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Subharmonic oscillations of a forced pendulum with time-dependent damping

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Abstract. We study the existence of subharmonic solutions for a forced pendulum equation with time-dependent damping. The proof is based on the theorem of the least action principle due to Mawhin and Willem (Critical point theory and hamiltonian systems. Springer, Berlin, 1989). Some results in the literature are generalized and improved.

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1. Introduction and main results

During the last few decades, the problem on the existence of periodic solutions for the forced pendulum equation

$$u'' + \beta \sin u = p(t) \tag{1}$$

has been studied by many researchers, and many interesting results have been obtained in literature, where $\beta > 0$ is a parameter and $p \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ is an external force. We just refer the reader to the classical monograph [12] for a very complete survey on this problem. Meanwhile, the problem on the existence of subharmonic solutions for Eq. (1) has also attracted the attention of scholars. See [4,8,20] and the references therein. More precisely, in [8], Fonda and Willem imposed certain non-degeneracy conditions on Eq. (1) in order to find subharmonic solutions of order m for every large prime number m. By using the Poincaré–Birkhoff fixed point theorem, Boscaggin and his coauthors in [4] proved that, for every $\beta > 0$ and for a large set of continuous and T-periodic functions with zero average, Eq. (1) has a subharmonic solution of order m for every large integer number m. In [20], by applying the critical point theory and an original decomposition technique, Yu obtained the following result.

Theorem 1. [20, Theorem 3.1] *Assume that $p \in L^2([0,T],\mathbb{R})$ is an odd periodic function such that, for some integer $m > 2$, $\gamma < \beta \le S_m^2\gamma$ and*

$$\|p\|^2_{L^2(0,T)} \le \frac{T\gamma^2(\beta-\gamma)(m^2-1)^2}{2\beta},$$

where $\gamma = (\frac{2\pi}{mT})^2$ and S_m is the smallest prime factor of m. Then Eq. (1) *has at least one subharmonic solution of order m.*

Later, Chen et al. [5] , Kong [9] and Xie et al. [19] generalized the above result to the forced pendulum equation with impulses.

Moreover, the problem about the existence of periodic solutions of the forced pendulum equation with constant friction coefficient has also attracted attention of many researchers in the last few decades. See, for example, the survey monographs [12,13] and the literature [1–3,7,10,15,18] and the references therein. Recently, some scholars have paid attention to the dynamical behaviors of the following forced pendulum equation with time-dependent damping:

$$u'' + h(t)u' + \beta \sin u = p(t), \tag{2}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a T-periodic function, $\beta > 0$ is a parameter and $p \in L^2([0,T],\mathbb{R})$. For example, Chu et al. in [6] and Sugie in [17] studied the stability of the equilibrium of Eq. (2) when the driving force $p \equiv 0$. Chu et al. in [6] gave a sufficient condition for the stability of the equilibrium of Eq. (2) by the computation of the corresponding Birkhoff normal forms. Based on Lyapunov's stability theory and phase plane analysis of the positive orbits of an equivalent planar system to Eq. (2), Sugie in [17] obtained a necessary and sufficient condition of the asymptotic stability for the equilibrium of Eq. (2). In [11], the authors studied the existence, multiplicity and stability of periodic solutions of Eq. (2) by using the the third-order approximation method and a suitable version of the Poincaré–Birkhoff theorem.

However, as far as we know, the problem on the existence of subharmonic solutions of Eq. (2) has not attracted attention in the literature. Motivated by this, in this paper, we will establish the conditions for the existence of subharmonic solutions to Eq. (2) by using the theorem of the least action principle due to Mawhin and Willem [14]. In general, the theorem of the least action principle cannot be applied to damped equations, which are in general dissipative. But, we find that such theorem can be applied to damped equations when the damping coefficient has zero mean. Therefore, throughout this paper, we assume that

$$\bar{h} = \frac{1}{T}\int_0^T h(s)\mathrm{d}s = 0.$$

Set

$$\sigma(h)(t) = \exp\left(\int_0^t h(s)\mathrm{d}s\right)$$

and assume that $\sigma(h) \in L^2([0,T],\mathbb{R})$. Obviously, $\sigma(h)$ is a T-periodic function. Moreover, we use the notations

$$\sigma(h)_* = \min_{t\in[0,T]} \sigma(h)(t), \quad \sigma(h)^* = \max_{t\in[0,T]} \sigma(h)(t).$$

Now, we first present an existence result of periodic solutions of Eq. (2).

Theorem 2. *Assume that*

$$\int_0^T \sigma(h)(s)p(s)\mathrm{d}s = 0.$$

Then Eq. (2) *has at least two geometrically distinct T-periodic solutions.*

Here we say that Eq. (2) has at least two geometrically distinct T-periodic solutions, if such T-periodic solutions do not differ by a multiple of 2π. The above theorem can be proved easily with a simple modification of the argument of [16].

We are now ready to state the main result of this paper.

Theorem 3. *For any integer $m \geq 2$ with*

$$S_m > \frac{\sigma(h)^*}{\sigma(h)_*},$$

assume that $\sigma(h)p$ is odd in t,

$$\frac{\sigma(h)^*\gamma}{\sigma(h)_*} < \beta < \frac{\sigma(h)_* S_m^2\gamma}{\sigma(h)^*}$$

and

$$\|p\|^2_{L^2(0,T)} \leq \frac{2T\Big[\beta\sigma(h)_* - \sigma(h)^*\gamma\Big]^2 \cdot \Big(\sigma(h)_* S_m^2\gamma - \beta\sigma(h)^*\Big)}{\beta\sigma(h)_*\sigma(h)^{*2}} := G(m). \tag{3}$$

Then Eq. (2) *has at least one subharmonic solution of order m.*

Theorem 3 will be proved in Sect. 3 by using the theorem of the least action principle due to Mawhin and Willem [14]. Let us recall that a mT-periodic solution of Eq. (2) is called subharmonic solution of order m if it is not lT periodic for any $l = 1,\ldots,m-1$. Moreover, by Theorem 3, we can easily obtain the following result for the case $h \equiv 0$.

Theorem 4. *Assume that p is an odd periodic function such that, for some integer $m \geq 2$, $\gamma < \beta < S_m^2\gamma$ and*

$$\|p\|^2_{L^2(0,T)} \leq \frac{2T\big(\beta-\gamma\big)^2 \cdot \big(S_m^2\gamma - \beta\big)}{\beta}.$$

Then Eq. (1) *has at least one subharmonic solution of order m.*

Compared with the results in literature [4,8,20], the novelties of our result for $h \equiv 0$ are as follows. First, we do not need to assume that m is a large integer or prime number as in literature [4,8]; Secondly, we obtain some new sufficient conditions for the existence of subharmonic solutions of Eq. (1) with minimal period mT for all $m \geq 2$ provided that β and the L^2-norm of p satisfy certain quantitative conditions, which different from the conditions in [20, Theorem 3.1] (see Theorems 1 and 4 for details).

Let us finally recall that, to the best of our knowledge, the results on the existence of subharmonic solutions with prescribed minimal period of the forced damped pendulum equation presented in this paper are the first ones available in the literature, and can be seen as natural extensions of the classical monograph [4,8,20] for the forced pendulum equation without time-dependent damping.

2. Preliminaries

We recall the following theorem of the least action principle due to Mawhin and Willem [14].

Lemma 1. [14] *Suppose that $\mathbb{H}$ is a reflexive Banach space and $\Psi : \mathbb{H} \to \mathbb{R}$ is weakly lower semi-continuous. Assume that Ψ is coercive,*

$$\Psi(u) \to +\infty \ \ as \ \|u\|_{\mathbb{H}} \to +\infty.$$

Then Ψ has at least one minimum.

Define a Hilbert space

$$\mathbb{H} = \{u : \mathbb{R} \to \mathbb{R}|\ u,\ u' \in L^2([0, mT], \mathbb{R}),\ u(t) = u(t + mT),\ t \in \mathbb{R}\},$$

endowed with the inner product

$$(u, v) = \int_0^{mT} [u(t)v(t) + u'(t)v'(t)]\mathrm{d}t,$$

where $m \geq 2$ is an integer. Define its norm by $\|u\|_{\mathbb{H}} = (u, u)^{\frac{1}{2}}$. For $u \in \mathbb{H}$, let

$$\tilde{u} = u(t) - \bar{u},$$

where $\bar{u} = \frac{1}{T}\int_0^T u(s)\mathrm{d}s$. By the following Wirtinger's inequality

$$\|\tilde{u}\|^2_{L^2(0,mT)} \leq \frac{m^2T^2}{4\pi^2}\|u'\|^2_{L^2(0,mT)}, \tag{4}$$

we have

$$\|\tilde{u}\|^2_{\mathbb{H}} \leq \big(1 + \frac{m^2T^2}{4\pi^2}\big)\|u'\|^2_{L^2(0,mT)}. \tag{5}$$

For any $u \in \mathbb{H}$, u has a Fourier series expansion

$$u(t) = \sum_{n=0}^{+\infty}\left(a_n \cos(\frac{\kappa n}{m}t) + b_n \sin\left(\frac{\kappa n}{m}t\right)\right),$$

where $\kappa = \frac{2\pi}{T}$. Moreover, we have

$$\|u\|_{L^2}^2 = \frac{mT}{2}\sum_{n=0}^{+\infty}(a_n^2 + b_n^2) < \infty \tag{6}$$

and

$$\|u'\|_{L^2}^2 = \frac{\kappa^2 T}{2m}\sum_{n=0}^{+\infty} n^2(a_n^2 + b_n^2) < \infty. \tag{7}$$

Define

$$\mathbb{H}^* = \{u \in \mathbb{H}|\ u(-t) = -u(t)\}.$$

Obviously, $\mathbb{H}^*$ is a closed subspace of $\mathbb{H}$. Moreover, it is obvious that $u \in \mathbb{H}^*$ if and only if

$$u(t) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{\kappa n}{m}t\right).$$

3. Proof of Theorem 3 and consequences

We can easily verify that Eq. (2) can be rewritten as its equivalent form

$$(\sigma(h)(t)u')' + \beta\sigma(h)(t)\sin u = \sigma(h)(t)p(t). \tag{8}$$

We consider the functional

$$\Psi(u) = \int_0^{mT}\left[\frac{\sigma(h)(t)(u')^2}{2} - \beta\sigma(h)(t)(1-\cos u) + \sigma(h)(t)p(t)u\right]\mathrm{d}t,$$

for any $u \in \mathbb{H}$. Obviously, Ψ is well defined on $\mathbb{H}$ and Fréchet differentiable. Let us compute the Fréchet derivative of Ψ

$$\Psi'(u)v = \int_0^{mT}\left[\sigma(h)(t)u'v' - \beta\sigma(h)(t)\sin(u)v + \sigma(h)(t)p(t)v\right]\mathrm{d}t,$$

for $v \in \mathbb{H}$. It is easy to see that the critical points of Ψ correspond to the mT-periodic solutions of Eq. (8), but mT is not necessarily the minimal period.

Lemma 2. *The functional Ψ is weakly lower semi-continuous on $\mathbb{H}^*$.*

Proof. If $\{u_n\} \in \mathbb{H}^*$ and $u_n \rightharpoonup u$, then $\{u_n\}$ converges uniformly to u on $[0, mT]$ and $u_n \to u$ on $L^2([0, mT])$. By the fact $\lim_{n\to\infty}\|u_n\|_{\mathbb{H}} \geq \|u\|_{\mathbb{H}}$ that

$$\begin{aligned}\liminf_{n\to\infty}\Psi(u_n) &= \liminf_{n\to\infty}\Bigg[\int_0^{mT}\frac{\sigma(h)(t)(u_n')^2}{2}\mathrm{d}t \\ &\quad - \int_0^{mT}\beta\sigma(h)(t)(1-\cos u_n)\mathrm{d}t + \int_0^{mT}\sigma(h)(t)p(t)u_n\mathrm{d}t\Bigg] \\ &\geq \int_0^{mT}\frac{\sigma(h)(t)(u')^2}{2}\mathrm{d}t - \int_0^{mT}\beta\sigma(h)(t)(1-\cos u)\mathrm{d}t. \\ &\quad + \int_0^{mT}\sigma(h)(t)p(t)u\mathrm{d}t = \Psi(u),\end{aligned}$$

which implies that Ψ is weakly lower semi-continuous on $\mathbb{H}^*$. □

Lemma 3. *The functional Ψ is coercive on $\mathbb{H}^*$.*

Proof. For any $u \in \mathbb{H}^*$, we have $\bar{u} = 0$. By (4) and (5), for all $u \in \mathbb{H}^*$, we have

$$\begin{aligned}\Psi(u) &= \int_0^{mT} \frac{\sigma(h)(t)(u')^2}{2}\mathrm{d}t - \int_0^{mT} \beta\sigma(h)(t)(1-\cos u)\mathrm{d}t \\ &\quad + \int_0^{mT} \sigma(h)(t)p(t)u\mathrm{d}t \\ &\geq \frac{\sigma(h)_*}{2}\|u'\|^2_{L^2(0,mT)} - 2mT\beta\sigma(h)^* - \|\sigma(h)p\|_{L^2(0,mT)}\|u\|_{L^2(0,mT)} \\ &\geq \frac{\sigma(h)_*}{2}\|u'\|^2_{L^2(0,mT)} - 2mT\beta\sigma(h)^* - \frac{2\pi}{mT}\|\sigma(h)p\|_{L^2}\|u'\|_{L^2(0,mT)} \\ &\geq \frac{2\pi^2\sigma(h)_*}{4\pi^2+m^2T^2}\|u\|^2_{\mathbb{H}} - 2mT\beta\sigma(h)^* - \frac{4\pi^2\|hp\|_{L^2(0,mT)}\|u\|_{\mathbb{H}}}{mT\sqrt{4\pi^2+m^2T^2}},\end{aligned}$$

which implies that

$$\lim_{\|u\|_{\mathbb{H}}\to\infty} \Psi(u) = +\infty.$$

Hence the functional Ψ is coercive. □

Lemma 4. *Assume that $\sigma(h)p$ is an odd function. If u is a critical point of Ψ on $\mathbb{H}^*$, then u is a critical point of Ψ on $\mathbb{H}$.*

Proof. If u is a critical point of Ψ on $\mathbb{H}^*$, then

$$\Psi'(u)v = 0, \ \text{for any } v \in \mathbb{H}^*.$$

For any $v \in \mathbb{H}$ satisfying $v \perp \mathbb{H}^*$, then v is even in t. Integrating by parts, we have

$$\begin{aligned}\Psi'(u)v &= \int_0^{mT} \sigma(h)(t)u'v'\mathrm{d}t - \int_0^{mT} \beta\sigma(h)(t)\sin(u)v\mathrm{d}t \\ &\quad + \int_0^{mT} \sigma(h)(t)p(t)v\mathrm{d}t \\ &= -\int_0^{mT} (\sigma(h)(t)u')'v\mathrm{d}t - \int_0^{mT} \beta\sigma(h)(t)\sin(u)v\mathrm{d}t \\ &\quad + \int_0^{mT} \sigma(h)(t)p(t)v\mathrm{d}t \\ &= -\int_0^{mT} [(\sigma(h)(t)u')' + \beta\sigma(h)(t)\sin(u) - \sigma(h)(t)p(t)]v\mathrm{d}t = 0.\end{aligned}$$

Therefore, it follows easily that $\Psi'(u)v = 0$ for all $v \in \mathbb{H}$. □

By Lemma 1–4, we know that Eq. (8) has a solution $\hat{u}(t)$ which minimizes Ψ on $\mathbb{H}^*$. Obviously, we know that $\hat{u} \not\equiv 0$.

Now we claim that the minimal period of $\hat{u}$ is mT. Otherwise, assume that $\hat{u}$ has the minimal period $\frac{mT}{l}$ for some positive integer $l > 1$. It follows from the minimal periods of critical points of Ψ are integer multiples of T that $\frac{m}{l}$ must be an integer. Furthermore, we have $l \geq S_m$. By Fourier expansion, we have

$$u(t) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{l\kappa n}{m}t\right).$$

By (4), we have

$$\begin{aligned}
\Psi(\hat{u}) &= \int_0^{mT} \frac{\sigma(h)(t)(\hat{u}')^2}{2}\mathrm{d}t - \int_0^{mT} \beta\sigma(h)(t)(1-\cos\hat{u})\mathrm{d}t \\
&\quad + \int_0^{mT} \sigma(h)(t)p(t)\hat{u}\mathrm{d}t \\
&\geq \frac{\sigma(h)_*}{2}\int_0^{mT} |\hat{u}'|^2\mathrm{d}t - \frac{\beta\sigma(h)^*}{2}\int_0^{mT} |\hat{u}|^2\mathrm{d}t - \sigma(h)^*\int_0^{mT} |p(t)\hat{u}|\mathrm{d}t \\
&\geq \frac{\sigma(h)_*}{2}\|\hat{u}'\|^2_{L^2_m} - \frac{\beta\sigma(h)^*}{2}\|\hat{u}\|^2_{L^2_m} - \sigma(h)^*\|p\|_{L^2_m}\|\hat{u}\|_{L^2_m} \\
&\geq \frac{2\pi^2\sigma(h)_* l^2}{m^2T^2}\|\hat{u}\|^2_{L^2_m} - \frac{\beta\sigma(h)^*}{2}\|\hat{u}\|^2_{L^2_m} - \sigma(h)^*\|p\|_{L^2_m}\|\hat{u}\|_{L^2_m} \\
&= \frac{\sigma(h)_* l^2\kappa^2 - \beta\sigma(h)^* m^2}{2m^2}\|\hat{u}\|^2_{L^2_m} - \sigma(h)^*\|p\|_{L^2_m}\|\hat{u}\|_{L^2_m} \\
&\geq -\frac{\sigma(h)^{*2}m^2\|p\|^2_{L^2_m}}{2[\sigma(h)_* l^2\kappa^2 - \beta\sigma(h)^* m^2]} \\
&\geq -\frac{\sigma(h)^{*2}m^3\|p\|^2_{L^2(0,T)}}{2[\sigma(h)_* S_m^2\kappa^2 - \beta\sigma(h)^* m^2]} \\
&= -\frac{\sigma(h)^{*2}m\|p\|^2_{L^2(0,T)}}{2[\sigma(h)_* S_m^2\gamma - \beta\sigma(h)^*]},
\end{aligned} \tag{9}$$

here $\gamma = (\frac{2\pi}{mT})^2$ and $\|\cdot\|_{L^2_m} = \|\cdot\|_{L^2(0,mT)}$.

On the other hand, let $\check{u}(t) = \alpha\sin\frac{\kappa}{m}t$ with

$$\alpha^2 = 8\left[1 - \frac{\sigma(h)^*\gamma}{\beta\sigma(h)_*}\right]. \tag{10}$$

Then $\check{u}$ is mT-periodic with minimal period mT. By the fact $\sigma(h)(t)p(t)$ is T-periodic and odd in t that

$$\int_0^{mT} \sigma(h)(t)p(t)\check{u}(t)\mathrm{d}t = 0.$$

Then by (6), (7) and (10), we have

$$\begin{aligned}
\Psi(\check{u}) &= \int_0^{mT} \frac{\sigma(h)(t)(\check{u}')^2}{2}\mathrm{d}t - \int_0^{mT} \beta\sigma(h)(t)(1-\cos\check{u})\mathrm{d}t \\
&\leq \frac{\sigma(h)^*}{2}\int_0^{mT} (\check{u}')^2\mathrm{d}t - \beta\sigma(h)_*\int_0^{mT} (\frac{|\check{u}|^2}{2} - \frac{|\check{u}|^4}{24})\mathrm{d}t \\
&\leq \frac{\sigma(h)^*}{2}\|\check{u}'\|^2_{L^2(0,mT)} - \frac{\beta\sigma(h)_*}{2}\|\check{u}\|^2_{L^2(0,mT)} + \frac{\beta\sigma(h)_*\alpha^4}{24}\int_0^{mT} \sin^4\frac{\kappa}{m}t\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\sigma(h)^*\kappa^2 T}{4m}\alpha^2 - \frac{\beta\sigma(h)_* mT}{4}\alpha^2 + \frac{\beta\sigma(h)_* mT\alpha^4}{64} \\
&= -\frac{\beta\sigma(h)_* m^2 T - \sigma(h)^*\kappa^2 T}{4m}\alpha^2 + \frac{\beta\sigma(h)_* mT}{64}\alpha^4 \\
&= -\frac{T\Big[\beta\sigma(h)_* m^2 - \sigma(h)^*\kappa^2\Big]^2}{\beta\sigma(h)_* m^3} \\
&= -\frac{mT\Big[\beta\sigma(h)_* - \sigma(h)^*\gamma\Big]^2}{\beta\sigma(h)_*}. \qquad (11)
\end{aligned}$$

By the choices of $\hat{u}$ and $\check{u}$, we know that $\Psi(\hat{u}) \leq \Psi(\check{u})$. Then it follows from (9) and (11) that

$$-\frac{mT\Big[\beta\sigma(h)_* - \sigma(h)^*\gamma\Big]^2}{\beta\sigma(h)_*} > -\frac{\sigma(h)^{*2} m\|p\|^2_{L^2(0,T)}}{2[\sigma(h)_* S_m^2\gamma - \beta\sigma(h)^*]},$$

which yields

$$\|p\|^2_{L^2(0,T)} > G(m),$$

this contradicts (4). Hence the proof of Theorem 3 is complete.

Finally, we present an existence result of the odd periodic solutions.

Theorem 5. *Assume that $\sigma(h)p$ is an odd function. Then* Eq. (8) *has at least one odd T-periodic solution.*

Proof. Let $m = 1$ in $\mathbb{H}$. By Lemma 2–4, we get that Ψ has at least one critical point on $\mathbb{H}^*$. Then Lemma 1 guarantees that Eq. (8) has at least one odd T-periodic solution. □

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