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Some characterizations of Reich and Chatterjea type nonexpansive mappings

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Abstract. We introduce two types of mappings, namely Reich type nonexpansive and Chatterjea type nonexpansive mappings, and derive some sufficient conditions under which these two types of mappings possess an approximate fixed point sequence (AFPS). We obtain the desired AFPS using the well-known $Schäefer$ iteration method. Along with these, we check some special properties of the fixed point sets of these mappings, such as closedness, convexity, remotality, unique remotality, etc. We also derive a nice interrelation between AFPS and maximizing sequence for both types of mappings. Finally, we will get some sufficient conditions under which the class of Reich type nonexpansive mappings reduces to that of nonexpansive maps.

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1. Introduction

It is well known that various nonlinear generalizations of the contraction mapping are of great significance in the literature. Nonexpansive mappings, asymptotically nonexpansive mappings are some examples of such generalizations. Let X be a normed linear space. A mapping $T : X \to X$ is said to be nonexpansive if the condition $\|Tx - Ty\| \leq \|x - y\|$ holds for all $x, y \in X$. A mapping $T : X \to X$ is said to be asymptotically nonexpansive if for all $x, y \in X$ and for all $n \in \mathbb{N}$, the condition $\|T^n x - T^n y\| \leq \alpha_n \|x - y\|$ holds for some sequence (α_n) with $\alpha_n \geq 1$ and $\lim_{n\to\infty} \alpha_n = 1$. We know that every nonexpansive mapping or asymptotically nonexpansive mapping on a non-empty closed, bounded, convex subset of a uniformly convex Banach space has at least one fixed point, see [1,6,23]. Subsequently, many authors have introduced several kinds of nonlinear mappings generalizing the class of nonexpansive mappings such as asymptotically pseudocontractive mappings, uniformly asymptotically regular mappings, uniformly asymptotically regular mappings with sequence, uniformly L-Lipschitzian mappings, etc.

In proving the existence of fixed point of the above mentioned mappings, many authors took the help of approximate fixed point sequence (AFPS). A sequence (x_n) in a normed linear space is said to be an AFPS if $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The importance of AFPS lies in the fact that, for the above classes of mappings, if a sequence is constructed and shown to be an AFPS by taking some sufficient conditions, then the limit of that sequence can be shown (by taking some additional mild condition or sometimes by taking no condition) to be a fixed point of that mappings. So, it turns out that the fixed point problem reduces to a problem of obtaining AFPS. Many authors have studied a number of methods for iterative AFPS, see [2,4,17,21]. Here we mention some of those iterative AFPS.

(a) The Mann iteration method: Here the sequence (x_n) is defined by
$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n,$$
where (λ_n) is a sequence of real numbers satisfying $0 \leq \lambda_n < 1$ for all $n \in \mathbb{N}$.

(b) The Krasnoselskij iteration method: Here the sequence (x_n) is defined by
$$x_{n+1} = \frac{1}{2}(x_n + Tx_n).$$

(c) The *Schäefer* iteration method: Here the sequence (x_n) is defined by
$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n,$$
where $\lambda \in (0, 1)$.

(d) The Halpern iteration method: Here the sequence (x_n) is defined by
$$x_{n+1} = (1 - \lambda_n)Tx_n + \lambda_n u,$$
where (λ_n) is a sequence in $[0, 1]$ and $u \in X$.

(e) The modified Mann iteration method: Here the sequence (x_n) is defined by
$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T^n x_n,$$
where (λ_n) is a sequence in $[0, 1]$.

With the help of the above iterative sequences, many authors obtained different types of interesting results showing the existence of AFPS for various kinds of generalized nonexpansive mappings. In [16], Reinermann proved the following result in a Hilbert space:

Theorem 1.1. *Let H be a Hilbert space, $K \subset H$ be non-empty closed, bounded and convex. Let T be an asymptotically nonexpansive mapping on K with $(k_n) \subset (0, \infty)$, $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, $\epsilon \leq \lambda_n \leq 1 - \epsilon$ for all $n \in \mathbb{N}$ and some $\epsilon > 0$; pick $x_0 \in K$, and define $x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T^n x_n$ for all $n \geq 0$. Then (x_n) is an AFPS for T.*

Later on, Rhoades extended Theorem 1.1 in uniformly convex Banach spaces and proved the following result:

Theorem 1.2 [17]. *Let X be a uniformly convex Banach space, $\phi \neq A \subset X$, and A be closed, convex and bounded. Let T be an asymptotically self-map of A with $(k_n) \geq 1$, $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$ for some $r > 1$, $\epsilon \leq 1-\alpha_n \leq 1-\epsilon$ for all $n > 0$ and some $\epsilon > 0$. Pick $x_0 \in A$ and define $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n$ for $n \geq 0$. Then (x_n) is an AFPS for T.*

Subsequently, many mathematicians have studied the AFPS for different types of nonexpansive mappings by considering several iterative sequences, (see [5,10,20] and the references therein). It is very interesting to notice that the most of the results of such kinds have been established in Hilbert spaces, uniformly convex Banach space, smooth reflexive Banach space, etc. In this connection see [11,14,22]. So, it is an appropriate question to ask whether it is possible to derive some results concerning AFPS in some more general spaces, more specifically in Banach spaces. One of the main purposes of this paper is to give an affirmative answer to this problem, i.e., by taking the underlying structure as Banach space. We proceed to our purpose by considering two new types of nonexpansive mappings, namely Reich type nonexpansive and Chatterjea type nonexpansive mappings. We show that if these two types of mappings satisfy some additional mild conditions, then there exist AFPS for the mappings. Further, we show that the respective approximate fixed point sequences converge strongly to a fixed point of the respective mappings. For more considerations on Reich contractions and Chatterjea contractions see [13,15], respectively [3]. For Hardy–Rogers contractions see [8].

Another purpose of this paper was to study some important properties regarding the farthest points of the domains of the mappings under consideration. In this context, we present some results showing that the set of fixed points of these types of mappings are closed, convex and remotal. We also show, by suitable examples, that the fixed point set of the mappings need not to be uniquely remotal. Alongside these, we obtain some interesting results related to maximizing sequence, farthest point map for these two types of mappings. In addition, we will derive some sufficient conditions for which the class of Reich type nonexpansive mappings reduces to the class of nonexpansive mappings. Finally, we pose an open problem requiring some sufficient conditions for which the class of Chatterjea type nonexpansive mappings reduces to that of nonexpansive mappings.

2. Preliminaries

Let X be a normed linear space, C a non-empty subset of X and $T : C \to C$ be a mapping. The mapping T is said to be a Reich type nonexpansive mapping if there exists non-negative real numbers a, b, c with $a + b + c = 1$, such that the condition

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|$$

holds for all $x, y \in C$. The mapping T is said to be a Chatterjea type nonexpansive mapping if there exists non-negative real numbers a, b, c with

$a+b+c=1$, such that the condition

$$\|Tx-Ty\| \leq a\|x-y\| + b\|x-Ty\| + c\|y-Tx\|$$

holds for all $x, y \in C$. In both cases, we say that T is a Reich type nonexpansive (Chatterjea type nonexpansive) mapping with coefficients (a, b, c).

In what follows, the following lemma will be necessary in our main results:

Lemma 2.1 [7]. *Let (z_n) and (w_n) be two bounded sequences in a Banach space X and let $\lambda \in (0,1)$. Let $z_{n+1} = \lambda w_n + (1-\lambda) z_n$ and suppose $\|w_{n+1} - w_n\| \leq \|z_{n+1} - z_n\|$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \|w_n - z_n\| = 0$.*

Next, we will recall some notions about remotal and uniquely remotal sets from [18] in a normed linear space. Let X be a real normed linear space and A a non-empty, bounded subset of X and $x \in X$. The farthest distance from x to A is defined by

$$\delta(x, A) = \sup\{\|x-a\| : a \in A\}.$$

The farthest distance from x to A may or may not attend at some elements of A. Let $F(x, A) = \{e \in A : \|x-e\| = \delta(x, A)\}$. The set A is said to be remotal if $F(x, A) \neq \phi$ for all $x \in X$ and is said to be uniquely remotal if $F(x, A)$ is singleton for each $x \in A$. A sequence $(x_n) \subset A$ is said to be maximizing in A if there exists $x \in X$ such that $\|x_n - x\| \to \delta(x, A)$ as $n \to \infty$. A non-empty set K is said to be M-compact [12] if every maximizing sequence (x_n) in K is compact. The Chebyshev radius of A [19] is defined by $r(A) = \inf\{\delta(x, A) : x \in X\}$. An element $c \in X$ is said to be Chebyshev center of A if $\delta(c, A) = r(A)$. A non-empty, bounded subset A of X is said to be centerable [19] if $\operatorname{diam}(A) = 2r(A)$, where $\operatorname{diam}(A)$ denotes the diameter of the set A.

3. Main results

In this section, we will give first some sufficient conditions under which a Reich nonexpansive mapping possesses AFPS.

Theorem 3.1. *Let X be a Banach space and C be a non-empty closed, convex, bounded subset of X. Let $T : C \to C$ be a Reich type nonexpansive mapping with coefficients (a, b, c) such that $c < 1$. Also, assume that for $x, y \in C$*

$$\frac{1-c}{6}\|x-Tx\| \leq \|x-y\| \Rightarrow \|Tx-Ty\| \leq \|x-y\|.$$

Then T has an AFPS in C. Moreover, the AFPS is asymptotically regular.

Proof. Since T is Reich type nonexpansive mapping with coefficients (a, b, c), it follows that a, b, c are non-negative real numbers with $a+b+c=1$, such that

$$\|Tx-Ty\| \leq a\|x-y\| + b\|x-Tx\| + c\|y-Ty\| \tag{3.1}$$

holds for all $x, y \in C$. Let $x_0 \in C$ be arbitrary but fixed. We consider the sequence (x_n) in X defined by $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$ for all $n \geq 2$ where

$\frac{1}{2} \leq \lambda < 1$. Since C is bounded and convex, it follows that (x_n) is a bounded sequence in C. Next, putting $x = x_n$, $y = x_{n+1}$ in (3.1), we get

$$\|Tx_n - Tx_{n+1}\| \leq a\|x_n - x_{n+1}\| + b\|x_n - Tx_n\| + c\|x_{n+1} - Tx_{n+1}\|.$$

Thus

$$\lambda\|Tx_n - Tx_{n+1}\| \leq a\lambda\|x_n - x_{n+1}\| + b\lambda\|x_n - Tx_n\| + c\lambda\|x_{n+1} - Tx_{n+1}\| \tag{3.2}$$

We have $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, so $\lambda\|Tx_n - x_n\| = \|x_{n+1} - x_n\|$. Similarly we have $\lambda\|Tx_{n+1} - x_{n+1}\| = \|x_{n+2} - x_{n+1}\|$.

Using the above relations in (3.2), we get

$$\lambda\|Tx_n - Tx_{n+1}\| \leq a\lambda\|x_n - x_{n+1}\| + b\|x_n - x_{n+1}\| + c\|x_{n+1} - x_{n+2}\|. \tag{3.3}$$

Again we have

$$\begin{aligned} x_{n+1} - x_{n+2} &= \lambda(Tx_n - Tx_{n+1}) + (1-\lambda)(x_n - x_{n+1}) \\ \Rightarrow \|x_{n+1} - x_{n+2}\| &\leq \lambda\|Tx_n - Tx_{n+1}\| + (1-\lambda)\|x_n - x_{n+1}\|. \end{aligned} \tag{3.4}$$

Using (3.4) in (3.3), we get that

$$\begin{aligned} \lambda\|Tx_n - Tx_{n+1}\| &\leq a\lambda\|x_n - x_{n+1}\| + b\|x_n - x_{n+1}\| + c\lambda\|Tx_n - Tx_{n+1}\| \\ &\quad + c(1-\lambda)\|x_n - x_{n+1}\|. \end{aligned}$$

Thus

$$\begin{aligned} (\lambda - c\lambda)\|Tx_n - Tx_{n+1}\| &\leq a\lambda\|x_n - x_{n+1}\| + b\|x_n - x_{n+1}\| \\ &\quad + c(1-\lambda)\|x_n - x_{n+1}\| \\ &< a\|x_n - x_{n+1}\| + b\|x_n - x_{n+1}\| + c\|x_n - x_{n+1}\| \\ &= \|x_n - x_{n+1}\|. \end{aligned}$$

Therefore, $(\lambda - c\lambda)\|Tx_n - Tx_{n+1}\| < \|x_n - x_{n+1}\|$. Then we have

$$\begin{aligned} \|x_n - Tx_{n+1}\| &\leq \|x_n - Tx_n\| + \|Tx_n - Tx_{n+1}\| \\ &= \frac{1}{\lambda}\|x_n - x_{n+1}\| + \|Tx_n - Tx_{n+1}\| \\ \Rightarrow (\lambda - c\lambda)\|x_n - Tx_{n+1}\| &\leq \frac{\lambda - c\lambda}{\lambda}\|x_n - x_{n+1}\| + (\lambda - c\lambda)\|Tx_n - Tx_{n+1}\| \\ &< \frac{\lambda - c\lambda}{\lambda}\|x_n - x_{n+1}\| + \|x_n - x_{n+1}\| \\ &= \frac{2\lambda - c\lambda}{\lambda}\|x_n - x_{n+1}\|. \end{aligned}$$

Hence

$$\frac{\lambda(\lambda - c\lambda)}{2\lambda - c\lambda}\|x_n - Tx_{n+1}\| < \|x_n - x_{n+1}\|. \tag{3.5}$$

Now since $\frac{1}{2} \leq \lambda < 1$ and $0 \leq c < 1$, we have $\frac{1-c}{4} < \frac{\lambda(\lambda - c\lambda)}{2\lambda - c\lambda}$. Then, from (3.5), we get

$$\frac{1-c}{4}\|x_n - Tx_{n+1}\| < \|x_n - x_{n+1}\|.$$

Therefore,

$$\begin{aligned}\frac{1-c}{4}\|x_n - Tx_n\| &\le \frac{1-c}{4}\|x_n - Tx_{n+1}\| + \frac{1-c}{4}\|Tx_n - Tx_{n+1}\| \\ &< \|x_n - x_{n+1}\| + \frac{1-c}{4(\lambda - c\lambda)}\|x_n - x_{n+1}\| \\ &\le \|x_n - x_{n+1}\| + \frac{1}{4\lambda}\|x_n - x_{n+1}\| \\ &\le \frac{3}{2}\|x_n - x_{n+1}\|,\end{aligned}$$

which implies that $\frac{1-c}{6}\|x_n - Tx_n\| < \|x_n - x_{n+1}\|$. Thus by given assumption, we get

$$\|Tx_n - Tx_{n+1}\| < \|x_n - x_{n+1}\|.$$

Hence using Lemma 2.1, we get $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, i.e., (x_n) is an AFPS of T. Further, we have

$$\|x_n - x_{n+1}\| = \lambda\|x_n - Tx_n\| \to 0 \text{ as } n \to \infty.$$

Therefore, the AFPS (x_n) is asymptotically regular also. □

In the next theorem, we show the existence of fixed point for Reich type nonexpansive mappings with the help of Theorem 3.1.

Theorem 3.2. *Under the assumptions of Theorem 3.1, T has a fixed point, provided $a < 1$.*

Proof. By Theorem 3.1, it follows that T has an AFPS (x_n), say, and this sequence is asymptotically regular also. Then, we have

$$\begin{aligned}\|x_n - x_m\| &\le \|x_n - Tx_n\| + \|Tx_n - Tx_m\| + \|x_m - Tx_m\| \\ &\le \|x_n - Tx_n\| + a\|x_n - x_m\| + b\|x_n - Tx_n\| + c\|x_m - Tx_m\| \\ &\quad + \|x_m - Tx_m\|,\end{aligned}$$

which implies that

$$\begin{aligned}(1-a)\|x_n - x_m\| &\le \|x_n - Tx_n\| + b\|x_n - Tx_n\| + c\|x_m - Tx_m\| \\ &\quad + \|x_m - Tx_m\| \to 0 \text{ as } n, m \to \infty.\end{aligned}$$

Therefore, (x_n) is a Cauchy sequence in C and hence convergent to some $z \in C$. Again, we see that

$$\begin{aligned}\|Tx_n - Tx_m\| &\le a\|x_n - x_m\| + b\|x_n - Tx_n\| + c\|x_m - Tx_m\| \\ &\to 0 \text{ as } n, m \to \infty.\end{aligned}$$

Thus (Tx_n) is also a Cauchy sequence in C. Then, we have

$$\|Tx_n - z\| \le \|Tx_n - x_n\| + \|x_n - z\| \to 0 \text{ as } n \to \infty.$$

Again we have

$$\|Tx_n - Tz\| \le a\|x_n - z\| + b\|x_n - Tx_n\| + c\|z - Tz\|.$$

Taking limit as $n \to \infty$ in above inequality, we get

$$\|z - Tz\| \le c\|z - Tz\|,$$

which gives $z = Tz$, i.e., z is a fixed point of T. □

The following theorem characterizes the fixed point set of Reich type nonexpansive mappings.

Theorem 3.3. *Let X be a Banach space and C be a non-empty subset of X. If T is Reich type nonexpansive mapping on C with coefficients (a, b, c) such that $c < 1$, then $Fix(T)$ is a closed subset of C.*

Proof. Let (z_n) be a sequence in $Fix(T)$ such that (z_n) converges to $z \in C$. Then we have

$$\begin{aligned} &\|Tz_n - Tz\| \leq a\|z_n - z\| + b\|z_n - Tz_n\| + c\|z - Tz\| \\ \Rightarrow &\|z_n - Tz\| \leq a\|z_n - z\| + c\|z - Tz\|. \end{aligned}$$

Taking limit as $n \to \infty$ in above inequality, we get

$$\|z - Tz\| \leq c\|z - Tz\|,$$

which gives $z = Tz$, i.e., $z \in Fix(T)$. This shows that $Fix(T)$ is a closed subset of C. □

In the next theorem, we give another characterization of the fixed point set of Reich type nonexpansive mapping by taking the underlying space as a Hilbert space in place of Banach space.

Theorem 3.4. *Let X be a Hilbert space and C be a non-empty subset of X. Let T be Reich type nonexpansive mapping on C with coefficients (a, b, c) with $c < 1$. Assume that $b \leq c$. Then $Fix(T)$ is a convex subset of C.*

Proof. Let $p, q \in Fix(T)$ be arbitrary and λ be a scalar with $0 \leq \lambda \leq 1$. Take $z = \lambda p + (1 - \lambda)q$. Then we have

$$\begin{aligned} &\|Tz - Tp\| \leq a\|z - p\| + b\|z - Tz\| + c\|p - Tp\| \\ \Rightarrow &\|Tz - p\| \leq a\|z - p\| + b\|z - p\| + b\|p - Tz\| \\ \Rightarrow &(1 - b)\|Tz - p\| \leq (1 - c)\|z - p\|. \end{aligned}$$

Thus

$$\|Tz - p\| \leq \|z - p\|. \tag{3.6}$$

Now by parallelogram law, we have

$$\left\|\frac{z-p}{2} + \frac{Tz-p}{2}\right\|^2 + \left\|\frac{z-p}{2} - \frac{Tz-p}{2}\right\|^2 = 2\left\{\left\|\frac{z-p}{2}\right\|^2 + \left\|\frac{Tz-p}{2}\right\|^2\right\}$$

$$\begin{aligned} \Rightarrow \left\|\frac{z-p}{2} + \frac{Tz-p}{2}\right\|^2 + \frac{1}{4}\|z - Tz\|^2 &= \frac{1}{2}\|z-p\|^2 + \frac{1}{2}\|Tz - p\|^2 \\ &\leq \frac{1}{2}\|z-p\|^2 + \frac{1}{2}\|z-p\|^2 \\ &= \|z-p\|^2. \end{aligned}$$

Then

$$\left\|\frac{z-p}{2} + \frac{Tz-p}{2}\right\|^2 \leq \|z-p\|^2 - \frac{1}{4}\|z - Tz\|^2,$$

which implies that

$$\left\| \frac{z+Tz}{2} - p \right\|^2 \leq \|z-p\|^2 - \frac{1}{4}\|z-Tz\|^2 = (1-\lambda)^2\|p-q\|^2 - \frac{1}{4}\|z-Tz\|^2.$$

Similarly, we can show that

$$\left\| \frac{z+Tz}{2} - q \right\|^2 \leq \lambda^2\|p-q\|^2 - \frac{1}{4}\|z-Tz\|^2.$$

If $z \neq Tz$, then we from above inequalities, we get

$$\left\| \frac{z+Tz}{2} - p \right\| < (1-\lambda)\|p-q\|,$$

and

$$\left\| \frac{z+Tz}{2} - q \right\| < \lambda\|p-q\|.$$

Then we have

$$\begin{aligned} \|p-q\| &\leq \left\| \frac{z+Tz}{2} - p \right\| + \left\| \frac{z+Tz}{2} - q \right\| \\ &< (1-\lambda)\|p-q\| + \lambda\|p-q\| = \|p-q\|, \end{aligned}$$

which is a contradiction. So, we must have $z = Tz$. i.e., $z \in Fix(T)$. Therefore, $Fix(T)$ is a convex set. □

Our next theorem gives sufficient condition for Chatterjea nonexpansive type mappings which guarantees that this class of mappings possess AFPS.

Theorem 3.5. *Let X be a Banach space and C be a non-empty closed, convex, bounded subset of X. Let $T : C \to C$ be a Chatterjea type nonexpansive mapping with coefficients (a, b, c), such that $b < 1$. Also assume that for $x, y \in C$*

$$\frac{1-b}{7}\|x-Ty\| \leq \|x-y\| \Rightarrow \|Tx-Ty\| \leq \|x-y\|.$$

Then T has an AFPS in C. Moreover, the AFPS is asymptotically regular.

Proof. Since T is a Chatterjea type nonexpansive mapping with coefficients (a, b, c), we have $a, b, c \geq 0$, $a + b + c = 1$ and

$$\|Tx-Ty\| \leq a\|x-y\| + b\|x-Ty\| + c\|y-Tx\| \tag{3.7}$$

for all $x, y \in C$. Let $x_0 \in C$ be arbitrary but fixed. We consider the sequence (x_n) in X defined by $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$ for all $n \geq 2$ where $\frac{1}{2} \leq \lambda < 1$. Since C is bounded and convex, it follows that (x_n) is a bounded sequence in C. Now putting $x = x_n$, $y = x_{n+1}$ in Eq. (3.7), we get

$$\begin{aligned} \|Tx_n - Tx_{n+1}\| &\leq a\|x_n - x_{n+1}\| + b\|x_n - Tx_{n+1}\| + c\|x_{n+1} - Tx_n\| \\ &\leq a\|x_n - x_{n+1}\| + b\|x_n - x_{n+1}\| + b\|x_{n+1} - Tx_{n+1}\| \\ &\quad + c\|x_{n+1} - x_n\| + c\|x_n - Tx_n\| \\ &= \|x_n - x_{n+1}\| + b\|x_{n+1} - Tx_{n+1}\| + c\|x_n - Tx_n\|. \end{aligned} \tag{3.8}$$

Since $x_{n+1} = \lambda T x_n + (1-\lambda)x_n$ for all $n \geq 2$, we have $\|x_n - Tx_n\| = \frac{1}{\lambda}\|x_n - x_{n+1}\|$ and $\|x_{n+1} - Tx_{n+1}\| = \frac{1}{\lambda}\|x_{n+1} - x_{n+2}\|$. Using these in Eq. (3.8), we get

$$\lambda\|Tx_n - Tx_{n+1}\| \leq \lambda\|x_n - x_{n+1}\| + b\|x_{n+1} - x_{n+2}\| + c\|x_n - x_{n+1}\|. \quad (3.9)$$

Now

$$\begin{aligned} x_{n+1} - x_{n+2} &= \lambda(Tx_n - Tx_{n+1}) + (1-\lambda)(x_n - x_{n+1}) \\ \Rightarrow \|x_{n+1} - x_{n+2}\| &\leq \lambda\|Tx_n - Tx_{n+1}\| + (1-\lambda)\|x_n - x_{n+1}\| \\ &\leq \lambda\|x_n - x_{n+1}\| + b\|x_{n+1} - x_{n+2}\| + c\|x_n - x_{n+1}\| \\ &\quad + (1-\lambda)\|x_n - x_{n+1}\| \text{ [using (3.9)]} \\ \Rightarrow \|x_{n+1} - x_{n+2}\| &\leq \frac{1+c}{1-b}\|x_n - x_{n+1}\|. \end{aligned} \quad (3.10)$$

Again, we have

$$\begin{aligned} &\lambda(Tx_n - Tx_{n+1}) = (x_{n+1} - x_{n+2}) + (\lambda - 1)(x_n - x_{n+1}) \\ &\quad \Rightarrow \lambda\|Tx_n - Tx_{n+1}\| \leq \|x_{n+1} - x_{n+2}\| + (1-\lambda)\|x_n - x_{n+1}\|. \end{aligned} \quad (3.11)$$

Therefore,

$$\begin{aligned} \|x_n - Tx_{n+1}\| &\leq \|x_n - Tx_n\| + \|Tx_n - Tx_{n+1}\| \\ &= \frac{1}{\lambda}\|x_n - x_{n+1}\| + \|Tx_n - Tx_{n+1}\| \\ \Rightarrow \lambda\|x_n - Tx_{n+1}\| &\leq \|x_n - x_{n+1}\| + \lambda\|Tx_n - Tx_{n+1}\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + (1-\lambda)\|x_n - x_{n+1}\| \text{ [using (3.11)]} \\ &\leq (2-\lambda)\|x_n - x_{n+1}\| + \frac{1+c}{1-b}\|x_n - x_{n+1}\| \text{ [using (3.10)]} \\ &< \left(\frac{3}{2} + \frac{2}{1-b}\right)\|x_n - x_{n+1}\|. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{2}\|x_n - Tx_{n+1}\| &\leq \lambda\|x_n - Tx_{n+1}\| \\ &\leq \left(\frac{3}{2} + \frac{2}{1-b}\right)\|x_n - x_{n+1}\|. \end{aligned}$$

Hence $\frac{1-b}{7}\|x_n - Tx_{n+1}\| < \|x_n - x_{n+1}\|$. Therefore, by given hypothesis, we get

$$\|Tx_n - Tx_{n+1}\| \leq \|x_n - x_{n+1}\|.$$

Thus, by Lemma 2.1, we have $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. So (x_n) is an AFPS of T. The fact that (x_n) is asymptotically regular can be easily seen from Theorem 3.1. □

Next, we prove a result concerning the existence of fixed points of such mappings using Theorem 3.5.

Theorem 3.6. *Suppose that all the conditions of Theorem 3.5 are satisfied. Further, assume that for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\|x - y\| + \|x - Ty\| + \|y - Tx\| < 3\epsilon + \delta \Rightarrow \|Tx - Ty\| \leq \frac{\epsilon}{2}. \tag{3.12}$$

Then T has a fixed point in C.

Proof. By Theorem 3.5, T has an AFPS (x_n), where $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$ and $\frac{1}{2} \leq \lambda < 1$. Here we take $\lambda = \frac{1}{2}$. So we get an approximate fixed point sequence (x_n) given by $x_{n+1} = \frac{1}{2}(Tx_n + x_n)$, and this sequence is asymptotically regular also. Next, we show that (x_n) is a Cauchy sequence. Let $\epsilon > 0$ be arbitrary. So there exists $\delta > 0$ such that Eq. (3.12) holds. Without loss of generality, we take $\delta < \epsilon$. Since, (x_n) is asymptotically regular, there exists $N \in \mathbb{N}$ such that

$$\|x_n - x_{n+1}\| < \frac{\delta}{4}$$

for all $n \geq N$. Next, we show by induction on p that

$$\|x_N - x_{N+p}\| < \epsilon \text{ for all } p \in \mathbb{N}. \tag{3.13}$$

Clearly Eq. (3.13) is true for $p = 1$. Let Eq. (3.13) be true for some $p \in \mathbb{N}$. Therefore,

$$\begin{aligned}
\|x_N - x_{N+p}\| &+ \|x_N - Tx_{N+p}\| + \|x_{N+p} - Tx_N\| \\
&\leq \|x_N - x_{N+p}\| + \|x_N - x_{N+p}\| \\
&\quad + \|x_{N+p} - Tx_{N+p}\| + \|x_{N+p} - x_N\| \\
&\quad + \|x_N - Tx_N\| \\
&= 3\|x_N - x_{N+p}\| + 2\|x_{N+p} - x_{N+p+1}\| \\
&\quad + 2\|x_N - x_{N+1}\| \\
&< 3\epsilon + \delta.
\end{aligned}$$

From Eq. (3.12), we get

$$\|Tx_N - Tx_{N+p}\| \leq \frac{\epsilon}{2}.$$

Again by the formation of (x_n), we get

$$\|x_{N+p+1} - x_{N+1}\| \leq \frac{1}{2}\|Tx_{N+p} - Tx_N\| + \frac{1}{2}\|x_{N+p} - x_N\| < \frac{3\epsilon}{4}.$$

Thus

$$\begin{aligned}
\|x_N - x_{N+p+1}\| &\leq \|x_N - x_{N+1}\| + \|x_{N+1} - x_{N+p+1}\| \\
&< \frac{\delta}{4} + \frac{3\epsilon}{4} < \epsilon.
\end{aligned}$$

Therefore, Eq. (3.13) is true for $p+1$. So Eq. (3.13) is true for all p. Continuing in a similar manner, we can show that

$$\|x_n - x_{n+p}\| < \epsilon \text{ for all } n \in \mathbb{N} \text{ and for all } p \in \mathbb{N}.$$

Therefore, (x_n) is a Cauchy sequence and hence convergent to some $z \in C$. Again,

$$\begin{aligned}\|Tx_n - Tx_m\| &\le a\|x_n - x_m\| + b\|x_n - Tx_m\| + c\|x_m - Tx_n\| \\ &\le a\|x_n - x_m\| + b\|x_n - x_m\| + b\|x_m - Tx_m\| + c\|x_m - x_n\| \\ &\quad + c\|x_n - Tx_n\| \\ &= \|x_n - x_m\| + b\|x_m - Tx_m\| + c\|x_n - Tx_n\| \to 0 \text{ as } n, m \to \infty.\end{aligned}$$

Therefore, (Tx_n) is a Cauchy sequence in C. Also, since $x_{n+1} = \frac{1}{2}(Tx_n + x_n)$, we have that $Tx_n = 2x_{n+1} - x_n \to z$ as $n \to \infty$. Again,

$$\begin{aligned}\|z - Tz\| &\le \|z - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tz\| \\ &\le \|z - x_n\| + \|x_n - Tx_n\| + a\|x_n - z\| + b\|x_n - Tz\| + c\|z - Tx_n\|.\end{aligned}$$

Letting $n \to \infty$ in above inequality, we get

$$\|z - Tz\| \le b\|z - Tz\|,$$

which gives $z = Tz$, i.e., z is a fixed point of T. □

In the next theorem, we check the closedness property of the fixed point set of Chatterjea type nonexpansive mappings.

Theorem 3.7. *Let X be a Banach space and C be a non-empty subset of X. If T is a Chatterjea type nonexpansive mapping on C with coefficients (a, b, c) such that $b < 1$, then $Fix(T)$ is a closed subset of C.*

Proof. Since T is Chatterjea type nonexpansive mapping with coefficients (a, b, c), we have $a, b, c \ge 0$ with $a + b + c = 1$ and

$$\|Tx - Ty\| \le a\|x - y\| + b\|x - Ty\| + c\|y - Tx\| \tag{3.14}$$

for all $x, y \in C$. Let (z_n) be a sequence in $Fix(T)$ converging to some $z \in C$. Then, we have

$$\begin{aligned}&\|Tz_n - Tz\| \le a\|z_n - z\| + b\|z_n - Tz\| + c\|z - Tz_n\| \\ \Rightarrow\ &\|z_n - Tz\| \le a\|z_n - z\| + b\|z_n - Tz\| + c\|z - z_n\|.\end{aligned}$$

Taking limit as $n \to \infty$ in above inequality, we get $\|z - Tz\| \le b\|z - Tz\|$. So $z = Tz$, i.e., $z \in Fix(T)$ and hence $Fix(T)$ is a closed set. □

In the upcoming theorem, we check the convexity property of the fixed point set of Chatterjea type nonexpansive mappings.

Theorem 3.8. *Let X be a Hilbert space and C be a non-empty subset of X. If T is a Chatterjea type nonexpansive mapping on C with coefficients (a, b, c) such that $c < 1$, then $Fix(T)$ is a convex subset of C.*

Proof. Let $x, y \in Fix(T)$ be any two points and take $z = \lambda x + (1 - \lambda)y$, where λ is a scalar with $0 \le \lambda \le 1$. Then we have

$$\begin{aligned}&\|Tz - Tx\| \le a\|z - x\| + b\|z - Tx\| + c\|x - Tz\| \\ \Rightarrow\ &\|Tz - x\| \le a\|z - x\| + b\|z - x\| + c\|x - Tz\| \\ \Rightarrow\ &(1 - c)\|Tz - x\| \le (a + b)\|z - x\| \\ \Rightarrow\ &\|Tz - x\| \le \|z - x\|.\end{aligned}$$

Next, using parallelogram law we have

$$\left\|\frac{z-x}{2}+\frac{Tz-x}{2}\right\|^2+\left\|\frac{z-x}{2}-\frac{Tz-x}{2}\right\|^2=2\left(\left\|\frac{z-x}{2}\right\|^2+\left\|\frac{Tz-x}{2}\right\|^2\right)$$

$$\begin{aligned}\Rightarrow\left\|\frac{z-x}{2}+\frac{Tz-x}{2}\right\|^2+\frac{1}{4}\|z-Tz\|^2&=\frac{1}{2}\|z-x\|^2+\frac{1}{2}\|Tz-x\|^2\\&\leq\frac{1}{2}\|z-x\|^2+\frac{1}{2}\|z-x\|^2\\\Rightarrow\left\|\frac{z-x}{2}+\frac{Tz-x}{2}\right\|^2&\leq\|z-x\|^2-\frac{1}{4}\|z-Tz\|^2\\\Rightarrow\left\|\frac{z+Tz}{2}-x\right\|^2&\leq(1-\lambda)^2\|x-y\|^2-\frac{1}{4}\|z-Tz\|^2.\end{aligned}$$

Similarly, we have

$$\left\|\frac{z+Tz}{2}-y\right\|^2\leq\lambda^2\|x-y\|^2-\frac{1}{4}\|z-Tz\|^2.$$

Now if $z\neq Tz$, then we have

$$\begin{aligned}\left\|\frac{z+Tz}{2}-x\right\|&<(1-t)\|x-y\|,\ \text{and}\\\left\|\frac{z+Tz}{2}-y\right\|&<t\|x-y\|.\end{aligned}$$

Then, we get

$$\begin{aligned}\|x-y\|&\leq\left\|\frac{z+Tz}{2}-x\right\|+\left\|\frac{z+Tz}{2}-y\right\|\\&<(1-t)\|x-y\|+t\|x-y\|\\&=\|x-y\|,\end{aligned}$$

which gives a contradiction. So we must have $z=Tz$, i.e., $z\in Fix(T)$. Therefore, $Fix(T)$ is a convex set. □

Next, we present the following example showing the existence of AFPS and of a fixed point for Reich type nonexpansive mappings.

Example 3.9. Let us consider the Banach space $\mathbb{R}$ equipped with the usual norm and take $C=[1,10]$. We define a function $T:C\to C$ by

$$Tx=\begin{cases}\frac{4}{3}x+8, & \text{if } 1\leq x\leq\frac{3}{2};\\10, & \text{if } x>\frac{3}{2}.\end{cases}$$

Choose, $a=\frac{1}{3}=b=c$.

Let $x,y\in C$ be arbitrary. Then the following three cases may arise:

Case I: Let $1\leq x,y\leq\frac{3}{2}$. Then, $Tx=\frac{4}{3}x+8$ and $Ty=\frac{4}{3}y+8$. Therefore,

$$\|Tx-Ty\|=\frac{4}{3}|x-y|\leq\frac{2}{3},$$

and

$$a\|x-y\|+b\|x-Tx\|+c\|y-Ty\| = \frac{1}{3}\left(|x-y|+8+\frac{x}{3}+8+\frac{y}{3}\right) \geq \frac{2}{3}.$$

Thus

$$\|Tx-Ty\| \leq a\|x-y\|+b\|x-Tx\|+c\|y-Ty\|.$$

Case II: Let $1 \leq x \leq \frac{3}{2}$ and $y > \frac{3}{2}$. Then, $Tx = \frac{4}{3}x+8$ and $Ty = 10$. Therefore,

$$\|Tx-Ty\| = \|\frac{4}{3}x-2\| = 2-\frac{4}{3}x,$$

and

$$\begin{aligned}&a\|x-y\|+b\|x-Tx\|+c\|y-Ty\|\\&=\frac{1}{3}\left(y-x+8+\frac{x}{3}+10-y\right) = \frac{1}{3}\left(18-\frac{2}{3}x\right).\end{aligned}$$

So,

$$\|Tx-Ty\| \leq a\|x-y\|+b\|x-Tx\|+c\|y-Ty\|.$$

Case III: Let $x, y > \frac{3}{2}$. Then, it is obvious that

$$\|Tx-Ty\| \leq a\|x-y\|+b\|x-Tx\|+c\|y-Ty\|.$$

Thus we see that

$$\|Tx-Ty\| \leq a\|x-y\|+b\|x-Tx\|+c\|y-Ty\|$$

for all $x, y \in C$, i.e., T is a Reich type nonexpansive mapping with coefficients $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. It is easy to note that T has an AFPS. Here the AFPS is (x_n), where $x_n = 10$ for all n. Also, T has a fixed point at $x = 10$.

Now we provide an example to validate Theorems 3.1 and 3.2.

Example 3.10. Let us consider the Banach space $\mathbb{R}$ equipped with the usual norm and take $C = [0, 1]$. Define a mapping $T : C \to C$ by $Tx = 1 - x$ for all $x \in C$. Here we choose $b = c = \frac{1}{2}$. Let $x, y \in C$ be two arbitrary points. Then the following three cases may arise:

Case I: Assume that $x, y \leq \frac{1}{2}$ and take $x > y$. Then

$$\begin{aligned}\|Tx-Ty\|-b\|x-Tx\|-c\|y-Ty\| &= (x-y)-\frac{1}{2}(1-2x)-\frac{1}{2}(1-2y)\\&= 2x-1 \leq 0\\&\Rightarrow \|Tx-Ty\| \leq b\|x-Tx\|+c\|y-Ty\|.\end{aligned}$$

Case II: Assume that $x, y > \frac{1}{2}$and take $x > y$. Then

$$\begin{aligned}\|Tx-Ty\|-b\|x-Tx\|-c\|y-Ty\| &= (x-y)-\frac{1}{2}(2x-1)-\frac{1}{2}(2y-1)\\&= 1-2y < 0\\&\Rightarrow \|Tx-Ty\| < b\|x-Tx\|+c\|y-Ty\|.\end{aligned}$$

Case III: Assume that $x \leq \frac{1}{2}$ and take $y > \frac{1}{2}$. Then

$$\begin{aligned}\|Tx - Ty\| - b\|x - Tx\| - c\|y - Ty\| &= (y - x) - \frac{1}{2}(1 - 2x) - \frac{1}{2}(2y - 1) \\ &= 0 \\ \Rightarrow \|Tx - Ty\| &= b\|x - Tx\| + c\|y - Ty\|.\end{aligned}$$

Thus T is a Reich type nonexpansive mapping with coefficients $(0, \frac{1}{2}, \frac{1}{2})$. Also, it can be easily verified that all the conditions of Theorems 3.1 and 3.2 hold good. So these two theorems, T has AFPS and fixed point. Indeed, the sequences $\left(\frac{1}{2} - \frac{1}{n}\right)$ and $\left(\frac{1}{2} + \frac{1}{n}\right)$ are approximate fixed point sequences of T, and $\frac{1}{2}$ is a fixed point of T.

Next, we present an example of a Chatterjea type nonexpansive mapping:

Example 3.11. Let us consider the Banach space $\mathbb{R}$ equipped with the usual norm, take $C = [0, 2]$ and define a mapping $T : C \to C$ by

$$Tx = \begin{cases} 2 & \text{if } x < \frac{1}{3}; \\ \frac{5}{3} & \text{if } x \geq \frac{1}{3}. \end{cases}$$

Choose $a = b = c = \frac{1}{3}$. Then for any $x, y \in C$, if $x, y < \frac{1}{3}$ or if $x, y \geq \frac{1}{3}$, it is obvious to check that

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\| + c\|y - Tx\|.$$

Next, suppose that $x < \frac{1}{3}$ and $y \geq \frac{1}{3}$. Then $\|Tx - Ty\| = \frac{1}{3}$ and

$$a\|x - y\| + b\|x - Ty\| + c\|y - Tx\| = \frac{1}{3}(y - x + \frac{5}{3} - x + 2 - y) \geq \frac{1}{3},$$

so

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\| + c\|y - Tx\|.$$

Therefore, T is a Chatterjea type nonexpansive mapping with coefficients $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Also, T has an AFPS $\left(\frac{5}{3} + \frac{1}{n+2}\right)_{n\in\mathbb{N}}$ and $Fix(T) = \{\frac{5}{3}\}$, which is obviously closed and convex.

Remark 3.12. Examples (3.9), (3.11) show that the classes of Reich and Chatterjea type nonexpansive mappings are larger than that of nonexpansive mappings. Now we will provide a sufficient condition under which class of Reich type nonexpansive mapping reduces to the class of nonexpansive mappings.

Theorem 3.13. *Let X be a finite dimensional Banach space, and C be nonempty subset of X. Let $T : C \to C$ be a Reich type nonexpansive mapping with coefficients (a, b, c) and assume that $diam(Fix(T)) > 0$. If $Fix(T)$ is centerable and contains its Chebyshev center, then T becomes nonexpansive.*

Proof. By Theorem 3.3, $Fix(T)$ is a closed subset of C. This implies that Fix(T) is compact as X is finite dimensional. So Fix(T) is M-compact. Since Fix(T) is centerable and contains its Chebyshev center, it follows from Lemma

6.1 of [19] that $Fix(T)$ attains its diameter. So there exists $z_1, z_2 \in Fix(T)$ such that $\|z_1 - z_2\| = \text{diam}(Fix(T)) > 0$. Therefore, we have

$$\begin{aligned}\|T(z_1) - T(z_2)\| &\leq a\|z_1 - z_2\| + b\|z_1 - T(z_1)\| + c\|z_2 - T(z_2)\| \\ &\Rightarrow \|z_1 - z_2\| \leq a\|z_1 - z_2\| \\ &\Rightarrow (1-a)\|z_1 - z_2\| = 0. \\ &\Rightarrow a = 1.\end{aligned}$$

This implies that $b = c = 0$. Thus T becomes nonexpansive. □

In Theorem 3.13, we give a sufficient condition under which the class of Reich type nonexpansive mapping reduces with the class of nonexpansive mappings, but it is still unknown when the class of Chatterjea type nonexpansive mappings reduces to the class of nonexpansive mappings. So. we propose the following open question:

Open question: Under what conditions does the class of Chatterjea type nonexpansive mappings reduce to the class of nonexpansive mappings?

In the following theorem, we give an interesting property of AFPS for Reich type and Chatterjea type nonexpansive mappings:

Theorem 3.14. *Let X be a finite dimensional Banach space and C be nonempty subset of X. Let $T : C \to C$ be a Reich type or a Chatterjea type nonexpansive mapping with coefficients (a, b, c) such that $\max\{b, c\} < 1$. If (x_n) is an AFPS for T, then for each $k \in \mathbb{N}, (T^k x_n)$ is an AFPS for T.*

Proof. We will prove this result by mathematical induction and we will divide the proofs into two cases:

Case 1: T is a Reich type nonexpansive mapping.

As $T : C \to C$ is a Reich type nonexpansive mappings, so for each $x, y \in C$, we have

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|, \tag{3.15}$$

where $a, b, c \geq 0,\ a + b + c = 1$ and $b, c < 1$. Taking $x = Tx_n$ and $y = x_n$ in Eq. (3.15), we get

$$\begin{aligned}&\|T(Tx_n) - Tx_n\| \leq a\|Tx_n - x_n\| + b\|Tx_n - T(Tx_n)\| + c\|x_n - Tx_n\| \\ &\Rightarrow \lim_{n\to\infty}(1-b)\|T(Tx_n) - Tx_n\| \leq 0 \ \left[\text{since } \lim_{n\to\infty}\|x_n - Tx_n\| = 0\right] \\ &\Rightarrow \lim_{n\to\infty}\|T(Tx_n) - Tx_n\| = 0 \text{ [since } b < 1].\end{aligned}$$

This shows that the sequence (Tx_n) is an AFPS for the mapping T. So the result is true for $k = 1$. Now assume that the result is true for $k = p$, i.e., the sequence $(T^p x_n)$ is an AFPS for the mapping T. Then putting $x = T^{p+1}x_n$ and $y = T^p x_n$ in Eq. (3.15), we get

$$\begin{aligned}&\|T(T^{p+1}x_n) - T^{p+1}x_n\| \leq a\|T^{p+1}x_n - T^p x_n\| + b\|T^{p+1}x_n - T^{p+2}x_n\| \\ &\qquad + c\|T^p x_n - T^{p+1}x_n\|. \\ &\Rightarrow \lim_{n\to\infty}(1-b)\|T(T^{p+1}x_n) - T^{p+1}x_n\| = 0 \text{ as } \lim_{n\to\infty}\|T^{p+1}x_n - T^p x_n\| = 0. \\ &\Rightarrow \lim_{n\to\infty}\|T(T^{p+1}x_n) - T^{p+1}x_n\| = 0 \text{ as } b < 1.\end{aligned}$$

This shows that the sequence $(T^{p+1}x_n)$ is an AFPS for the mapping T. So the result is true for $k = p + 1$. By mathematical induction we have for each $k \in \mathbb{N}$, the sequence $(T^k x_n)$ is an AFPS for the mapping T.

Case 2: T is a Chatterjea type nonexpansive mapping.

As $T : C \to C$ is a Chatterjea type nonexpansive mapping, so for each $x, y \in C$, we have

$$\|T(x) - T(y)\| \leq a\|x - y\| + b\|x - T(y)\| + c\|y - T(x)\|, \tag{3.16}$$

where $a, b, c \geq 0,\ a + b + c = 1$ and $b, c < 1$. At first, we show that the sequence (Tx_n) is an AFPS for the mapping T. Taking $x = Tx_n$ and $y = x_n$ in Eq. (3.16), we get

$$\begin{aligned}
&\|T(Tx_n) - Tx_n\| \leq a\|Tx_n - x_n\| + b\|Tx_n - Tx_n\| + c\|x_n - T(Tx_n)\|\\
&\quad \leq a\|Tx_n - x_n\| + c\|x_n - Tx_n\| + c\|Tx_n - T(Tx_n)\|\\
&\quad \Rightarrow \lim_{n\to\infty}(1 - c)\|T(Tx_n) - Tx_n\| = 0 \text{ as } \lim_{n\to\infty}\|x_n - Tx_n\| = 0\\
&\quad \Rightarrow \lim_{n\to\infty}\|T(Tx_n) - Tx_n\| = 0 \text{ as } c < 1.
\end{aligned}$$

This shows that the sequence (Tx_n) is an AFPS for the mapping T. So the result is true for $k = 1$. Next, we assume that the result is true for $k = p$, i.e., the sequence $(T^p x_n)$ is an AFPS for the mapping T. Then putting $x = T^{p+1}x_n$ and $y = T^p x_n$ in Eq. (3.16), we get

$$\begin{aligned}
&\|T(T^{p+1}x_n) - T^{p+1}x_n\| \leq a\|T^{p+1}x_n - T^p x_n\| + b\|T^{p+1}x_n - T^{p+1}x_n\|\\
&\qquad + c\|T^p x_n - T^{p+2}x_n\|\\
&\quad \leq a\|T^{p+1}x_n - T^p x_n\| + c\|T^p x_n - T^{p+1}x_n\|\\
&\qquad + c\|T^{p+1}x_n - T^{p+2}x_n\|\\
&\quad \Rightarrow \lim_{n\to\infty}(1 - c)\|T(T^{p+1}x_n) - T^{p+1}x_n\| = 0 \text{ as } \lim_{n\to\infty}\|T^p x_n - T^{p+1}x_n\| = 0\\
&\quad \Rightarrow \lim_{n\to\infty}\|T(T^{p+1}x_n) - T^{p+1}x_n\| = 0 \text{ as } c < 1.
\end{aligned}$$

This shows that the sequence $(T^{p+1}x_n)$ is an AFPS for the mapping T. So the result is true for $k = p + 1$. By mathematical induction we have for each $k \in \mathbb{N}$, the sequence $(T^k x_n)$ is an AFPS for the mapping T. □

In this part we will characterize the fixed point set $Fix(T)$ and the domain set C of the two newly introduced mappings.

Theorem 3.15. *Let X be a finite dimensional normed linear space, and C be a non-empty, closed, bounded subset of X. Let $T : C \to C$ be a Reich type or a Chatterjea type nonexpansive mapping with coefficients (a, b, c) such that $\max\{b, c\} < 1$. Then C is remotal but not uniquely remotal and $Fix(T)$ is remotal but may not be uniquely remotal.*

Proof. We will prove this result for Chatterjea type nonexpansive mappings. The proof for Reich type nonexpansive mappings follows similarly. Let $T : C \to C$ be a Chatterjea type nonexpansive mapping. As X is finite dimensional and C is closed and bounded, so from [18] we can conclude that C is remotal. But C cannot be uniquely remotal. Because C is a closed and bounded set in a finite dimensional space X, C is compact. If C is uniquely remotal then by Klee's result [9], C must be a singleton. Since C is not a singleton, the set C cannot be uniquely remotal.

For the fixed point set we have already proved in Theorem 3.7, that $Fix(T)$ is a closed set. Also since $Fix(T) \subset C$, $Fix(T)$ is bounded. Since X is finite dimensional, it follows that $Fix(T)$ is compact. So, from Theorem B of [18], we can conclude that $Fix(T)$ is remotal. But $Fix(T)$ may not be uniquely remotal because if $Fix(T)$ is uniquely remotal, then by Klee's result, the fixed point set $Fix(T)$ must be a singleton. But, the identity mapping $I : C \to C$ is a Chatterjea type nonexpansive mapping for $a = b = c = \frac{1}{3}$ and $Fix(I) = C$, which is not a singleton. □

Remark 3.16. Let X be a finite dimensional normed linear space, and C be non-empty, closed, bounded subset of X. Let $T : C \to C$ be a Reich type or a Chatterjea type nonexpansive mapping. If $Fix(T)$ is uniquely remotal, then T has a unique fixed point.

In the succeeding part, we prove a result regarding fixed points of farthest point map on a normed linear space.

Theorem 3.17. *Let C be a non-empty, bounded subset of a normed linear space X, and $T : C \to C$ be a mapping. If diam$(Fix(T)) > 0$ and $Fix(T)$ is uniquely remotal, then the farthest point map $F : X \to Fix(T)$ is fixed point free.*

Proof. Since $\text{diam}(Fix(T)) > 0$, $Fix(T)$ is non-empty, bounded subset of X and $Fix(T)$ contains more than one point. As $Fix(T)$ is uniquely remotal so the farthest point map $F : X \to Fix(T)$ is well defined. Suppose F has a fixed point. So there exists $x \in X$ such that $F(x) = x$. So $x \in \text{Fix}(T)$ and $\|x - F(x)\| = 0$. This implies $D(x, \text{Fix}(T)) = \sup\{\|x - a\| : a \in Fix(T)\} = 0$ as $\|x - F(x)\| = D(x, \text{Fix}(T))$. But this is not possible as $\text{diam}(Fix(T)) > 0$. So the farthest point map F is fixed point free. □

From the above theorem, we immediately have the following corollary:

Corollary 3.18. *Let X be a normed linear space and C be non-empty, bounded subset of X. Let $T : C \to C$ be Reich type or Chatterjea type nonexpansive mapping. If diam$(Fix(T)) > 0$ and $Fix(T)$ is uniquely remotal, then the farthest point map $F : X \to Fix(T)$ is fixed point free.*

In the upcoming lemma, we present a relation between AFPS and maximizing sequence.

Lemma 3.19. *Let C be non-empty, bounded, closed subset of a finite dimensional normed linear space X, and Let $T : C \to C$ be a mapping. If a sequence (x_n) in C is an AFPS for T, then (x_n) is maximizing if and only if (Tx_n) is maximizing.*

Proof. We first assume that (x_n) is maximizing. So there exists $x_0 \in X$ such that $\|x_n - x_0\| \to D(x_0, C) = \sup\{\|x_0 - a\| : a \in C\}$ as $n \to \infty$. Since C is closed and bounded set in X and X is finite dimensional, it follows from Theorem B of [18] that C is remotal. So there exists $c_0 \in C$ such that $\|x_0 - c_0\| = D(x_0, C)$. So $\|x_n - x_0\| \to \|x_0 - c_0\|$ as $n \to \infty$. Now

$$\|Tx_n - x_0\| \leq \|Tx_n - x_n\| + \|x_n - x_0\|$$
$$\Rightarrow \lim_{n\to\infty} \|Tx_n - x_0\| \leq \|x_0 - c_0\| \text{ as } \lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Also

$$\|x_n - x_0\| \leq \|Tx_n - x_n\| + \|Tx_n - x_0\|$$
$$\Rightarrow \lim_{n\to\infty} \|x_n - x_0\| \leq \lim_{n\to\infty} \|Tx_n - x_0\| \text{ as } \lim_{n\to\infty} \|Tx_n - x_n\| = 0$$
$$\Rightarrow \|x_0 - c_0\| \leq \lim_{n\to\infty} \|Tx_n - x_0\|.$$

So we have

$$\lim_{n\to\infty} \|Tx_n - x_0\| = \|x_0 - c_0\| = D(x_0, C).$$

This proves that the sequence (Tx_n) is maximizing.
Next, suppose that (Tx_n) is maximizing. So there exists $x \in X$ such that $\|Tx_n - x\| \to D(x, C)$ as $n \to \infty$. As C is remotal so there exists $c \in C$ such that $\|x - c\| = D(x, C)$. So $\|Tx_n - x\| \to \|x - c\|$ as $n \to \infty$. Then,

$$\|x_n - x\| \leq \|Tx_n - x_n\| + \|Tx_n - x\|$$
$$\Rightarrow \lim_{n\to\infty} \|x_n - x\| \leq \|x - c\|.$$

Since $\|Tx_n - x\| \leq \|Tx_n - x_n\| + \|x_n - x\|$, we get that $\|x - c\| \leq \lim_{n\to\infty} \|x_n - x\|$. Thus, we have

$$\lim_{n\to\infty} \|x_n - x\| = \|x - c\| = D(x, C).$$

This proves that the sequence (x_n) is maximizing. □

The following corollary is an immediate consequence of the above lemma:

Corollary 3.20. *Let X be a finite dimensional normed linear space and C be non-empty, closed, bounded subset of X. Let $T : C \to C$ be a Reich type or Chatterjea type nonexpansive mapping. If a sequence (x_n) in C is an AFPS for T, then (x_n) is maximizing if and only if (Tx_n) is maximizing.*

In the upcoming result, we present a theorem relating to preserveness of a maximizing sequence and an AFPS.

Theorem 3.21. *Let X be a finite dimensional Banach space and C be non-empty, closed, bounded subset of X. Let $T : C \to C$ be a Reich type or Chatterjea type nonexpansive mapping with (a, b, c) such that $b, c < 1$. If a sequence (x_n) in C is an AFPS and maximizing for T , then for each $k \in \mathbb{N}$, the sequence $(T^k x_n)$ is also AFPS and maximizing.*

Proof. First let T be a Reich type nonexpansive mapping, and the sequence (x_n) be an AFPS and maximizing for T. Then from Theorem 3.14, we can say that the sequence (Tx_n) is an AFPS for T, and also from Corollary 3.20, we can say the sequence (Tx_n) is maximizing. So the result is true for $k = 1$. Now suppose the result is true for $k = p$. So the sequence $(T^p x_n)$ is an AFPS and maximizing for T. Similarly from Theorem 3.14 and Corollary 3.20, we can say that the sequence $(T^{p+1}x_n)$ is an AFPS and maximizing for T. Therefore, by mathematical induction we have for each $k \in \mathbb{N}$, the sequence $(T^k x_n)$ is an AFPS and maximizing for T.

The proof for the case of Chatterjea type nonexpansive mappings is similar to that of Reich type and so omitted. □

Finally, we prove a result concerning the preserving property of M-compactness of Reich type or Chatterjea nonexpansive mappings.

Theorem 3.22. *Let X be a finite dimensional normed linear space and C be non-empty, bounded subset of X. Let $T : C \to C$ be a Reich type or Chatterjea type nonexpansive mapping. If every sequence (x_n) in C is an AFPS for T and T is continuous, then C is M-compact if and only if $T(C)$ is M-compact.*

Proof. Let T be a Reich type nonexpansive mapping, and suppose that C is M-compact. Let (y_n) be a maximizing sequence in $T(C)$. So $y_n = Tx_n\ \forall\ n \in \mathbb{N}$, where $x_n \in C$. So (x_n) is a sequence in C and since (x_n) is an AFPS for T, by corollary 3.20, (x_n) is a maximizing sequence in C. As C is M-compact, so (x_n) has a convergent subsequence (x_{n_k}) in C. As T is continuous, so the sequence (Tx_{n_k}), which is a subsequence of the sequence (y_n), will converge to a point in $T(C)$. So the sequence (y_n) is compact. This proves that the set $T(C)$ is M-compact.

Conversely, suppose that the set $T(C)$ is M-compact. Let (x_n) be a maximizing sequence in C. Then (Tx_n) is a sequence in $T(C)$. Since the sequence (x_n) is an AFPS for T, by Corollary 3.20, we can say that the sequence (Tx_n) is a maximizing sequence in $T(C)$. As $T(C)$ is M-compact so (Tx_n) has a convergent subsequence (Tx_{n_k}) in $T(C)$. Since T is continuous (x_{n_k}) converges to a point in C. So the sequence (x_n) has a convergent subsequence in C. This proves that C is M-compact.

The proof for the case of Chatterjea type nonexpansive mappings is similar to the above proof and it is skipped again. □

Remark 3.23. The conditions in Theorem 3.22 are sufficient but not necessary, which can be seen from the Example 3.11. In Example 3.11, T is a Chatterjea type nonexpansive mapping, and $C = [0, 2]$ and $T(C) = \{2, \frac{5}{3}\}$ are M-compact subsets of $\mathbb{R}$. But for the mapping T, the sequence $(\frac{1}{n+3})$ is not an AFPS and also T is not continuous at $x = \frac{1}{3}$.

The conditions in Theorem 3.22 are sufficient but not necessary for Reich type nonexpansive mapping also, as can be seen from the Example (3.9). In Example (3.9), T is Reich type nonexpansive mapping and here $C = [1, 10]$ and $T(C) = [\frac{28}{3}, 10]$ are M-compact subsets of $\mathbb{R}$. But for the mapping T, the sequence $(\frac{3}{2} - \frac{1}{n})$ is not an AFPS but T is continuous.

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