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Ulam stability of a successive approximation equation

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Abstract. Let T be a bounded linear operator acting on a Banach space X. We obtain some results on Ulam stability for the linear difference equation $x_{n+1} = Tx_n + a_n$ associated with an iterative process for the linear equation $x - Tx = y$. As applications, we get some stability results for the case when X is a finite-dimensional space and for the case when T is a Fredholm operator.

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1. Introduction

One of the most important methods of determining solutions of an equation is the successive approximation method. It is connected with the existence of a fixed point of an operator and with a recurrent sequence converging to the fixed point. In this paper, we deal with Ulam stability of the difference equation which defines the sequence of successive approximations. In what follows, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers, X a Banach space over $\mathbb{K}$, and $T : X \to X$ a bounded linear operator. By $\mathbb{N} = \{0, 1, 2, \ldots\}$, we denote the set of all nonnegative integers.

Consider the equation:

$$x - Tx = y, \tag{1}$$

where y is a given element in X. Under appropriate conditions on T, Eq. (1) admits a solution x^* which is the limit of the sequence $(x_n)_{n\geq 0}$ defined by the difference equation:

$$x_{n+1} = Tx_n + y, \quad n \in \mathbb{N} \tag{2}$$

for some $x_0 \in X$, called the sequence of successive approximations. In this connection, see also the early paper [24] by S. Reich.

In what follows, we consider the linear difference equation:

$$x_{n+1} = Tx_n + a_n, \quad n \in \mathbb{N}, \tag{3}$$

where $(a_n)_{n\geq 0}$ is a given sequence in X. We study its Ulam stability which concerns the behavior of the solutions of the equation (3) under perturbations. For various results on difference equations, we refer the reader to [12,13].

Definition 1.1. Equation (3) is called Ulam stable if there exists $L \geq 0$, such that for every $\varepsilon > 0$ and any $(x_n)_{n\geq 0}$ in X satisfying the relation:

$$\|x_{n+1} - Tx_n - a_n\| \leq \varepsilon, \quad n \in \mathbb{N}, \tag{4}$$

there exists a sequence $(y_n)_{n\geq 0}$ in X, such that:

$$y_{n+1} = Ty_n + a_n, \quad n \in \mathbb{N}, \tag{5}$$

$$\|x_n - y_n\| \leq L\varepsilon, \quad n \in \mathbb{N}. \tag{6}$$

A sequence $(x_n)_{n\geq 0}$ which satisfies (4) for some $\varepsilon > 0$ is called approximate solution of Eq. (3).

In other words, we say that Eq. (3) is Ulam stable if, for every approximate solution of it, there exists an exact solution close to it. The number L from (6) is called an Ulam constant of Eq. (3). In what follows, we will denote by L_R the infimum of all Ulam constants of (3). If L_R is an Ulam constant for (3), then we call it *the best Ulam constant* or *the Ulam constant* of Eq. (3). In general, the infimum of all Ulam constants of an equation is not an Ulam constant of that equation (see [7,22]). If, in the above definition, the number ε is replaced by a sequence of positive numbers $(\varepsilon_n)_{n\geq 0}$, we get the notion of *generalized stability* in Ulam sense. Let S^+ be the set of all sequences of nonnegative numbers, $\mathcal{E} \subseteq S^+$, and $U : \mathcal{E} \to S^+$ an operator.

Definition 1.2. We say that Eq. (3) is ($\mathcal{E}$, U)-stable or generalized stable in Ulam sense, if for every sequence $\varepsilon = (\varepsilon_n)_{n\geq 0}$ in $\mathcal{E}$ and every sequence $(x_n)_{n\geq 0}$ in X satisfying:

$$\|x_{n+1} - Tx_n - a_n\| \leq \varepsilon_n, n \in \mathbb{N},$$

there exists a sequence $(y_n)_{n\geq 0}$ in X, such that:

$$y_{n+1} = Ty_n + a_n, \quad n \in \mathbb{N},$$

$$\|x_n - y_n\| \leq (U\varepsilon)_n, \quad n \in \mathbb{N}.$$

The problem of stability of functional equations was formulated by Ulam [25] in 1940 for the equation of the homomorphism of a metric group. The first answer to Ulam's problem was given, a year later, by Hyers [14] for the Cauchy functional equation in Banach spaces. Since then, the topic was intensively studied by many authors, we can merely mention here a few papers on Ulam stability of functional equations as [2,7,15,19]. Recall also the results obtained in [6,16,18] on Ulam stability of some second-order functional equations connected with Fibonacci and Lucas sequences.

Some results on Ulam stability for the linear difference equations in Banach spaces were obtained by Brzdek, Popa, and Xu in [8–10,20]. Buse et al. [5,11] proved that a discrete system $X_{n+1} = AX_n$, $n \in \mathbb{N}$, where A is a

$m \times m$ complex matrix, is Ulam stable if and only if A possesses a discrete dichotomy. Recently, Baias and Popa obtained results on Ulam stability of linear difference equations of order one and two, and determined the best Ulam constant in [3,4]. Popa and Rasa obtained an explicit representation of the best Ulam constant of some classical operators in approximation theory in [21,23].

2. Main results

Recall first some classical results which will be used in the sequel. Let $T : X \to X$ be a linear and bounded operator and consider the geometric series:

$$\sum_{n=0}^{\infty} T^n = I + T + T^2 + \dots \tag{7}$$

Theorem 2.1 [17, Théorèm 1, Section 4.2]. *For any linear and bounded operator $T : X \to X$, there exists:*

$$lim \sqrt[n]{\|T^n\|} = \rho. \tag{8}$$

Moreover, the series (7) is absolutely convergent for $\rho < 1$ and divergent for $\rho > 1$.

Theorem 2.2 [17, Corollaire, Section 4.2]. *The series (7) is absolutely convergent if and only if there exists $p \in \mathbb{N}$, such that:*

$$\|T^p\| < 1. \tag{9}$$

In what follows, we present some results on Ulam stability and generalized Ulam stability for Eq. (3). The following lemma is useful in the sequel.

Lemma 2.3. *If $(x_n)_{n\geq 0}$ satisfies Eq. (3), then:*

$$x_n = T^n x_0 + \sum_{k=1}^{n} T^{n-k} a_{k-1}, \quad n \geq 1. \tag{10}$$

Proof. Induction on n. □

The first result on generalized Ulam stability of (3) is contained in the next theorem.

Theorem 2.4. *Suppose that T is an invertible operator and let $(\varepsilon_n)_{n\geq 0}$ be a sequence of positive numbers, such that the series:*

$$\sum_{n=1}^{\infty} \|T^{-n}\| \varepsilon_{n-1} \tag{11}$$

is convergent. Then, for every sequence $(x_n)_{n\geq 0}$ in X satisfying:

$$\|x_{n+1} - Tx_n - a_n\| \leq \varepsilon_n, \quad n \in \mathbb{N}, \tag{12}$$

there exists a sequence $(y_n)_{n\geq 0}$ *in* X *with the properties:*

$$y_{n+1} = Ty_n + a_n, \quad n \in \mathbb{N}, \tag{13}$$

$$\|x_n - y_n\| \leq \sum_{k=0}^{\infty} \|T^{-k-1}\|\varepsilon_{n+k}, \quad n \in \mathbb{N}. \tag{14}$$

Moreover, if

$$\sup_{n\geq 1} \frac{1}{\varepsilon_{n-1}} \sum_{k=0}^{\infty} \|T^{-k-1}\|\varepsilon_{n+k} < \infty, \tag{15}$$

then the sequence $(y_n)_{n\geq 1}$ *satisfying (13), (14) is unique.*

Proof. **Existence.** Suppose that $(x_n)_{n\geq 0}$ satisfies relation (12) and let:

$$x_{n+1} - Tx_n - a_n = b_n, \quad n \geq 0.$$

Then, $\|b_n\| \leq \varepsilon_n$, $n \geq 0$, and taking into account (10), we get:

$$x_n = T^n(x_0 + \sum_{k=1}^{n} T^{-k}(a_{k-1} + b_{k-1})), \quad n \geq 1.$$

Since:

$$\|T^{-n}b_{n-1}\| \leq \|T^{-n}\| \cdot \|b_{n-1}\| \leq \varepsilon_{n-1}\|T^{-n}\|, \quad n \geq 1,$$

it follows that the series:

$$\sum_{n=1}^{\infty} T^{-n}b_{n-1}$$

is convergent, according to the comparison test for series with positive terms. Let

$$\sum_{n=1}^{\infty} T^{-n}b_{n-1} = s, \quad s \in X.$$

Define the sequence $(y_n)_{n\geq 0}$ by the relation:

$$y_{n+1} = Ty_n + a_n, \quad n \geq 0, \quad y_0 = x_0 + s.$$

Then, in view of Lemma 2.3, it follows:

$$y_n = T^n\left(y_0 + \sum_{k=1}^{n} T^{-k}a_{k-1}\right), \quad n \geq 1.$$

Consequently:

$$\begin{aligned} x_n - y_n &= T^n\left(x_0 - y_0 + \sum_{k=1}^{n} T^{-k}b_{k-1}\right) = T^n\left(-s + \sum_{k=1}^{n} T^{-k}b_{k-1}\right) \\ &= -T^n\left(\sum_{k=0}^{\infty} T^{-n-k-1}b_{n+k}\right) = -\sum_{k=0}^{\infty} T^{-k-1}b_{n+k}, \quad n \geq 1. \end{aligned}$$

Hence:

$$\begin{aligned}\|x_n - y_n\| &\le \sum_{k=0}^{\infty} \|T^{-k-1} b_{n+k}\| \le \sum_{k=0}^{\infty} \|T^{-k-1}\| \|b_{n+k}\| \\ &\le \sum_{k=0}^{\infty} \|T^{-k-1}\| \varepsilon_{n+k}, \quad n \in \mathbb{N}.\end{aligned}$$

Uniqueness. Suppose that for a sequence $(x_n)_{n\ge 0}$ satisfying (12), there exist two sequences $(y_n)_{n\ge 0}$, $(z_n)_{n\ge 0}$ satisfying (13) and (14). Then:

$$\|y_n - z_n\| \le \|y_n - x_n\| + \|x_n - z_n\| \le 2\sum_{k=0}^{\infty} \|T^{-k-1}\| \varepsilon_{n+k}, \quad n \ge 0.$$

On the other hand, taking account of Lemma 2.3, it follows:

$$y_n - z_n = T^n(y_0 - z_0) \text{ or equivalently } y_0 - z_0 = T^{-n}(y_n - z_n).$$

Hence:

$$\begin{aligned}\|y_0 - z_0\| &\le \|T^{-n}(y_n - z_n)\| \le \|T^{-n}\| \|y_n - z_n\| \\ &\le 2\|T^{-n}\| \sum_{k=0}^{\infty} \|T^{-k-1}\| \varepsilon_{n+k} \\ &= 2\|T^{-n}\| \varepsilon_{n-1} \cdot \frac{1}{\varepsilon_{n-1}} \sum_{k=0}^{\infty} \|T^{-k-1}\| \varepsilon_{n+k}, \quad n \ge 1. \qquad (16)\end{aligned}$$

The convergence of the series (11) implies that:

$$\lim_{n\to\infty} \|T^{-n}\| \varepsilon_{n-1} = 0.$$

Therefore, according to (15) and (16), we get $y_0 = z_0$, and so $y_n = z_n$, for all $n \in \mathbb{N}$. □

Corollary 2.5. *Suppose that T is an invertible operator and $(\varepsilon_n)_{n\ge 0}$ is a sequence of positive numbers, such that there exists $q \in (0,1)$ with the property:*

$$\|T^{-1}\| \le q \frac{\varepsilon_n}{\varepsilon_{n+1}}, \quad n \in \mathbb{N}. \qquad (17)$$

Then, for every sequence $(x_n)_{n\ge 0}$ in X satisfying the relation (12), there exists a sequence $(y_n)_{n\ge 0}$ in X with the properties (13) and:

$$\|x_n - y_n\| \le \frac{q}{1-q} \varepsilon_{n-1}, \quad n \ge 1.$$

Proof. The series $\sum_{n=1}^{\infty} \|T^{-n}\| \varepsilon_{n-1}$ is convergent. Indeed:

$$\begin{aligned}\limsup \frac{\|T^{-n-1}\| \varepsilon_n}{\|T^{-n}\| \varepsilon_{n-1}} &\le \limsup \frac{\|T^{-n}\| \|T^{-1}\| \varepsilon_n}{\|T^{-n}\| \varepsilon_{n-1}} \\ &= \limsup \frac{\varepsilon_n}{\varepsilon_{n-1}} \|T^{-1}\| \le q < 1.\end{aligned}$$

Then, according to Theorem 2.4, for every sequence $(x_n)_{n\geq 0}$ satisfying (12), there exists a sequence $(y_n)_{n\geq 0}$:

$$y_{n+1} = Ty_n + a_n,\ n \in \mathbb{N}, y_0 = x_0 + s,$$

such that

$$\|x_n - y_n\| \leq \sum_{k=0}^{\infty} \|T^{-k-1}\|\varepsilon_{n+k}, \quad n \in \mathbb{N}.$$

Taking account of (17), we get:

$$\begin{aligned}
\varepsilon_n\|T^{-1}\| &\leq q\varepsilon_{n-1} \\
\varepsilon_{n+1}\|T^{-2}\| &\leq \varepsilon_{n+1}\|T^{-1}\|\|T^{-1}\| \leq q\varepsilon_n\|T^{-1}\| \leq q^2\varepsilon_{n-1} \\
&\dots \\
\varepsilon_{n+k}\|T^{-k-1}\| &\leq q^{k+1}\varepsilon_{n-1}.
\end{aligned}$$

Hence:

$$\begin{aligned}
\sum_{k=0}^{\infty} \|T^{-k-1}\|\varepsilon_{n+k} &\leq \left(\sum_{k=0}^{\infty} q^{k+1}\right)\varepsilon_{n-1} \\
&= \frac{q}{1-q}\varepsilon_{n-1}, \quad n \geq 1.
\end{aligned}$$

The corollary is proved. □

Corollary 2.6. *Let T be an invertible operator with $\|T^{-p}\| < 1$ for some $p \in \mathbb{N}$. Then, Eq. (3) is Ulam stable with the Ulam constant:*

$$L = \sum_{n=1}^{\infty} \|T^{-n}\|.$$

Proof. The condition $\|T^{-p}\| < 1$, leads to the convergence of the series $\sum_{n=1}^{\infty} \|T^{-n}\|$, in view of Theorem 2.2. The conclusion of the corollary follows letting $\varepsilon_n = \varepsilon$, $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \|T^{-n}\| = L$ in Theorem 2.4.

Uniqueness holds since for $\varepsilon_n = \varepsilon$, $n \geq 0$, the condition (15) is satisfied. □

Remark 2.7. If, in Corollary 2.6, we take $p = 1$, i.e., $\|T^{-1}\| < 1$, the conclusion holds with:

$$L = \frac{\|T^{-1}\|}{1 - \|T^{-1}\|}.$$

Indeed: $\sum_{n=1}^{\infty} \|T^{-n}\| \leq \sum_{n=1}^{\infty} \|T^{-1}\|^n = \frac{\|T^{-1}\|}{1-\|T^{-1}\|}$.

Similar results can be obtained replacing the condition on the operator T^{-1} with conditions on T in the previous theorems and corollaries.

Theorem 2.8. *Let $(\varepsilon_n)_{n\geq 0}$ be a sequence of positive numbers and suppose that there exists $q \in (0, 1)$, such that:*

$$\|T\| \leq q\frac{\varepsilon_{n+1}}{\varepsilon_n}, \quad n \in \mathbb{N}. \tag{18}$$

Then, for every sequence $(x_n)_{n\geq 0}$ *in* X *satisfying:*

$$\|x_{n+1} - Tx_n - a_n\| \leq \varepsilon_n, \quad n \in \mathbb{N}, \tag{19}$$

there exists a sequence $(y_n)_{n\geq 0}$ *in* X *with the properties:*

$$\begin{aligned} &y_{n+1} = Ty_n + a_n, \quad n \in \mathbb{N}, \\ &\|x_n - y_n\| \leq \frac{1}{1-q}\varepsilon_{n-1}, \quad n \geq 1. \end{aligned} \tag{20}$$

Proof. Let

$$x_{n+1} - Tx_n - a_n = b_n, \quad n \in \mathbb{N},$$

for some sequence $(x_n)_{n\geq 0}$ satisfying (19). Then, $\|b_n\| \leq \varepsilon_n$, $n \in \mathbb{N}$, and according to Lemma 2.3, we get:

$$x_n = T^n x_0 + \sum_{k=1}^{n} T^{n-k}(a_{k-1} + b_{k-1}), \quad n \geq 1.$$

Consider the sequence $(y_n)_{n\geq 0}$ given by (20) with $y_0 = x_0$, and then:

$$y_n = T^n x_0 + \sum_{k=1}^{n} T^{n-k} a_{k-1}. \tag{21}$$

Consequently:

$$\begin{aligned} \|x_n - y_n\| &= \|\sum_{k=1}^{n} T^{n-k} b_{k-1}\| \leq \sum_{k=1}^{n} \|T^{n-k} b_{k-1}\| \\ &\leq \sum_{k=1}^{n} \|T\|^{n-k} \|b_{k-1}\| \leq \sum_{k=1}^{n} \|T\|^{n-k} \varepsilon_{k-1}, \quad n \geq 1. \end{aligned}$$

On the other hand, in view of (18), it follows:

$$\frac{\varepsilon_{n-1}}{\varepsilon_{k-1}} = \frac{\varepsilon_{n-1}}{\varepsilon_{n-2}} \cdot \frac{\varepsilon_{n-2}}{\varepsilon_{n-3}} \cdot \ldots \cdot \frac{\varepsilon_k}{\varepsilon_{k-1}} \geq \frac{1}{q^{n-k}} \|T\|^{n-k}, \quad n \geq k \geq 1,$$

and

$$\begin{aligned} \|x_n - y_n\| &\leq \sum_{k=1}^{n} q^{n-k} \varepsilon_{n-1} \\ &= (1 + q + \ldots + q^{n-1})\varepsilon_{n-1} \\ &\leq \frac{1}{1-q}\varepsilon_{n-1}, \quad n \geq 1. \end{aligned}$$

□

Theorem 2.9. *Suppose* $\|T^p\| < 1$ *for some* $p \in \mathbb{N}$*. Then, Eq. (3) is Ulam stable with the Ulam constant:*

$$L = \sum_{n=0}^{\infty} \|T^n\|.$$

Proof. Let $\varepsilon > 0$ and let $(x_n)_{n\geq 0}$ be a sequence in X satisfying:

$$x_{n+1} - Tx_n - a_n = b_n, \quad n \in \mathbb{N},$$

with $\|b_n\| < \varepsilon$, $n \in \mathbb{N}$. Then, in view of Lemma 2.3, we get:

$$x_n = T^n x_0 + \sum_{k=1}^{n} T^{n-k}(a_{k-1} + b_{k-1}), \quad n \geq 1.$$

Define the sequence $(y_n)_{n\geq 0}$ by $y_{n+1} = Ty_n + a_n$, $n \in \mathbb{N}$, $y_0 = x_0$. Then:

$$y_n = T^n y_0 + \sum_{k=1}^{n} T^{n-k} a_{k-1}, \quad n \geq 1,$$

and

$$\begin{aligned}\|x_n - y_n\| &= \|\sum_{k=1}^{n} T^{n-k} b_{k-1}\| \leq \sum_{k=1}^{n} \|T^{n-k}\| \|b_{k-1}\| \\ &\leq \varepsilon \sum_{k=1}^{n} \|T^{n-k}\| \leq \varepsilon \sum_{k=0}^{\infty} \|T^k\| = L\varepsilon, \quad n \geq 1.\end{aligned}$$

□

Remark 2.10. If, in Corollary 2.9, we take $p = 1$, i.e., $\|T\| < 1$, then the conclusion holds with:

$$L = \frac{1}{1 - \|T\|}.$$

Proof. Indeed, according to Corollary 2.9, we obtain:

$$\sum_{n=0}^{\infty} \|T^n\| \leq \sum_{n=0}^{\infty} \|T\|^n = \frac{1}{1 - \|T\|}.$$

□

Finally, we present a nonstability result for Eq. (3). Taking into account that the stability results hold in general for $\|T\| < 1$ or $\|T^{-1}\| < 1$, we will consider for nonstability results the case $\|T\| = 1$.

Theorem 2.11. *Suppose that $\|T\| = 1$ and there exists $u_0 \in X$, such that:*

$$\lim_{n\to\infty} \|T^n u_0\| > 0. \tag{22}$$

Then, for every $\varepsilon > 0$, there exists a sequence $(x_n)_{n\geq 0}$ in X satisfying:

$$\|x_{n+1} - Tx_n - a_n\| \leq \varepsilon, \quad n \in \mathbb{N},$$

such that for every sequence $(y_n)_{n\geq 0}$ given by the recurrence:

$$y_{n+1} = Ty_n + a_n, \quad n \in \mathbb{N}, \ y_0 \in X,$$

we have:

$$\sup_{n\in\mathbb{N}} \|x_n - y_n\| = +\infty,$$

i.e., Eq. (3) is not Ulam stable.

Proof. The sequence $(\|T^n u_0\|)_{n\geq 0}$ is decreasing (see Remark 2.14) and (22) shows that $\|T^n u_0\| > 0$, $n \in \mathbb{N}$. Let $\varepsilon > 0$ and consider the sequence $(x_n)_{n\geq 0}$ defined by the relation:

$$x_{n+1} = Tx_n + a_n + \frac{T^{n+1}u_0}{\|T^{n+1}u_0\|}\varepsilon, \quad n \in \mathbb{N}.$$

Then, in view of Lemma 2.3, we get:

$$x_n = T^n x_0 + \sum_{k=1}^{n} T^{n-k} a_{k-1} + \varepsilon \left(\sum_{k=1}^{n} \frac{1}{\|T^k u_0\|} \right) T^n u_0, \quad n \geq 1.$$

On the other hand:

$$\|x_{n+1} - Tx_n - a_n\| = \varepsilon, \quad n \in \mathbb{N},$$

and hence, $(x_n)_{n\geq 0}$ is an approximate solution of Eq. (3). Let $(y_n)_{n\geq 0}$ be an arbitrary sequence in X, $y_{n+1} = Ty_n + a_n$, $n \geq 0$, $y_0 \in X$. Then:

$$y_n = T^n y_0 + \sum_{k=1}^{n} T^{n-k} a_{k-1}, \quad n \geq 1;$$

therefore:

$$x_n - y_n = T^n(x_0 - y_0) + \varepsilon \left(\sum_{k=1}^{n} \frac{1}{\|T^k u_0\|} \right) T^n u_0, \quad n \in \mathbb{N}.$$

The sequence $(T^n(x_0 - y_0))_{n\geq 0}$ is bounded, since:

$$\|T^n(x_0 - y_0)\| \leq \|T^n\|\|x_0 - y_0\| \leq \|T\|^n \|x_0 - y_0\| = \|x_0 - y_0\|, n \in \mathbb{N}.$$

Taking account of

$$\lim_{k\to\infty} \frac{1}{\|T^k u_0\|} > 0,$$

we get $\sum\limits_{k=1}^{\infty} \frac{1}{\|T^k u_0\|} = \infty$.

It follows:

$$\begin{aligned}\lim_{n\to\infty} \|x_n - y_n\| &= \lim_{n\to\infty} \left\|T^n(x_0 - y_0) + \varepsilon \left(\sum_{k=1}^{n} \frac{1}{\|T^k u_0\|} \right) T^n u_0 \right\| \\ &\geq \lim_{n\to\infty} \left| \|T^n(x_0 - y_0)\| - \varepsilon \left(\sum_{k=1}^{n} \frac{1}{\|T^k u_0\|} \right) \|T^n u_0\| \right| \\ &= +\infty.\end{aligned}$$

□

Remark 2.12. Every linear and bounded operator T, which has an eigenvalue λ, $|\lambda| = 1$, satisfies the condition (22).

Indeed, there exists $u_0 \neq 0$, such that $Tu_0 = \lambda u_0$. Then, it is easy to check that $T^n u_0 = \lambda^n u_0$, for all $n \in \mathbb{N}$, and the condition (22) is satisfied.

Remark 2.13. There exist operators T which do not satisfy condition (22). Indeed, if T is nilpotent, there exists $p \geq 1$, such that $T^p = 0$; therefore, $T^n u_0 = 0$ for all $n \geq p$.

Remark 2.14. Let $\|T\| = 1$, $u_0 \in X$. Then:

$$\|T^{n+1}u_0\| = \|T(T^n u_0)\| \le \|T^n u_0\|,$$

i.e., the sequence $(\|T^n u_0\|)_{n\ge 0}$ is decreasing and convergent. Suppose that there exists $p \in \mathbb{N}$, such that $\|T^p\| < 1$. Then:

$$\|T^{np}u_0\| = \|(T^p)^n u_0\| \le \|T^p\|^n \|u_0\|,$$

and so $\lim_{n\to\infty} T^{np}u_0 = 0$. This implies $\lim_{n\to\infty} T^n u_0 = 0$.

Briefly, if $\lim_{n\to\infty} T^n u_0 \neq 0$, as in Theorem 2.11, then $\|T^n\| = 1$, for all $n \in \mathbb{N}$. An example is presented below.

Example 1. Let $\mathcal{C}[0,1]$ be the Banach space of continuous, real-valued functions defined on [0,1], endowed with the supremum norm. Consider the Bernstein operator:

$$B_m : \mathcal{C}[0,1] \to \mathcal{C}[0,1], \quad B_m f(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f(\frac{k}{m}).$$

Then, for a fixed m, each operator B_m^n is linear, positive, and reproduces the constant function 1. Therefore, $\|B_m^n\| = 1$, $n \in \mathbb{N}$. Moreover:

$$\lim_{n\to\infty} B_m^n f(x) = (1-x)f(0) + xf(1), \quad f \in \mathcal{C}[0,1]$$

uniformly on $[0,1]$; see, e.g, Example 3.2.7, with $b = 0$, in [1]. Hence $\lim_{n\to\infty} B_m^n f = 0$ if and only if $f(0) = f(1) = 0$.

Consequently, for $T = B_m$, Eq. (3) is not Ulam stable.

On the other hand, the following result shows the stability of the equation (3) when T is a nilpotent operator.

Theorem 2.15. *Let $T : X \to X$ be a nilpotent operator, i.e., there exists $p \ge 1$, such that $T^p = 0$. Then, for every $\varepsilon > 0$ and every sequence $(x_n)_{n\ge 0}$ in X satisfying:*

$$\|x_{n+1} - Tx_n - a_n\| \le \varepsilon, \quad n \in \mathbb{N}, \tag{23}$$

there exists a sequence $(y_n)_{n\ge 0}$ in X with the properties:

$$y_{n+1} = Ty_n + a_n$$
$$\|x_n - y_n\| \le L\varepsilon, \quad n \ge p,$$

where $L = 1 + \|T\| + \cdots + \|T\|^{p-1}$.

Proof. Let

$$x_{n+1} - Tx_n - a_n = b_n, \quad n \in \mathbb{N}$$

for some sequence $(x_n)_{n\ge 0}$ satisfying (23). Then, in view of Lemma 2.3, we get:

$$x_n = T^n x_0 + \sum_{k=1}^{n} T^{n-k}(a_{k-1} + b_{k-1}), \quad n \ge 1.$$

Since T is a nilpotent operator, we have:

$$x_n = \sum_{k=1}^{p} T^{p-k}(a_{n-p+k-1} + b_{n-p+k-1}), \quad n \geq p.$$

Define $(y_n)_{n\geq 0}$ by $y_{n+1} = Ty_n + a_n$, $n \in \mathbb{N}$, $y_0 = x_0$. Then:

$$y_n = T^n x_0 + \sum_{k=1}^{n} T^{n-k} a_{k-1}, \quad n \geq 1,$$

or equivalently:

$$y_n = \sum_{k=1}^{p} T^{p-k} a_{n-p+k-1}, \quad n \geq p.$$

Therefore:

$$x_n - y_n = \sum_{k=1}^{p} T^{p-k} b_{n-p+k-1} = b_{n-1} + b_{n-2}T + \cdots + b_{n-p}T^{p-1}, \quad n \geq p,$$

and

$$\begin{aligned} \|x_n - y_n\| &\leq \|b_n\| + \|b_{n-1}\|\|T\| + \ldots + \|b_{n-p}\|\|T^{p-1}\| \\ &\leq \varepsilon(1 + \|T\| + \|T\|^2 + \cdots + \|T\|^{p-1}) \\ &= \varepsilon L, \quad n \geq p. \end{aligned}$$

□

3. Applications

Let $X = \mathbb{K}^p$ be endowed with the Euclidean norm ($\|x\| = \sqrt{|x_1|^2 + \cdots + |x_p|^2}$, $x = (x_1, x_2, \ldots, x_p) \in \mathbb{K}^p$) and $T : \mathbb{K}^p \to \mathbb{K}^p$, $Tx = Ax$, where A is a square matrix of order p with entries in $\mathbb{K}$. Suppose that A is normal, denote by $\lambda_1, \ldots, \lambda_p$ and $\Lambda_1, \ldots, \Lambda_p$ the eigenvalues of A and A^*A, respectively. Recall that A^* denotes the conjugate transposed of A. Suppose that:

$$|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_p| \text{ and } \Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_p.$$

Then (see [17]) $\|T\| = \sqrt{\Lambda_p}$, and, if A is an invertible matrix, $\|T^{-1}\| = \frac{1}{\sqrt{\Lambda_1}}$. Moreover, if A is a self-adjoint, invertible matrix, we obtain $\|T\| = |\lambda_p|$, $\|T^{-1}\| = \frac{1}{|\lambda_1|}$. Consequently, we get the following result on Ulam stability.

Theorem 3.1. *Let $\varepsilon > 0$ and $(x_n)_{n\geq 0}$ be a sequence in $\mathbb{K}^p$ satisfying:*

$$\|x_{n+1} - Ax_n - a_n\| \leq \varepsilon, \quad n \in \mathbb{N}.$$

i) If $\Lambda_p < 1$, then there exists a sequence $(y_n)_{n\geq 0}$ in $\mathbb{K}^p$, such that:

$$\begin{aligned} &y_{n+1} = Ay_n + a_n, \\ &\|x_n - y_n\| \leq \frac{\varepsilon}{1 - \sqrt{\Lambda_p}}, \ n \in \mathbb{N}. \end{aligned}$$

ii) If A is an invertible matrix and $\Lambda_1 > 1$, then there exists a sequence $(y_n)_{n\geq 0}$ in $\mathbb{K}^p$, such that:

$$y_{n+1} = Ay_n + a_n,$$
$$\|x_n - y_n\| \leq \frac{\varepsilon}{\sqrt{\Lambda_1} - 1}, \quad n \in \mathbb{N}.$$

Proof. The result follows from Remark 2.7 and Remark 2.10. □

Remark 3.2. If A is a self-adjoint, invertible matrix Theorem 3.1 holds with $|\lambda_1|, |\lambda_p|$ instead of $\sqrt{\Lambda_1}$, $\sqrt{\Lambda_p}$, respectively.

Consider the linear operator $T : L^2[a, b] \to L^2[a, b]$ defined by:

$$(Tx)(s) = \int_a^b K(s,t)x(t)dt,$$

where $K : [a, b] \times [a, b] \to \mathbb{R}$ is symmetric, square-measurable and:

$$\int_a^b \int_a^b |K(s,t)|^2 dsdt = L^2 < \infty.$$

Then, T is a continuous operator and $\|T\| = \frac{1}{|\lambda_1|}$, where λ_1 is the eigenvalue of K of least absolute value (see [17]). Then, for the linear difference equation:

$$x_{n+1} = Tx_n + a_n, x_0 \in L^2[a, b], \quad n \in \mathbb{N},$$

where $(a_n)_{n\geq 0}$ is a sequence in $L^2[a, b]$; we get the following stability result.

Theorem 3.3. *Suppose that $|\lambda_1| > 1$. Then, for every $\varepsilon > 0$ and every $(x_n)_{n\geq 0}$ in $L^2[a, b]$ satisfying*

$$\|x_{n+1} - Tx_n - a_n\| \leq \varepsilon, \quad n \in \mathbb{N},$$

there exists a sequence $(y_n)_{n\geq 0}$, such that:

$$y_{n+1} = Ty_n + a_n, y_0 \in L^2[a, b], \quad n \in \mathbb{N},$$
$$\|x_n - y_n\| \leq \frac{\varepsilon|\lambda_1|}{|\lambda_1| - 1}, \quad n \in \mathbb{N}.$$

Proof. The result is a simple consequence of Remark 2.10 □

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