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# Existence of nontrivial solutions for a nonlinear second order periodic boundary value problem with derivative term

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**Abstract.** In this paper, we study the existence of nontrivial solutions to the following nonlinear differential equation with derivative term:

$$\begin{cases} u''(t) + a(t)u(t) = f\big(t, u(t), u'(t)\big), & t \in [0, \omega], \\ u(0) = u(\omega), \quad u'(0) = u'(\omega), \end{cases}$$

where $a$: $[0, \omega] \to \mathbb{R}^+ \big(\mathbb{R}^+ = [0, +\infty)\big)$ is a continuous function with $a(t) \not\equiv 0$, $f$: $[0, \omega] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and may be sign-changing and unbounded from below. Without making any nonnegative assumption on nonlinearity, using the first eigenvalue corresponding to the relevant linear operator and the topological degree, the existence of nontrivial solutions to the above periodic boundary value problem is established in $C^1[0, \omega]$. Finally, an example is given to demonstrate the validity of our main result.

**Mathematics Subject Classification.** Primary 34C25; Secondary 34B15.

**Keywords.** Nontrivial solution, Spectral radius, Topological degree, Fixed point.

## 1. Introduction

Due to wide applications in physics and engineering, second-order periodic boundary value problems (PBVPs) have been extensively studied by many authors, see [1–12] and relevant references therein.

In [1], the following problem was discussed by Atici and Guseinov

$$\begin{cases} -\big(p(t)u'(t)\big)' + q(t)u(t) = f\big(t, u(t)\big), & t \in [0, \omega], \\ u(0) = u(\omega), \quad p(0)u'(0) = p(\omega)u'(\omega), \end{cases}$$

where $p(x)$ and $q(x)$ are real-valued measurable functions defined on $[0, \omega]$ satisfying $p(x) > 0, q(x) \geq 0, q(x) \neq 0$ almost everywhere, and

$$\int_0^\omega \frac{dx}{p(x)} < +\infty, \int_0^\omega q(x)dx < +\infty,$$

$f$: $[0,\omega]\times\mathbb{R}\to\mathbb{R}$ is continuous and $f(t,x)\ge 0$ for $(t,x)\in[0,\omega]\times[0,\infty]$. If there exist numbers $0<r<R<+\infty$ such that for all $t\in[0,\omega]$,

$$f(t,x)\le\frac{1}{\omega M}x \text{ for } 0\le x\le r \text{ and } f(t,x)\ge\frac{M}{\omega m^2}x \text{ for } R\le x<+\infty,$$

where $m=\min_{t,s\in[0,\omega]}G(t,s), M=\max_{t,s\in[0,\omega]}G(t,s)$ and $G(t,s)$ is the Green's function according to its linear problem, the authors established the existence of positive solutions.

Graef et al. [2] investigated the existence of positive solutions, under

$$\lim_{u\to 0}\frac{f(u)}{u}=+\infty,\ \lim_{u\to+\infty}\frac{f(u)}{u}=0 \text{ or } \lim_{u\to 0}\frac{f(u)}{u}=0,\ \lim_{u\to+\infty}\frac{f(u)}{u}=+\infty$$

with $f$ convex and nondecreasing, to

$$\begin{cases} u''(t)+a(t)u(t)=g(t)f(u(t)), & t\in[0,2\pi],\\ u(0)=u(2\pi), \quad u'(0)=u'(2\pi), \end{cases}$$

where $f:[0,+\infty)\to[0,+\infty), g:[0,2\pi]\to[0,+\infty)$ are continuous such that $\min_{t\in[0,2\pi]}g(t)>0$, and the Green's function is nonnegative.

Hai [3] proved the existence of positive solutions to

$$\begin{cases} u''(t)+a(t)u(t)=\lambda g(t)f(u(t)), & t\in[0,2\pi],\\ u(0)=u(2\pi), \quad u'(0)=u'(2\pi) \end{cases}$$

for all $\lambda>0$, where $a:[0,2\pi]\to[0,+\infty)$ is continuous with $a(t)\le 1/4$ for all $t$ and $a(t)\not\equiv 0$, $f:[0,+\infty)\to[0,+\infty)$ is continuous, $g\in L^1(0,2\pi)$ with $g\ge 0$ and $g\not\equiv 0$ on any subinterval of $(0,2\pi)$.

Li and Liang in [4] obtained the existence of positive solutions for

$$\begin{cases} u''(t)+a(t)u(t)=f(t,u(t)), & t\in[0,\omega],\\ u(0)=u(\omega), \quad u'(0)=u'(\omega), \end{cases} \tag{1.1}$$

where $f:[0,\omega]\times[0,+\infty)\to[0,+\infty)$ and $a:[0,\omega]\to[0,+\infty)$ are continuous with $a(t)\not\equiv 0$, provided that a constant $0<M\le(\pi/\omega)^2$ and

$$\limsup_{u\to 0^+}\max_{t\in[0,\omega]}\frac{f(t,u)}{u}<\lambda_1<\liminf_{u\to+\infty}\min_{t\in[0,\omega]}\frac{f(t,u)}{u}$$

or

$$\liminf_{u\to 0^+}\min_{t\in[0,\omega]}\frac{f(t,u)}{u}>\lambda_1>\limsup_{u\to+\infty}\max_{t\in[0,\omega]}\frac{f(t,u)}{u}.$$

Torres [5] gave classical conditions that guarantee the nonnegativity of the Green's function $G(t,s)$ of the linear problem

$$\begin{cases} u''(t)+a(t)u(t)=0, \text{ a.e. } t\in(0,\omega),\\ u(0)=u(\omega), \quad u'(0)=u'(\omega). \end{cases} \tag{1.2}$$

The following best Sobolev constants were used:

$$K(q)=\begin{cases} \frac{2\pi}{q\omega^{1+\frac{2}{q}}}\left(\frac{2}{2+q}\right)^{1-\frac{2}{q}}\left(\frac{\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{q}\right)}\right)^2, & 1\le q<+\infty,\\ \frac{4}{\omega}, \quad q=+\infty, \end{cases} \tag{1.3}$$

where $\Gamma$ is the Gamma function. Let $a \in L^p(0,\omega)(1 \le p \le +\infty)$ and $a \succ 0$ which means that $a(t) \ge 0$ for a.e. $t \in (0,\omega)$ and $a(t) > 0$ for $t$ in a subset of positive measure. Then,

$$G(t,s) \ge (>) 0 \,\text{for}\, (t,s) \in [0,\omega] \times [0,\omega] \,\text{when}\, \|a\|_p \le (<) K(2p^*), \quad (1.4)$$

where $1/p + 1/p^* = 1$. Define

$$\Delta = \{a \in L^p(0,\omega) : a \succ 0, \|a\|_p < K(2p^*) \,\text{for some}\, 1 \le p \le +\infty\}. \quad (1.5)$$

For $a \in \Delta$, the explicit expression of $G(t,s)$ was found by Ma et al. in [6] which shows that it is symmetrical, i.e., $G(t,s) = G(s,t), \forall t,s \in [0,\omega]$.

For $\omega = 1$, PBVP (1.1) was studied in [7] when $a \in L^1[0,1] \bigcap \Delta$ and $f : [0,1] \times (0,+\infty) \to \mathbb{R}$ with a bound below. Under some other conditions, the existence of positive solutions was obtained.

By means of the fixed point theory, Liu et al. in [8] established the existence of nontrivial solutions for PBVP (1.1) in which $a : [0,\omega] \to [0,+\infty)$ is a continuous function with $a(t) \not\equiv 0$, and $f$: $[0,\omega] \times \mathbb{R} \to \mathbb{R}$ is continuous under constraints associated with the first eigenvalue corresponding to the relevant linear operator.

In the works mentioned above, all the nonlinearities are independence of the derivative term $u'$. By Leray–Schauder fixed point theorem, Li and Guo in [9] considered the existence of solution to

$$\begin{cases} u''(t) = f(t,u(t),u'(t)), \quad t \in [0,2\pi], \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{cases}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous and satisfies the following conditions:

($F_1$) $f(-t,-x,y) = -f(t,x,y), \forall (t,x,y) \in \mathbb{R}^3$;

($F_2$) there exist nonnegative constants $a$ and $b$ satisfying $a + b < 1$ and a positive constant $C_0$ such that $f(t,x,y)x \ge -ax^2 - by^2 - C_0, (t,x,y) \in \mathbb{R}^3$;

($F_3$) the following Nagumo condition is satisfied, that is, for any given $M > 0$, there is a positive continuous function $g_M(\rho)$ on $\mathbb{R}^+ = [0,+\infty)$ satisfying $\int_0^{+\infty} \frac{\rho d\rho}{g_M(\rho)+1} = +\infty$ such that $|f(t,x,y)| \le g_M(|y|), (t,x,y) \in [0,2\pi] \times [-M,M] \times \mathbb{R}$.

Inspired by the references cited above and [13–17], we in this paper explore the existence of nontrivial solutions to the following periodic boundary value problem with the nonlinearity dependent on derivative term

$$\begin{cases} u''(t) + a(t)u(t) = f\big(t,u(t),u'(t)\big), \quad t \in [0,\omega], \\ u(0) = u(\omega), \quad u'(0) = u'(\omega), \end{cases} \quad (1.6)$$

where $a : [0,\omega] \to \mathbb{R}^+$ is a continuous function with $a(t) \not\equiv 0$, $f : [0,\omega] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and may be sign-changing and unbounded from below. Without making any nonnegative assumption on the nonlinearity, using the first eigenvalue corresponding to the relevant linear operator and the topological degree, the existence of nontrivial solutions is established in $C^1[0,\omega]$. As far as we know, this kind of PBVP has achieved fewer results.

## 2. Preliminaries

Let $E = C^1[0,\omega]$ be the Banach space of all continuously differentiable functions on $[0,\omega]$ with the norm $\|u\|_{C^1} = \max\{\|u\|_C, \|u'\|_C\}$ for all $u \in E$. Set $P = \{u \in C[0,\omega] : u(t) \geq 0, t \in [0,\omega]\}$, it is clear that $P$ is a solid cone in $C[0,\omega]$, that is, the interior point set of $P$ is nonempty. Thus $P$ is a total cone in $C[0,\omega]$, i.e. $C[0,\omega] = \overline{P-P}$, which means that the set $P - P = \{u - v : u, v \in P\}$ is dense in $C[0,\omega]$ (see [18,19]).

To state our main theorem in this paper, we make the following hypotheses:

($C_1$) $a : [0,\omega] \to \mathbb{R}^+$ is a continuous function with $a(t) \not\equiv 0$ and $\|a\|_C < \left(\frac{\pi}{\omega}\right)^2 = K(2)$, where $K$ is defined by (1.3);

($C_2$) $f\colon [0,\omega] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

**Lemma 2.1.** *If ($C_1$) and ($C_2$) hold, then PBVP (1.6) is equivalent to*

$$u(t) = \int_0^\omega G(t,s) f(s, u(s), u'(s))ds,$$

*where $G(t,s)$ is the Green's function of (1.2).*

**Lemma 2.2.** [5,6] *If ($C_1$) holds, then $G(t,s)$ has the following properties:*

(1) *$G(t,s) = G(s,t) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$;*

(2) *Let $l_1 = \min_{0\leq t,s\leq\omega} G(t,s)$ and $l_2 = \max_{0\leq t,s\leq\omega} G(t,s)$, then $l_2 > l_1 > 0$;*

(3) *Let $l_3 = \sup_{0\leq t,s\leq\omega} \left|\frac{\partial G(t,s)}{\partial t}\right|$, then $l_3 > 0$;*

(4) *For $c = \frac{l_1}{l_2} \in (0,1)$, $G(t,s) \geq cG(\tau,s), \quad \forall\, t,s,\tau \in [0,\omega]$.*

*Remark 2.3.* By ($C_1$), $a \in L^\infty[0,\omega]$ and thus $a \in \Delta$ defined by (1.4) for $p = +\infty$ and $p^* = 1$. Lemma 2.1 follows from (1.4). If $a(t) \equiv m^2$ , $m \in (0, \frac{1}{2})$ and $\omega = 2\pi$, we can obtain that $G(t,s)$ defined as follows satisfies Lemma 2.2.

$$G(t,s) = \begin{cases} \frac{\sin m(t-s) + \sin m(2\pi - t + s)}{2m(1-\cos 2m\pi)}, 0 \leq s \leq t \leq 2\pi, \\ \frac{\sin m(s-t) + \sin m(2\pi - s + t)}{2m(1-\cos 2m\pi)}, 0 \leq t \leq s \leq 2\pi, \end{cases}$$

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} \frac{\cos m(t-s) - \cos m(2\pi - t + s)}{2(1-\cos 2m\pi)}, 0 \leq s \leq t \leq 2\pi, \\ \frac{-\cos m(s-t) + \cos m(2\pi - s + t)}{2(1-\cos 2m\pi)}, 0 \leq t \leq s \leq 2\pi, \end{cases}$$

$$l_1 = \frac{\sin 2m\pi}{2m(1-\cos 2m\pi)}, \quad l_2 = \frac{\sin m\pi}{m(1-\cos 2m\pi)}, \quad l_3 = \frac{1}{2}.$$

Define a linear operator $L : C[0,\omega] \to C[0,\omega]$ by

$$(Lu)(t) = \int_0^\omega G(t,s)u(s)ds \tag{2.1}$$

and an operator $A : E \to E$ as follows:

$$(Au)(t) = \int_0^\omega G(t,s) f(t, u(t), u'(t))ds, \quad u \in E. \tag{2.2}$$

Clearly, $A : E \to E$ is a completely continuous operator, and the existence of solutions of (1.6) is equivalent to the fixed points of $A$. Moreover,

$L: C[0,\omega] \to C[0,\omega]$ is a completely continuous linear operator, satisfying $L(P) \subset P$. Since $G(t,s) > 0$ for $(t,s) \in [0,\omega] \times [0,\omega]$, the spectral radius $r(L)$ of $L$ is positive by Lemma 2.2. The Krein-Rutman theorem [20] then asserts that there exists $\varphi \in P \backslash \{0\}$ corresponding to the first eigenvalue $\lambda_1 = \frac{1}{r(L)}$ of $L$ such that

$$\lambda_1 L\varphi = \varphi. \tag{2.3}$$

It is easy to see that $\varphi \in E$. According to (2.1), (2.3) and Lemma 2.2 (1), one has

$$\int_0^\omega \varphi(t)(Lu)(t)dt = \frac{1}{\lambda_1} \int_0^\omega \varphi(t)u(t)dt \quad \text{for all}\, u \in E. \tag{2.4}$$

Let $\delta = c\int_0^\omega \varphi(t)dt$, so $\delta > 0$.

Choose a subcone $P_1$ of $P$ given by

$$P_1 = \left\{ u \in P : \int_0^\omega \varphi(t)u(t)dt \geq \delta \|u\|_C \right\}.$$

**Lemma 2.4.** $L(P) \subset P_1$.

*Proof.* Suppose $u \in P$, the following equation can be obtained in light of Lemma 2.2:

$$\begin{aligned}\int_0^\omega \varphi(t)(Lu)(t)dt &= \int_0^\omega \varphi(t) \int_0^\omega G(t,s)u(s)dsdt \\ &\geq c\int_0^\omega \varphi(t)dt \int_0^\omega G(\tau,s)u(s)ds = \delta(Lu)(\tau),\end{aligned}$$

where $\tau \in [0,\omega]$. Obviously, $\int_0^\omega \varphi(t)(Lu)(t)dt \geq \delta\|Lu\|_C$.

**Lemma 2.5.** [18,19] *Let $E$ be a Banach space and $\Omega$ be a bounded open set in $E$ with $0 \in \Omega$. Suppose that $A : \Omega \to E$ is a completely continuous operator. If*

$$Au \neq \tau u, \quad \forall u \in \partial\Omega, \quad \tau \geq 1,$$

*then the topological degree* $\deg(I - A, \Omega, 0) = 1$.

**Lemma 2.6.** [18,19] *Let $E$ be a Banach space and $\Omega$ be a bounded open set in $E$. Suppose that $A : \Omega \to E$ is a completely continuous operator. If there exists $u_0 \neq 0$ such that*

$$u - Au \neq \mu u_0, \quad \forall u \in \partial\Omega, \quad \mu \geq 0,$$

*then the topological degree* $\deg(I - A, \Omega, 0) = 0$.

Three sets are given for the sake of simplicity for later writing as follows:

$$\begin{aligned}&B_r = \{u \in E : \|u\|_{C^1} < r\}, \quad \partial B_r = \{u \in E : \|u\|_{C^1} = r\}, \\ &\overline{B}_r = \{u \in E : \|u\|_{C^1} \leq r\} \,\text{for}\, r > 0.\end{aligned}$$

## 3. Main result

**Theorem 3.1.** Under the hypotheses $(C_1)$-$(C_2)$ suppose that

$(C_3)$ there exists a constant $\alpha > 1$ such that $\liminf_{x+\alpha|y|\to+\infty} \frac{f(t,x,y)}{x+\alpha|y|} > \lambda_1$ uniformly on $t \in [0,\omega]$;

$(C_4)$ $\limsup_{x+|y|\to-\infty} \frac{f(t,x,y)}{x+|y|} < \lambda_1$ uniformly on $t \in [0,\omega]$;

$(C_5)$ there exist nonnegative constants $a \geq 0, b \geq 0$ and $r > 0$ satisfying

$$\omega(a+b)\max\{l_2, l_3\} < 1, \tag{3.1}$$

such that

$$|f(t,x,y)| \leq a|x| + b|y|, \tag{3.2}$$

for all $(t,x,y) \in [0,\omega] \times [-r,r]^2$, where $\lambda_1$ is the first eigenvalue of the operator $L$ defined by (2.1) and $l_2, l_3$ are as Lemma 2.2.

If the following Nagumo condition is fulfilled, i.e.

$(C_6)$ for any $M > 0$ there is a positive continuous function $H_M(\rho)$ on $\mathbb{R}^+$ satisfying

$$\int_0^{+\infty} \frac{\rho d\rho}{H_M(\rho)+1} = +\infty, \tag{3.3}$$

such that

$$|f(t,x,y)| \leq H_M(|y|), \forall (t,x,y) \in [0,\omega] \times [-M,M] \times \mathbb{R}, \tag{3.4}$$

then PBVP (1.6) has at least one nontrivial solution.

*Proof.* (i) First, we prove that $Au \neq \tau u$ for $u \in \partial B_r$ and $\tau \geq 1$. In fact, if there exist $u_1 \in \partial B_r$ and $\tau_0 \geq 1$ such that $Au_1 = \tau_0 u_1$, then we deduce from Lemma 2.1, (3.1), (3.7) and $-r \leq u_1(t) \leq r, -r \leq u_1'(t) \leq r, \forall t \in [0,\omega]$ that

$$\begin{aligned}\|u_1\|_C &= \frac{1}{\tau_0}\max_{0\leq t\leq\omega}\left|\int_0^\omega G(t,s)f(s,u_1(s),u_1'(s))ds\right| \\ &\leq \int_0^\omega l_2|f\big(s,u_1(s),u_1'(s)\big)|ds \leq \int_0^\omega l_2\big(a|u_1(s)|+b|u_1'(s)|\big)ds \\ &\leq \omega(a+b)l_2\|u_1\|_{C^1} < \|u_1\|_{C^1} = r\end{aligned}$$

and

$$\begin{aligned}\|u_1'\|_C &= \frac{1}{\tau_0}\max_{0\leq t\leq\omega}\left|\int_0^\omega \frac{\partial G(t,s)}{\partial t} f\big(s,u_1(s),u_1'(s)\big)ds\right| \\ &\leq \frac{1}{\tau_0}\max_{0\leq t\leq\omega}\int_0^\omega \left|\frac{\partial G(t,s)}{\partial t}\right||f\big(s,u_1(s),u_1'(s)\big)|ds \\ &\leq \int_0^\omega l_3\big(a|u_1(s)|+b|u_1'(s)|\big)ds \leq \omega(a+b)l_3\|u_1\|_{C^1} < \|u_1\|_{C^1} = r.\end{aligned}$$

Hence $\|u_1\|_{C^1} < r$, which contradicts $u_1 \in \partial B_r$.

Therefore, it follows from Lemma 2.5 that

$$\deg(I-A, B_r, 0) = 1. \tag{3.5}$$

(ii) It follows from $(C_3)$ and $(C_4)$ that there exist $\varepsilon_0 \in (0, \lambda_1)$ and $X_0 > 0$ such that

$$f(t,x,y) \geq (\lambda_1 + \varepsilon_0)(x + \alpha|y|) \,\text{for}\, x + \alpha|y| > X_0 \,\text{and}\, t \in [0,\omega]$$

and

$$f(t,x,y) \geq (\lambda_1 - \varepsilon_0)(x + |y|) \,\text{for}\, x + |y| < -X_0 \,\text{and}\, t \in [0,\omega].$$

Let

$$\Omega = \mathbb{R}^2 \backslash \left(\{(x,y) : x + \alpha|y| > X_0\} \cup \{(x,y) : x + |y| < -X_0\}\right) = \Omega_1 \cup \Omega_2,$$

where $\Omega_1 = \Omega \cap \{(x,y) : x + |y| \geq 0\}, \Omega_2 = \Omega \cap \{(x,y) : x + |y| \leq 0\}$, it is easy to see that $\Omega$, $\Omega_1$ and $\Omega_2$ are closed bounded sets in $\mathbb{R}^2$.

Take $C_1 > 0$ such that

$$-C_1 \leq \min_{t\in[0,\omega],(x,y)\in\Omega_1} \{f(t,x,y) - (\lambda_1 + \varepsilon_0)\,(x + \alpha|y|)\},$$

hence

$$f(t,x,y) \geq (\lambda_1 + \varepsilon_0)(x + \alpha|y|) - C_1 \geq (\lambda_1 + \varepsilon_0)x - C_1 \tag{3.6}$$

and

$$f(t,x,y) \geq (\lambda_1 + \varepsilon_0)(x + \alpha|y|) - C_1 \geq (\lambda_1 - \varepsilon_0)x - C_1 \tag{3.7}$$

on $\{(x,y) : x + \alpha|y| > X_0\} \cup \Omega_1$ and $t \in [0,\omega]$.

Take $C_2 > 0$ such that

$$-C_2 \leq \min_{t\in[0,\omega],(x,y)\in\Omega_2} \left\{f(t,x,y) - (\lambda_1 - \varepsilon_0)\,(x + |y|)\right\},$$

hence

$$f(t,x,y) \geq (\lambda_1 - \varepsilon_0)(x + |y|) - C_2 \geq (\lambda_1 - \varepsilon_0)x - C_2 \tag{3.8}$$

and

$$f(t,x,y) \geq (\lambda_1 - \varepsilon_0)(x + |y|) - C_2 \geq (\lambda_1 + \varepsilon_0)x - C_2 \tag{3.9}$$

on $\{(x,y) : x + |y| < -X_0\} \cup \Omega_2$ and $t \in [0,\omega]$.

Let $C = \max\{C_1, C_2\}$. We can derive from (3.7) and (3.8) that

$$f(t,x,y) \geq (\lambda_1 - \varepsilon_0)x - C, \quad \forall (t,x,y) \in [0,\omega] \times \mathbb{R}^2, \tag{3.10}$$

from (3.6) and (3.9) that

$$f(t,x,y) \geq (\lambda_1 + \varepsilon_0)x - C, \quad \forall (t,x,y) \in [0,\omega] \times \mathbb{R}^2. \tag{3.11}$$

(iii) Let $\varphi_0(t) \equiv 1, \forall t \in [0,\omega]$, then

$$\int_0^\omega \varphi(t)\varphi_0(t)dt \geq c\int_0^\omega \varphi(t)dt = \delta\|\varphi_0\|_C,$$

and hence $\varphi_0 \in P_1$. Let

$$D = \left\{u \in C^1[0,\omega] :\, \text{there exists some}\, \mu \geq 0 \,\text{such that}\, u = Au + \mu\varphi_0\right\}.$$

We claim that there is $M > 0$ such that $\|u\|_C \leq M, \forall u \in D$. Indeed, if $u_0 \in D$, then there is $\mu_0 \geq 0$ such that by (3.11)

$$u_0(t) = (Au_0)(t) + \mu_0\varphi_0(t)$$

$$\begin{aligned}
&= \int_0^\omega G(t,s) f(s, u_0(s), u_0'(s)) ds + \mu_0 \varphi_0(t) \\
&\geq \int_0^\omega G(t,s) f(s, u_0(s), u_0'(s)) ds \\
&\geq \int_0^\omega G(t,s) \left( (\lambda_1 + \varepsilon_0) u_0(s) - C \right) ds \\
&= (\lambda_1 + \varepsilon_0)(Lu_0)(t) - C(L\varphi_0)(t),
\end{aligned}$$

i.e.,

$$u_0(t) \geq (\lambda_1 + \varepsilon_0)(Lu_0)(t) - C(L\varphi_0)(t). \tag{3.12}$$

Multiplying (3.12) by $\varphi(t)$ on both sides and integrating over $[0, \omega]$, from (2.4) we obtain

$$\begin{aligned}
\int_0^\omega u_0(t)\varphi(t)dt &\geq (\lambda_1 + \varepsilon_0) \int_0^\omega (Lu_0)(t)\varphi(t)dt - C \int_0^\omega (L\varphi_0)(t)\varphi(t)dt \\
&= \left(1 + \frac{\varepsilon_0}{\lambda_1}\right) \int_0^\omega u_0(t)\varphi(t)dt - C \int_0^\omega (L\varphi_0)(t)\varphi(t)dt.
\end{aligned}$$

Thus by (2.4) we have

$$\int_0^\omega u_0(t)\varphi(t)dt \leq \frac{\lambda_1 C}{\varepsilon_0} \int_0^\omega (L\varphi_0)(t)\varphi(t)dt = \frac{C}{\varepsilon_0} \int_0^\omega \varphi(t)dt. \tag{3.13}$$

By the definition of $D$,

$$\begin{aligned}
&u_0(t) - (\lambda_1 - \varepsilon_0)(Lu_0)(t) + C(L\varphi_0)(t) \\
&\quad = (Au_0)(t) + \mu_0\varphi_0(t) - (\lambda_1 - \varepsilon_0)(Lu_0)(t) + C(L\varphi_0)(t) \\
&\quad = L[(Fu_0) - (\lambda_1 - \varepsilon_0)u_0 + C\varphi_0](t) + \mu_0\varphi_0(t),
\end{aligned}$$

where $(Fu_0)(t) = f(t, u_0(t), u_0'(t))$. In light of (3.10), one has

$$(Fu_0) - (\lambda_1 - \varepsilon_0)u_0 + C\varphi_0 \in P,$$

which implies that $L[(Fu_0) - (\lambda_1 - \varepsilon_0)u_0 + C\varphi_0] \in P_1$ from Lemma 2.4. So we have

$$u_0 - (\lambda_1 - \varepsilon_0)Lu_0 + CL\varphi_0 \in P_1.$$

Therefore, by (2.4) and (3.13), we have

$$\begin{aligned}
&\|u_0 - (\lambda_1 - \varepsilon_0)Lu_0 + CL\varphi_0\|_C \\
&\quad \leq \frac{1}{\delta} \int_0^\omega \varphi(t)[u_0 - (\lambda_1 - \varepsilon_0)Lu_0 + CL\varphi_0](t)dt \\
&\quad = \frac{1}{\delta\lambda_1} \left( \varepsilon_0 \int_0^\omega \varphi(t)u_0(t)dt + C \int_0^\omega \varphi(t)dt \right) \\
&\quad \leq \frac{1}{\delta\lambda_1} \left( C \int_0^\omega \varphi(t)dt + C \int_0^\omega \varphi(t)dt \right) \leq \frac{2C\omega}{\delta\lambda_1} \|\varphi\|_C.
\end{aligned}$$

Since $(\lambda_1 - \varepsilon_0)r(L) < 1$, the operator $I - (\lambda_1 - \varepsilon_0)L$ has the bounded inverse operator $(I - (\lambda_1 - \varepsilon_0)L)^{-1}$ in $C[0, \omega]$. Therefore, there is $M > 0$ such that $\|u\|_C \leq M, \forall u \in D$.

(iv) By (3.3), it is easy to see that there exists $M_1 > 2M$ such that

$$\int_0^{M_1} \frac{\rho d\rho}{H_M(\rho) + B} > 2M, \tag{3.14}$$

where $B = \frac{\pi^2}{\omega^2} M$.

Let $R > \max\{r, M_1\}$ and we will show that

$$u - Au \neq \mu\varphi_0, \quad \forall u \in \partial B_R, \quad \mu \geq 0. \tag{3.15}$$

If it does not hold, there exist $u_2 \in \partial B_R$ and $\mu_1 \geq 0$ such that

$$u_2 - Au_2 = \mu_1 \varphi_0, \tag{3.16}$$

thus $u_2 \in D$, $\|u_2\|_C \leq M$. We can derive from (3.4) and (3.16) that

$$\begin{aligned} u_2''(t) &= -a(t)u_2(t) + f\big(t, u_2(t), u_2'(t)\big) \\ &\leq \|a\|_C \|u_2\|_C + |f\big(t, u_2(t), u_2'(t)\big)| \\ &\leq \frac{\pi^2}{\omega^2} M + |f\big(t, u_2(t), u_2'(t)\big)| \leq H_M(|u_2'(t)|) + B. \end{aligned} \tag{3.17}$$

In the same way,

$$\begin{aligned} -u_2''(t) &= a(t)u_2(t) - f\big(t, u_2(t), u_2'(t)\big) \\ &\leq \|a\|_C \|u_2\|_C + |f\big(t, u_2(t), u_2'(t)\big)| \\ &\leq \frac{\pi^2}{\omega^2} M + |f\big(t, u_2(t), u_2'(t)\big)| \leq H_M(|u_2'(t)|) + B. \end{aligned} \tag{3.18}$$

We will use (3.14), (3.17) and (3.18) to show that $\|u_2'\|_C \leq M_1$.

Let $u_2'(t) \not\equiv 0$. Since $u_2(0) = u_2(\omega)$, there exist $t_0 \in (0, \omega)$ and $t_1 \in [0, \omega]$ with $t_1 \neq t_0$ such that

$$u_2'(t_0) = 0, \|u_2'\|_C = |u_2'(t_1)| > 0. \tag{3.19}$$

Case 1. $u_2'(t_1) > 0, t_0 < t_1$. Set

$$s_1 = \sup\{s \in [t_0, t_1) : u_2'(s) = 0\}. \tag{3.20}$$

Then $s_1 < t_1$ and by the definition of supremum,

$$u_2'(t) > 0, t \in (s_1, t_1]; \quad u_2'(s_1) = 0. \tag{3.21}$$

Hence, for every $t \in [s_1, t_1]$, by (3.17) we have

$$\frac{u_2'(t) u_2''(t)}{H_M(u_2'(t)) + B} \leq u_2'(t), t \in [s_1, t_1]. \tag{3.22}$$

Then integrating the inequality (3.22) over $[s_1, t_1]$ and making the variable transformation $\rho = u_2'(t)$, we have

$$\int_0^{u_2'(t_1)} \frac{\rho d\rho}{H_M(\rho) + B} = \int_{u_2'(s_1)}^{u_2'(t_1)} \frac{\rho d\rho}{H_M(\rho) + B} \leq u_2(t_1) - u_2(s_1) \leq 2M.$$

From this inequality and (3.14) it follows that $u_2'(t_1) \leq M_1$. Hence, $\|u_2'\|_C = u_2'(t_1) \leq M_1$.

Case 2. $u_2'(t_1) > 0, t_0 > t_1$. Set

$$s_2 = \inf\{s \in (t_1, t_0] : u_2'(s) = 0\}. \tag{3.23}$$

Then $t_1 < s_2$ and by the definition of infimum,

$$u_2'(t) > 0, t \in [t_1, s_2); \quad u_2'(s_2) = 0. \tag{3.24}$$

Hence, for every $t \in [t_1, s_2]$, by (3.18) we have

$$\frac{-u_2'(t)u_2''(t)}{H_M(u_2'(t)) + B} \leq u_2'(t), t \in [t_1, s_2]. \tag{3.25}$$

Then integrating the inequality (3.25) over $[t_1, s_2]$ and making the variable transformation $\tilde{\rho} = u_2'(t)$, we have

$$\int_0^{u_2'(t_1)} \frac{\tilde{\rho} d\tilde{\rho}}{H_M(\tilde{\rho}) + B} = -\int_{u_2'(t_1)}^{u_2'(s_2)} \frac{\tilde{\rho} d\tilde{\rho}}{H_M(\tilde{\rho}) + B} \leq u_2(s_2) - u_2(t_1) \leq 2M.$$

From this inequality and (3.14) it follows that $u_2'(t_1) \leq M_1$. Hence, $\|u_2'\|_C = u_2'(t_1) \leq M_1$.

Case 3. $u_2'(t_1) < 0, t_0 < t_1$. Set

$$s_3 = \sup\{s \in [t_0, t_1) : u_2'(s) = 0\}. \tag{3.26}$$

Then $s_3 < t_1$ and by the definition of supremum,

$$u_2'(t) < 0, t \in (s_3, t_1]; \quad u_2'(s_3) = 0. \tag{3.27}$$

Hence, for every $t \in [s_3, t_1]$, by (3.18) we have

$$\frac{-u_2'(t)u_2''(t)}{H_M(-u_2'(t)) + B} \geq u_2'(t), t \in [s_3, t_1]. \tag{3.28}$$

Then integrating the inequality (3.28) over $[s_3, t_1]$ and making the variable transformation $\tau = -u_2'(t)$, we have

$$-\int_0^{-u_2'(t_1)} \frac{\tau d\tau}{H_M(\tau) + B} = -\int_{-u_2'(s_3)}^{-u_2'(t_1)} \frac{\tau d\tau}{H_M(\tau) + B} \geq u_2(t_1) - u_2(s_3) \geq -2M.$$

From this inequality and (3.14), it follows that $-u_2'(t_1) \leq M_1$. Hence, $\|u_2'\|_C = -u_2'(t_1) \leq M_1$.

Case 4. $u_2'(t_1) < 0, t_0 > t_1$. Set

$$s_4 = \inf\{s \in (t_1, t_0] : u_2'(s) = 0\}. \tag{3.29}$$

Then $t_1 < s_4$ and by the definition of infimum,

$$u_2'(t) < 0, t \in [t_1, s_4); \quad u_2'(s_4) = 0. \tag{3.30}$$

Hence, for every $t \in [t_1, s_4]$, by (3.17) we have

$$\frac{-u_2'(t)u_2''(t)}{H_M(-u_2'(t)) + B} \leq -u_2'(t), \ t \in [t_1, s_4]. \tag{3.31}$$

Then integrating the inequality (3.31) over $[t_1, s_4]$ and making the variable transformation $\tilde{\tau} = -u_2'(t)$, we have

$$\int_0^{-u_2'(t_1)} \frac{\tilde{\tau} d\tilde{\tau}}{H_M(\tilde{\tau}) + B} = -\int_{-u_2'(t_1)}^{-u_2'(s_4)} \frac{\tilde{\tau} d\tilde{\tau}}{H_M(\tilde{\tau}) + B} \leq u_2(t_1) - u_2(s_4) \leq 2M,$$

From this inequality and (3.14) it follows that $-u_2'(t_1) \leq M_1$. Hence, $\|u_2'\|_C = -u_2'(t_1) \leq M_1$.

In summary, $\|u_2'\|_C \le M_1$ and thus $\|u_2\|_{C^1} \le M_1$, which is a contradiction to $\|u_2\|_{C^1} = R > M_1$. Therefore, (3.15) holds.

According to Lemma 2.6, one has

$$\deg(I - A, B_R, 0) = 0. \tag{3.32}$$

From (3.7) and (3.32) we have

$$\deg(I - A, B_R\backslash\overline{B}_r, 0) = \deg(I - A, B_R, 0) - \deg(I - A, B_r, 0) = -1.$$

As a result, $A$ has at least one fixed point on $B_R\backslash\overline{B}_r$, which means that PBVP (1.6) has at least one nontrivial solution.

*Example.* Consider the following PBVP

$$\begin{cases} u''(t) + \frac{1}{16}u(t) = f(u(t), u'(t)), & t \in [0, 2\pi], \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{cases} \tag{3.33}$$

where

$$f(x, y) = \begin{cases} \frac{1}{72}(x + 2|y|), \, x + 2|y| \le 0, \\ (x + 2|y|)^2, \quad x + 2|y| > 0. \end{cases}$$

Obviously, $a(t) = \frac{1}{16}$ and $f(x, y)$ satisfy $(C_1)$ and $(C_2)$, respectively. It is easy to see that $l_2 = 2\sqrt{2}$ and $l_3 = \frac{1}{2}$.

It follows from (2.1) and Lemma 2.2 that for $u \in C[0, \omega]$,

$$(Lu)(t)| \le \int_0^\omega |G(t, s)u(s)|ds \le 2\pi l_2\|u\|_C$$

and the spectral radius $r(L) \le \|L\| \le 2\pi l_2$, thus $\lambda_1 \ge \frac{1}{2\pi l_2} = \frac{1}{4\sqrt{2}\pi}$.

Obviously, for $\alpha = 2$, we have

$$\liminf_{x+2|y|\to+\infty} \frac{f(x, y)}{x + 2|y|} = \liminf_{x+2|y|\to+\infty} \frac{(x + 2|y|)^2}{x + 2|y|} = +\infty > \lambda_1.$$

If $x + 2|y| \le 0$, then $x + |y| \le x + 2|y| \le 0$ and $|x + 2|y|| \le |x + |y||$, hence

$$\limsup_{x+|y|\to-\infty} \frac{f(x, y)}{x + |y|} = \limsup_{x+|y|\to-\infty} \frac{x + 2|y|}{72(x + |y|)} \le \frac{1}{72} < \lambda_1;$$

if $x + 2|y| > 0$, then

$$\limsup_{x+|y|\to-\infty} \frac{f(x, y)}{x + |y|} = \limsup_{x+|y|\to-\infty} \frac{(x + 2|y|)^2}{x + |y|} \le 0 < \lambda_1.$$

Therefore, $(C_3)$ and $(C_4)$ are satisfied.

Take $a = b = \frac{1}{36}, r = \frac{1}{180}$, and thus $2\pi(a + b)\max\{l_2, l_3\} = \frac{2\sqrt{2}\pi}{9} < 1$. If $x + 2|y| \le 0$,

$$|f(x, y)| \le \frac{|x|}{72} + \frac{|y|}{36} \le a|x| + b|y|;$$

if $x + 2|y| > 0$ and $(x, y) \in [-r, r]^2$,

$$|f(x, y)| \le (|x| + 2|y|)^2 \le 5r|x| + 4r|y| \le a|x| + b|y|.$$

So $(C_5)$ holds.

For any $M > 0$ define $H_M(\rho) = (M + 2\rho)^2 + 1$ on $\mathbb{R}^+$, it is clear that (3.3) holds. If $-1 \leq x + 2|y| \leq 0$,

$$|f(x,y)| = \frac{1}{72}|x + 2|y|| \leq 1 \leq H_M(|y|);$$

if $x + 2|y| < -1$ and $(x, y) \in [-M, M] \times \mathbb{R}$,

$$|f(x,y)| = \frac{1}{72}|x + 2|y|| \leq |x + 2|y||^2 \leq (|x| + 2|y|)^2 \leq H_M(|y|);$$

if $x + 2|y| > 0$ and $(x, y) \in [-M, M] \times \mathbb{R}$,

$$|f(x,y)| = |x + 2|y||^2 \leq (|x| + 2|y|)^2 \leq H_M(|y|).$$

Therefore, (3.4) also holds. By Theorem 3.1, we know that PBVP (3.4) has at least one nontrivial solution.

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