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Best proximity point theorems in topological spaces

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Abstract. Let A, B be nonempty subsets of a metric space X and $f : A \to B$ be a mapping. A point $x_0 \in A$ is a best proximity point of f if $d(x, f(x)) = \text{dist}(A, B) := \{d(a, b) : a \in A, b \in B\}$. It is worth mentioning that the metric function d plays a vital role in defining the notion of best proximity points. In this manuscript, we introduced a notion of best proximity points in arbitrary topological spaces and established few best proximity point theorems. Our main result generalizes the well-known Edelstein's fixed point theorem for contractive mappings.

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1. Introduction

Let A be a nonempty subset of a metric space X. A mapping $f : A \to X$ is *contractive* if $d(f(x), f(y)) < d(x, y)$, for all $x, y \in A$ with $x \neq y$. A point $x \in A$ is a *fixed point* of f if $f(x) = x$. Edelstein [1] proved that every contractive self mapping $f : A \to A$, where A is a compact subset of X, has a unique fixed point in A. Edelstein's theorem drew attention of many researchers to obtain various extensions and generalization of it. For more details of Edelstein's theorem, one may refer [2,3]. Note that the metric space structure is essential to define contractive type mappings. In this connection, see also page 3 in the book by Goebel and Reich [4]. In [5], Liepinš obtained an interesting extension of Edelstein's theorem. The author extended the notion of contractive mapping to an arbitrary topological space and obtained fixed point theorems on topological spaces.

On the other hand, the necessary condition for the existence of a fixed point of a nonself mapping $f : A \to X$ is $f(A) \cap A \neq \emptyset$. Suppose that the necessary condition fails. Then $f(x) \neq x$, for all $x \in A$. Then we are looking for a point $x \in A$ for which the displacement $d(x, f(x))$ is optimum. Best Approximation and Best Proximity point theorems are developed in this direction. It is worth mentioning that the metric notion is an essential tool for finding optimum solutions.

In this manuscript, motivated by Liepins' theorem, we extended the notion of contractive mapping to an arbitrary topological space, which need not have a fixed point in general. Also, we generalize the notion of best proximity points to topological spaces and obtained sufficient conditions for the existence of a best proximity points for such class of cyclic mappings. Our main result provides an extended version of the well-known Edelsein's fixed point theorem for contractive mappings in metric spaces.

2. Definitions and notations

Let us consider a topological space X and a mapping $f : X \to X$. Fix $x_0 \in X$. The orbit $O(x_0, f)$ of f starting at x_0 is defined as

$$O(x_0, f) := \{f^n(x_0) : n \in \mathbb{N} \cup \{0\}\},$$

where $f^0(x_0) = x_0$ and $f^{n+1}(x_0) = f(f^n(x_0))$, for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $g : X \to X$ is a function. Then the image of $O(x_0, f)$ under g is $g(O(x_0, f)) = \{g(f^n(x_0)) : n \in \mathbb{N} \cup \{0\}\}$. The limit point set $\mathrm{Lim}\,\{x_n\}$ of a sequence $\{x_n\}$ in a topological space X is defined as

$$\mathrm{Lim}\{x_n\} := \bigcap_{n=1}^{\infty} \overline{\{x_m : m \geq n\}},$$

where $\overline{A}$ denote the closure of A in X. It is easy to verify that if $g : X \to X$ is a continuous mapping and $x \in \mathrm{Lim}\; O(x_0, f)$, then $g(x) \in \mathrm{Lim}\; g(O(x_0, f))$.

Definition 1. Let A, B be nonempty subsets of a topological space X. Let $g : X \times X \to \mathbb{R}$ be a continuous function. Put $D_g(A, B) = \inf\{|g(x, y)| : x \in A, y \in B\}$.

When X is a metric space and $g = d$ the metric function on X, then $D_g(A, B)$ is nothing but the distance between the sets A and B , which we usually denote by $\mathrm{dist}(A, B)$. A mapping $f : A \cup B \to A \cup B$ is cyclic if $f(A) \subseteq B, f(B) \subseteq A$. Let us define a generalized notion of cyclic contractive mapping:

Definition 2. A cyclic mapping $f : A \cup B \to A \cup B$ is topologically r-contractive with respect to g if for any $(u, v) \in A \times B($ or $B \times A)$,

$$|g(fu, fv)| < |g(u, v)|, \text{if } |g(u, v)| > D_g(A, B),$$
$$|g(fu, fv)| = |g(u, v)|, \text{if } |g(u, v)| = D_g(A, B).$$

In metric space setting, the notion of topologically r-contractive mapping reduces to notion of the cyclic contractive mapping introduced by Raju [6]. The following example shows that a cyclic mapping $f : A \cup B \to A \cup B$ may be topologically r-contractive with respect to a mapping $g : X \times X \to \mathbb{R}$ but need not with respect to some other mapping $h : X \times X \to \mathbb{R}$.

Example 2.1. Consider $\mathbb{R}^2$ with usual topology. Let $A := \{(0,x) : -1 \leq x \leq 1\}$ and $B := \{(1,y) : -1 \leq y \leq 1\}$. Define a cyclic mapping $f : A \cup B \to A \cup B$ as

$$f(t,x) := \begin{cases} (1, \frac{x}{2}), & \text{if } t = 0, \\ (0,x), & \text{if } t = 1. \end{cases}$$

Let $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$g((x,y),(u,v)) = yv, \text{ for all } (x,y),(u,v) \in \mathbb{R}^2$$
$$h((x,y),(u,v)) = \min\{y,v\}, \text{ for all } (x,y),(u,v) \in \mathbb{R}^2$$

Clearly, g and h are continuous functions which are not metric on $\mathbb{R}^2$. Note that $D_g(A,B) = 0$ and $D_h(A,B) = 0$. It is easy to verify that the function f is topologically r-contractive with respect to g, but not with respect to h, since $h((0,1),(1,\frac{1}{3})) = \min\{1,\frac{1}{3}\} = \frac{1}{3} > 0 = D_h(A,B)$ and $h(f(0,1),f(1,\frac{1}{3})) = h((0,\frac{1}{2}),(1,\frac{1}{3})) = \min\{\frac{1}{2},\frac{1}{3}\} = \frac{1}{3} = h((0,1),(1,\frac{1}{3}))$.

A point $u \in A \cup B$ is a best proximity point of f with respect to g if $|g(x,f(x))| = D_g(A,B)$. The following P-property plays vital role in proving the uniqueness of the best proximity points of a given cyclic mapping.

Definition 3. Let A, B be nonempty subsets of a topological space X and $g : X \times X \to \mathbb{R}$. We say that the pair (A,B) satisfy the topological P-property (simply we call P-property) with respect to g if whenever $x_1, x_2 \in A$ and $y_1, y_2 \in B$ with

$$\left.\begin{array}{l} |g(x_1,y_1)| = D_g(A,B) \\ |g(x_2,y_2)| = D_g(A,B) \end{array}\right\} \implies |g(x_1,x_2)| = |g(y_1,y_2)|.$$

Note that, if X is a metric space and g is a metric function on X, then the above definition reduces to the usual p-property, introduced in [7], which is used to obtain best proximity points for nonself mappings. The following example shows that a pair (A,B) of subsets of a topological space may have P-property with respect to a mapping $g : X \times X \to \mathbb{R}$ but need not with respect to some other mapping $h : X \times X \to \mathbb{R}$.

Example 2.2. Let $A := \{0\} \times [0,1]$, $B := \{1\} \times [0,1]$ be subsets of $\mathbb{R}^2$ with usual topology. Let $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be mappings given as in Example 2.1. We can see that the pair (A,B) has P-property with respect to g and not with respect to h, since $h((0,1),(1,0)) = 0 = D_h(A,B)$ and $h((0,\frac{1}{2}),(1,0)) = 0 = D_h(A,B)$ but $\frac{1}{2} = h((0,1),(0,\frac{1}{2})) \neq h((1,0),(1,0)) = 0$.

Remark 2.1. Let X be a topological space. Suppose that the continuous mapping $g : X \times X \to \mathbb{R}$ has the property $g(x,y) = 0 \Leftrightarrow x = y$. Then for any nonempty subset K of a topological space X, the pair (K,K) has P-property with respect to g.

3. Best proximity point theorems on topological spaces

Let us begin with the following best proximity point theorem on a topological space:

Theorem 3.1. *Let A, B be nonempty subsets of a topological space X and $f : A \cup B \to A \cup B$ be a continuous topologically r-contractive mapping with respect to a continuous mapping $g : X \times X \to \mathbb{R}$. Then, for any $u \in A \cup B$, the set $\operatorname{Lim} O(u, f)$ is either empty or consist of a point x which satisfy $|g(x, f(x))| = D_g(A, B)$.*

Proof. Let $u \in A \cup B$. Suppose that $\operatorname{Lim} O(u, f)$ is nonempty and choose $y \in \operatorname{Lim} O(u, f)$. Since f is continuous,

$$f(y) \in \operatorname{Lim} f(O(u, f)) \subseteq \operatorname{Lim} O(u, f).$$

Define $h : A \cup B \to \mathbb{R}$ by $h(x) := |g(x, f(x))|$, for all $x \in A \cup B$. Since f, g are continuous, h is continuous. Hence,

$$h(y) \in \operatorname{Lim}\ h(O(u, f)), \text{and } h(f(y)) \in \operatorname{Lim}\ h(O(u, f)). \tag{1}$$

Let us assume that $|g(f^n(u), f^{n+1}(u))| \neq D_g(A, B)$, for all $n \in \mathbb{N}$. Since f is topologically r- contractive with respective g, we have

$$h(f^n(u)) = |g(f^n(u), f^{n+1}(u))| > |g(f^{n+1}(u), f^{n+2}(u))| = h(f^{n+1}(u)).$$

i.e., $\{h(f^n(u))\}$ is a decreasing sequence of nonnegative real numbers. Hence $\{h(f^n(u))\}$ converges to a unique real number. i.e., $\operatorname{Lim} h(O(u, f))$ is a singleton set. From (1), $h(y) = h(f(y))$. i.e., $|g(y, f(y))| = |g(f(y), f^2(y))|$. Hence $|g(y, f(y))|$ must be equal to $D_g(A, B)$. i.e., y is a best proximity point of f with respect to g.
Suppose that $|g(f^{n_0}(u), f^{n_0+1}(u))| = D_g(A, B)$, for some $n_0 \in \mathbb{N}$. Then $\{h(f^n(u))\}$ is an eventually constant sequence with constant value $D_g(A, B)$. Hence $\operatorname{Lim} h(O(u, f))$ is singleton set. The same conclusion follows by the preceding argument. □

Let us illustrate the above result by the following example:

Example 3.1. Consider $\mathbb{R}^2$ with usual topology. Let $A := \{(0, t) : 0 \leq t \leq 1\}$ and $B := \{(1, s) : 0 \leq s \leq 1\}$. Let $f : A \cup B \to A \cup B$ be defined as

$$f(x, y) := \begin{cases} (1, \frac{y}{2}) & \text{if, } x = 0 \\ (0, \frac{y}{2}) & \text{if, } x = 1 \end{cases}$$

Let $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a mapping defined as $g((x, y), (u, v)) = yv$, for all $(x, y), (u, v) \in \mathbb{R}^2$. Clearly f and g are continuous mapping. The following inequality shows that f is a topologically r-contractive mapping with respect to g. Note that $D_g(A, B) = 0$. Whenever $|g((0, t), (1, s))| > 0, (i.e., t \neq 0$ and $s \neq 0)$

$$\begin{aligned} |g((0,t),(1,s))| &= \left| g\left(\left(1, \frac{t}{2}\right), \left(0, \frac{s}{2}\right) \right) \right| \\ &= \frac{ts}{4} < ts \\ &= |g((0,t),(1,s))| \end{aligned}$$

If $|g((0,t),(1,s))| = 0 = D_g(A,B)$ (i.e., $t = 0$ or $s = 0$), then $|g(f(0,t), f(1,s))| = 0 = D_g(A,B)$.

Clearly, for any $\mathbf{u} = (x,y) \in A \cup B$, $\operatorname{Lim} O(\mathbf{u}, f) = \{(0,0),(1,0)\}$ and (0,0) , (1,0) are best proximity points of f with respect to g. i.e., $|g((0,0), f(0,0))| = 0 = D_g(A,B), |g((1,0), f(1,0))| = 0 = D_g(A,B)$.

We obtained the following fixed point theorem due to Liepins as a corollary to Theorem 3.1:

Corollary 3.1. [5] *Let K be a nonempty subset of a topological space X and $g : X \times X \to \mathbb{R}$ be a continuous mapping satisfying that $|g(x,y)| = 0$ if and only if $x = y$, for all $x, y \in K$. Let $f : K \to K$ be a continuous mapping which satisfy that $|g(f(x), f(y))| < |g(x,y)|$, for all $x, y \in K$ with $x \neq y$. Then, for any $u \in K$, the set $\operatorname{Lim} O(u,f)$ is either empty or a singleton set $\{x\}$, which satisfy $x = f(x)$.*

Proof. Taking $A = B = K$ in Theorem 3.1 and since $D_g(K,K) = 0$, we have $\operatorname{Lim} O(u,f)$ is either empty or contains a point x_0 satisfying $|g(x_0, f(x_0))| = D_g(K,K) = 0$. i.e., $x_0 = f(x_0)$.

Suppose $\operatorname{Lim} O(u,f) \neq \emptyset$. Let us prove that it is a singleton set.

Assume that $x_0, y_0 \in$ Lim $O(u,f)$ with $f(x_0) = x_0, y_0 = f(y_0)$. If $x_0 \neq y_0$, then $|g(x_0,y_0)| = |g(f(x_0), f(y_0))| < |g(x_0,y_0)|$ a contradiction. Hence $x_0 = y_0$. i.e., $\operatorname{Lim} O(u,f)$ is a singleton set, for any $u \in K$. □

Now, let us prove the existence of a unique best proximity point of the given cyclic mapping $f : A \cup B \to A \cup B$.

Theorem 3.2. *Let A, B be nonempty subsets of a topological space X and $g : X \times X \to \mathbb{R}$ be a continuous mapping satisfying the following conditions:*

(1) *$g(x,y) = g(y,x)$, for all $x \in A, y \in B$*
(2) *$g(x_1,x_2) = 0 \Leftrightarrow x_1 = x_2$, for all $x_1, x_2 \in A$ (or $x_1, x_2 \in B$)*

Let $f : A \cup B \to A \cup B$ be a continuous topologically r-contractive mapping with respect to g. Suppose that the pair (A,B) has P-property with respect to g. Then for any $u \in A \cup B$, the set $\operatorname{Lim} O(u,f)$ is either empty or a singleton set $\{x\}$ in A, which satisfy $|g(x, f(x))| = D_g(A,B)$.

Proof. By Theorem 3.1, we conclude that the set $\operatorname{Lim} O(u,f)$ is either empty or contains a best proximity point of f with respect to g. Suppose there are two points in A such that $x_0, y_0 \in$ Lim $O(u,f)$ satisfying

$$|g(x_0, f(x_0))| = D_g(A,B) \tag{2}$$
$$|g(y_0, f(y_0))| = D_g(A,B) \tag{3}$$

Since f is topologically r-contractive with respect to g and using (2), we have $|g(f^2(x_0), f(x_0))| = |g(f(x_0), f^2(x_0))| = D_g(A,B)$. Hence, $|g(f^2(x_0), f(x_0))| = D_g(A,B) = |g(x_0, f(x_0))|$. By P-property, we have $|g(x_0, f^2(x_0)|$ $= |g(f(x_0), f(x_0))| = 0$. Thus $x_0 = f^2(x_0)$. In similar manner, we can show $f^2(y_0) = y_0$.

Suppose that $|g(f(x_0), y_0)| > D_g(A, B)$. Then,

$$|g(f(x_0), y_0)| = |g(f(x_0), f^2(y_0))| \\ \leq |g(x_0, f(y_0))| = |g(f^2(x_0), f(y_0))| < |g(f(x_0), y_0)|,$$

a contradiction. Hence $|g(f(x_0), y_0)| = D_g(A, B)$. By P-property, $|g(x_0, y_0)| = |g(f(x_0), f(x_0))| = 0$. Hence $x_0 = y_0$. i.e., the best proximity point of f with respect to g is unique in A. □

Now let us define the notion of relatively nonexpansive mapping on a topological space.

Definition 4. Let A, B be nonempty subsets of a topological space X and $g : X \times X \to \mathbb{R}$ be a continuous mapping. A cyclic mapping $f : A \cup B \to A \cup B$ is said to be topologically r-nonexpansive if $|g(f(x), f(y))| \leq |g(x, y)|$ for all $x \in A, y \in B$ (or $x \in B$, $y \in A$).

The following lemma plays vital role in Theorem 3.3, which provides sufficient conditions for the existence of best proximity points of a topologically r-nonexpansive mappings:

Lemma 3.1. *Let A, B be nonempty subsets of a topological space and $f : A \cup B \to A \cup B$ be a continuous cyclic mapping. Let $g : X \times X \to \mathbb{R}$ be a continuous mapping satisfying $g(x, y) = g(y, x)$ for all $x, y \in X$. Suppose there is a family $\mathscr{H}$ of cyclic mappings satisfying:*

(1) *for each $\epsilon > 0$, there is a mapping $h \in \mathscr{H}$ such that*

$$|g(x, h(x))| < D_g(A, B) + \varepsilon, \textit{for all } x \in A \cup B.$$

(2) *for each $h \in \mathscr{H}$, the mapping $h \circ f$ has a fixed point.*

(3) *there exists a point $z \in A \cup B$ such that*

$$|g(z, f(z))| = \inf\{|g(x, f(x))| : x \in A \cup B\}.$$

Then there exist $x_0 \in A \cup B$, such that $|g(x_0, f(x_0))| = D_g(A, B)$.

Proof. Let $\varepsilon > 0$. By (1), there is a mapping $h \in \mathscr{H}$ such that $|g(x, h(x))| < D_g(A, B) + \varepsilon$, for all $x \in A \cup B$. In particular,

$$|g(f(x), h(f(x)))| < D_g(A, B) + \varepsilon, \text{for all } x \in A \cup B.$$

By (2), there is a point $z_0 \in A \cup B$ such that $h(f(z_0)) = z_0$. Then by above equation, we have $|g(z_0, f(z_0))| < D_g(A, B) + \varepsilon$. Now, by (3), there is a point $x_0 \in A \cup B$ such that $D_g(A, B) \leq |g(x_0, f(x_0))| = \inf\{|g(x, f(x))| : x \in A \cup B\} \leq |g(z_0, f(z_0))| < D_g(A, B) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $|g(x_0, f(x_0))| = D_g(A, B)$. □

Let us prove a best proximity point theorem for a topologically r-nonexpansive mappings

Theorem 3.3. *Let A, B be nonempty subsets of a topological space X and $f : A \cup B \to A \cup B$ be a cyclic continuous mapping. Suppose that $\mathscr{H}$ is a family of injective cyclic continuous mappings on $A \cup B$. Let $g : X \times X \to \mathbb{R}$ and $g_1 : X \times X \to \mathbb{R}$ be two continuous functions satisfying the following conditions:*

(1) *$g(x,y)=g(y,x)$, for all $x,y\in X$,*
(2) *for each $\varepsilon>0$, there is a mapping $h\in\mathscr{H}$ such that*

$$|g(x,h(x))|<D_g(A,B)+\varepsilon,\ \textit{for all } x\in A\cup B,$$

(3) *there exist a point $z\in A\cup B$ such that*

$$|g(z,f(z))|=\inf\{|g(x,f(x))|:x\in A\cup B\},$$

(4) *each $h\in\mathscr{H}$ is topologically r-contractive with respect to g_1,*
(5) *f is topologically r-nonexpansive with respect to g_1,*
(6) *for each $h\in\mathscr{H}$, there is a point $x\in A\cup B$ such that*

$$\mathrm{Lim}\ O(x,h^2\circ f)\cap \mathrm{Lim}\ O(x,h)\neq\emptyset,$$

(7) *the pair (A,B) has P- property with respect to g_1,*
(8) *$g_1(x,y)=0$ if and only if $x=y$.*

Then there exist $x_0\in A\cup B$, such that $|g(x_0,f(x_0))|=D_g(A,B)$.

Proof. For fixed $h\in\mathscr{H}$ it is easy to verify that the mapping $(h^2\circ f)$ is a topologically r-contractive with respect to g_1. By (6) and Theorem 3.1, choose $x_0\in \mathrm{Lim}\ O(x,h^2\circ f)\cap \mathrm{Lim} O(x,h)$ which satisfies

$$|g_1(x_0,h^2(f(x_0)))|=D_{g_1}(A,B)=|g_1(x_0,h(x_0))|$$

Since (A,B) has P- property with respect to g_1 and by (7), we have

$$|g_1(h^2(f(x_0)),h(x_0))|=|g(x_0,x_0)|=0.$$

i.e., $h^2(f(x_0))=h(x_0)$. Since h is injective , $h(f(x_0))=x_0$. i.e., $h\circ f$ has a fixed point for any $h\in\mathscr{H}$. Hence the conclusion follows from Lemma 2.1. □

4. Conclusions

It is well-known fact that the metric space structure is essential to define the notion of best proximity points. In this manuscript, we generalized the notion of best proximity points to arbitrary topological spaces and obtained sufficient conditions for the existence of best proximity points for a given cyclic mapping in a topological space. Our results generalize the contractive fixed point theorem due to Edelstein.

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