslash D_g$, such that $\tilde{f}$ is KS-integrable w.r.t. g. Suppose condition (3) is fulfilled.

Then the function $u : [0, T] \to \mathbb{R}$ given by

$$u(t) = \frac{1}{\alpha(t)} \int_0^T \alpha(s)\tilde{g}(t, s)\tilde{f}(s)\mathrm{d}g(s),$$

where

$$\tilde{g}(t, s) = \frac{1}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \begin{cases} \alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^t \tilde{b}(r)\mathrm{d}g(r)}, & if\ 0 \le s \le t \le T \\ e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)}, & if\ 0 \le t < s \le T \end{cases}$$

is a solution of the periodic Stieltjes differential problem (2).

Proof. Note that hypothesis (3) ensures (together with the LS-integrability w.r.t. g of b) the LS-integrability of $\tilde{b}$ w.r.t. g.
Let $t \in [0, T]\backslash D_g$ be a point where g is differentiable and the identical function is g-differentiable.
One can see that

$$u(t) = \frac{1}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \left[\frac{\alpha(T)}{\alpha(t)} \int_0^t \alpha(s) e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^t \tilde{b}(r)\mathrm{d}g(r)} \cdot \tilde{f}(s)\mathrm{d}g(s) \right.$$
$$\left. + \frac{1}{\alpha(t)} \int_t^T \alpha(s) e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)} \cdot \tilde{f}(s)\mathrm{d}g(s) \right].$$

By making use of Proposition 5, $s \to \int_s^t \tilde{b}(r)\mathrm{d}g(r)$ is continuous at t (because g is continuous at t), therefore $s \to e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)} \cdot \alpha(s)\tilde{f}(s)$ is continuous at t.

We are able to apply Lemma 12 and use the fact that α is constant on a neighborhood of t:

$$\begin{aligned} u_g'(t) = & \frac{1}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \\ & \cdot \left[\frac{\alpha(T)}{\alpha(t)} \int_0^t \alpha(s) e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^t \tilde{b}(r)\mathrm{d}g(r)} \cdot \tilde{f}(s) \cdot (-\tilde{b}(t))\mathrm{d}g(s) \right. \\ & + \frac{\alpha(T)}{\alpha(t)} e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_t^t \tilde{b}(r)\mathrm{d}g(r)} \cdot \alpha(t)\tilde{f}(t) \cdot g'(t) \cdot t_g' \\ & + \frac{1}{\alpha(t)} \int_t^T e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)} \cdot \alpha(s)\tilde{f}(s) \cdot (-\tilde{b}(t))\mathrm{d}g(s) \\ & \left. - \frac{1}{\alpha(t)} e^{-\int_t^t \tilde{b}(r)\mathrm{d}g(r)} \cdot \alpha(t)\tilde{f}(t) \cdot g'(t) \cdot t_g' \right] \\ = & -b(t)u(t) + f(t) \end{aligned}$$

since $\int_t^t \tilde{b}(r)\mathrm{d}g(r) = 0$ (t being a continuity point of g).
Note that when calculating the partial g-derivative w.r.t. the argument t

of the function under the integral sign, we used Lemma 11.i) together with Proposition 8 (due to the continuity of $\tilde{b}$ at the point t), for instance

$$\frac{\partial}{\partial_g t} e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)} = e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)} \cdot \frac{\partial(-\int_s^t \tilde{b}(r)\mathrm{d}g(r))}{\partial_g t} = e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)} \cdot (-\tilde{b}(t)).$$

Consider next $t \in D_g \backslash \{t_1, ..., t_k\}$.
Denoting by

$$F(t) = \frac{1}{\alpha(t)} \int_0^t \alpha(s) e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)} \tilde{f}(s)\mathrm{d}g(s)$$

respectively

$$G(t) = \frac{1}{\alpha(t)} \int_t^T \alpha(s) e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)} \tilde{f}(s)\mathrm{d}g(s)$$

as in Lemma 15, we are able to write

$$u(t) = \frac{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)}}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \cdot F(t) + \frac{1}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \cdot G(t)$$

and so we can compute the g-derivative:

$$\begin{aligned} u'_g(t) &= \frac{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)}}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \cdot F'_g(t) + \frac{1}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \cdot G'_g(t) \\ &= \frac{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)}}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \cdot \left[\frac{e^{-\tilde{b}(t)\cdot\mu_g(\{t\})} - 1}{\mu_g(\{t\})} \cdot F(t) + e^{-\tilde{b}(t)\cdot\mu_g(\{t\})} \cdot \tilde{f}(t) \right] \\ &\quad + \frac{1}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \cdot \left[\frac{e^{-\tilde{b}(t)\cdot\mu_g(\{t\})} - 1}{\mu_g(\{t\})} \cdot G(t) - e^{-\tilde{b}(t)\cdot\mu_g(\{t\})} \cdot \tilde{f}(t) \right] \\ &= \frac{e^{-\tilde{b}(t)\mu_g(\{t\})} - 1}{\mu_g(\{t\})} \cdot u(t) + e^{-\tilde{b}(t)\mu_g(\{t\})} \cdot \tilde{f}(t). \end{aligned}$$

It follows that

$$\begin{aligned} u'_g(t) &= \frac{e^{\log(1-b(t)\mu_g(\{t\}))} - 1}{\mu_g(\{t\})} \cdot u(t) + e^{\log(1-b(t)\mu_g(\{t\}))} \cdot \frac{f(t)}{1 - b(t)\mu_g(\{t\})} \\ &= -b(t) \cdot u(t) + f(t). \end{aligned}$$

At each $t = t_i, i = 1, ..., k$,

$$\begin{aligned} u'_g(t) &= \frac{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)}}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \cdot F'_g(t) + \frac{1}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \cdot G'_g(t) \\ &= \frac{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)}}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \cdot \left[\frac{-e^{-\tilde{b}(t)\cdot\mu_g(\{t\})} - 1}{\mu_g(\{t\})} \cdot F(t) - e^{-\tilde{b}(t)\cdot\mu_g(\{t\})} \cdot \tilde{f}(t) \right] \\ &\quad + \frac{1}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \cdot \left[\frac{-e^{-\tilde{b}(t)\cdot\mu_g(\{t\})} - 1}{\mu_g(\{t\})} \cdot G(t) + e^{-\tilde{b}(t)\cdot\mu_g(\{t\})} \cdot \tilde{f}(t) \right] \\ &= \frac{-e^{-\tilde{b}(t)\mu_g(\{t\})} - 1}{\mu_g(\{t\})} \cdot u(t) - e^{-\tilde{b}(t)\mu_g(\{t\})} \cdot \tilde{f}(t), \end{aligned}$$

so

$$u_g'(t) = \frac{-e^{\log(-1+b(t)\mu_g(\{t\}))} - 1}{\mu_g(\{t\})} \cdot u(t) - e^{\log(-1+b(t)\mu_g(\{t\}))} \cdot \frac{f(t)}{1 - b(t)\mu_g(\{t\})}$$
$$= -b(t) \cdot u(t) + f(t).$$

Finally,

$$u(0) = \frac{1}{\alpha(0)} \int_0^T \alpha(s)\tilde{g}(0,s)\tilde{f}(s)\mathrm{d}g(s)$$
$$= \frac{1}{\alpha(T)e^{\int_0^T b(r)\mathrm{d}g(r)} - 1} \int_0^T \alpha(s)e^{\int_0^s b(r)\mathrm{d}g(r)} \cdot \tilde{f}(s)\mathrm{d}g(s)$$

and

$$u(T) = \frac{1}{\alpha(T)} \int_0^T \alpha(s)\tilde{g}(T,s)\tilde{f}(s)\mathrm{d}g(s)$$
$$= \frac{1}{\alpha(T)e^{\int_0^T b(r)\mathrm{d}g(r)} - 1} \int_0^T \alpha(s)e^{\int_0^T b(r)\mathrm{d}g(r) - \int_s^T b(r)\mathrm{d}g(r)} \cdot \tilde{f}(s)\mathrm{d}g(s)$$
$$= \frac{1}{\alpha(T)e^{\int_0^T b(r)\mathrm{d}g(r)} - 1} \int_0^T \alpha(s)e^{\int_0^s b(r)\mathrm{d}g(r)} \cdot \tilde{f}(s)\mathrm{d}g(s);$$

therefore,

$$u(0) = u(T).$$

□

Remark 18. If f is LS-integrable w.r.t. g, then the LS-integrability (thus, the KS-integrability as well) w.r.t. g of $\tilde{f}$ is a simple consequence of condition (3) since

$$|\tilde{f}(t)| \leq \max\left(1, \frac{1}{\delta}\right) \cdot |f(t)| \quad \forall t \in [0,T].$$

The reciprocal assertion is also valid:

Theorem 19. *If a function $u : [0,T] \to \mathbb{R}$ is a solution in the sense of Definition 14 of the periodic Stieltjes differential problem (2), then*

$$u(t) = \frac{1}{\alpha(t)} \int_0^T \alpha(s)\tilde{g}(t,s)\tilde{f}(s)\mathrm{d}g(s), \quad \forall\, t \in [0,T].$$

Proof. Let u be a solution of problem (2).
Take first $t \in [0,T]\backslash D_g$ a point where the equation is satisfied. Then since α is constant in a neighborhood of t,

$$\left(\alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)} u(t)\right)_g'$$
$$= \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}\tilde{b}(t) \cdot u(t) + \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)} \cdot u_g'(t)$$
$$= \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)} f(t) = \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)} \cdot \tilde{f}(t).$$

Let now $t \in D_g \backslash \{t_1, ..., t_k\}$. Applying Lemma 15, one gets

$$\begin{aligned}
&\left(\alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}u(t)\right)'_g \\
&= \alpha(t)u'_g(t)\cdot e^{\int_0^{t+} \tilde{b}(r)\mathrm{d}g(r)} + \alpha(t)u(t)\cdot e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}\cdot \frac{e^{\tilde{b}(t)\mu_g(\{t\})}-1}{\mu_g(\{t\})} \\
&= \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}\cdot\left[u'_g(t)\cdot e^{\int_t^{t+} \tilde{b}(r)\mathrm{d}g(r)} + u(t)\cdot \frac{e^{\tilde{b}(t)\mu_g(\{t\})}-1}{\mu_g(\{t\})}\right] \\
&= \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}\cdot\left(u'_g(t)\cdot e^{\tilde{b}(t)\mu_g(\{t\})} + u(t)\cdot \frac{e^{\tilde{b}(t)\mu_g(\{t\})}-1}{\mu_g(\{t\})}\right) \\
&= \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}\cdot\Bigg((-b(t)u(t)+f(t))\cdot\frac{1}{1-b(t)\mu_g(\{t\})} \\
&\quad + u(t)\cdot\frac{\frac{1}{1-b(t)\mu_g(\{t\})}-1}{\mu_g(\{t\})}\Bigg) \\
&= \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}\cdot\tilde{f}(t).
\end{aligned}$$

Also, at any point $t = t_i, i \in \{1, ..., k\}$,

$$\begin{aligned}
&\left(\alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}u(t)\right)'_g \\
&= \frac{\alpha(t+)e^{\int_0^{t+} \tilde{b}(r)\mathrm{d}g(r)}u(t+) - \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}u(t)}{\mu_g(\{t\})} \\
&= \frac{-\alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}e^{\int_t^{t+} \tilde{b}(r)\mathrm{d}g(r)}u(t+) - \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}u(t)}{\mu_g(\{t\})} \\
&= \frac{-\alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}e^{\tilde{b}(t)\mu_g(\{t\})}u(t+) - \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}u(t)}{\mu_g(\{t\})}.
\end{aligned}$$

Writing now

$$u(t+) = u'_g(t)\mu_g(\{t\}) + u(t)$$

and taking into account that

$$u'_g(t) = -b(t)u(t) + f(t)$$

and the definition of $\tilde{b}$, one gets

$$\begin{aligned}
&\left(\alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}u(t)\right)'_g \\
&= \frac{-\alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}e^{\tilde{b}(t)\mu_g(\{t\})}(u'_g(t)\mu_g(\{t\}) + u(t)) - \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}u(t)}{\mu_g(\{t\})} \\
&= \frac{-\alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}}{\mu_g(\{t\})}\left[e^{\tilde{b}(t)\mu_g(\{t\})}((-b(t)u(t)+f(t))\mu_g(\{t\}) + u(t)) + u(t)\right]
\end{aligned}$$

$$= \frac{-\alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}}{\mu_g(\{t\})}\Big[e^{\tilde{b}(t)\mu_g(\{t\})}((-b(t)\mu_g(\{t\})+1)u(t)$$
$$+ e^{\tilde{b}(t)\mu_g(\{t\})}f(t)\mu_g(\{t\})+u(t)\Big]$$
$$= \frac{-\alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}}{\mu_g(\{t\})}\Big[e^{-\log(-1+b(t)\mu_g(\{t\})}((-b(t)\mu_g(\{t\})+1)u(t)$$
$$+ e^{-\log(-1+b(t)\mu_g(\{t\})}f(t)\mu_g(\{t\})+u(t)\Big]$$
$$= \alpha(t)e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}\cdot \tilde{f}(t).$$

From here, the result is achieved by integrating on $[0,t]$ w.r.t. g the equality

$$\left(\alpha(s)e^{\int_0^s \tilde{b}(r)\mathrm{d}g(r)}u(s)\right)_g' = \alpha(s)e^{\int_0^s \tilde{b}(r)\mathrm{d}g(r)}\cdot \tilde{f}(s)$$

and imposing the boundary conditions as in the classical case of periodic differential problems (where $g(t)=t$). While integrating the g-derivative in the Kurzweil-Stieltjes sense we apply the fundamental theorem for this type of integral [12, Theorem 6.2] (this is possible since $t\mapsto e^{\int_0^t \tilde{b}(r)\mathrm{d}g(r)}$ and u and α are left-continuous and constant on any subinterval where g is constant). □

Remark 20. If for every $t\in[0,T]$

$$1-b(t)\mu_g(t)>0$$

then the set $\{t\in D_g : 1-b(t)\mu_g(t)<0\}$ is empty and $\alpha(t)=1$ on the whole interval; therefore, the calculus is much simpler. In this case, b can be assumed to be only KS-integrable w.r.t. g.

3.2. Existence result for the nonlinear problem

We go further to studying the nonlinear problem (1):

$$\begin{cases} u_g'(t)+b(t)u(t)=f(t,u(t)),\ t\in[0,T] \\ u(0)=u(T). \end{cases}$$

We shall apply Schaeffer's fixed point theorem.

Theorem 21. *Let S be a normed linear space and the operator $A:S\to S$ be continuous and compact. If the set*

$$\{x\in S : x=\lambda Ax \text{ for some } \lambda\in(0,1)\}$$

is bounded, then the operator has a fixed point.

Theorem 22. *Let $g:[0,T]\to\mathbb{R}$ be a nondecreasing left-continuous function which, μ_g-a.e. on the set of continuity points, has the property that g is differentiable and the identical function is g-differentiable.*
Let $b:[0,T]\to\mathbb{R}$ be LS-integrable w.r.t. g, continuous on $[0,T]\backslash D_g$ and suppose condition (3) is fulfilled.
Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ satisfy the following hypotheses:

- *f is continuous on $([0,T]\backslash D_g)\times\mathbb{R}$ and for every $t\in D_g$, $f(t,\cdot)$ is continuous;*

- *for every $u \in \mathbb{R}$, $f(\cdot, u)$ is μ_g-measurable;*
- *there exists a function $\overline{\phi}$ KS-integrable w.r.t. g such that*

$$|f(t,u)| \leq \overline{\phi}(t)$$

for every $t \in [0,T], u \in \mathbb{R}$.

Then the periodic Stieltjes differential equation (1) possesses solutions.

Proof. Let S_g be the subspace of the space of regulated real-valued maps defined on $[0,T]$ consisting of those functions which are continuous on $[0,T]\backslash D_g$.

Consider the operator $A : S_g \to S_g$ given by

$$Au = \frac{1}{\alpha(t)} \int_0^T \alpha(s)\tilde{g}(t,s)\tilde{f}(s,u(s))\mathrm{d}g(s), \ t \in [0,T]$$

with $\tilde{g}$ as in Theorem 17 and

$$\tilde{f}(t,u) = \frac{f(t,u)}{1 - b(t)\mu_g(\{t\})}.$$

It is well defined: $\tilde{f}(\cdot, u(\cdot))$ is KS-integrable w.r.t. g for each $u \in S_g$ since it is μ_g-measurable and

$$\left|\tilde{f}(t,u(t))\right| \leq \phi(t) = \max\left(1, \frac{1}{\delta}\right) \cdot \overline{\phi}(t) \ \forall t \in [0,T], u \in S_g.$$

Also, whenever $u \in S_g$, i.e. u is regulated and continuous on $[0,T]\backslash D_g$, Au has the same feature by Proposition 5 due to the fact that α is constant in a neighborhood of $t \in [0,T]\backslash D_g$, since

$$\begin{aligned} Au(t) = & \frac{1}{\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1} \\ & \cdot \left[\frac{\alpha(T)}{\alpha(t)} \int_0^t \alpha(s)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^t \tilde{b}(r)\mathrm{d}g(r)} \cdot \tilde{f}(s,u(s))\mathrm{d}g(s) \right. \\ & \left. + \frac{1}{\alpha(t)} \int_t^T \alpha(s)e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)} \cdot \tilde{f}(s,u(s))\mathrm{d}g(s)\right]. \end{aligned}$$

We shall see that A satisfies the hypotheses of Schaeffer's fixed point theorem.

First, let us prove that it is continuous. Let $(u_n)_n \subset S_g$ converge uniformly to u. As f is continuous with respect to the second argument,

$$\tilde{f}(s,u_n(s)) \to \tilde{f}(s,u(s)), \ \forall \ s \in [0,T].$$

We are now able to apply a dominated convergence result, e.g. [20, Theorem 6.8.6], since $-\phi(s) \leq \tilde{f}(s,u_n(s)) \leq \phi(s)$ for every $n \in \mathbb{N}$; we get, for all $t \in [0,T]$,

$$\int_0^T \alpha(s)\tilde{g}(t,s)\tilde{f}(s,u_n(s))\mathrm{d}g(s) \to \int_0^T \alpha(s)\tilde{g}(t,s)\tilde{f}(s,u(s))\mathrm{d}g(s),$$

so $Au_n(t) \to Au(t)$.

We check, using Lemma 2, that the convergence is uniform.

Take $0 \leq t < t' \leq T$.

We can see that for each n,

$$|Au_n(t) - Au_n(t')| \leq \left| \frac{1}{\alpha(t)} \int_0^T \alpha(s)(\tilde{g}(t,s) - \tilde{g}(t',s))\tilde{f}(s,u_n(s))\mathrm{d}g(s) \right| + \left| \left(\frac{1}{\alpha(t)} - \frac{1}{\alpha(t')} \right) \int_0^T \alpha(s)\tilde{g}(t',s)\tilde{f}(s,u_n(s))\mathrm{d}g(s) \right|.$$

We note that $|\alpha(t)| = 1$ for each $t \in [0,T]$, so we can see that

$$\begin{aligned}
&\left| \frac{1}{\alpha(t)} \int_0^T \alpha(s)(\tilde{g}(t,s) - \tilde{g}(t',s))\tilde{f}(s,u_n(s))\mathrm{d}g(s) \right| \\
&\leq \frac{1}{\left|\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1\right|} \\
&\quad \left[\left| \alpha(T) \int_0^t \alpha(s)(e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^t \tilde{b}(r)\mathrm{d}g(r)} - e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^{t'} \tilde{b}(r)\mathrm{d}g(r)})\tilde{f}(s,u_n(s))\mathrm{d}g(s) \right| \right. \\
&\quad + \left| \int_{t'}^T \alpha(s)(e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)} - e^{-\int_s^{t'} \tilde{b}(r)\mathrm{d}g(r)})\tilde{f}(s,u_n(s))\mathrm{d}g(s) \right| \\
&\quad \left. + \left| \int_t^{t'} \alpha(s)(e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)} - \alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^{t'} \tilde{b}(r)\mathrm{d}g(r)})\ \tilde{f}(s,u_n(s))\mathrm{d}g(s) \right| \right] \\
&= \frac{1}{\left|\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1\right|} \\
&\quad \left[\left| \int_0^t \alpha(s)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^t \tilde{b}(r)\mathrm{d}g(r)}(1 - e^{-\int_t^{t'} \tilde{b}(r)\mathrm{d}g(r)})\tilde{f}(s,u_n(s))\mathrm{d}g(s) \right| \right. \\
&\quad + \left| \int_{t'}^T \alpha(s)e^{-\int_s^{t'} \tilde{b}(r)\mathrm{d}g(r)}(e^{\int_t^{t'} \tilde{b}(r)\mathrm{d}g(r)} - 1)\tilde{f}(s,u_n(s))\mathrm{d}g(s) \right| \\
&\quad + \left| \int_t^{t'} \alpha(s)(e^{-\int_s^t \tilde{b}(r)\mathrm{d}g(r)} - \alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^{t'} \tilde{b}(r)\mathrm{d}g(r)}) \right. \\
&\quad \left. \left. \tilde{f}(s,u_n(s))\mathrm{d}g(s) \right| \right].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left| \left(\frac{1}{\alpha(t)} - \frac{1}{\alpha(t')} \right) \int_0^T \alpha(s)\tilde{g}(t',s)\tilde{f}(s,u_n(s))\mathrm{d}g(s) \right| \\
&\quad = |\alpha(t) - \alpha(t')| \left| \int_0^T \alpha(s)\tilde{g}(t',s)\tilde{f}(s,u_n(s))\mathrm{d}g(s) \right| \\
&\quad \leq \frac{|\alpha(t) - \alpha(t')|}{|\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1|} \left[\left| \int_0^{t'} \alpha(s)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^{t'} \tilde{b}(r)\mathrm{d}g(r)}\tilde{f}(s,u_n(s))\mathrm{d}g(s) \right| \right. \\
&\qquad \left. + \left| \int_{t'}^T \alpha(s)e^{-\int_s^{t'} \tilde{b}(r)\mathrm{d}g(r)}\tilde{f}(s,u_n(s))\mathrm{d}g(s) \right| \right] \\
&\quad \leq \frac{|\alpha(t) - \alpha(t')|}{|\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1|} \left[\int_0^{t'} \left| e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^{t'} \tilde{b}(r)\mathrm{d}g(r)}\tilde{f}(s,u_n(s)) \right| \mathrm{d}g(s) \right.
\end{aligned}$$

$$+ \int_{t'}^{T} \left| e^{-\int_s^{t'} \tilde{b}(r)\mathrm{d}g(r)} \tilde{f}(s, u_n(s)) \right| \mathrm{d}g(s) \Bigg].$$

But

$$\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^t \tilde{b}(r)\mathrm{d}g(r) = \int_0^s \tilde{b}(r)\mathrm{d}g(r) + \int_t^T \tilde{b}(r)\mathrm{d}g(r)$$

and

$$\int_0^T \tilde{b}(r)\mathrm{d}g(r) - \int_s^{t'} \tilde{b}(r)\mathrm{d}g(r) = \int_0^s \tilde{b}(r)\mathrm{d}g(r) + \int_{t'}^T \tilde{b}(r)\mathrm{d}g(r).$$

The map $(s', s'') \in [0,T] \times [0,T] \to e^{\int_{s'}^{s''} \tilde{b}(s)\mathrm{d}g(s)}$ is regulated in both arguments, therefore bounded. If we note by

$$M = \sup_{(s',s'')\in[0,T]\times[0,T]} e^{\int_{s'}^{s''} \tilde{b}(s)\mathrm{d}g(s)},$$

we get

$$\begin{aligned}
&|Au_n(t) - Au_n(t')| \\
&\quad \leq \frac{M}{\left|\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1\right|} \left[M \int_0^t \left| 1 - e^{-\int_t^{t'} \tilde{b}(r)\mathrm{d}g(r)} \right| \cdot |\tilde{f}(s, u_n(s))| \mathrm{d}g(s) \right. \\
&\qquad + \int_{t'}^T \left| e^{\int_t^{t'} \tilde{b}(r)\mathrm{d}g(r)} - 1 \right| \cdot |\tilde{f}(s, u_n(s))| \mathrm{d}g(s) \\
&\qquad \left. + (1+M) \int_t^{t'} \left| \tilde{f}(s, u_n(s)) \right| \mathrm{d}g(s) \right] \\
&\qquad + \frac{|\alpha(t) - \alpha(t')|}{|\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1|} \left[M^2 \int_0^{t'} |\tilde{f}(s, u_n(s))| \mathrm{d}g(s) \right. \\
&\qquad \left. + M \int_{t'}^T |\tilde{f}(s, u_n(s))| \mathrm{d}g(s) \right],
\end{aligned}$$

so

$$\begin{aligned}
&|Au_n(t) - Au_n(t')| \\
&\quad \leq \frac{M}{\left|\alpha(T)e^{\int_0^T \tilde{b}(r)\mathrm{d}g(r)} - 1\right|} \left[M \left| 1 - e^{-\int_t^{t'} \tilde{b}(r)\mathrm{d}g(r)} \right| \cdot \int_0^T \phi(s)\mathrm{d}g(s) \right. \\
&\qquad + \left| e^{\int_t^{t'} \tilde{b}(r)\mathrm{d}g(r)} - 1 \right| \cdot \int_0^T \phi(s)\mathrm{d}g(s) + (1+M) \int_t^{t'} \phi(s)\mathrm{d}g(s) \\
&\qquad \left. + (M+1)|\alpha(t) - \alpha(t')| \int_0^T \phi(s)\mathrm{d}g(s) \right].
\end{aligned}$$

But

$$\left|1-e^{-\int_t^{t'}\tilde{b}(r)\mathrm{d}g(r)}\right| = \left|e^{\int_0^t\tilde{b}(r)\mathrm{d}g(r)}\left(e^{-\int_0^t\tilde{b}(r)\mathrm{d}g(r)}-e^{-\int_0^{t'}\tilde{b}(r)\mathrm{d}g(r)}\right)\right|$$
$$\leq M\left|e^{-\int_0^t\tilde{b}(r)\mathrm{d}g(r)}-e^{-\int_0^{t'}\tilde{b}(r)\mathrm{d}g(r)}\right|$$

and $e^{-\int_0^{\cdot}\tilde{b}(r)\mathrm{d}g(r)}$ is regulated.

A similar calculus can be made for $\left|e^{\int_t^{t'}\tilde{b}(r)\mathrm{d}g(r)}-1\right|$, while

$$\int_t^{t'}\phi(s)\mathrm{d}g(s)=\int_0^{t'}\phi(s)\mathrm{d}g(s)-\int_0^t\phi(s)\mathrm{d}g(s)$$

and $\int_0^{\cdot}\phi(s)\mathrm{d}g(s)$ and α are regulated.

Remark 3 yields now that the sequence is equiregulated whence, by Lemma 2, $(Au_n)_n$ converges uniformly to Au, i.e. the operator A is continuous.

Let us next prove that the operator is compact. Take $B\subset S_g$ be a bounded set. Then, in the same manner as before, it can be seen that $\{Au : u\in B\}$ is equiregulated.

It is pointwise bounded as well. Indeed, fix $t\in[0,T]$. Then for every $u\in B$,

$$\begin{aligned}|Au(t)| &\leq \int_0^T|\tilde{g}(t,s)|\cdot\left|\tilde{f}(s,u(s))\right|\mathrm{d}g(s)\\ &\leq \frac{\max(M,M^2)}{\left|\alpha(T)e^{\int_0^T\tilde{b}(r)\mathrm{d}g(r)}-1\right|}\int_0^T|\tilde{f}(s,u(s))|\mathrm{d}g(s)\\ &\leq \frac{\max(M,M^2)}{\left|\alpha(T)e^{\int_0^T\tilde{b}(r)\mathrm{d}g(r)}-1\right|}\int_0^T\phi(s)\mathrm{d}g(s);\end{aligned}$$

therefore, the set is pointwise (in fact, even uniformly) bounded.

Lemma 1 implies that $\{Au : u\in B\}$ is relatively compact, thus A is a compact operator.

Let us now see that the set

$$\{u\in S_g : u=\lambda Au \text{ for some } \lambda\in(0,1)\}$$

is bounded.

Let u be an arbitrary element of this set. One can find $\lambda\in(0,1)$ such that

$$u(t)=\frac{\lambda}{\alpha(t)}\int_0^T\alpha(s)\tilde{g}(t,s)\tilde{f}(s,u(s))\mathrm{d}g(s),\ t\in[0,T].$$

It follows, as before, that

$$\begin{aligned}\|u\|_C &\leq \lambda\frac{\max(M,M^2)}{\left|\alpha(T)e^{\int_0^T\tilde{b}(r)\mathrm{d}g(r)}-1\right|}\int_0^T\phi(s)\mathrm{d}g(s)\\ &\leq \frac{\max(M,M^2)}{\left|\alpha(T)e^{\int_0^T\tilde{b}(r)\mathrm{d}g(r)}-1\right|}\int_0^T\phi(s)\mathrm{d}g(s)\end{aligned}$$

and the boundedness is achieved.

Schaeffer's fixed point theorem yields that the operator has fixed points, which are solutions to our problem. This follows exactly as in Theorem 17 since $\tilde{b}$ and $\tilde{f}(\cdot, u(\cdot))$ satisfy the Stieltjes integrability conditions and, by Lemma 16, the continuity on $[0,T]\backslash D_g$. □

Remark 23. The large applicability of our result is a motivation for the less general assumptions on the function f (comparing with the existence results available in particular cases, e.g. [3,14,17,33] or [32]).
In counterbalance, this is, as far as the authors know, the first existence result for periodic Stieltjes differential boundary value problems (and, consequently, for periodic measure boundary value problems). In particular, new results can be deduced for dynamic boundary value problems on time scales or for periodic impulsive differential equations allowing a countable number of impulses (i.e. the impulse moments can accumulate) which cannot be studied through the theory of impulsive differential equations.

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