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New fixed point results for F-contractions of Hardy–Rogers type in b-metric spaces with applications

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Abstract. The purpose of this paper is to introduce the notions of extended F-contraction of Hardy–Rogers type, extended F-contraction of Suzuki–Hardy–Rogers type and generalized F-weak contraction of Hardy–Rogers type and to establish some new fixed point results for such kind of mappings in the setting of complete b-metric spaces. These fixed point results improve (and/or) extend those obtained in Vetro (Nonlinear Anal Model Control 21(4):531–546, 2016) and Lukács and Kajántó (Fixed Point Theory 19(1):321–334, 2018) since some conditions made therein are removed or weakened. In addition, some illustrative examples are provided to show the usability of the obtained results. As an application of our results, we obtain the existence and uniqueness of solutions for certain functional, integral and differential equations.

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1. Introduction

The well-known Banach's contraction mapping principle [4] is the most significant fundamental fixed point result. Since this principle has a lot of applications in different branches of mathematics, several authors have extended, generalized and improved it in many directions by considering different forms of mappings or various types of spaces. In the paper [46], an interesting generalization of Banach contraction principle is given by introducing the concept of F-contraction. After that, the notion of F-contraction of Hardy–Rogers is introduced in [12] as a generalization of F-contraction in complete metric spaces. One of the most prevalent generalization of the metric spaces was given in the article [3] through the notion of b-metric spaces. In our paper, we utilize these two last notions to introduce new types of F-contractions of Hardy–Rogers in the setting of b-metric spaces and to prove some fixed point results. Roughly speaking, we extend and improve (respectively, improve) some results in [43] (respectively, in [23]). The work [43] has been later enriched and published as a book chapter [44] in which the authors (F. Vetro and C. Vetro) have proposed an important review concerning F-contraction conditions in the setting of metric and Banach spaces. More precisely, the results obtained in our paper extended the aforementioned results in b-metric spaces and contain less conditions imposed on the function F. Moreover, the consequences of our main results are improved and generalized versions of some results appearing in literature.

The article is organized as follows. In Sect. 2, we recollect some known definitions and results concerning b-metric spaces and various types of F-contractions. In Sect. 3, we define the notions of extended F-contraction of Hardy–Rogers type, extended F-contraction of Suzuki–Hardy–Rogers type and generalized F-weak contraction of Hardy–Rogers type. Using these concepts, we prove new fixed point theorems in the setting of complete b-metric spaces and we give some examples to illustrate the validity of the obtained results. In Sect. 4, we present three different applications and in each one we prove the existence and uniqueness of solutions for some classes of equations. In the first application, we deal with functional equations arising in dynamic programming. The second one concerns nonlinear Volterra integral equations. The last application is devoted to the study of a boundary value problem for the second-order differential equation.

2. Preliminaries

In this section, we recall some known definitions and results which will be used in the sequel. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}$ the sets of positive integers and real numbers, respectively. We also write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Henceforth, X will denote a nonempty set and the Picard sequence of a self-mapping $T : X \to X$ based on an arbitrary $x_0 \in X$ is given by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$, where T^n denotes the n^{th}-iterates of T.

2.1. *b*-Metric spaces

In 1989, Bakhtin [3] introduced the concept of b-metric spaces as a generalization of the metric spaces in the sense that the triangle inequality contains a suitable constant $s \geq 1$ (see also Czerwik [13]). Since then, several published papers have dealt with b-metric spaces and fixed point theory in the setting of b-metric spaces (see, e.g., [1,2,6,8–10,16,32,34,42] and some related references therein). For more details concerning some technical and useful tools in the context of b-metric spaces, the reader may consult [1] and [32]. Note that the topological framework of a b-metric space with the topology induced by its convergence was studied in [2].

We will first recall the definition of a b-metric space.

Definition 2.1. (*See* [14]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping $\sigma : X \times X \to [0, \infty)$ is said to be a b-metric if, for all $x, y, z \in X$, the following conditions hold:

(b_1) $\sigma(x, y) = 0$ if and only if $x = y$;
(b_2) $\sigma(x, y) = \sigma(y, x)$;
(b_3) $\sigma(x, z) \leq s[\sigma(x, y) + \sigma(y, z)]$.

The pair (X, σ) is called a b-metric space with constant $s \geq 1$.

It is obvious from the above definition that the class of b-metric spaces is larger than that of metric spaces, since a b-metric space is a metric space when $s = 1$ but the converse is not true. The following classical examples illustrate this fact.

Example 2.2. (See [1,42]) Let (X, d) be a metric space and let the mapping $\sigma_d : X \times X \to [0, \infty)$ be defined by

$$\sigma_d(x, y) = (d(x, y))^p, \quad \text{for all } x, y \in X,$$

where $p > 1$ is a fixed real number. Then (X, σ_d) is a b-metric space with $s = 2^{p-1}$.

In particular, if $X = \mathbb{R}$, $d(x, y) = |x - y|$ is the usual Euclidean metric and

$$\sigma_d(x, y) = (x - y)^2, \quad \text{for all } x, y \in \mathbb{R},$$

then $(\mathbb{R}, \sigma_d)$ is a b-metric with $s = 2$. However, $(\mathbb{R}, \sigma_d)$ is not a metric space on $\mathbb{R}$ since (b_3) does not hold. Indeed,

$$\sigma_d(-2, 2) = 16 > 8 = 4 + 4 = \sigma_d(-2, 0) + \sigma_d(0, 2).$$

Example 2.3. (See [21]) Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that

$$\int_0^1 |f(x)|^2 \, \mathrm{d}x < \infty.$$

Define $D : X \times X \to [0, \infty)$ by

$$D(f, g) = \int_0^1 |f(x) - g(x)|^2 \, \mathrm{d}x.$$

Then D satisfies the following properties

1. $D(f, g) = 0$ if and only if $f = g$,
2. $D(f, g) = D(g, f)$, for any $f, g \in X$,
3. $D(f, g) \leq 2(D(f, h) + D(h, g))$, for any points $f, g, h \in X$.

Clearly, (X, D) is a b-metric space with $s = 2$ but is not a metric space. For example, take $f(x) = 0$, $g(x) = 1$ and $h(x) = \dfrac{1}{2}$, for all $x \in [0, 1]$. Then

$$D(0, 1) = 1 > \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = D\left(0, \frac{1}{2}\right) + D\left(\frac{1}{2}, 1\right).$$

We present now the concepts of convergence, Cauchy sequence and completeness in b-metric spaces.

Definition 2.4. (*See* [8–10]) Let (X, σ) be a b-metric space. Then a sequence $\{x_n\}$ in X is called

(a) convergent if and only if there exists $x \in X$ such that $\lim_{n\to\infty}\sigma(x_n, x) = 0$ and in this case we write $\lim_{n\to\infty} x_n = x$;
(b) Cauchy if and only if $\lim_{n,m\to\infty}\sigma(x_n, x_m) = 0$.

Definition 2.5. (*See* [8–10]) The b-metric space (X, σ) is said complete if every Cauchy sequence in X converges in X.

Remark 2.6. (See [8–10]) In a b-metric space, the following assertions hold:

(i) a convergent sequence has a unique limit;
(ii) each convergent sequence is Cauchy.

Lemma 2.7. (See [16, Lemma 2.1]) *Let (X, σ) be a b-metric space with constant $s \geq 1$ and $\{x_n\}$, $\{y_n\}$ two sequences such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ in (X, σ). Then $\lim_{n\to\infty}\sigma(x_n, y_n) = 0$ if and only if $x = y$.*

It is worth recalling that a b-metric is generally not continuous (see, e.g., [19, Example 3.3]). The following lemmas are very useful to manage this problem.

Lemma 2.8. (See [1,23]) *Let (X, σ) be a b-metric space with constant $s \geq 1$ and $\{x_n\}$ be a convergent sequence in X with $\lim x_n = x$. Then for each $y \in X$, we have*

$$\frac{1}{s}\sigma(x, y) \leq \liminf_{n\to\infty}\sigma(x_n, y) \leq \limsup_{n\to\infty}\sigma(x_n, y) \leq s\sigma(x, y).$$

Lemma 2.9. (See [32, Lemma 1.7]) *Let (X, σ) be a b-metric space with constant $s \geq 1$ and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty}\sigma(x_n, x_{n+1}) = 0. \tag{2.1}$$

If $\{x_n\}$ is not a Cauchy sequence in (X, σ), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that the following items hold:

$$\begin{aligned}
\varepsilon \leq \liminf_{k\to\infty}\sigma\left(x_{m(k)}, x_{n(k)}\right) \leq \limsup_{k\to\infty}\sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon;\\
\frac{\varepsilon}{s} \leq \liminf_{k\to\infty}\sigma\left(x_{m(k)}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty}\sigma\left(x_{m(k)}, x_{n(k)+1}\right) \leq s^2\varepsilon;\\
\frac{\varepsilon}{s} \leq \liminf_{k\to\infty}\sigma\left(x_{m(k)+1}, x_{n(k)}\right) \leq \limsup_{k\to\infty}\sigma\left(x_{m(k)+1}, x_{n(k)}\right) \leq s^2\varepsilon;\\
\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty}\sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty}\sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq s^3\varepsilon.
\end{aligned}$$

Inspired by the works in [30], we can state the following lemma.

Lemma 2.10. *Let all the conditions of Lemma 2.9 be satisfied. Then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such* *that the following items hold:*

$$\begin{aligned} \varepsilon^+ &\le \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \le \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \le s\varepsilon^+; \\ \frac{\varepsilon}{s} &\le \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)+1}\right) \le \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)+1}\right) \le s^2\varepsilon; \\ \frac{\varepsilon}{s} &\le \liminf_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)}\right) \le \limsup_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)}\right) \le s^2\varepsilon; \\ \frac{\varepsilon}{s^2} &\le \liminf_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \le \limsup_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \le s^3\varepsilon. \end{aligned}$$

Proof. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of positive integers such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and $\sigma\left(x_{m(k)}, x_{n(k)}\right) > \varepsilon$. Due to (2.1), this implies that $\sigma\left(x_{m(k)}, x_{n(k)-1}\right) \le \varepsilon$ for all $k \ge 1$. Using the relaxed triangle inequality (b_3), we have

$$\begin{aligned} \sigma\left(x_{m(k)}, x_{n(k)}\right) &\le s\sigma\left(x_{m(k)}, x_{n(k)-1}\right) + s\sigma\left(x_{n(k)-1}, x_{n(k)}\right) \\ &\le s\varepsilon + s\sigma\left(x_{n(k)-1}, x_{n(k)}\right). \end{aligned}$$

This leads to

$$\frac{1}{s}\sigma\left(x_{m(k)}, x_{n(k)}\right) \le \varepsilon + \sigma\left(x_{n(k)-1}, x_{n(k)}\right).$$

Since $\sigma\left(x_{n(k)-1}, x_{n(k)}\right) > 0$, then by taking limit superior as $k \to \infty$ with (2.1), we get

$$\frac{1}{s}\limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \le \varepsilon^+,$$

or, equivalently,

$$\limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \le s\varepsilon^+. \tag{2.2}$$

On the other hand, we have

$$\frac{1}{k} + \sigma\left(x_{m(k)}, x_{n(k)}\right) > \frac{1}{k} + \varepsilon, \quad \text{for all } k \ge 1.$$

Taking the limit inferior as $k \to \infty$, we have

$$\liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \ge \varepsilon^+. \tag{2.3}$$

From (2.2) and (2.3), we obtain the first item of Lemma 2.10. Since the remaining items are the same as in Lemma 2.9, the proof is completed. □

Remark 2.11. Taking $s = 1$ (the case corresponding to a metric space (X, d)) in Lemma 2.10, we find Lemma 2.2 in [30]. More precisely, the above items become as follows:

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon^+$$

and

$$\begin{aligned} \lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)}\right) &= \lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right) \\ &= \lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \varepsilon. \end{aligned}$$

In 2015, An et al. [2] proved the following result:

Proposition 2.12. (See [2, Proposition 3.11]) *Let (X, σ) be a b-metric space with constant $s \geq 1$. If σ is continuous with respect in one variable, then σ is continuous in other variable.*

Obviously, we observe from the above result that if σ is not continuous with respect one variable, then σ is not continuous in each variable (refer to [2, Examples 3.9, 3.10].

We end this subsection by giving an example which illustrates some preceding properties concerning b-metric spaces.

Example 2.13. Let $X = [0, \infty)$. Let $\sigma : X \times X \to [0, \infty)$ be a mapping defined by

$$\sigma(x, y) = \begin{cases} d(x, y), & xy \neq 0, \\ 4d(x, y), & xy = 0, \end{cases}$$

where $d(x, y) = |x - y|$. Then the following hold:

(1) (X, σ) is a complete b-metric space with constant $s = 4$.
(2) σ is not a metric on X.
(3) σ is not continuous in each variable.

Proof. (1) We start to prove that (X, σ) is a b-metric space with constant $s = 4$. Clearly, (b_1) and (b_2) are satisfied. For (b_3), we can easily observe that for any $x, y \in X$,

$$d(x, y) \leq \sigma(x, y) \leq 4d(x, y). \tag{2.4}$$

We consider then the following cases.
Case 1 Suppose that $xy \neq 0$. Then using (2.4), for any $z \in X$, we obtain

$$\begin{aligned} \sigma(x, y) = d(x, y) &\leq d(x, z) + d(z, y) \\ &\leq \sigma(x, z) + \sigma(z, y) \leq 4(\sigma(x, z) + \sigma(z, y)). \end{aligned}$$

Case 2 Assume that $xy = 0$. Also, through (2.4), we have for any $z \in X$

$$\begin{aligned} \sigma(x, y) = 4d(x, y) &\leq 4d(x, z) + 4d(z, y) \\ &\leq 4(\sigma(x, z) + \sigma(z, y)). \end{aligned}$$

Next, since (X, d) is a complete metric space, the completeness of (X, σ) follows immediately from (2.4).

(2) Indeed, σ is not a metric on X since we have

$$\sigma(0, 2) = 8 > 5 = 4 + 1 = \sigma(0, 1) + \sigma(1, 2).$$

(3) Let $x_n = \dfrac{1}{n}$ for each $n \in \mathbb{N}$. We have

$$\lim_{n\to\infty} \sigma\left(\frac{1}{n}, 0\right) = \lim_{n\to\infty} \frac{4}{n} = 0.$$

Then $\lim_{n\to\infty} x_n = 0$ in (X, σ). On the other hand, we have

$$\lim_{n\to\infty} \sigma(x_n, 1) = 1 \neq 4 = \sigma(0, 1).$$

This, together with Proposition 2.12, proves that σ is not continuous in each variable. ☐

2.2. F-contractions

Now, let us review some results concerning F-contractions related to the existing literature. In 2012, Wardowski [46] introduced the notion of F-contraction as follows:

Definition 2.14. (*See* [46]) Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \qquad (2.5)$$

where $\mathcal{F}$ is the family of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

- (F_1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$, if $\alpha < \beta$, then $F(\alpha) < F(\beta)$.
- (F_2) For each sequence $\{\alpha_n\}$ of positive numbers, the following holds:
$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty.$$
- (F_3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Remark 2.15. (See [46]) Let $\alpha > 0$. Let the following functions $F_1(\alpha) = \ln \alpha$, $F_2(\alpha) = \ln \alpha + \alpha$, $F_3(\alpha) = \dfrac{-1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln(\alpha^2 + \alpha)$. Then F_1, F_2, F_3 and $F_4 \in \mathcal{F}$.

Remark 2.16. (See [46]) Clearly, if F is an increasing function (not necessary strictly increasing), inequality (2.5) implies that T is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y), \quad \forall x, y \in X, \ x \neq y.$$

Hence, every F-contraction is a continuous mapping.

Wardowski's result is given as follows:

Theorem 2.17. (See [46, Theorem 2.1]) *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.*

Remark 2.18. (See [46]) Wardowski showed that T is a Banach contraction [4] by taking $F(\alpha) = \ln \alpha$ in (2.5).

In [35], Secelean showed that condition (F_2) can be replaced by an equivalent and more easier one (noted (F_2'): $\inf F = -\infty$). Afterwards, Piri and Kumam [26] established Wardowski's theorem using (F_2') and the continuity instead of (F_2) and (F_3), respectively. Later, Wardowski [47] proved a fixed point theorem concerning F-contractions when τ is taken as a function. In this work, the author used a relaxed version of (F_2) and dropped also condition (F_3). In 2018, Lukács and Kajántó [23] extended Wardowski's theorem in the setting of b-metric spaces and omitted condition (F_2). Very recently, some authors proved (in different ways) the original results of Wardowski without both conditions (F_2) and (F_3) (see, e.g., [24, Remark 3.7], [30, Corollary 3.21 and Theorem 4.1]). It is also worth mentioning that many others papers dealing with various types of F-contractions can be found in the literature (see, e.g., [11,15,17,20,24,25,27–29,33,36–38,40,41,44,45] and references therein).

2.2.1. *F*-contractions of Hardy–Rogers type. We present here various types of F-contractions of Hardy–Rogers and some related works which will be needed for stating our results in the sequel.

In 2014, Wardowski and Dung [48] proved the following result.

Theorem 2.19. (See [48, Corollary 2.5]) *Let (X,d) be a complete metric space. Assume that there exist $\tau > 0$ and $F \in \mathcal{F}$ such that $T : X \to X$ satisfies*

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F\left(B_T^d(x,y)\right),$$

where

$$B_T^d(x,y) = ad(x,y) + bd(x,Tx) + cd(y,Ty) + e\left[d(x,Ty) + d(y,Tx)\right]$$

with $a,b,c,e \geq 0$ and $a+b+c+2e < 1$. If T or F is continuous, then

(1) *T has a unique fixed point $x^* \in X$.*
(2) *For all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.*

Afterwards, Cosentino and Vetro [12] introduced a new notion, namely the notion of F-contraction of Hardy–Rogers type given below.

Definition 2.20. (*See* [12]) Let (X,d) be a metric space. A self-mapping T on X is called an F-contraction of Hardy–Rogers type if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x,y \in X$,

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F\left(Q_T^d(x,y)\right),$$

where

$$Q_T^d(x,y) = \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + Ld(y,Tx)$$

with $\alpha,\beta,\gamma,\delta,L \geq 0$, $\alpha+\beta+\gamma+2\delta = 1$ and $\gamma \neq 1$.

The authors in [12] obtained the following fixed point result.

Theorem 2.21. (See [12, Theorem 3.1]) *Let (X,d) be a complete metric space and let $T : X \to X$ be an F-contraction of Hardy–Rogers type. Then T has a fixed point. Moreover, if $\alpha + \delta + L \leq 1$, then the fixed point of T is unique.*

Later, Vetro [43] proved some new results about F-contraction of Hardy–Rogers type. Before enunciating these results, we need to introduce some notations and definitions. Let us note $\mathbb{S}$ the family of all functions $\tau : (0,\infty) \to (0,\infty)$ satisfying the following property:

$$\liminf_{t \to \eta^+} \tau(t) > 0, \text{ for all } \eta \geq 0.$$

Let us consider also the following condition:

(F_3') : F is continuous on $(0,\infty)$.

Henceforth, we denote by $\mathfrak{F}$ the set of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying the conditions (F_1), (F_2) and (F_3') and by $\mathbb{F}$ the set of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying the conditions (F_1) and (F_2).

Vetro [43] generalized the notion of F-contraction of Hardy–Rogers type as follows:

Definition 2.22. (*See* [43, Definition 3]) Let (X,d) be a complete metric space. A self-mapping T on X is called an F-contraction of Hardy–Rogers type if there exist $F\in\mathbb{F}$ and $\tau\in\mathbb{S}$ such that for all $x,y\in X$,

$$d\left(Tx,Ty\right)>0\Rightarrow \tau\left(d\left(x,y\right)\right)+F\left(d\left(Tx,Ty\right)\right)\le F\left(Q_T^d(x,y)\right),$$

where $\alpha,\beta,\gamma,\delta,L\ge 0$, $\alpha+\beta+\gamma+2\delta=1$, $\gamma\ne 1$ and $\alpha+\delta+L\le 1$.

Also, the author introduced the notion of F-contraction of Suzuki–Hardy–Rogers type given below.

Definition 2.23. (*See* [43, Definition 3]) Let (X,d) be a complete metric space. A self-mapping T on X is called an F-contraction of Suzuki–Hardy–Rogers type if there exist $F\in\mathbb{F}$ and $\tau\in\mathbb{S}$ such that for all $x,y\in X$ with $Tx\ne Ty$,

$$\frac{1}{2s}\sigma\left(x,Tx\right)<\sigma\left(x,y\right)\Rightarrow \tau\left(d\left(x,y\right)\right)+F\left(d\left(Tx,Ty\right)\right)\le F\left(Q_T^d(x,y)\right),$$

where $\alpha,\beta,\gamma,\delta,L\ge 0$, $\alpha+\beta+\gamma+2\delta=1$, $\gamma\ne 1$ and $\alpha+\delta+L\le 1$.

The first Vetro's result is the following:

Theorem 2.24. (See [43, Theorem 1]) *Let (X,d) be a complete metric space. If T is an F-contraction of Hardy–Rogers type and F is continuous (i.e., $F\in\mathfrak{F}$), then T has a unique fixed point.*

The second Vetro's result (Suzuki-type version) is given as follows:

Theorem 2.25. (See [43, Theorem 2]) *Let (X,d) be a complete metric space. If T is an F-contraction of Suzuki–Hardy–Rogers type and F is continuous (i.e., $F\in\mathfrak{F}$), then T has a unique fixed point.*

Remark 2.26. In [43, Remark 3], Vetro proved that if (F_3') is weaken to the condition that F is upper semicontinuous on $(0,\infty)$, then Theorem 2.24 holds for the strict inequality $\alpha+\delta+L<1$.

Vetro established also the following corollaries.

Corollary 2.27. (See [43, Corollary 1]) *Let (X,d) be a complete metric space and let T be a self-mapping on X. Assume that there exist an upper semi-continuous $F\in\mathbb{F}$ and $\tau\in\mathbb{S}$ such that for all $x,y\in X$ with $Tx\ne Ty$,*

$$\tau\left(d\left(x,y\right)\right)+F\left(d\left(Tx,Ty\right)\right)\le F\left(\beta d\left(x,Tx\right)+\gamma d\left(y,Ty\right)\right),$$

where $\beta,\gamma\in[0,\infty)$ satisfying $\beta+\gamma=1$ and $\gamma\ne 1$. Then T has a unique fixed point in X.

Corollary 2.28. (See [43, Corollary 3]) *Let (X,d) be a complete metric space and let T be a self-mapping on X. Assume that there exist a continuous $F\in\mathbb{F}$ and $\tau\in\mathbb{S}$ such that for all $x,y\in X$ with $Tx\neq Ty$,*

$$\tau\left(d\left(x,y\right)\right)+F\left(d\left(Tx,Ty\right)\right)\leq F\left(\alpha d\left(x,y\right)+\beta d\left(x,Tx\right)+\gamma d\left(y,Ty\right)\right),$$

where $\alpha,\beta,\gamma\in[0,\infty)$ satisfying $\alpha+\beta+\gamma=1$ and $\gamma\neq1$. Then T has a unique fixed point in X.

Consistent with [11] and [23], what follows is needed to deal with more results concerning F-contraction of Hardy–Rogers type.

In 2015, Cosentino et al. [11] introduced the following condition (noted (F_4) in [11, Definition 3.1]):

Let $s\geq1$. If $\{\alpha_n\}\subset(0,\infty)$ is a sequence such that $\tau+F\left(s\alpha_n\right)\leq F\left(\alpha_{n-1}\right)$, for all $n\in\mathbb{N}$ and some $\tau>0$, then $\tau+F\left(s^n\alpha_n\right)\leq F\left(s^{n-1}\alpha_{n-1}\right)$, for all $n\in\mathbb{N}$.

In the same context, Lukács and Kajántó [23] defined a new class of functions (noted $\mathcal{F}_{s,\tau}$) satisfying an easier condition than (F_4). Their definition is given below.

Definition 2.29. (*See* [23, Definition 2.7]) Let $s\geq1$ and $\tau>0$. We say that $F\in\mathbb{F}^*$ belongs to $\mathcal{F}_{s,\tau}$ if it is also satisfies

$(\mathrm{F}_{s,\tau})$ if $\inf F=-\infty$ and $x,y,z\in(0,\infty)$ are such that $\tau+F\left(sx\right)\leq F\left(y\right)$ and $\tau+F\left(sy\right)\leq F\left(z\right)$ then

$$\tau+F\left(s^2x\right)\leq F\left(sy\right),$$

where $\mathbb{F}^*$ is the set of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying the conditions (F_1) and (F_3).

Next, the authors in [23] introduced the notion of F-weak contraction of Hardy–Rogers type in the setting of b-metric spaces as follows:

Definition 2.30. (*See* [23, Definition 5.1]) Let (X,σ) be a b-metric space with constant $s\geq1$, $a,b,c,e,f\geq0$ real numbers and $T:X\to X$ an operator. If there exist $\tau>0$ and $F\in\mathcal{F}_{s,\tau}$ such that for all $x,y\in X$ the inequality $\sigma\left(Tx,Ty\right)>0$ implies

$$\tau+F\left(s\sigma\left(Tx,Ty\right)\right)\leq F\left(A_T^\sigma\left(x,y\right)\right),$$

where

$$A_T^\sigma\left(x,y\right)=a\sigma\left(x,y\right)+b\sigma\left(x,Tx\right)+c\sigma\left(y,Ty\right)+e\sigma\left(x,Ty\right)+f\sigma\left(y,Tx\right),$$

then T is called an F-weak contraction of Hardy–Rogers type.

In [23], Lukács and Kajántó showed that if F is an increasing function, then $(\mathrm{F}_{s,\tau})$ is equivalent to (F_4) (see [23, Proposition 2.8]) and proved the fixed point result below.

Theorem 2.31. (See [23, Theorem 5.2]) *Suppose that (X,σ) is a b-metric space with constant $s\geq1$ and $T:X\to X$ is an F-weak contraction of Hardy–Rogers type. If either $a+b+c+(s+1)\,e<1$ or $a+b+c+(s+1)\,f<1$ holds, then every $x_0\in X$, the sequence $x_{n+1}=Tx_n$ converges to a fixed point of T. Moreover, if $a+e+f<s$ holds as well, then T has exactly one fixed point.*

3. Main results

In this section, we essentially improve (and/or) extend the aforementioned results: Theorem 2.24, Theorem 2.25, Remark 2.26 and Theorem 2.31 in the setting of b-metric spaces. It is worth mentioning that in our following results, the b-metric need not to be continuous.

For convenience, we set

$$\mathcal{F}_c = \{F : (0,\infty) \to \mathbb{R} : \ F \text{ is nondecreasing continuous function}\}.$$

Let $\omega \geq 1$ be a given real number. We denote by $\mathcal{S}_\omega$ the family of all functions $\tau : (0,\infty) \to (0,\infty)$ which satisfy the following condition:

$$\liminf_{t \to r} \tau(t) > 0, \quad \text{where } r \in \left[\eta^+, \eta^+\omega\right], \text{ for all } \eta > 0. \tag{A_ω}$$

Remark 3.1. Obviously, if $\omega = 1$, condition (A_ω) becomes as follows:

$$\liminf_{t \to \eta^+} \tau(t) > 0, \text{ for all } \eta > 0. \tag{A_1}$$

Henceforth, we denote by $\mathcal{S}_1$ the set $\mathcal{S}_\omega$ when $\omega = 1$. Clearly, we have $\mathcal{S}_\omega \subseteq \mathcal{S}_1$.

Example 3.2. Let the following function $G : (0,\infty) \to \mathbb{R}$, $G(x) = \dfrac{-1}{x+1}$. Clearly, $G \in \mathcal{F}_c$ but it does not satisfy condition (F_2). Indeed, if $\alpha_n = \dfrac{1}{n}, n \in \mathbb{N}$, we have $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty} G(\alpha_n) = -1 \neq -\infty$. More precisely, we have $\mathfrak{F} \subset \mathcal{F}_c$.

Example 3.3. (See [40, Example 2.2])

(a) Let $\tau > 0$ be a fixed real number and $\tau_1(t) = \tau$ for all $t \in (0,\infty)$. Then $\tau_1 \in \mathcal{S}_\omega$.
(b) Let $\tau_2(t) = \ln(1+t)$ for all $t \in (0,\infty)$. Then $\tau_2 \in \mathcal{S}_\omega$.
(c) Let $\tau_3(t) = \varrho t$ for all $t \in (0,\infty)$, where $\varrho > 0$. Then $\tau_3 \in \mathcal{S}_\omega$.

Remark 3.4. Since $\tau_2 \notin \mathbb{S}$, it is easy to see that $\mathbb{S} \subset \mathcal{S}_1$.

Motivated by the works in [23] and [43], we refine the notions of F-contraction of Hardy–Rogers type, F-contraction of Suzuki–Hardy–Rogers type and F-weak contraction of Hardy–Rogers type by introducing new notions in the context of b-metric spaces, namely the notions of extended F-contraction of Hardy–Rogers type, extended F-contraction of Suzuki–Hardy–Rogers type and generalized F-weak contraction of Hardy–Rogers type.

Before stating and proving our main results, we start to prove the following useful lemma (see also the works in [30] and [39]).

Lemma 3.5. *Let $\kappa \geq 1$ be a given real number. Let $\{t_n\} \subset (0,\infty)$ be a sequence and let $\phi, \psi : (0,\infty) \to \mathbb{R}$ be two functions satisfying the following conditions:*

(i) $\psi(\kappa t_n) \leq \phi(t_{n-1})$, *for all* $n \in \mathbb{N}$;
(ii) ψ *is nondecreasing;*
(iii) $\phi(t) < \psi(t)$, *for all* $t > 0$;

(*iv*) $\limsup_{t\to\eta^+} \phi(t) < \psi(\eta^+)$, *for all* $\eta > 0$.

Then $\lim_{n\to\infty} t_n = 0$.

Proof. First, we note that the right limit of ψ exists since ψ is nondeceasing. Through (*i*) and (*iii*), we have

$$\psi(\kappa t_n) \le \phi(t_{n-1}) < \psi(t_{n-1}), \quad \text{for all } n \in \mathbb{N}.$$

Taking into account condition (*ii*), it follows that

$$\kappa t_n < t_{n-1}, \quad \text{for all } n \in \mathbb{N}.$$

As $\kappa \ge 1$, the last inequality implies that $\{t_n\}$ is a strictly decreasing sequence of positive numbers. Hence, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} t_n = r^+.$$

Now, we show that $r = 0$. Arguing by contradiction, we assume that $r > 0$. Again by (*i*) and (*ii*), we have

$$\psi(t_n) \le \phi(t_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Taking the upper limit as $n \to \infty$ in (3.1), we get

$$\psi(r^+) = \lim_{n\to\infty} \psi(t_n) \le \limsup_{n\to\infty} \phi(t_{n-1}) \le \limsup_{t\to r^+} \phi(t),$$

which contradicts (*iv*). Thus, $r = 0$, that is, $\lim_{n\to\infty} t_n = 0$. □

We prove now the following proposition which plays an important role in the proofs of our results.

Proposition 3.6. *Let* (X, σ) *be a b-metric space with constant* $s \ge 1$ *and let* λ *be a given real number such that* $1 \le \lambda \le s$. *Let* $T : X \to X$ *be a mapping and* $\{x_n\}$ *the Picard sequence of* T *based on an arbitrary* $x_0 \in X$. *Assume that there exist a nondecreasing function* F *and* $\tau \in \mathcal{S}_1$ *such that for all* $z \in X$ *with* $Tz \ne T^2z$,

$$\begin{aligned}&\tau(\sigma(z, Tz)) + F\left(\lambda\sigma\left(Tz, T^2z\right)\right)\\ &\le F((d_1 + d_2)\sigma(z, Tz) + d_3\sigma\left(Tz, T^2z\right) + d_4\sigma\left(z, T^2z\right)),\end{aligned} \tag{P}$$

where d_1, d_2, d_3, d_4 *are nonnegative real numbers satisfying*

$$d_1 + d_2 + d_3 + 2d_4 s = \frac{\lambda}{s} \text{ and } d_3 \ne \frac{\lambda}{s}. \tag{D}$$

Then $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$.

Proof. Let us put $\sigma_n := \sigma(x_n, x_{n+1})$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, the proof is immediately finished. Hence, we assume that

$$x_n \ne x_{n+1}, \quad \text{for all } n \in \mathbb{N}_0.$$

This means that $Tx_{n-1} \ne T^2x_{n-1}$ for all $n \in \mathbb{N}$. Applying the inequality (P) with $z = x_{n-1}$, we have for all $n \in \mathbb{N}$

$$\begin{aligned}&\tau(\sigma_{n-1}) + F(\lambda\sigma_n)\\ &\le F((d_1 + d_2)\sigma_{n-1} + d_3\sigma_n + d_4\sigma(x_{n-1}, x_{n+1})).\end{aligned} \tag{3.2}$$

Using the relaxed triangle inequality (b_3), we get

$$\sigma\left(x_{n-1}, x_{n+1}\right) \leq s\left(\sigma_{n-1}+\sigma_n\right), \quad \text{for all } n \in \mathbb{N}.$$

So, (3.2) turns into

$$\begin{aligned} &\tau\left(\sigma_{n-1}\right)+F\left(\lambda \sigma_n\right) \\ &\leq F\left(\left(d_1+d_2+d_4 s\right) \sigma_{n-1}+\left(d_3+d_4 s\right) \sigma_n\right), \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3.3}$$

Since F is nondecreasing and $\tau(t)>0, \forall t>0$, it follows that

$$\lambda \sigma_n<\left(d_1+d_2+d_4 s\right) \sigma_{n-1}+\left(d_3+d_4 s\right) \sigma_n, \quad \text{for all } n \in \mathbb{N}.$$

This implies that

$$\left(\lambda-d_3-d_4 s\right) \sigma_n<\left(d_1+d_2+d_4 s\right) \sigma_{n-1}, \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

Since

$$\lambda-d_3-d_4 s \geq \frac{\lambda}{s}-d_3-d_4 s,$$

inequality (3.4) gives

$$\left(\frac{\lambda}{s}-d_3-d_4 s\right) \sigma_n<\left(d_1+d_2+d_4 s\right) \sigma_{n-1}, \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

Since $d_1+d_2+d_3+2 d_4 s=\frac{\lambda}{s}$ and $d_3 \neq \frac{\lambda}{s}$, we get

$$\frac{\lambda}{s}-d_3-d_4 s>0.$$

Therefore, inequality (3.5) yields

$$\sigma_n<\frac{d_1+d_2+d_4 s}{\frac{\lambda}{s}-d_3-d_4 s} \sigma_{n-1}=\sigma_{n-1}, \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

As F is nondecreasing, then by substituting (3.6) into (3.3) and using again $d_1+d_2+d_3+2 d_4 s=\frac{\lambda}{s}$ with $1 \leq \lambda \leq s$, we obtain

$$\begin{aligned} F\left(\lambda \sigma_n\right) & \leq F\left(\left(d_1+d_2+d_4 s\right) \sigma_{n-1}+\left(d_3+d_4 s\right) \sigma_{n-1}\right)-\tau\left(\sigma_{n-1}\right) \\ & =F\left(\frac{\lambda}{s} \sigma_{n-1}\right)-\tau\left(\sigma_{n-1}\right) \\ & \leq F\left(\sigma_{n-1}\right)-\tau\left(\sigma_{n-1}\right). \end{aligned} \tag{3.7}$$

This leads to

$$F\left(\lambda \sigma_n\right) \leq F\left(\sigma_{n-1}\right)-\tau\left(\sigma_{n-1}\right), \quad \text{for all } n \in \mathbb{N}. \tag{3.8}$$

Taking $\psi(t)=F(t)$ and $\phi(t)=F(t)-\tau(t)$ for all $t \in(0, \infty)$, inequality (3.8) can be written in the following form:

$$\psi\left(\lambda \sigma_n\right) \leq \phi\left(\sigma_{n-1}\right), \quad \text{for all } n \in \mathbb{N}.$$

As F is nondecreasing, then in view of the last inequality and using the fact that $\tau \in \mathcal{S}_1$ (i.e., (A_1) holds), it is easy to see that all the conditions of Lemma 3.5 with $(\kappa=\lambda \geq 1)$ are satisfied. Thus, $\lim_{n \rightarrow \infty} \sigma_n=0$ and the proof is finished. □

Remark 3.7. As in [43, Proposition 1, inequality (6)], Proposition 3.6 also furnishes that the sequence $\{\sigma_n\}$ is a strictly decreasing (see inequality (3.6)) when $\sigma_n > 0$, for all $n \in \mathbb{N}_0$.

Remark 3.8. Proposition 3.6 extends and improves [43, Proposition 1]. In fact, taking $s = 1$ (which yields $\lambda = 1$ as well) in Proposition 3.6 (it corresponds to the case of metric spaces), we find [43, Proposition 1]. Moreover, condition (F_2) from [43, Proposition 1] is omitted. Otherwise, for the function τ, we have used the condition that $\tau \in \mathcal{S}_1$ instead of the condition that $\tau \in \mathbb{S}$. This is a slightly weaker condition since $\mathbb{S} \subset \mathcal{S}_1$. In addition, we also change the condition that F is strictly increasing from [43, Proposition 1] into the weaker condition that F is nondecreasing (i.e., the strictness of the monotonicity of F is not necessary).

3.1. Extended *F*-contraction of Hardy–Rogers type

In this subsection and for the sake of readability, we present our results gradually to point out the different techniques used in some steps of our proofs in the case where the only omitted condition is (F_2) and in the case where we assume only the condition that F is nondecreasing.

Let (X, σ) be a b-metric space with constant $s \geq 1$. Throughout this subsection, we denote, for all $x, y \in X$,

$$Q_T^{\sigma}(x,y) = \alpha\sigma\left(x,y\right) + \beta\sigma\left(x,Tx\right) + \gamma\sigma\left(y,Ty\right) + \delta\sigma\left(x,Ty\right) + L\sigma\left(y,Tx\right),$$

where $\alpha, \beta, \gamma, \delta, L$ are nonnegative real numbers. If $s = 1$, we write $Q_T^d(x,y)$ instead of $Q_T^{\sigma}(x,y)$, where d is a metric on X.

We begin this subsection with the following definitions.

Definition 3.9. Let (X, σ) be a b-metric space with constant $s \geq 1$. A mapping $T : X \to X$ is said to be an *extended F-contraction of Hardy–Rogers type* if there exist $F : (0, \infty) \to \mathbb{R}$ and $\tau \in \mathcal{S}_{\omega}$ such that for all $x, y \in X$,

$$\sigma\left(Tx,Ty\right) > 0 \Rightarrow \tau\left(\sigma\left(x,y\right)\right) + F\left(\sigma\left(Tx,Ty\right)\right) \leq F\left(Q_T^{\sigma}(x,y)\right). \tag{3.9}$$

Remark 3.10. If F is nondecreasing, it is easy to see from Definition 3.9 that every T which is an extended F-contraction of Hardy–Rogers type satisfies the following condition

$$\sigma\left(Tx,Ty\right) < Q_T^{\sigma}(x,y), \tag{3.10}$$

for all $x, y \in X$ with $Tx \neq Ty$.

Consistent with [43], we give analogously the definition of an extended F-contraction of Suzuki–Hardy–Rogers type in b-metric spaces.

Definition 3.11. Let (X, σ) be a b-metric space with constant $s \geq 1$. A mapping $T : X \to X$ is said to be an *extended F-contraction of Suzuki–Hardy–Rogers type* if there exist $F : (0, \infty) \to \mathbb{R}$ and $\tau \in \mathcal{S}_{\omega}$ such that for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{1}{2s}\sigma\left(x,Tx\right) < \sigma\left(x,y\right) \Rightarrow \tau\left(\sigma\left(x,y\right)\right) + F\left(\sigma\left(Tx,Ty\right)\right) \leq F\left(Q_T^{\sigma}(x,y)\right). \tag{3.11}$$

Remark 3.12. If F is nondecreasing, it is easy also to see from Definition 3.11 that every T which is an extended F-contraction of Suzuki–Hardy–Rogers type satisfies the following condition:

$$\sigma(Tx, Ty) < Q_T^{\sigma}(x, y), \tag{3.12}$$

for all $x, y \in X$ with $Tx \neq Ty$ and $\frac{1}{2s}\sigma(x, Tx) < \sigma(x, y)$.

Now, we are ready to state and prove our main results. The following theorem is one of them and it is an extension and an improvement of Theorem 2.24.

Theorem 3.13. *Let (X, σ) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ an extended F-contraction of Hardy–Rogers type with $F \in \mathcal{F}_c$. Suppose that either $(\mathcal{H}_s^1)$ or $(\mathcal{H}_s^2)$ holds, where*

$(\mathcal{H}_s^1)$ $\alpha + \beta + \gamma + 2\delta s = \frac{1}{s}$ *and* $\gamma \neq \frac{1}{s}$,

$(\mathcal{H}_s^2)$ $\alpha + \beta + \gamma + 2Ls = \frac{1}{s}$ *and* $\beta \neq \frac{1}{s}$.

Furthermore, we assume that $s^2\alpha + s^3(\delta + L) \leq 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.

Proof. First, we will show that T has at most one fixed point. Assume that x^* and y^* are two distinct fixed points of T, that is, $Tx^* = x^* \neq y^* = Ty^*$. Then

$$\sigma(Tx^*, Ty^*) = \sigma(x^*, y^*) > 0.$$

1. If $\alpha + \delta + L > 0$, from (3.9) (with $x = x^*$ and $y = y^*$), we obtain

$$\begin{aligned}\tau(\sigma(x^*, y^*)) + F(\sigma(x^*, y^*)) &\leq F((\alpha + \delta + L)\sigma(x^*, y^*)) \\ &\leq F((s^2\alpha + s^3(\delta + L))\sigma(x^*, y^*)) \\ &\leq F(\sigma(x^*, y^*)).\end{aligned}$$

   The last inequality yields $\tau(\sigma(x^*, y^*)) \leq 0$, which is a contradiction.
2. If $\alpha + \delta + L = 0$, from (3.10) (with $x = x^*$ and $y = y^*$), we have

$$\sigma(x^*, y^*) < Q_T^{\sigma}(x^*, y^*) = (\alpha + \delta + L)\sigma(x^*, y^*) = 0,$$

   which is a contradiction.

Thus, in both cases, we get a contradiction. Hence, T has at most one fixed point.

Next, we prove the existence of a fixed point. Let $\{x_n\}$ be the Picard sequence based on an arbitrary $x_0 \in X$. If there exists $n_0 \in \mathbb{N}_0$, such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is the fixed point of T and the proof is completed. If $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}_0$, we have

$$\sigma_n := \sigma(x_n, x_{n+1}) = \sigma(Tx_{n-1}, Tx_n) > 0, \quad \text{for all } n \in \mathbb{N}. \tag{3.13}$$

From the hypothesis of the theorem, we consider the following cases:

Case 1 If $\left(\mathcal{H}_s^1\right)$ holds. Owing to (3.13), we can apply the contractive condition (3.9) with $x = x_{n-1}$ and $y = x_n$. Hence, we get for all $n \in \mathbb{N}$

$$\begin{aligned} &\tau\left(\sigma\left(x_{n-1}, x_n\right)\right) + F\left(\sigma\left(Tx_{n-1}, Tx_n\right)\right) \\ &\leq F((\alpha+\beta)\,\sigma\left(x_{n-1}, x_n\right) + \gamma\sigma\left(x_n, Tx_n\right) + \delta\sigma\left(x_{n-1}, Tx_n\right)). \end{aligned} \tag{3.14}$$

Putting $x_{n-1} = z$ in (3.14) and using the fact that

$$Tz = Tx_{n-1} = x_n \neq x_{n+1} = T^2x_{n-1} = T^2z,$$

the inequality (3.14) turns into (P) with $d_1 = \alpha$, $d_2 = \beta$, $d_3 = \gamma$, $d_4 = \delta$ and $\lambda = 1$. Therefore, by virtue of ($\mathcal{S}_\omega \subseteq \mathcal{S}_1$) and Proposition 3.6 with $\lambda = 1$, we have $\lim_{n\to\infty}\sigma_n = 0$.

Case 2 If $\left(\mathcal{H}_s^2\right)$ holds. From (3.13), we can also apply (3.9) with $x = x_n$ and $y = x_{n-1}$. So, using the symmetry condition (b_2), we get for all $n \in \mathbb{N}$

$$\begin{aligned} &\tau\left(\sigma\left(x_{n-1}, x_n\right)\right) + F\left(\sigma\left(Tx_{n-1}, Tx_n\right)\right) \\ &\leq F((\alpha+\gamma)\,\sigma\left(x_{n-1}, x_n\right) + \beta\sigma\left(x_n, Tx_n\right) + L\sigma\left(x_{n-1}, Tx_n\right)). \end{aligned} \tag{3.15}$$

Similarly, as in *Case 1*, inequality (3.15) turns into (P) with $d_1 = \alpha$, $d_2 = \gamma$, $d_3 = \beta$, $d_4 = L$ and $\lambda = 1$. Again, according to Proposition 3.6 with $\lambda = 1$, we have $\lim_{n\to\infty}\sigma_n = 0$.

Consequently, in both cases, we obtain

$$\lim_{n\to\infty}\sigma_n = \lim_{n\to\infty}\sigma\left(x_n, x_{n+1}\right) = 0. \tag{3.16}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary, i.e., $\{x_n\}$ is not a Cauchy sequence. Then from (3.16) and the first item of Lemma 2.10, there exist $\varepsilon > 0$ and two sequences $\{m(k)\}, \{n(k)\}$ of positive integers such that

$$\varepsilon^+ \leq \liminf_{k\to\infty}\sigma\left(x_{m(k)}, x_{n(k)}\right) \leq \limsup_{k\to\infty}\sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon^+.$$

Thus, we infer that there exists $k_0 \in \mathbb{N}$ such that $\left\{\sigma\left(x_{m(k)}, x_{n(k)}\right)\right\}$ is bounded for all $k \geq k_0$ and thereby it has a convergent subsequence. It follows that there exist a real number l and a subsequence $\{k\,(p)\}_{p\geq k_0}$ of $\{k\}_{k\geq k_0}$ such that

$$\lim_{p\to\infty}\sigma\left(x_{m(k(p))}, x_{n(k(p))}\right) = l \tag{3.17}$$

with

$$0 < \varepsilon^+ \leq \liminf_{k\to\infty}\sigma\left(x_{m(k)}, x_{n(k)}\right) \leq l \leq \limsup_{k\to\infty}\sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon^+. \tag{3.18}$$

On the other hand, using (b_3), we get for all $p \geq k_0$

$$\begin{aligned} &\sigma\left(x_{m(k(p))}, x_{n(k(p))}\right) \\ &\leq s\sigma\left(x_{m(k(p))}, x_{m(k(p))+1}\right) + s\sigma\left(x_{m(k(p))+1}, x_{n(k(p))}\right) \\ &\leq s\sigma\left(x_{m(k(p))}, x_{m(k(p))+1}\right) + s^2\sigma\left(x_{m(k(p))+1}, x_{n(k(p))+1}\right) \\ &+s^2\sigma\left(x_{n(k(p))}, x_{n(k(p))+1}\right) \\ &= s\sigma_{m(k(p))} + s^2\sigma\left(x_{m(k(p))+1}, x_{n(k(p))+1}\right) + s^2\sigma_{n(k(p))}. \end{aligned}$$

This leads to

$$\begin{aligned}&\sigma\left(x_{m(k(p))+1},x_{n(k(p))+1}\right)\\&\geq\frac{1}{s^2}\left(\sigma\left(x_{m(k(p))},x_{n(k(p))}\right)-s\sigma_{m(k(p))}-s^2\sigma_{n(k(p))}\right),\end{aligned}\tag{3.19}$$

for all $p\geq k_0$.

Taking the lower limit as $p\to\infty$ in (3.19) and using (3.16), we obtain

$$\liminf_{p\to\infty}\sigma\left(x_{m(k(p))+1},x_{n(k(p))+1}\right)\geq\frac{l}{s^2}.\tag{3.20}$$

Consequently, there exists $N\geq k_0$ such that

$$\sigma\left(Tx_{m(k(p))},Tx_{n(k(p))}\right)=\sigma\left(x_{m(k(p))+1},x_{n(k(p))+1}\right)>0,\quad\text{for all }p\geq N.\tag{3.21}$$

For convenience, we set

$$\begin{aligned}a_p&=\sigma\left(x_{m(k(p))},x_{n(k(p))}\right), & b_p&=\sigma\left(x_{m(k(p))+1},x_{n(k(p))+1}\right),\\c_p&=\sigma\left(x_{m(k(p))},x_{n(k(p))+1}\right), & d_p&=\sigma\left(x_{n(k(p))},x_{m(k(p))+1}\right).\end{aligned}$$

Therefore, it follows from (3.21) that the contractive inequality (3.9) can be applied with $x=x_{m(k(p))}$ and $y=x_{n(k(p))}$. Hence, for all $p\geq N$, we have

$$\tau\left(a_p\right)+F\left(b_p\right)\leq F\left(\alpha a_p+\beta\sigma_{m(k(p))}+\gamma\sigma_{n(k(p))}+\delta c_p+Ld_p\right).$$

Using (b_3), the monotonicity of F and $s^2\alpha+s^3\left(\delta+L\right)\leq 1$, we get

$$\begin{aligned}&\tau\left(a_p\right)+F\left(b_p\right)\\&\leq F\left(\left(\alpha+s\left(\delta+L\right)\right)a_p+\left(\beta+sL\right)\sigma_{m(k(p))}+\left(\gamma+s\delta\right)\sigma_{n(k(p))}\right)\\&\leq F\left(\frac{1}{s^2}a_p+\left(\beta+sL\right)\sigma_{m(k(p))}+\left(\gamma+s\delta\right)\sigma_{n(k(p))}\right),\end{aligned}$$

for all $p\geq N$.

Now, combining the above inequality with (3.17) and (3.20) through the fact that $F\in\mathcal{F}_c$, we obtain the following chain of inequalities

$$\begin{aligned}&\liminf_{t\to l}\tau\left(t\right)+F\left(\frac{l}{s^2}\right)\\&\leq\liminf_{p\to\infty}\tau\left(a_p\right)+F\left(\frac{l}{s^2}\right)\\&\leq\liminf_{p\to\infty}\tau\left(a_p\right)+F\left(\liminf_{p\to\infty}b_p\right)\\&=\liminf_{p\to\infty}\tau\left(a_p\right)+\liminf_{p\to\infty}F\left(b_p\right)=\liminf_{p\to\infty}\left[\tau\left(a_p\right)+F\left(b_p\right)\right]\\&\leq\lim_{p\to\infty}F\left(\frac{1}{s^2}a_p+\left(\beta+sL\right)\sigma_{m(k(p))}+\left(\gamma+s\delta\right)\sigma_{n(k(p))}\right)\\&=F\left(\frac{l}{s^2}\right).\end{aligned}$$

Having in mind (3.18), the preceding inequality implies that

$$\liminf_{t\to l}\tau\left(t\right)\leq 0,\quad\text{where}\quad l\in\left[\varepsilon^+,\varepsilon^+s\right],$$

which is a contradiction with (A_ω) $(\eta=\varepsilon>0,\omega=s\geq 1)$. This contradiction shows that $\{x_n\}$ is a Cauchy sequence. By completeness of (X,σ), $\{x_n\}$

converges to some point x^* in X, that is,

$$\lim_{n\to\infty} \sigma(x_n, x^*) = 0. \tag{3.22}$$

Finally, we show that x^* is a fixed point of T, that is, $Tx^* = x^*$. Assume on the contrary, i.e., $\sigma(x^*, Tx^*) > 0$. Then through (3.22), there exists $n_0 \in \mathbb{N}$ such that

$$\sigma(x_n, x^*) \leq \frac{\sigma(x^*, Tx^*)}{2s}, \quad \forall n \geq n_0. \tag{3.23}$$

On the other hand, by (b_3), we have

$$\sigma(x^*, Tx^*) \leq s\sigma(x^*, Tx_n) + s\sigma(Tx_n, Tx^*). \tag{3.24}$$

Using (3.23), inequality (3.24) yields

$$\begin{aligned}\sigma(Tx_n, Tx^*) &\geq \frac{1}{s}\left(\sigma(x^*, Tx^*) - s\sigma(x^*, Tx_n)\right)\\ &= \frac{1}{s}\sigma(x^*, Tx^*) - \sigma(x^*, x_{n+1})\\ &\geq \frac{\sigma(x^*, Tx^*)}{2s} > 0,\end{aligned} \tag{3.25}$$

for all $n \geq n_0$.

Taking into account (3.25), we can apply (3.10) with $x = x_n$ and $y = x^*$. Hence, for all $n \geq n_0$, (3.24) gives

$$\begin{aligned}\sigma(x^*, Tx^*) &\leq s\sigma(x^*, Tx_n) + s\sigma(Tx_n, Tx^*).\\ &< s\sigma(x^*, Tx_n) + s\alpha\sigma(x_n, x^*) + s\beta\sigma(x_n, Tx_n)\\ &\quad + s\gamma\sigma(x^*, Tx^*) + s\delta\sigma(x_n, Tx^*) + sL\sigma(x^*, Tx_n)\\ &= s(1+L)\sigma(x^*, x_{n+1}) + s\alpha\sigma(x_n, x^*)\\ &\quad + s\beta\sigma(x_n, Tx_n) + s\delta\sigma(x_n, Tx^*) + s\gamma\sigma(x^*, Tx^*).\end{aligned}$$

The above inequality leads to

$$\begin{aligned}(1-s\gamma)\sigma(x^*, Tx^*) < s(1+L)\sigma(x^*, x_{n+1}) + s\alpha\sigma(x_n, x^*)\\ + s\delta\sigma(x_n, Tx^*) + s\beta\sigma(x_n, Tx_n),\end{aligned} \tag{3.26}$$

for all $n \geq n_0$.

Taking the limit superior as $n \to \infty$ in (3.26) and using Lemma 2.8, (3.16) and (3.22), we get

$$(1-s\gamma)\sigma(x^*, Tx^*) \leq s^2\delta\sigma(x^*, Tx^*). \tag{3.27}$$

In a similar way, we can also apply (3.10) with $x = x^*$, $y = x_n$ and we obtain

$$(1-s\beta)\sigma(x^*, Tx^*) \leq s^2L\sigma(x^*, Tx^*). \tag{3.28}$$

Again, according to the hypothesis of the theorem, we consider the following cases:

Case 1 If $(\mathcal{H}_s^1)$ holds. In this case, we have $1 - s\gamma > 0$ and $\gamma + s\delta < \frac{1}{s}$. Consequently, (3.27) implies that

$$\sigma(x^*, Tx^*) \leq \frac{s^2\delta}{(1-s\gamma)}\sigma(x^*, Tx^*) < \sigma(x^*, Tx^*),$$

which is a contradiction.

Case 2 If $\left(\mathcal{H}_s^2\right)$ holds. In this case, we have $1 - s\beta > 0$ and $\beta + sL < \frac{1}{s}$. Hence, (3.28) yields

$$\sigma\left(x^*, Tx^*\right) \leq \frac{s^2 L}{(1 - s\beta)} \sigma\left(x^*, Tx^*\right) < \sigma\left(x^*, Tx^*\right),$$

which is a contradiction. Therefore, whether $\left(\mathcal{H}_s^1\right)$ or $\left(\mathcal{H}_s^2\right)$ holds, we obtain a contradiction. So, we have $Tx^* = x^*$ and this completes the proof of the theorem. □

Remark 3.14. By inspecting the proof of Theorem 3.13 and taking into account Remark 2.11, we observe that if $s = 1$ (its corresponds to the case of metric spaces), it suffices to use the condition that $\tau \in \mathcal{S}_1$ instead of the condition that $\tau \in \mathcal{S}_\omega$.

In the sequel, $\left(\mathcal{H}_s^1\right)$ and $\left(\mathcal{H}_s^2\right)$ denote the hypotheses given in Theorem 3.13. Also, if $s = 1$, $\left(\mathcal{H}_s^1\right)$ and $\left(\mathcal{H}_s^2\right)$ are noted $\left(\mathcal{H}_1^1\right)$ and $\left(\mathcal{H}_1^2\right)$, respectively.

Remark 3.15. Theorem 3.13 extends and greatly improves Theorem 2.24. Actually, by taking $s = 1$ in Theorem 3.13 with the hypothesis $\left(\mathcal{H}_1^1\right)$, we recover Theorem 2.24. In addition, we show that Theorem 2.24 can be proved also through the hypothesis $\left(\mathcal{H}_1^2\right)$. Moreover, condition (F_2) from Theorem 2.24 is omitted and the condition that $\tau \in \mathbb{S}$ is weaken to the condition that $\tau \in \mathcal{S}_1$ (see Remark 3.14). Besides these, we have shown implicitly from the proof of Theorem 3.13 that the strictness of the monotonicity of F and $\liminf_{t \to 0^+} \tau(t) > 0$ are superfluous conditions for all $s \geq 1$.

If $s = 1$, then by taking $\delta = L = 0$ in Theorem 3.13 and taking into account Remark 3.14, we obtain the following result.

Corollary 3.16. *Let (X, d) be a complete metric space and let T be a self-mapping on X. Assume that there exist $F \in \mathcal{F}_c$ and $\tau \in \mathcal{S}_1$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\tau\left(d(x, y)\right) + F\left(d(Tx, Ty)\right) \leq F\left(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty)\right),$$

where $\alpha, \beta, \gamma \in [0, \infty)$ satisfying $\alpha + \beta + \gamma = 1$ and $\gamma \neq 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to x^*.

Remark 3.17. Corollary 3.16 improves Corollary 2.28. Indeed, condition (F_2) from Theorem 2.24 is deleted and the condition that $\tau \in \mathbb{S}$ is weaken to the condition that $\tau \in \mathcal{S}_1$. In addition, Corollary 2.28 remains true without the strictness of the monotonicity of F.

Remark 3.18. Corollary 3.16 generalizes and improves [26, Theorem 2.1]. In fact, by taking $\alpha = 1, \beta = \gamma = 0$ in Corollary 3.16 and $\tau(t) = \tau > 0$ for all $t \in (0, \infty)$, we find Theorem 2.1 of Piri and Kumam [26]. Corollary 3.16 shows that condition (F_2') can be omitted from [26, Theorem 2.1]. Besides these, the strictness of the monotonicity of F is not necessary.

The following example illustrates the usability of Theorem 3.13.

Example 3.19. Let $X = [0, 7]$. Let $T : X \to X$ be a mapping given by

$$Tx = \begin{cases} 7, & \text{if } x \in]0, 7], \\ 6, & \text{if } x = 0. \end{cases}$$

Let $\sigma : X \times X \to [0, \infty)$ be the mapping defined by

$$\sigma(x, y) = (x - y)^2, \quad \text{for all } x, y \in X.$$

By Example 2.2, (X, σ) is a complete b-metric space with constant $s = 2$.

First, we observe that

$$\sigma(Tx, Ty) = 1 > 0 \Leftrightarrow [(x \in]0, 7] \wedge y = 0) \vee (y \in]0, 7] \wedge x = 0)].$$

Let $x, y \in X$ and denote

$$M(x, y) = \frac{1}{8}\sigma(x, y) + \frac{1}{8}\sigma(y, Ty) + \frac{1}{16}\sigma(x, Ty).$$

Now, we consider the following cases:

Case 1 If $x \in]0, 7]$ and $y = 0$. In this case, we obtain

$$\begin{aligned} \frac{\sigma(x, 0)}{294} - \frac{1}{\sigma(Tx, T0) + 1} &\leq \frac{49}{294} - \frac{1}{2} = -\frac{1}{3} \\ &< -\frac{8}{36} = -\frac{8}{\sigma(0, T0)} \\ &\leq -\frac{1}{M(x, 0)} \\ &< -\frac{1}{M(x, 0) + 1}. \end{aligned} \tag{3.29}$$

Case 2 If $y \in]0, 7]$ and $x = 0$. In this case, we get

$$\begin{aligned} \frac{\sigma(0, y)}{294} - \frac{1}{\sigma(T0, Ty) + 1} &\leq \frac{49}{294} - \frac{1}{2} = -\frac{1}{3} \\ &< -\frac{16}{49} = -\frac{16}{\sigma(0, Ty)} \\ &\leq -\frac{1}{M(0, y)} \\ &< -\frac{1}{M(0, y) + 1}. \end{aligned} \tag{3.30}$$

From (3.29) and (3.30), we deduce

$$\frac{\sigma(x, y)}{294} - \frac{1}{\sigma(Tx, Ty) + 1} < -\frac{1}{M(x, y) + 1},$$

for all $x, y \in X$ with $\sigma(Tx, Ty) = 1 > 0$.

Therefore, by choosing $F(t) = -\frac{1}{t+1}$, $\tau(t) = \frac{t}{294}$, for all $t \in (0, \infty)$, we see that T is an extended F-contraction of Hardy–Rogers type with $\alpha = \frac{1}{8}, \gamma = \frac{1}{8}, \delta = \frac{1}{16}$, $\beta = L = 0$ and all the conditions of Theorem 3.13 (with $(\mathcal{H}_s^1)$) are satisfied. Hence, T has a unique fixed point x^* (here $x^* = 7$). Notice that F does not satisfy condition (F_2) and $\tau \notin \mathbb{S}$.

The following example shows that Theorem 3.13 greatly improves Theorem 2.24.

Example 3.20. Let $X = [10, 20]$ be endowed with the euclidean metric $d = |.|$ and $T : X \to X$ a mapping defined as follows:

$$Tx = \begin{cases} 20, & \text{if } x \in]10, 20], \\ 19, & \text{if } x = 10. \end{cases}$$

Obviously, T is not an F-contraction since T is not continuous (see Remark 2.16).

Let $x, y \in X$ and denote

$$D(x, y) = \frac{1}{2} d(x, Tx) + \frac{1}{4} d(y, Tx).$$

First, we have

$$d(Tx, Ty) = 1 > 0 \Leftrightarrow [(x = 10 \wedge y \in]10, 20]) \vee (y = 10 \wedge x \in]10, 20])].$$

Second, T is an extended F-contraction of Hardy–Rogers type. In fact, we distinguish the following cases:

Case 1 If $(x = 10 \wedge y \in]10, 20])$. In this case, we get

$$\begin{aligned} 1 + D(10, y) &\geq 1 + \frac{1}{2} d(10, T10) = \frac{11}{2} \\ &\geq \left(\frac{d(10, y)}{20} \right) (1 + d(T10, Ty)). \end{aligned}$$

Case 2 If $(y = 10 \wedge x \in]10, 20])$. In this case, we obtain

$$\begin{aligned} 1 + D(x, 10) &\geq 1 + \frac{1}{4} d(10, Tx) = \frac{10}{4} \\ &\geq \left(\frac{d(x, 10)}{20} + 1 \right) (1 + d(Tx, T10)). \end{aligned}$$

In view of the above cases, we deduce that

$$1 + D(x, y) \geq \left(\frac{d(x, y)}{20} + 1 \right) (1 + d(Tx, Ty)).$$

By passing to logarithms, we get

$$\tau(d(x, y)) + F(d(Tx, Ty)) \leq F(D(x, y)),$$

where $F(t) = \ln(t + 1)$ and $\tau(t) = \ln\left(\frac{t}{20} + 1\right)$, for all $t \in (0, \infty)$.

Consequently, T is an extended F-contraction of Hardy–Rogers type with $\beta = \frac{1}{2}, L = \frac{1}{4}$ and $\alpha = \gamma = \delta = 0$. Then we conclude that Theorem 3.13 holds true (with $(\mathcal{H}_1^2)$) and $x^* = 20$ is the unique fixed point of T. On the other hand, Theorem 2.24 is not applicable in this case since $\alpha + \beta + \gamma + 2\delta = \frac{1}{2} \neq 1$, $\tau \notin \mathbb{S}$ and F does not satisfy condition (F_2).

The following example shows that Corollary 3.16 improves Corollary 2.28.

Example 3.21. Let $X = [0, 10]$ be endowed with the euclidean metric $d = |.|$ and $T : X \to X$ a mapping given by

$$Tx = \begin{cases} 10, & \text{if } x \in]0, 10], \\ 9, & \text{if } x = 0. \end{cases}$$

First, since T is not continuous, T is not an F-contraction by Remark 2.16.

Let $x, y \in X$ and denote

$$G(x, y) = \frac{1}{3}\left[d(x, y) + d(x, Tx) + d(y, Ty)\right].$$

On the one hand, it easy to see that

$$d(Tx, Ty) = 1 > 0 \Leftrightarrow \left[(x \in]0, 10] \wedge y = 0) \vee (y \in]0, 10] \wedge x = 0)\right].$$

On the other hand, in both cases $[(x \in]0, 10] \wedge y = 0) \vee (y \in]0, 10] \wedge x = 0)]$, we have

$$\begin{aligned} 1 + G(x, y) &\geq 1 + \frac{1}{3} d(0, T0) = 4. \\ &= 2\left(d(Tx, Ty) + 1\right). \end{aligned}$$

Consequently, we obtain

$$\tau(d(x, y)) + F(d(Tx, Ty)) \leq F(G(x, y))$$

for all $x, y \in X$ and $\sigma(Tx, Ty) = 1 > 0$, where $\tau(t) = \ln(2)$ and $F(t) = \ln(t + 1)$, for all $t \in (0, \infty)$. Hence, we claim that all the conditions of Corollary 3.16 are fulfilled with $\alpha = \beta = \gamma = \frac{1}{3}$. So, T has a unique fixed point (which is 10). However, Corollary 2.28 can not be applied since F does not satisfy condition (F_2).

Our second result is devoted to prove a fixed point theorem concerning an extended F-contraction of Suzuki–Hardy–Rogers type in the setting of b-metric spaces. The following theorem is an extension and an improvement of Theorem 2.25.

Theorem 3.22. *Let (X, σ) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ an extended F-contraction of Suzuki–Hardy–Rogers type with $F \in \mathcal{F}_c$. Suppose that either $\left(\mathcal{H}_s^1\right)$ or $\left(\mathcal{H}_s^2\right)$ holds. Furthermore, we assume that $s^2\alpha + s^3(\delta + L) \leq 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.*

Proof. First, we show that T has at most one fixed point. Assume that x^* and y^* are two distinct fixed points of T, that is, $Tx^* = x^* \neq y^* = Ty^*$. So, we have

$$\sigma(Tx^*, Ty^*) = \sigma(x^*, y^*) > 0,$$

which implies that

$$\frac{1}{2s}\sigma(x^*, Tx^*) = 0 < \sigma(x^*, y^*). \tag{3.31}$$

Therefore, through the contractive inequalities (3.11) and (3.12), the uniqueness of the fixed point is obtained similarly as in the proof of Theorem 3.13.

Let $\{x_n\}$ be the Picard sequence based on an arbitrary $x_0 \in X$. As in the proof of Theorem 3.13, without loss of generality, we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. Thus,

$$\sigma_n = \sigma\left(x_n, x_{n+1}\right) = \sigma\left(x_n, Tx_n\right) > 0.$$

Hence,

$$\frac{1}{2s}\sigma\left(x_n, Tx_n\right) < \sigma\left(x_n, Tx_n\right).$$

In addition, we have also

$$Tz = Tx_n = x_{n+1} \neq x_{n+2} = T^2x_n = T^2z.$$

Then following the same steps as those used in the proof of Theorem 3.13, we get according to the case $\left(\mathcal{H}_s^1\right)$ (respectively, $\left(\mathcal{H}_s^2\right)$), that the contractive inequality (3.11) with $x = x_n$ and $y = Tx_n$ (respectively, with $x = Tx_n$ and $y = x_n$) turns into (P) with $z = x_n$, $\lambda = 1$ and $d_1 = \alpha$, $d_2 = \beta$, $d_3 = \gamma$, $d_4 = \delta$ (respectively, $d_1 = \alpha$, $d_2 = \gamma$, $d_3 = \beta$, $d_4 = L$). Accordingly, by Proposition 3.6 with $\lambda = 1$, we obtain

$$\lim_{n\to\infty} \sigma_n = 0 \tag{3.32}$$

and $\{\sigma_n\}$ is a strictly decreasing (see Remark 3.7).

Now we claim that $\{x_n\}$ is a Cauchy sequence. We argue by contradiction by supposing that $\{x_n\}$ is not a Cauchy sequence. By (3.32) and having in mind the process of proof of Theorem 3.13, there exist $\varepsilon > 0$, $k_0 \in \mathbb{N}$ and two subsequences $\left\{x_{m(k(p))}\right\}_{p\geq k_0}, \left\{x_{n(k(p))}\right\}_{p\geq k_0}$ of positive integers such that

$$\lim_{p\to\infty} \sigma\left(x_{m(k(p))}, x_{n(k(p))}\right) = l, \tag{3.33}$$

where $0 < \varepsilon^+ \leq l \leq s\varepsilon^+$.

Again, as in the proof of Theorem 3.13, there exists $N \geq k_0$ such that

$$\sigma\left(Tx_{m(k(p))}, Tx_{n(k(p))}\right) > 0, \quad \text{for all } p \geq N. \tag{3.34}$$

On the one hand, from (3.33), there exists $p_0 \geq k_0$ such that

$$\sigma\left(x_{m(k(p))}, x_{n(k(p))}\right) \geq \frac{l}{2}, \quad \text{for all } p \geq p_0.$$

On the other hand, by (3.32), there exists $p_1 \geq k_0$ such that

$$\sigma\left(x_{m(k(p))}, x_{m(k(p))+1}\right) \leq \frac{l}{2}, \quad \text{for all } p \geq p_1.$$

Thus, setting $p_2 = \max\{p_0, p_1\}$, we obtain

$$\begin{aligned} \frac{1}{2s}\sigma\left(x_{m(k(p))}, Tx_{m(k(p))}\right) &= \frac{1}{2s}\sigma\left(x_{m(k(p))}, x_{m(k(p))+1}\right) \\ &\leq \frac{l}{4s} < \frac{l}{2} \leq \sigma\left(x_{m(k(p))}, x_{n(k(p))}\right), \end{aligned} \tag{3.35}$$

for all $p \geq p_2$.

Consequently, in view of (3.34) and (3.35), the contractive condition (3.11) can be applied with $x = x_{m(k(p))}$ and $y = x_{n(k(p))}$ for all $p \geq p_3 = \max\{N, p_2\}$. Hence, following the same method as the one used in the proof of Theorem 3.13, we find a contradiction. In other words, $\{x_n\}$ is a Cauchy sequence. As (X,σ) is complete, $\{x_n\}$ converges to some point x^* in X, that is,

$$\lim_{n\to\infty}\sigma(x_n,x^*)=0.$$

Next, we show that x^* is a fixed point of T, that is, $Tx^*=x^*$. Assume on the contrary, i.e., $\sigma(x^*,Tx^*)>0$. We now prove that, for every $n\in\mathbb{N}$

$$\frac{1}{2s}\sigma(x_n,Tx_n)<\sigma(x_n,x^*) \text{ or } \frac{1}{2s}\sigma\left(Tx_n,T^2x_n\right)<\sigma(Tx_n,x^*). \tag{3.36}$$

Arguing by contradiction, we assume that there exists $m\in\mathbb{N}$ such that

$$\frac{1}{2s}\sigma(x_m,Tx_m)\geq\sigma(x_m,x^*) \text{ and } \quad \frac{1}{2s}\sigma\left(Tx_m,T^2x_m\right)\geq\sigma(Tx_m,x^*). \tag{3.37}$$

Using (b_3) and (3.37) with the fact that $\{\sigma_n\}$ is a strictly decreasing, we get

$$\begin{aligned}\sigma(x_m,Tx_m)&\leq s\sigma(x_m,x^*)+s\sigma(x^*,Tx_m)\\&\leq\frac{1}{2}\sigma(x_m,Tx_m)+\frac{1}{2}\sigma\left(Tx_m,T^2x_m\right)\\&<\frac{1}{2}\sigma(x_m,Tx_m)+\frac{1}{2}\sigma(x_m,Tx_m)\\&=\sigma(x_m,Tx_m).\end{aligned}$$

This is a contradiction. Hence, (3.36) holds. Through the same arguments as those used in the proof of Theorem 3.13, it follows that there exists $n_1\in\mathbb{N}$ such that

$$\sigma(Tx_n,Tx^*)\geq\frac{\sigma(Tx^*,x^*)}{2s}>0 \tag{3.38}$$

and

$$\sigma\left(T^2x_n,Tx^*\right)\geq\frac{\sigma(Tx^*,x^*)}{2s}>0, \tag{3.39}$$

for all $n\geq n_1$.

Therefore, from the proof of Theorem 3.13, the first case in (3.36) (respectively, the second case) with (3.38) (respectively, (3.39)), allow us to apply (3.12) with ($x=x_n$, $y=x^*$ or $x=x^*$, $y=x_n$) (respectively, with ($x=Tx_n$, $y=x^*$ or $x=x^*$, $y=Tx_n$) for all $n\geq n_1$. Hence, in a similar way as in the proof of Theorem 3.13, we arrive at a contradiction. So, $Tx^*=x^*$ and the proof is completed.

Remark 3.23. Due to the same reasons mentioned in Remark 3.15, Theorem 3.22 extends and greatly improves Theorem 2.25.

Remark 3.24. By taking $s=1$ with $\alpha=1,\beta=\gamma=\delta=L=0$ in Theorem 3.22 and taking into account the same arguments as those given in Remark 3.18, we recover, generalize and improve Theorem 2.2 of Piri and Kumam [26].

The following example shows that Theorem 3.22 greatly improves Theorem 2.25.

Example 3.25. Let $X = \{0, 3, 7\}$ be endowed with the metric d defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Also define the mapping $T : X \to X$ by $T(0) = T(3) = 3$ and $T(7) = 0$.

Clearly, we have

$$d(Tx, Ty) = 3 > 0 \Leftrightarrow \begin{bmatrix} (x = 0 \wedge y = 7) \vee (x = 7 \wedge y = 0) \\ \vee (x = 3 \wedge y = 7) \vee (x = 7 \wedge y = 3) \end{bmatrix}. \tag{3.40}$$

On the other hand, in each of the above cases (3.40), we obtain

$$\frac{1}{2} d(x, Tx) < 4 \leq d(x, y),$$

and yields

$$\begin{aligned} \frac{d(x, y)}{8} + d(Tx, Ty) &= \frac{d(x, y)}{8} + 3 \\ &\leq \frac{d(x, y)}{8} + \frac{3d(x, y)}{4} \\ &= \frac{7d(x, y)}{8} \leq Z(x, y), \end{aligned} \tag{3.41}$$

where $Z(x, y) = \frac{7}{8} d(x, y) + \frac{1}{16} d(y, Tx)$.

Equivalently, inequality (3.41) takes the form

$$\tau(d(x, y)) + F(d(Tx, Ty)) \leq F(Z(x, y)),$$

where $F(t) = t$ and $\tau(t) = \frac{t}{8}$, for all $t \in (0, \infty)$.

Therefore, T is an extended F-contraction of Suzuki–Hardy–Rogers type with $\alpha = \frac{7}{8}$ and $L = \frac{1}{16}$, $\beta = \gamma = \delta = 0$ and all the conditions of Theorem 3.22 are satisfied (with $(\mathcal{H}_1^2)$). Hence, T has a unique fixed point x^* (which is 3). Note that Theorem 2.25 is not applicable since $\alpha + \beta + \gamma + 2\delta = \frac{7}{8} \neq 1$, $\tau \notin \mathbb{S}$ and F does not satisfy condition (F_2).

Our third result extends and greatly improves the result stated in Remark 2.26. In the following theorem, we prove a fixed point result concerning an extended F-contraction of Hardy–Rogers type in the setting of b-metric spaces without both conditions (F_2) and "F is upper semicontinuous".

Theorem 3.26. *Let (X, σ) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ satisfying the contractive condition (3.9) with $F : (0, \infty) \to \mathbb{R}$ a nondecreasing function and $\tau \in \mathcal{S}_1$. Suppose that either $(\mathcal{H}_s^1)$ or $(\mathcal{H}_s^2)$ holds. Furthermore, we assume that $s^2\alpha + s^3(\delta + L) < 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to x^*.*

Proof. In this proof, for the sake of avoiding repetition, many details are omitted here and readers are referred essentially to the proof of Theorem 3.13.

The uniqueness part is obtained similarly as in Theorem 3.13. Let $\{x_n\}$ be the Picard sequence based on an arbitrary $x_0 \in X$. Also, as in the proof of Theorem 3.13, without loss of generality, we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. Hence, we have

$$\sigma_n = \sigma\left(x_n, x_{n+1}\right) > 0.$$

Following the same steps as those used in the proof of Theorem 3.13, we obtain

$$\lim_{n\to\infty} \sigma_n = 0. \tag{3.42}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence. Arguing by contradiction, we assume that $\{x_n\}$ is not a Cauchy sequence. By (3.42) and recalling again the process of proof of Theorem 3.13, there exist $\varepsilon > 0$, $k_0 \in \mathbb{N}$ and two subsequences $\left\{x_{m(k(p))}\right\}_{p\geq k_0}, \left\{x_{n(k(p))}\right\}_{p\geq k_0}$ of positive integers such that

$$\lim_{p\to\infty} \sigma\left(x_{m(k(p))}, x_{n(k(p))}\right) = l, \tag{3.43}$$

where $0 < \varepsilon^+ \leq l \leq s\varepsilon^+$.

Again, as in the proof of Theorem 3.13, we have

$$\liminf_{p\to\infty} \sigma\left(x_{m(k(p))+1}, x_{n(k(p))+1}\right) \geq \frac{l}{s^2}. \tag{3.44}$$

Let us put

$$\mu_s = \frac{l\left(1 - A_s\right)}{s^2 B_s}, \tag{3.45}$$

where

$$A_s = s^2\alpha + s^3\left(\delta + L\right) \tag{3.46}$$

and

$$B_s = 1 + \alpha + \beta + \gamma + 2s\left(\delta + L\right). \tag{3.47}$$

From the fact that $s^2\alpha + s^3(\delta + L) < 1$, we get $A_s < 1$ and $\mu_s > 0$. This implies, using (3.42), that there exist $j_1, j_2 \geq k_0$ such that

$$\begin{aligned} \sigma_{m(k(p))} &= \sigma\left(x_{m(k(p))}, x_{m(k(p))+1}\right) \leq \mu_s, \quad \text{for all } p \geq j_1, \\ \sigma_{n(k(p))} &= \sigma\left(x_{n(k(p))}, x_{n(k(p))+1}\right) \leq \mu_s, \quad \text{for all } p \geq j_2. \end{aligned} \tag{3.48}$$

On the other hand, by virtue of (3.43) and $\mu_s > 0$, it follows that there exists $j_3 \geq k_0$ such that

$$\sigma\left(x_{m(k(p))}, x_{n(k(p))}\right) \leq l + \mu_s, \quad \text{for all } p \geq j_3. \tag{3.49}$$

Since $B_s > 1$ (otherwise, if $B_s = 1$, we get $\alpha = \beta = \gamma = \delta = L = 0$, which contradicts $\left(\mathcal{H}_s^1\right)$ or $\left(\mathcal{H}_s^2\right)$), we have $\mu_s < \dfrac{l}{s^2}$. Then in view of (3.44) and $\mu_s > 0$, there exists $j_4 \geq k_0$ such that

$$\sigma\left(Tx_{m(k(p))}, Tx_{n(k(p))}\right) > \frac{l - s^2\mu_s}{s^2} > 0, \quad \text{for all } p \geq j_4. \tag{3.50}$$

Using (3.50), the relaxed triangle inequality (b_3) and the monotonicity of F with keeping the same notations as those used in the proof of Theorem 3.13, the contractive inequality (3.9) with $x = x_{m(k(p))}$ and $y = x_{n(k(p))}$ gives for all $p \geq j_4$

$$\begin{aligned} &\tau(a_p) + F(b_p) \\ &\leq F\left((\alpha + s(\delta + L))\, a_p + (\beta + sL)\, \sigma_{m(k(p))} + (\gamma + s\delta)\, \sigma_{n(k(p))}\right). \end{aligned} \tag{3.51}$$

Setting $j = \max\{j_1, j_2, j_3, j_4\}$ and using (3.45)–(3.51) through again the monotonicity of F, we arrive at

$$\begin{aligned} \tau(a_p) + F\left(\frac{l - s^2\mu_s}{s^2}\right) &\leq F\left(\frac{A_s}{s^2} a_p + (\beta + sL)\, \sigma_{m(k(p))} + (\gamma + s\delta)\, \sigma_{n(k(p))}\right) \\ &\leq F\left(\frac{A_s}{s^2}(l + \mu_s) + (\beta + sL)\, \mu_s + (\gamma + s\delta)\, \mu_s\right) \\ &= F\left(\frac{lA_s}{s^2} + (B_s - 1)\, \mu_s\right) \\ &= F\left(\frac{l - s^2\mu_s}{s^2}\right), \end{aligned}$$

for all $p \geq j$.

The above inequality implies that $\tau(a_p) \leq 0$, for all $p \geq j$, which is a contradiction. In other words, $\{x_n\}$ is a Cauchy sequence. By completeness of (X, σ), $\{x_n\}$ converges to some point x^* in X, that is,

$$\lim_{n \to \infty} \sigma(x_n, x^*) = 0. \tag{3.52}$$

Following the same method as the one used in the proof of Theorem 3.13, we obtain also that x^* is a fixed point, i.e., $Tx^* = x^*$. This completes the proof of the theorem. □

Remark 3.27. Theorem 3.26 extends and greatly improves the result stated in Remark 2.26 on several sides. First, by taking $s = 1$ in Theorem 3.26 with the hypothesis $(\mathcal{H}_1^1)$, we recover the result given in Remark 2.26. Second, Theorem 3.26 shows that both conditions (F_2) and "F is upper semicontinuous" can be omitted from the result stated in Remark 2.26. Third, we show that the result given in Remark 2.26 can be proved also through the hypothesis $(\mathcal{H}_1^2)$. Fourth, the condition that $\tau \in \mathbb{S}$ is weaken to the condition that $\tau \in \mathcal{S}_1$ and the condition that F is strictly increasing is changed into the weaker condition that F is nondecreasing (i.e., the strictness of the monotonicity of F is not necessary).

By combining the proofs of Theorem 3.22 and Theorem 3.26, it is easy to state and prove the following theorem.

Theorem 3.28. *Let (X, σ) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ satisfying the contractive condition (3.11) with $F : (0, \infty) \to \mathbb{R}$ a nondecreasing function and $\tau \in \mathcal{S}_1$. Suppose that either $(\mathcal{H}_s^1)$ or $(\mathcal{H}_s^2)$ holds. Furthermore, we assume that $s^2\alpha + s^3(\delta + L) < 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to x^*.*

Remark 3.29. Regarding the proof of Theorem 3.28, inequality (3.35) in the proof of Theorem 3.22 will be written in the following form:

$$\frac{1}{2s}\sigma\left(x_{m(k(p))}, Tx_{m(k(p))}\right) = \frac{1}{2s}\sigma\left(x_{m(k(p))}, x_{m(k(p))+1}\right)$$
$$\leq \frac{\mu_s}{2s} < \frac{l}{2s^3} \leq \frac{l}{2} \leq \sigma\left(x_{m(k(p))}, x_{n(k(p))}\right),$$

for infinitely many values of p.

Putting $\alpha = \delta = L = 0$, Theorem 3.26 reduces to the following corollary.

Corollary 3.30. *Let (X, σ) be a complete b-metric space with constant $s \geq 1$ and let T be a self-mapping on X. Assume that there exist a nondecreasing function $F : (0, \infty) \to \mathbb{R}$ and $\tau \in \mathcal{S}_1$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\tau\left(d\left(x,y\right)\right) + F\left(d\left(Tx,Ty\right)\right) \leq F\left(\beta d\left(x,Tx\right) + \gamma d\left(y,Ty\right)\right),$$

where $\beta, \gamma \in [0, \infty)$ satisfying $\beta + \gamma = \frac{1}{s}$ and $\gamma \neq \frac{1}{s}$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.

Remark 3.31. Corollary 3.30 is a proper extension and an improvement of Corollary 2.27. In fact, by taking $s = 1$ in Corollary 3.30, we recover Corollary 2.27. Moreover, we show that both conditions (F_2) and "F is upper semicontinuous" from Corollary 2.27 can be removed. In addition, the condition that $\tau \in \mathbb{S}$ is changed into the slightly weaker condition that $\tau \in \mathcal{S}_1$ and Corollary 2.27 remains valid without the strictness of the monotonicity of F.

Example 3.32. Let $X = [0, 4]$. Let $T : X \to X$ be a mapping given by

$$Tx = \begin{cases} 3, & \text{if } x \in]0,4], \\ \frac{5}{2}, & \text{if } x = 0. \end{cases} \tag{3.53}$$

Let $\sigma : X \times X \to [0, \infty)$ be the mapping defined by

$$\sigma\left(x,y\right) = \left(x-y\right)^2, \quad \text{for all } x, y \in X.$$

As in Example 3.19, (X, σ) is a complete b-metric space with constant $s = 2$.

First, we easily obtain

$$\sigma\left(Tx,Ty\right) = \frac{1}{4} > 0 \Leftrightarrow \left[\left(x \in]0,4] \wedge y = 0\right) \vee \left(y \in]0,4] \wedge x = 0\right)\right].$$

Let $x, y \in X$ and denote

$$N\left(x,y\right) = \frac{1}{4}\left[\sigma\left(x,Tx\right) + \sigma\left(y,Ty\right)\right].$$

Next, in both cases $[(x \in]0,4] \wedge y = 0) \vee (y \in]0,4] \wedge x = 0)]$, we get

$$N\left(x,y\right) \geq \frac{1}{4}\sigma\left(0,T0\right) = \frac{25}{16}.$$

On the other hand, using the following inequality

$$h + \frac{1}{h} \geq 2, \quad \text{for all } h > 0,$$

we obtain

$$\frac{\sigma(x,y)}{17} + \ln(\sigma(Tx,Ty)+1) \le \frac{16}{17} + \ln\left(\frac{1}{4}+1\right) < 2 \le N(x,y) + \frac{1}{N(x,y)},$$

for all $x, y \in X$ with $Tx \neq Ty$.

As $\sigma(Tx,Ty) < 1$ and $N(x,y) > 1$, then by choosing $\tau(t) = \frac{t}{17}$, $t \in (0,\infty)$ and $F : (0,\infty) \to \mathbb{R}$ given as follows:

$$F(t) = \begin{cases} \ln(t+1), & \text{if } t \in]0,1], \\ t + \frac{1}{t}, & \text{if } t > 1, \end{cases}$$

we see that all the hypotheses of Corollary 3.30 are satisfied for $\beta = \gamma = \frac{1}{4}$. Consequently, T has a unique fixed point x^* (here $x^* = 3$). Notice that F is not upper semicontinuous at $t = 1$ and does not satisfy condition (F_2).

The following example illustrates that Corollary 3.30 generalizes Corollary 2.27.

Example 3.33. Let $X = [0,4]$ be equipped with the euclidean distance $d = |.|$ and $T : X \to X$ defined by (3.53).

In view of Remark 2.16, T is not an F-contraction because T is not continuous.

First, we get

$$d(Tx,Ty) = \frac{1}{2} > 0 \Leftrightarrow [(x \in]0,4] \wedge y = 0) \vee (y \in]0,4] \wedge x = 0)].$$

Let $x, y \in X$ and denote

$$N_d(x,y) = \frac{1}{2}[d(x,Tx) + d(y,Ty)].$$

Next, in both cases $[(x \in]0,4] \wedge y = 0) \vee (y \in]0,4] \wedge x = 0)]$, we obtain

$$N_d(x,y) \ge \frac{1}{2} d(0,T0) = \frac{5}{4}.$$

On the other hand, we get

$$\frac{d(x,y)}{3} - \frac{1}{d(Tx,Ty)+1} \le \frac{4}{3} - \frac{2}{3} = \frac{2}{3} < \ln\left(\frac{9}{4}\right) = \ln\left(1 + \frac{5}{4}\right) \le \ln(1 + N_d(x,y)),$$

for all $x, y \in X$ with $Tx \neq Ty$.

Since $d(Tx,Ty) < 1$ and $N_d(x,y) > 1$, then by choosing $\tau(t) = \frac{t}{3}$, $t \in (0,\infty)$ and $F : (0,\infty) \to \mathbb{R}$ given as follows:

$$F(t) = \begin{cases} -\frac{1}{t+1}, & \text{if } t \in]0,1], \\ \ln(t+1), & \text{if } t > 1, \end{cases}$$

we see that all the conditions of Corollary 3.30 are satisfied for $\beta = \gamma = \frac{1}{2}$. Accordingly, T has a unique fixed point x^* (which is 3). However, Corollary 2.27 can not be applied since F is not upper semicontinuous at $t = 1$ and does not satisfy condition (F_2). Besides these, the function $\tau \notin \mathbb{S}$.

Remark 3.34. Notice that Example 3.33 ($\alpha = \delta = L = 0$, $\beta = \gamma = \frac{1}{2}$) allow us also to show that Theorem 3.26 greatly improves the result stated in Remark 2.26.

3.2. Generalized F-weak contraction of Hardy–Rogers type

In this subsection, we do several improvements in Theorem 2.31. For the sake of readability, we keep some notations used in [23]. Throughout this subsection, (X, σ) represents a b-metric space with constant $s \geq 1$. We recall again (see Definition 2.30), for all $x, y \in X$

$$A_T^\sigma(x,y) = a\sigma(x,y) + b\sigma(x,Tx) + c\sigma(y,Ty) + e\sigma(x,Ty) + f\sigma(y,Tx),$$

where a, b, c, e, f are nonnegative real numbers. If $s = 1$, we write $A_T^\sigma(x,y) = A_T^d(x,y)$, where d is a metric on X.

Before sating our result, we introduce the following definition.

Definition 3.35. Let (X, σ) be a b-metric space with constant $s \geq 1$. A mapping $T : X \to X$ is said to be a *generalized F-weak contraction of Hardy–Rogers type* if there exist a nondecreasing function $F : (0, \infty) \to \mathbb{R}$ and $\tau \in \mathcal{S}_1$ such that for all $x, y \in X$,

$$\sigma(Tx,Ty) > 0 \Rightarrow \tau(\sigma(x,y)) + F(s\sigma(Tx,Ty)) \leq F(A_T^\sigma(x,y)). \tag{3.54}$$

Remark 3.36. It is easy to see from Definition 3.35 that every T which is a generalized F-weak contraction of Hardy–Rogers type satisfies the following condition:

$$\sigma(Tx,Ty) < \frac{1}{s} A_T^\sigma(x,y), \tag{3.55}$$

for all $x, y \in X$ with $Tx \neq Ty$.

Now, we are ready to state and prove our fourth result.

Theorem 3.37. *Let (X, σ) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ a generalized F-weak contraction of Hardy–Rogers type. Suppose that either $(\mathcal{A}_s^1)$ or $(\mathcal{A}_s^2)$ holds, where*

$(\mathcal{A}_s^1)$ $a + b + c + (s+1)e < 1$,
$(\mathcal{A}_s^2)$ $a + b + c + (s+1)f < 1$.

Furthermore, we assume that $sa + s^2(e+f) < 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to x^*.

Proof. The uniqueness part is obtained similarly as in Theorem 3.13. Let $\{x_n\}$ be the Picard sequence based on an arbitrary $x_0 \in X$. Again, as in Theorem 3.13 and without loss of generality, we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. Hence, we have

$$\sigma_n = \sigma(x_n, x_{n+1}) > 0.$$

Following the same steps as those used in Theorem 2.31 with (3.54) and (3.55), we obtain analogously

$$F\left(s\sigma_n\right) \le F\left(\sigma_{n-1}\right) - \tau\left(\sigma_{n-1}\right), \quad \text{for all } n \in \mathbb{N}. \tag{3.56}$$

Note that the above part of the proof is proved without conditions (F_3) and ($\mathrm{F}_{s,\tau}$) (see Definition 2.29).

Now, by taking $\psi\left(t\right) = F\left(t\right)$ and $\phi\left(t\right) = F\left(t\right) - \tau\left(t\right)$ for all $t \in (0, \infty)$, inequality (3.56) can be written in the following form:

$$\psi\left(s\sigma_n\right) \le \phi\left(\sigma_{n-1}\right), \quad \text{for all } n \in \mathbb{N}.$$

Since F is nondecreasing, then in view of the above inequality and using the fact that $\tau \in \mathcal{S}_1$, it is easy to see that all the conditions of Lemma 3.5 are satisfied for $\kappa = s \ge 1$. Thus, $\lim_{n\to\infty}\sigma_n = 0$.

Now we prove that $\{x_n\}$ is a Cauchy sequence. Arguing by contradiction, we assume that $\{x_n\}$ is not a Cauchy sequence. Again, by the process of proof of Theorem 3.26 and using

$$\mu_s^* = \frac{l\left(1 - A_s^*\right)}{sB_s^*},$$

where

$$A_s^* = sa + s^2\left(e + f\right), \; B_s^* = a + b + c + s\left[1 + 2\left(e + f\right)\right],$$

instead of $\mu_s = \dfrac{l\left(1 - A_s\right)}{s^2 B_s}$, we get

$$\tau\left(a_p\right) + F\left(\frac{l - s^2\mu_s^*}{s}\right) \le F\left(\frac{l - s^2\mu_s^*}{s}\right), \tag{3.57}$$

for infinitely many values of p.

This is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. By completeness of (X, σ), $\{x_n\}$ converges to some point x^* in X, that is,

$$\lim_{n\to\infty} \sigma\left(x_n, x^*\right) = 0. \tag{3.58}$$

The rest of the proof still the same as in Theorem 2.31 and the fact that x^* is a fixed point of T is proven in a similar way using (3.55). Thus, the proof of the theorem is finished. □

Remark 3.38. It is worth noticing that (3.57) is well defined since $l - s^2\mu_s^* > 0$. This last fact comes from $B_s^* > s$ (otherwise, if $B_s^* = s$, we get $\alpha = \beta = \gamma = \delta = L = 0$ which contradicts inequality (3.55)).

Remark 3.39. Compared with Theorem 2.31, it is clear that Theorem 3.37 gives some improvements. Actually, τ is taken as a function in Theorem 3.37. Moreover, Theorem 3.37 shows that both conditions (F_3) and ($\mathrm{F}_{s,\tau}$) from Theorem 2.31 are dropped and replaced by the condition that $sa + s^2\left(e + f\right) < 1$. This latter condition is quite simple and ensures simultaneously, with the remaining common hypotheses of Theorem 2.31 and Theorem 3.37, the existence and uniqueness of the fixed point. However, Theorem 3.37 does not cover totally Theorem 2.31, since the condition that $a + e + f < s$ (in Theorem 2.31) which is only used in the uniqueness part is slightly weaker than the condition that $sa + s^2(e+f) < 1$. Besides, the strictness of the monotonicity of F is not necessary.

Remark 3.40. By inspecting the proofs of Theorem 3.37 and Theorem 2.31, we can also obtain $\lim_{n\to\infty}\sigma_n = 0$ in a straightforward manner using an adapted version of Proposition 3.6 ((D) is changed into $d_1 + d_2 + d_3 + (s+1)\,d_4 < \frac{\lambda}{s}$). The desired result is obtained by taking $\lambda = s$.

By taking $s = 1$ and $\tau(t) = \tau > 0$, $t \in (0,\infty)$ in Theorem 3.37, we obtain the following result.

Corollary 3.41. *Let (X,d) be a complete metric space and let T be a self-mapping on X. Assume that there exist a nondecreasing function $F : (0,\infty) \to \mathbb{R}$ and $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\tau + F(d(Tx,Ty)) \leq F\left(A_T^d(x,y)\right).$$

Suppose that either $\left(\mathcal{A}_1^1\right)$ or $\left(\mathcal{A}_1^2\right)$ holds, where

$\left(\mathcal{A}_1^1\right)$ $a + b + c + 2e < 1$,
$\left(\mathcal{A}_1^2\right)$ $a + b + c + 2f < 1$.

Furthermore, we assume that $a + e + f < 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.

Remark 3.42. Corollary 3.41 generalizes and greatly improves Theorem 2.19 in the following sense.

1. By taking $f = e$ in Corollary 3.41, we recover Theorem 2.19.
2. The assumption that T or F is continuous is removed.
3. Both conditions (F_2) and (F_3) are omitted.
4. The strictness of the monotonicity of F is not necessary.

The following example illustrates that Corollary 3.41, generalizes and greatly improves Theorem 2.19.

Example 3.43. Let $X = [0,5]$ be equipped with the euclidean distance $d = |.|$ and $T : X \to X$ a mapping defined by

$$Tx = \begin{cases} 5, & \text{if } x \in]0,5], \\ \frac{9}{2}, & \text{if } x = 0. \end{cases}$$

Obviously, we get

$$d(Tx,Ty) = \frac{1}{2} > 0 \Leftrightarrow [(x \in]0,5] \wedge y = 0) \vee (y \in]0,5] \wedge x = 0)]. \tag{3.59}$$

Let $x, y \in X$ and denote

$$\mathfrak{D}(x,y) = \frac{1}{8}d(x,y) + \frac{1}{4}d(x,Tx) + \frac{1}{4}d(y,Ty) + \frac{1}{16}(d(x,Ty) + d(y,Tx)).$$

Next, in each of the above cases (3.59), we obtain

$$\mathfrak{D}(x,y) \geq \frac{1}{4}d(0,T0) = \frac{9}{8}.$$

On the other hand, using $h+\frac{1}{h}\geq 2$, $\forall h>0$, we get

$$\frac{22}{9}-\frac{1}{\left(d\left(Tx,Ty\right)\right)^{2}+1+\left(-1\right)^{q}}\leq\frac{22}{9}-\frac{4}{9}=2$$
$$\leq\mathfrak{D}\left(x,y\right)+\frac{1}{\mathfrak{D}\left(x,y\right)},$$

for all $x,y\in X$ with $Tx\neq Ty$ and $q\in\mathbb{N}_0$.

As $d\left(Tx,Ty\right)<1$ and $\mathfrak{D}\left(x,y\right)>1$, then by choosing $\tau=\frac{22}{9}$ and $F:(0,\infty)\to\mathbb{R}$ defined by

$$F\left(t\right)=\begin{cases}-\dfrac{1}{t^{2}+1+\left(-1\right)^{q}}, & \text{if } 0<t\leq 1,\quad q\in\mathbb{N}_0,\\ t+\dfrac{1}{t}, & \text{if } t>1,\end{cases}$$

it is easy to see that all the conditions of Corollary 3.41 are fulfilled for $a=\frac{1}{8}, b=c=\frac{1}{4}$ and $e=f=\frac{1}{16}$. Consequently, T has a unique fixed point x^* (which is 5). However, Theorem 2.19 cannot be applied since neither T nor F is continuous. Moreover, F does not satisfy condition (F_2) when q is even and does not satisfy condition (F_3) when q is odd. In other words, Corollary 3.41 is greatly superior to Theorem 2.19.

In what follows, we give an another proof of Theorem 1-(a) of Hardy–Rogers [18] (see also Reich [31]).

Corollary 3.44. (See [18, Theorem 1-(a)]) *Let (X,d) be a complete metric space and T a self-mapping on X satisfying for all $x,y\in X$,*

$$d\left(Tx,Ty\right)\leq\theta_1 d\left(x,y\right)+\theta_2 d\left(x,Tx\right)+\theta_3 d\left(y,Ty\right)+\theta_4 d\left(x,Ty\right)+\theta_5 d\left(y,Tx\right),\tag{3.60}$$

where $\theta_i, i=1,\ldots,5$ are nonnegative numbers such that $\theta=\sum_{i=1}^{5}\theta_i<1$. Then T has a unique fixed point.

Proof. First, we prove that T has at most one fixed point. Assume that x^* and y^* are two fixed points of T, i.e., $Tx^*=x^*\neq y^*=Ty^*$. Using (3.60) with $x=x^*$ and $y=y^*$, we get when $\theta\neq 0$ (the case $\theta=0$ is trivial)

$$0<d\left(x^*,y^*\right)\leq\theta d\left(x^*,y^*\right)<d\left(x^*,y^*\right).$$

It is a contradiction. Accordingly, T has at most one fixed point.

If $d\left(Tx,Ty\right)>0$ with $x,y\in X$, we have $\theta>0$ (otherwise, $\theta_i=0,\forall i=1,\ldots,5$ and from (3.60), this yields $d\left(Tx,Ty\right)=0$, which is a contradiction). Thus, choosing $\rho\in]\theta,1[$, we can write

$$d\left(Tx,Ty\right)\leq\rho\left[ad\left(x,y\right)+bd\left(x,Tx\right)+cd\left(y,Ty\right)+ed\left(x,Ty\right)+fd\left(y,Tx\right)\right],\tag{3.61}$$

where

$$a=\frac{\theta_1}{\rho},\quad b=\frac{\theta_2}{\rho},\quad c=\frac{\theta_3}{\rho},\quad e=\frac{\theta_3}{\rho},\quad f=\frac{\theta_4}{\rho}.$$

In addition, we have

$$a+b+c+e+f=\frac{\theta}{\rho}<1. \tag{3.62}$$

By taking $F(t)=\ln(t)$, $t\in(0,\infty)$ and $\tau(t)=\ln\left(\frac{1}{\rho}\right)>0$, (3.61) turns into

$$\begin{aligned}&\tau+F(d(Tx,Ty))\\&\leq F[ad(x,y)+bd(x,Tx)+cd(y,Ty)+ed(x,Ty)+fd(y,Tx)],\end{aligned}$$

for all $x,y\in X$ with $Tx\neq Ty$ and a,b,c,e,f are nonnegative real numbers satisfying (3.62). Then we distinguish the following cases:

(i) If $e\leq f$, from (3.62), we obtain $a+b+c+2e<1$. Therefore, Corollary 3.41 with $\left(\mathcal{A}_1^1\right)$ ensures that T has a fixed point.

(ii) If $e>f$, (3.62) implies that $a+b+c+2f<1$. Consequently, the desired result follows from Corollary 3.41 with $\left(\mathcal{A}_1^2\right)$.

□

4. Applications

4.1. Existence and uniqueness of bounded solutions of functional equations in dynamic programming

In this subsection, we study the existence and uniqueness of the bounded solution of the following functional equation occurring in dynamic programming of multistage decision processes (see, e.g., [5] and [7]):

$$u(x)=\sup_{y\in D}\{f(x,y)+G(x,y,u(\varphi(x,y)))\},\qquad x\in W, \tag{4.1}$$

where $f:W\times D\to\mathbb{R}$ and $G:W\times D\times\mathbb{R}\to\mathbb{R}$ are bounded, $\varphi:W\times D\to W$. We assume that W and D are Banach spaces. In this framework, W (respectively, D) is called the state space (respectively, the decision space). Furthermore, φ is the transformation of process and $u(x)$ represents the optimal return function with initial state x.

Let $X=B(W)$ denotes the space of all bounded real-valued functions on W. Now, we endow X with σ defined by

$$\sigma(h,k)=\sup_{x\in W}|h(x)-k(x)|^p,\quad p\geq 1,$$

for all $h,k\in X$. Hence, (X,σ) is a complete b-metric space with $s=2^{p-1}\geq 1$. Indeed, from Example 2.2, we can deduce that (X,σ) is a b-metric space with $s=2^{p-1}\geq 1$. Also, it is easy to see that every Cauchy sequence $\{h_n\}$ in X converges uniformly to a bounded function h^*, which allows us to obtain the completeness of X.

We also define the mapping $T:X\to X$ by

$$(Tu)(x)=\sup_{y\in D}\{f(x,y)+G(x,y,u(\varphi(x,y)))\}, \tag{4.2}$$

for all $u\in X$ and $x\in W$. Since f and G are bounded, it is easy to see that T is well defined.

Let $p \geq 1$ and let $\Psi : (0, \infty) \to (0, \infty)$ be defined by

$$\Psi(t) = \begin{cases} \dfrac{(3t)^{\frac{1}{p}}}{2^{1+\frac{3}{p}}}, & \text{if } 0 < t \leq 1, \\ \dfrac{1}{2^{1-\frac{1}{p}}}, & \text{if } t > 1. \end{cases}$$

Let $h, k \in X$. Denote

$$\chi_p(h, k) := \xi M(h, k),$$

where $M(h, k) = \sigma(h, Th) + \sigma(k, Tk)$ and $\xi = \dfrac{1}{2^p}$, $p \geq 1$.

Now, we are ready to state and prove our next result.

Theorem 4.1. *Let $p \geq 1$. Let T be the self-mapping on X defined by (4.2) and assume that the following condition is satisfied:*

($\mathcal{K}$): For all $h, k \in X$ with $Th \neq Tk$,

$$|G(x, y, h(z)) - G(x, y, k(z))| \leq \Psi(\chi_p(h, k)),$$

where $x, z \in W$ and $y \in D$. Then the functional equation (4.1) has a unique bounded solution.

Proof. Let λ be an arbitrary positive number, $x \in W$ and $h, k \in X$ with $Th \neq Tk$. Then there exist $y_1, y_2 \in D$ such that

$$(Th)(x) < f(x, y_1) + G(x, y_1, h(\varphi(x, y_1))) + \frac{\lambda^{\frac{1}{p}}}{2}, \tag{4.3}$$

$$(Tk)(x) < f(x, y_2) + G(x, y_2, k(\varphi(x, y_2))) + \frac{\lambda^{\frac{1}{p}}}{2}. \tag{4.4}$$

Again, by definition of T, we have

$$(Th)(x) \geq f(x, y_2) + G(x, y_2, h(\varphi(x, y_2))), \tag{4.5}$$

$$(Tk)(x) \geq f(x, y_1) + G(x, y_1, k(\varphi(x, y_1))). \tag{4.6}$$

Utilizing (4.3) and (4.6) together with ($\mathcal{K}$), one can get

$$\begin{aligned} (Th)(x) - (Tk)(x) &< G(x, y_1, h(\varphi(x, y_1))) - G(x, y_1, k(\varphi(x, y_1))) + \frac{\lambda^{\frac{1}{p}}}{2} \\ &\leq |G(x, y_1, h(\varphi(x, y_1))) - G(x, y_1, k(\varphi(x, y_1)))| + \frac{\lambda^{\frac{1}{p}}}{2} \\ &\leq \Psi(\chi_p(h, k)) + \frac{\lambda^{\frac{1}{p}}}{2}. \end{aligned} \tag{4.7}$$

Analogously, from (4.4) and (4.5) together with ($\mathcal{K}$), we have

$$(Tk)(x) - (Th)(x) < \Psi(\chi_p(h, k)) + \frac{\lambda^{\frac{1}{p}}}{2}. \tag{4.8}$$

Combining (4.7) and (4.8), we deduce

$$|(Th)(x) - (Tk)(x)| < \Psi(\chi_p(h, k)) + \frac{\lambda^{\frac{1}{p}}}{2}.$$

Using the following inequality

$$(a+b)^p \leq 2^{p-1}\left(a^p+b^p\right), \qquad a,b>0,$$

it follows that

$$\left|(Th)(x)-(Tk)(x)\right|^p < 2^{p-1}\left[\Psi\left(\chi_p(h,k)\right)\right]^p+\frac{\lambda}{2}.$$

The above inequality yields

$$\sigma(Th,Tk) < 2^{p-1}\left[\Psi\left(\chi_p(h,k)\right)\right]^p+\frac{\lambda}{2}. \tag{4.9}$$

Now, we discuss the two possible cases:
Case 1 If $0<\chi_p(h,k)\leq 1$. In this case, (4.9) turns into

$$\sigma(Th,Tk) < \frac{3\chi_p(h,k)}{16}+\frac{\lambda}{2}. \tag{4.10}$$

As (4.10) does not depend on $x\in W$ and $\lambda>0$ is taken arbitrarily, we have

$$\sigma(Th,Tk) \leq \frac{3\chi_p(h,k)}{16}, \tag{4.11}$$

which follows that

$$\sigma(Th,Tk) < 1. \tag{4.12}$$

On the other hand, using (b_3), we have

$$\begin{aligned}\sigma(h,k) &\leq s\sigma(h,Th)+s^2\sigma(Th,Tk)+s^2\sigma(Tk,k)\\ &\leq s^2\sigma(h,Th)+s^2\sigma(k,Tk)+s^2\sigma(Th,Tk)\\ &\leq s^2M(h,k)+s^2\sigma(Th,Tk)\\ &\leq s^2\frac{\chi_p(h,k)}{\xi}+s^2\sigma(Th,Tk).\end{aligned}$$

Keeping in mind $s=2^{p-1}$ and $\xi=\dfrac{1}{2^p}$, the last inequality leads to

$$\sigma(h,k) \leq 2s^3\chi_p(h,k)+s^2\sigma(Th,Tk). \tag{4.13}$$

Owing to (4.10) and (4.13), we get

$$\begin{aligned}\frac{\sigma(h,k)}{16s^3} &\leq \frac{\chi_p(h,k)}{8}+\frac{\sigma(Th,Tk)}{16s}\\ &< \frac{\chi_p(h,k)}{8}+\sigma(Th,Tk)\\ &< \frac{\chi_p(h,k)}{8}+\frac{3\chi_p(h,k)}{16}+\frac{\lambda}{2}\\ &= \frac{5\chi_p(h,k)}{16}+\frac{\lambda}{2}.\end{aligned} \tag{4.14}$$

Taking into account (4.14) and using the following compound inequality

$$\frac{a}{1+a} < \ln(1+a) < a, \quad \text{for all } a>0,$$

we obtain

$$\begin{aligned}\frac{\sigma(h,k)}{16s^3}+\ln(1+\sigma(Th,Tk)) &< \frac{5\chi_p(h,k)}{16}+\frac{\lambda}{2}+\sigma(Th,Tk)\\ &< \frac{\chi_p(h,k)}{2}+\lambda\\ &\leq \frac{\chi_p(h,k)}{1+\chi_p(h,k)}+\lambda\\ &< \ln(1+\chi_p(h,k))+\lambda.\end{aligned}$$

Since the last inequality does not depend on $x\in W$ and $\lambda>0$ is taken arbitrarily, we get

$$\frac{\sigma(h,k)}{16s^3}+\ln(1+\sigma(Th,Tk))\leq \ln(1+\chi_p(h,k)). \tag{4.15}$$

In addition, by virtue of (4.11) and (4.13) with the fact that $0<\chi_p(h,k)\leq 1$, we get

$$\sigma(h,k)\leq 2s^3+\frac{3s^2}{16}=2^{3p-2}+3\times 2^{2p-6}. \tag{4.16}$$

Case 2 If $\chi_p(h,k)>1$. In this case, (4.9) takes the form

$$\sigma(Th,Tk)<1+\frac{\lambda}{2}. \tag{4.17}$$

Again, as (4.17) does not depend on $x\in W$ and $\lambda>0$ is taken arbitrarily, we have

$$\sigma(Th,Tk)\leq 1. \tag{4.18}$$

From (4.17) and using the following inequalities

$$\ln(1+b)<b,\ b+\frac{1}{b}\geq 2,\quad \text{for all } b>0,$$

one gets

$$\begin{aligned}1+\ln(1+\sigma(Th,Tk)) &< 1+\sigma(Th,Tk)\\ &< 2+\frac{\lambda}{2}\\ &< \chi_p(h,k)+\frac{1}{\chi_p(h,k)}+\lambda.\end{aligned} \tag{4.19}$$

Since (4.19) does not depend on $x\in W$ and $\lambda>0$ is taken arbitrarily, we have

$$1+\ln(1+\sigma(Th,Tk))\leq \chi_p(h,k)+\frac{1}{\chi_p(h,k)}. \tag{4.20}$$

Therefore, bearing in mind inequalities (4.12), (4.16) and (4.18), inequalities (4.15) and (4.20) allow us to obtain

$$\begin{aligned}\tau(\sigma(h,k))+F(\sigma(Th,Tk)) &\leq F(\chi_p(h,k))\\ &= F\left(\frac{1}{2^p}M(h,k)\right),\end{aligned}$$

for all $h, k \in X$ and $Th \neq Tk$ with $F : (0, \infty) \to \mathbb{R}$ given by

$$F(t) = \begin{cases} \ln(t+1), & \text{if } t \in]0,1], \\ t + \dfrac{1}{t}, & \text{if } t > 1 \end{cases}$$

and $\tau : (0, \infty) \to (0, \infty)$ given as follows:

$$\tau(t) = \begin{cases} \dfrac{t}{2^{3p+1}}, & \text{if } t \in]0, C_p], \\ 1, & \text{otherwise,} \end{cases}$$

where $C_p = 2^{3p-2} + 3 \times 2^{2p-6}$.

Hence, all the conditions of Corollary 3.30 are satisfied with $\beta = \gamma = \dfrac{1}{2^p}$, $p \geq 1$. Consequently, T has a unique fixed point u^* in $X = B(W)$. Thus, the functional equation (4.1) has a unique bounded solution. □

4.2. Application to nonlinear Volterra integral equations

Inspired by the approaches developed in the works [25,33,38], we apply Theorem 3.26 to prove the existence and uniqueness of a solution for the following integral equation of Volterra type :

$$u(t) = g(t) + \int_0^t K(t, r, u(r))\, \mathrm{d}r, \quad t \in [0, a], \tag{4.21}$$

where $a > 0$, $K : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ and $g : [0, a] \to \mathbb{R}$.

Let us consider $X = \mathcal{C}([0, a], \mathbb{R})$ be the set of all continuous functions $u : [0, a] \to \mathbb{R}$. It is well known that X equipped with Bielecki's norm

$$\|u\| = \sup_{t \in [0,a]} e^{-t} |u(t)|$$

is a Banach space. Thus, X endowed with the distance associated with Bielecki's norm

$$d(u, v) = \sup_{t \in [0,a]} e^{-t} |u(t) - v(t)|, \quad \text{for all } u, v \in X$$

is a complete metric space. Now, we define

$$\sigma(u, v) = (d(u, v))^2 = \sup_{t \in [0,a]} e^{-2t} (u(t) - v(t))^2, \quad \text{for all } u, v \in X. \tag{4.22}$$

Clearly, (X, σ) is a complete b-metric space with constant $s = 2$.

Theorem 4.2. *Suppose that the following hypotheses hold:*

(H_1) *the functions K and g are continuous;*
(H_2) *for all $r, t \in [0, a]$ and for all $z, \omega \in \mathbb{R}$, we have*

$$|K(t, r, z) - K(t, r, w)| \leq \frac{\sqrt{2}\,|z - w|}{\sqrt{a((z - w)^2 + 16)}}.$$

Then integral equation (4.21) has a unique solution in X.

Proof. Endow $X = \mathcal{C}([0,a],\mathbb{R})$ with the b-metric defined by (4.22) and consider the mapping $T : X \to X$ as follows:

$$(Tu)(t) = g(t) + \int_0^t K(t,r,u(r))\,\mathrm{d}r, \quad u \in X,\ t \in [0,a].$$

Obviously, under the assumptions of the theorem, T is well defined.

Let $u, v \in X$ such that $Tu \neq Tv$. Using Cauchy–Schwarz inequality and assumption (H_2), we obtain

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)|^2 &\leq \left(\int_0^t 1\mathrm{d}r\right) \int_0^t |K(t,r,u(r)) - K(t,r,v(r))|^2\,\mathrm{d}r \\
&\leq \int_0^t \frac{2(u(r)-v(r))^2}{(u(r)-v(r))^2+16}\mathrm{d}r \\
&\leq \int_0^t \frac{2(u(r)-v(r))^2}{(u(r)-v(r))^2 e^{-2r}+16}\mathrm{d}r \\
&= \int_0^t \frac{2(u(r)-v(r))^2 e^{-2r} e^{2r}}{(u(r)-v(r))^2 e^{-2r}+16}\mathrm{d}r \\
&\leq \frac{2\sigma(u,v)}{\sigma(u,v)+16} \int_0^t e^{2r}\mathrm{d}r \\
&\leq \frac{\sigma(u,v)\, e^{2t}}{\sigma(u,v)+16}.
\end{aligned}$$

This implies that

$$((Tu)(t) - (Tv)(t))^2 e^{-2t} \leq \frac{\sigma(u,v)}{\sigma(u,v)+16}.$$

Taking the supremum with respect to $t \in [0,a]$ in the above inequality, we have

$$\sigma(Tu,Tv) \leq \frac{\sigma(u,v)}{\sigma(u,v)+16}. \tag{4.23}$$

Using (4.23), after routine calculations, one can get

$$\begin{aligned}
\ln\left(\frac{\sigma(u,v)}{16}+1\right) + \sigma(Tu,Tv) + \lfloor \sigma(Tu,Tv) \rfloor &\leq \frac{\sigma(u,v)}{8} + \left\lfloor \frac{\sigma(u,v)}{8} \right\rfloor \\
&\leq \mathcal{N}(u,v) + \lfloor \mathcal{N}(u,v) \rfloor,
\end{aligned} \tag{4.24}$$

where $\lfloor x \rfloor$ denotes the integral part of x and

$$\mathcal{N}(u,v) = \frac{1}{8}d(x,y) + \frac{1}{4}d(x,Tx) + \frac{1}{16}d(y,Ty) + \frac{1}{64}\left(d(x,Ty) + d(y,Tx)\right).$$

By choosing $F(t) = t + \lfloor t \rfloor$ and $\tau(t) = \ln\left(\frac{t}{16}+1\right)$, for all $t \in (0,\infty)$, inequality (4.24) can be equivalently written as

$$\tau(\sigma(u,v)) + F(\sigma(Tu,Tv)) \leq F(\mathcal{N}(u,v)).$$

Therefore, all the conditions of Theorem 3.26 are satisfied with $\alpha = \frac{1}{8}$, $\beta = \frac{1}{4}, \gamma = \frac{1}{16}$ and $L = \delta = \frac{1}{64}$. Hence, T has unique fixed point u^* in

$X = \mathcal{C}([0,a],\mathbb{R})$. Then integral equation (4.21) has a unique solution u^* in $\mathcal{C}([0,a],\mathbb{R})$. □

4.3. Application to second-order differential equations

Motivated and inspired by the works [33] and [45], we discuss the existence and uniqueness of solutions of the following two-point boundary value problem for the second order differential equation:

$$\begin{cases} -\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = f(t, x(t)), \quad t \in I, \\ x(0) = x(1) = 0, \end{cases} \tag{4.25}$$

where $I = [0,1]$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $X = \mathcal{C}(I,\mathbb{R})$ be the space of all continuous functions $x : I \to \mathbb{R}$. It is well known that X endowed with

$$\sigma_\infty(x,y) = \sup_{t\in I}\left\{(x(t) - y(t))^2\right\}, \quad \text{for all } x, y \in X$$

is a complete b-metric space with constant $s = 2$.

Theorem 4.3. *Suppose that the following condition is satisfied:*

(W) *for all* $z, \omega \in \mathbb{R}$ *and for all* $r \in I$, *we have*

$$|f(r,z) - f(r,w)| \le \sqrt{\ln\left(\frac{(z-w)^2}{16} + 1\right)}.$$

Then problem (4.25) *has a unique solution* $x^* \in \mathcal{C}^2(I,\mathbb{R})$.

Proof. It is known that problem (4.25) is equivalent to the following integral equation:

$$x(t) = \int_0^1 G(t,r) f(r, x(r))\,\mathrm{d}r, \quad \forall t \in I, \tag{4.26}$$

where G is the Green function associated to problem (4.25), given by

$$G(t,r) = \begin{cases} t(1-r), \, 0 \le t \le r \le 1, \\ r(1-t), \, 0 \le r \le t \le 1. \end{cases}$$

Consequently, $x \in \mathcal{C}^2(I,\mathbb{R})$ is a solution of problem (4.25) if and only if $x \in \mathcal{C}(I,\mathbb{R})$ is a solution of the integral equation (4.26). Now, we can define the mapping $T : X \to X$ as follows:

$$Tx(t) = \int_0^1 G(t,r) f(r, x(r))\,\mathrm{d}r, \quad \forall t \in I, \ \forall x \in X.$$

Then finding a unique fixed point $x^* \in X$ of T is equivalent to establishing the existence and uniqueness of solutions of problem (4.25).

Let $x, y \in X$ such that $Tx \neq Ty$. From assumption (W), we get

$$\begin{aligned} ((Tx)(t) - (Ty)(t))^2 &\le \left[\int_0^1 G(t,r)\sqrt{\ln\left(\tfrac{(x(r)-y(r))^2}{16} + 1\right)}\mathrm{d}r\right]^2 \\ &\le \ln\left(\tfrac{\sigma_\infty(x,y)}{16} + 1\right)\left(\sup_{t\in I}\int_0^1 G(t,r)\,\mathrm{d}r\right)^2 \\ &= \tfrac{1}{64}\ln\left(\tfrac{\sigma_\infty(x,y)}{16} + 1\right). \end{aligned}$$

In the above inequality, we have used that $\sup_{t\in I}\int_0^1 G(t,r)\,\mathrm{d}r = \frac{1}{8}$. Hence,

$$\sigma_\infty(Tx,Ty) \le \frac{1}{64}\ln\left(\frac{\sigma_\infty(x,y)}{16}+1\right). \tag{4.27}$$

Utilizing (4.27), after routine calculations, we obtain

$$\frac{\sigma_\infty(x,y)}{16} + \sigma_\infty(Tx,Ty) \le \frac{\sigma_\infty(x,y)}{8} \le \mathcal{M}(x,y),$$

where

$$\mathcal{M}(u,v) = \frac{1}{8}(d(x,y)+d(x,Tx)+d(y,Ty)) + \frac{1}{32}(d(x,Ty)+d(y,Tx)),$$

or, equivalently,

$$\tau(\sigma(u,v)) + F(\sigma(Tu,Tv)) \le F(\mathcal{M}(u,v)).$$

Hence, all the conditions of Theorem 3.13 are satisfied with $F(t)=t$, $\tau(t) = \frac{t}{16}$, for all $t\in(0,\infty)$, $\alpha=\beta=\gamma=\frac{1}{8}$ and $L=\delta=\dfrac{1}{32}$. Therefore, T has unique fixed point u^* in $X=\mathcal{C}(I,\mathbb{R})$. Then problem (4.25) has a unique solution u^* in $\mathcal{C}^2(I,\mathbb{R})$. □

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