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Caristi–Kirk and Oettli–Théra ball spaces and applications

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Abstract. Based on the theory of ball spaces introduced by Kuhlmann and Kuhlmann, we introduce and study Caristi–Kirk and Oettli–Théra ball spaces. We show that if the underlying metric space is complete, then these have a very strong property: every ball contains a singleton ball. This fact provides quick proofs for several results which are equivalent to the Caristi–Kirk fixed point theorem, namely Ekeland's variational principles, the Oettli–Théra theorem, Takahashi's theorem and the flower petal theorem.

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1. Introduction

1.1. General setting

The literature on complete metric spaces contains remarkable results such as the Theorem of Caristi and Kirk ([2] and [7]), Ekeland's principle ([4]), Takahashi's theorem ([17]) and the flower petal theorem ([16]). These theorems are known to be equivalent (see, e.g., [15,16] and Remark 3 below). Their statements can be found in Sect. 4.

In [15], Oettli and Théra introduced an alternative approach to the Caristi–Kirk Theorem and showed it to be equivalent to what was later (in publications such as [14]) called Oettli–Théra Theorem (Theorem 29 below).

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The concept of a ball space was first introduced by F.-V. and K. Kuhlmann in [10,11]. In [12], taking *balls* to be sets previously used by J.-P. Penot in [16], they provided a way to prove the Caristi–Kirk Fixed Point Theorem (FPT) using ball space techniques.

In this paper, the authors aim to further develop the language of ball spaces and describe new applications of their theory. Translating known results equivalent to the Caristi–Kirk FPT in the language of ball spaces opens up the possibility to employ the developed theory to prove them in a simple manner (see, Sects. 3 and 4). While Propositions 16 and 20 below also provide new statements equivalent to the Caristi–Kirk FPT, it is worth pointing out that the main idea of this paper is not simply to add another theorem to the set of equivalent results. Instead, the authors focus on employing the ball space framework to express a common principle which connects the already known equivalent theorems. This principle, whose core can be found in Theorem 2 below, sheds new light on the existing theory and helps to acquire a better understanding thereof. Finally, the authors wish to encourage the reader to discover and further explore the theory of ball spaces and to look for new possibilities to broaden the range of its applications.

1.2. Ball spaces

As in [12], by a *ball space* we mean a pair $(X, \mathcal{B})$, where X is a nonempty set and $\mathcal{B} \subseteq \mathcal{P}(X)$ is a nonempty family of nonempty subsets of X. An element $B \in \mathcal{B}$ is called a *ball*. If no confusion arises, we will write $\mathcal{B}$ in place of $(X, \mathcal{B})$ when speaking of a ball space.

A *nest of balls* in a ball space $\mathcal{B}$ is a nonempty family $\mathcal{N}$ of balls from $\mathcal{B}$ which is totally ordered by inclusion. We say that a ball space $\mathcal{B}$ is *spherically complete* if for every nest of balls $\mathcal{N} \subseteq \mathcal{B}$ we have $\bigcap \mathcal{N} \neq \emptyset$. Further details about ball spaces can be found in [3].

Definition 1. A ball space $(X, \mathcal{B})$ is *strongly contractive* if there is a function that associates with every $x \in X$ some ball $B_x \in \mathcal{B}$ such that, for every $x, y \in X$, the following conditions hold:

(1) $x \in B_x$;
(2) if $y \in B_x$, then $B_y \subseteq B_x$;
(3) if $y \in B_x \setminus \{x\}$, then $B_y \subsetneq B_x$.

This particular type of ball spaces has a remarkable property stated in the following theorem.

Theorem 2. *In every spherically complete, strongly contractive ball space every ball B_x contains a singleton ball. In other words, there exists $a \in B_x$ such that $B_a = \{a\}$.*

Proof. Let $\mathcal{B}$ be a strongly contractive, spherically complete ball space and $B_x \in \mathcal{B}$ any ball. Consider the family

$$\mathcal{A} = \{\mathcal{N} \subseteq \mathcal{P}(B_x) \mid \mathcal{N} \text{ is a nest of balls in } \mathcal{B}\}.$$

This family is partially ordered by inclusion and nonempty since $\{B_x\} \in \mathcal{A}$. If we have a chain of nests in $\mathcal{A}$, the union of that chain is again a nest of balls in $\mathcal{A}$, hence an upper bound of the chain. By Zorn's Lemma, we obtain the existence of a maximal nest $\mathcal{M} \in \mathcal{A}$. Since the space is spherically complete, there exists an element $a \in \bigcap \mathcal{M}$. Since $a \in B$ for every $B \in \mathcal{M}$, by condition (2) of Definition 1 also $B_a \subseteq B$ for every $B \in \mathcal{M}$ and so $B_a \subseteq \bigcap \mathcal{M}$. This means that $\mathcal{M} \cup \{B_a\}$ is a nest of balls in $\mathcal{A}$ which contains $\mathcal{M}$. By maximality of $\mathcal{M}$ we obtain $\mathcal{M} \cup \{B_a\} = \mathcal{M}$, i.e., $B_a \in \mathcal{M}$. Now we wish to show that B_a is a singleton. Suppose that there exists an element $b \in B_a \setminus \{a\}$. Then $B_b \subsetneq B_a$ (in particular, $B_a \not\subseteq B_b$) and so $B_b \notin \mathcal{M}$. But this means that $\mathcal{M} \cup \{B_b\}$ is a nest of balls that properly contains $\mathcal{M}$, which contradicts the maximality of $\mathcal{M}$. Therefore, $B_a = \{a\}$. $\square$

Remark 3. In the proof of Theorem 2, we have used Zorn's Lemma. Our theorem will play a crucial role in the proofs of the results mentioned at the beginning of the introduction. Nevertheless, it is known that some of these theorems (e.g., Caristi–Kirk FPT and basic Ekeland's variational principle) can be proved without employing the full axiom of choice AC (see [5,9] and also Remark 17 below). In fact, their proofs can be carried out within the Zermelo–Fraenkel axiom system ZF plus the axiom of dependent choice DC. The latter is a weaker form of AC which covers the usual mathematical induction but not transfinite induction. It is worth noting that some authors (e.g., [6,13]) claim that the Caristi–Kirk FPT can be proved within ZF only. This implies that the equivalence between basic Ekeland's variational principle and the Caristi–Kirk FPT, which holds in ZF+DC, does *not* hold in ZF, as the former implies DC in ZF [8]. However, in view of the number of possible applications in settings other than metric spaces (see [3]), it is reasonable to expect that several of them will require AC.

2. Caristi–Kirk and Oettli–Théra ball spaces

In this section, we will be working with a nonempty metric space (X, d).

2.1. Caristi–Kirk ball spaces

Consider a function $\varphi : X \to \mathbb{R}$, a point $x \in X$ and the following set:

$$B_x^\varphi = \{y \in X \mid d(x, y) \leq \varphi(x) - \varphi(y)\}.$$

Since $B_x^\varphi \neq \emptyset$ (see (1) of Lemma 4 below), we may think of this set as a ball and consider the ball space $(X, \mathcal{B}^\varphi)$, where

$$\mathcal{B}^\varphi := \{B_x^\varphi \mid x \in X\}.$$

We will call the function φ a *Caristi–Kirk function on* X if it is lower semicontinuous, that is,

$$\forall_{y \in X} \liminf_{x \to y} \varphi(x) \geq \varphi(y),$$

and bounded from below, that is,

$$\inf_{x \in X} \varphi(x) > -\infty.$$

The corresponding balls B_x^φ had appeared in [16] and were later given the name *Caristi–Kirk balls* in [12]. The collection $\mathcal{B}^\varphi$ is the induced *Caristi–Kirk ball space*. For brevity, we will write “CK” in place of “Caristi–Kirk”.

A number of remarkable properties of the balls defined above, given in the following lemma, can be found in [12, Lemma 5].

Lemma 4. *Take a metric space (X, d) and any function $\varphi : X \to \mathbb{R}$. Then the following assertions hold.*

(1) *For every $x \in X$, $x \in B_x^\varphi$.*
(2) *If $y \in B_x^\varphi$, then $B_y^\varphi \subseteq B_x^\varphi$; if in addition $x \neq y$, then $B_y^\varphi \subsetneq B_x^\varphi$ and $\varphi(y) < \varphi(x)$.*
(3) *If φ is lower semicontinuous, then all CK balls B_x are closed in the topology induced by the metric.*

Lemma 4 immediately yields the following result.

Corollary 5. *The CK ball space $\mathcal{B}^\varphi$ is strongly contractive.*

Another important fact about CK ball spaces can also be found in [12, Proposition 3]:

Proposition 6. *Let (X, d) be a metric space. Then the following statements are equivalent:*

(i) *The metric space (X, d) is complete.*
(ii) *Every CK ball space $(X, \mathcal{B}^\varphi)$ is spherically complete.*
(iii) *For every continuous function $\varphi : X \to \mathbb{R}$ bounded from below, the CK ball space $(X, \mathcal{B}^\varphi)$ is spherically complete.*

2.2. Oettli–Théra ball spaces

The following notions can be found in [15].

Definition 7. We say that a function $\phi : X \times X \to (-\infty, +\infty]$ is an *Oettli–Théra function on X* if the following properties hold:

(a) $\phi(x, \cdot) : X \to (-\infty, +\infty]$ is lower semicontinuous for all $x \in X$;
(b) $\phi(x, x) = 0$ for all $x \in X$;
(c) $\phi(x, y) \leq \phi(x, z) + \phi(z, y)$ for all $x, y, z \in X$;
(d) there exists $x_0 \in X$ s.t. $\inf_{x \in X} \phi(x_0, x) > -\infty$.

If an element $x_0 \in X$ satisfies property (d), we will call it an *Oettli–Théra element for ϕ in X*. For brevity, we will write “OT” in place of “Oettli–Théra”.

Definition 8. Let ϕ be an OT function on X.

(i) For $x \in X$, we will call the following set an *OT ball*:
$$B_x^\phi := \{y \in X \mid d(x, y) \leq -\phi(x, y)\}.$$
(ii) By (1) of Lemma 10 below each OT ball is nonempty, so this gives rise to a ball space $(X, \mathcal{B}^\phi)$, where
$$\mathcal{B}^\phi := \{B_x \mid x \in X\}.$$

(iii) Fix any OT element $x_0 \in X$ and set

$$\mathcal{B}^{\phi}_{x_0} := \{B_x \mid x \in B_{x_0}\}.$$

We will refer to the ball space $(B_{x_0}, \mathcal{B}^{\phi}_{x_0})$ as the *OT ball space generated by* x_0.

If no confusion arises as to which OT function is considered, we will write B_x in place of B^{ϕ}_x.

In this subsection, if an OT element $x_0 \in X$ has been fixed, we will write for brevity B_0 in place of B_{x_0}.

It is worth noting that if we are given a CK function φ, we may define ϕ by

$$\phi(x, y) := \varphi(y) - \varphi(x). \tag{1}$$

The following fact is straightforward to prove.

Fact 9. *If φ is a CK function, then the function $\phi : X \times X \to \mathbb{R}$ defined in (1) is an OT function. Moreover, every $x \in X$ is an OT element for ϕ.*

As we know from Corollary 5, any CK ball space is strongly contractive. A similar result can be shown for OT ball spaces.

Lemma 10. *Take a metric space (X, d) and $\phi : X \times X \to \mathbb{R}$ a function satisfying (b) and (c) in Definition 7. Then the following assertions hold, for every $x \in X$.*

(1) *$x \in B_x$.*
(2) *If $y \in B_x$, then $B_y \subseteq B_x$.*
(3) *If $y \in B_x \backslash \{x\}$, then $B_y \subsetneq B_x$ and $\phi(x, y) < \phi(y, x)$.*

Proof. (1): Indeed, $d(x, x) = -\phi(x, x) = 0$.

(2): Take $y \in B_x$, i.e.,

$$d(x, y) \leq -\phi(x, y).$$

Take any $z \in B_y$, then

$$d(y, z) \leq -\phi(y, z).$$

By condition (c) for an OT function, we have

$$d(x, z) \leq d(x, y) + d(y, z) \leq -\phi(x, y) - \phi(y, z) \leq -\phi(x, z),$$

so $z \in B_x$ and, as a result, $B_y \subseteq B_x$.

(3): Let $y \in B_x$ and $y \neq x$. We wish to show that $x \notin B_y$. Suppose that $x \in B_y$. Then $d(y, x) \leq -\phi(y, x)$ and by conditions (b) and (c) for an OT function we obtain

$$0 < d(y, x) + d(x, y) \leq -\phi(y, x) - \phi(x, y) \leq -\phi(y, y) = 0,$$

contradiction. Thus, $x \notin B_y$ and so $B_y \subsetneq B_x$. Clearly, this also implies

$$-\phi(y, x) < d(x, y) \leq -\phi(x, y).$$

$\square$

Lemma 10 instantly yields the following corollary.

Corollary 11. *For an OT function ϕ on X, the ball space $\mathcal{B}^\phi$ is strongly contractive. Furthermore, for a fixed OT element x_0 for ϕ in X, the OT ball space $(B_0, \mathcal{B}^\phi_{x_0})$ is also strongly contractive and consists exactly of the OT balls which are contained in B_0.*

As stated in Fact 9, for the OT function ϕ defined in (1) every $x \in X$ is an OT element. While this does not have to be true in general for any OT function ϕ, this property turns out to be 'hereditary' in the following sense.

Lemma 12. *Let ϕ be an OT function. If $x_0 \in X$ is an OT element for ϕ in X and $x \in B_0$, then also x is an OT element for ϕ in X.*

Proof. Let $r \in \mathbb{R}$ be such that

$$\inf_{y \in X} \phi(x_0, y) \geq r.$$

Take any $x \in B_0$. Note that $0 \leq d(x_0, x) \leq -\phi(x_0, x)$. For every $y \in X$, we have

$$r \leq \phi(x_0, y) \leq \phi(x_0, x) + \phi(x, y),$$

so

$$\phi(x, y) \geq r - \phi(x_0, x).$$

In particular,

$$\inf_{y \in X} \phi(x, y) \geq r - \phi(x_0, x) \geq r.$$

□

As stated in Proposition 6, there is an equivalence between completeness of a metric space and spherical completeness of the respective CK ball spaces. A similar result can be shown for the OT ball spaces. For that we will need to state an auxiliary lemma first.

Lemma 13. *Let (X, d) be a metric space, ϕ an OT function on X and x_0 an OT element for ϕ in X. Moreover, let $\mathcal{N} \subseteq \mathcal{B}^\phi_{x_0}$ be a nest of balls and write $\mathcal{N} = \{B_x \mid x \in A\}$ for some set $A \subseteq B_0$. Then for every $x, y \in A$, we have*

$$d(x, y) \leq |\phi(x_0, x) - \phi(x_0, y)|. \tag{2}$$

Moreover, the following statements are equivalent for every $x, y \in A$:

(i) $y \in B_x$,
(ii) $\phi(x, y) \leq \phi(y, x)$,
(iii) $\phi(x_0, y) \leq \phi(x_0, x)$.

Proof. For every $x, y \in A$ either $x \in B_y$ or $y \in B_x$ since $\mathcal{N}$ is a nest, so

$$d(x, y) \leq \max\{-\phi(x, y), -\phi(y, x)\}. \tag{3}$$

If the above maximum is equal to $-\phi(x, y)$, we have

$$d(x, y) \leq -\phi(x, y) \leq \phi(x_0, x) - \phi(x_0, y) \leq |\phi(x_0, x) - \phi(x_0, y)|.$$

Similarly, if the maximum is equal to $-\phi(y, x)$, we have

$$d(x, y) \leq -\phi(y, x) \leq \phi(x_0, y) - \phi(x_0, x) \leq |\phi(x_0, x) - \phi(x_0, y)|.$$

Either way, we deduce (2).

To prove $(i) \Leftrightarrow (ii)$ assume that $y \in B_x$. If $y = x$, then (ii) is trivial. If $y \neq x$, then by assertion (3) of Lemma 10, we have

$$-\phi(y,x) < -\phi(x,y).$$

Hence, (ii) follows. Conversely, if $y \notin B_x$ (in particular, $y \neq x$), then $x \in B_y \backslash \{y\}$. As a result, again by assertion (3) of Lemma 10, $-\phi(x,y) < -\phi(y,x)$.

To prove $(i) \Leftrightarrow (iii)$ we proceed as follows. If $y \in B_x$, then

$$0 \leq d(x,y) \leq -\phi(x,y) \leq -\phi(x_0,y) + \phi(x_0,x),$$

thus $\phi(x_0,x) \geq \phi(x_0,y)$. For the converse, if $y \notin B_x$, then $x \in B_y$ and so

$$0 < d(x,y) \leq -\phi(y,x) \leq -\phi(x_0,x) + \phi(x_0,y),$$

hence $\phi(x_0,x) < \phi(x_0,y)$. □

Proposition 14. *Let (X,d) be a metric space. Then the following statements are equivalent:*

(i) *The metric space (X,d) is complete.*
(ii) *The OT ball space $(B^{\phi}_{x_0}, \mathcal{B}^{\phi}_{x_0})$ is spherically complete for every OT function ϕ on X and every OT element x_0 for ϕ in X.*

Proof. Suppose that for every OT function ϕ and every OT element x_0 for ϕ in X the ball space $(B_0, \mathcal{B}^{\phi}_{x_0})$ is spherically complete. We wish to show that the ball space $(X, \mathcal{B}^{\varphi})$ is spherically complete for every CK function φ on X, which by Proposition 6 will yield the completeness of the space X.

Take any CK function φ on X, consider the ball space $(X, \mathcal{B}^{\varphi})$ and fix any nest of balls $\mathcal{N}$ in $\mathcal{B}^{\varphi}$. Pick some $B^{\varphi}_{x_0} \in \mathcal{N}$ and consider the nest

$$\mathcal{N}_0 = \{B \in \mathcal{N} \mid B \subseteq B^{\varphi}_{x_0}\}.$$

By Fact 9, x_0 is an OT element for the OT function ϕ defined as in (1). Any ball in $\mathcal{N}_0$ is of the form $B^{\varphi}_x \subseteq B^{\varphi}_{x_0}$ for some $x \in X$. Therefore, since $B^{\varphi}_x = B^{\phi}_x$ for all $x \in X$, by Corollary 11, $\mathcal{N}_0$ is a nest in the ball space $(B_{x_0}, \mathcal{B}^{\phi}_{x_0})$. By assumption, we then obtain that $\emptyset \neq \bigcap \mathcal{N}_0 = \bigcap \mathcal{N}$.

For the converse, assume that X is complete. Fix any OT function ϕ on X and any OT element x_0 for ϕ in X. Take a nest of balls $\mathcal{N}$ in the ball space $\mathcal{B}^{\phi}_{x_0}$ and write $\mathcal{N} = \{B_x \mid x \in A\}$ for some set $A \subseteq B_0$. By assumption on x_0 there exists

$$r := \inf_{x \in A} \phi(x_0,x) \in \mathbb{R}.$$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements in A such that

$$\lim_{n\to\infty} \phi(x_0,x_n) = r.$$

Then $(\phi(x_0,x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence because it converges to r. By (2) of Lemma 13 the sequence $(x_n)_{n\in\mathbb{N}}$ is also Cauchy. Since X is complete, $(x_n)_{n\in\mathbb{N}}$ converges to an element $z \in X$. We want to show that $z \in \bigcap \mathcal{N}$ or, equivalently, that $z \in B_x$ for every $x \in A$. Fix an arbitrary element $x \in A$.

If $\phi(x_0, x) = r$ (in particular, the infimum is achieved), then applying (2) of Lemma 13 yields $x = z$, because

$$d(x_n, x) \leq |\phi(x_0, x_n) - \phi(x_0, x)| = |\phi(x_0, x_n) - r| \to 0,$$

showing that x is a limit of $(x_n)_{n\in\mathbb{N}}$. Hence, in this case, we obtain that $z \in B_x$ trivially. Therefore, we may assume that $\phi(x_0, x) > r$. Then from the definition of $(x_n)_{n\in\mathbb{N}}$ we obtain the existence of $N \in \mathbb{N}$ such that, for every $n \geq N$, we have $\phi(x_0, x_n) \leq \phi(x_0, x)$. This, by Lemma 13, is equivalent to $\phi(x, x_n) \leq \phi(x_n, x)$. Therefore, for every $n \geq N$,

$$d(x, x_n) \leq \max\{-\phi(x, x_n), -\phi(x_n, x)\} = -\phi(x, x_n),$$

where the first inequality is deduced similar to (3). Taking the limes superior on both sides and using that $\phi(x, \cdot)$ is lower semicontinuous, we obtain

$$d(x, z) \leq \limsup_{n\to\infty} -\phi(x, x_n) \leq -\phi(x, z),$$

so that $z \in B_x$. Since $x \in A$ was an arbitrary element, we conclude that $z \in \bigcap \mathcal{N}$ as claimed. $\square$

Remark 15. Proposition 14 does in general not hold for the ball space $\mathcal{B}^\phi$ in place of $\mathcal{B}^\phi_{x_0}$. Denote by $(\mathbb{N}_0, |\cdot|)$ the metric space of nonnegative integers with the standard metric. Define

$$\phi : \mathbb{N}_0 \times \mathbb{N}_0 \to (-\infty, +\infty]$$

as

$$\phi(n, m) = \begin{cases} n - m & \text{if } n > 0 \text{ or } n = m = 0, \\ +\infty & \text{if } 0 = n < m. \end{cases}$$

To show that ϕ is an OT function as in Definition 7 proceed as follows.

Condition (b) is asked in the definition of ϕ and $0 \in \mathbb{N}_0$ is an OT element for ϕ. Condition (a) follows from the fact that if $k \to m$ in $\mathbb{N}_0$, then $k = m$ ultimately. Thus, it suffices to show (c), that is,

$$\phi(n, m) \leq \phi(n, k) + \phi(k, m) \tag{4}$$

for all $n, m, k \in \mathbb{N}_0$. We will proceed by case distinction.

(1) Assume first that $n > 0$. If $k = 0$ and $m > 0$, then $\phi(k, m) = +\infty$. If $k = m = 0$, then $\phi(n, m) = \phi(n, k) + \phi(0, 0)$. If $k > 0$, then

$$\phi(n, k) + \phi(k, m) = n - k + k - m = n - m = \phi(n, m).$$

For all $m \in \mathbb{N}_0$. Thus, (4) holds in all cases.

(2) Assume now that $n = m = 0$. If $k = n = m = 0$, there is nothing to show. If $k > 0$, then $\phi(n, k) = +\infty$ and (4) holds.

(3) Finally, assume that $0 = n < m$. If $k = 0$, then $\phi(k, m) = +\infty$ and (4) follows. If $k > 0$, then $\phi(n, k) = +\infty$ and this implies (4).

Although ϕ is an OT function on the complete space $\mathbb{N}_0$, the ball space $(\mathbb{N}_0, \mathcal{B}^\phi)$ is not spherically complete. To see this, fix $n > 0$.

If $m < n$, then

$$|n - m| > 0 > m - n = -\phi(n, m),$$

which implies $m \notin B_n^\phi$. On the other hand, if $m \geq n$, then

$$|n-m| = m-n = -\phi(n,m),$$

so that $m \in B_n^\phi$. Therefore,

$$B_n^\phi = \{m \in \mathbb{N}_0 \;:\; |n-m| \leq -\phi(n,m)\} = \{m \in \mathbb{N}_0 \;:\; m \geq n\};$$

hence, the nest

$$\mathcal{N} = \{B_n^\phi \mid n > 0\}$$

has empty intersection.

Let us finally observe that the OT ball space $(B_{x_0}^\phi, \mathcal{B}_{x_0}^\phi)$ induced by the OT element $x_0 = 0$ is trivially spherically complete since it consists of only one singleton ball, that is, $\{x_0\}$.

The example above can be generalized to the metric space $(\mathbb{R}, |\cdot|)$ as well. One may consider the following OT function:

$$\phi(x,y) = \begin{cases} x-y & \text{if } x > 0 \text{ and } y \geq 0 \text{ or if } x = y, \\ +\infty & \text{otherwise.} \end{cases}$$

This yields $B_x^\phi = [x, +\infty)$ for $x > 0$ and $B_x^\phi = \{x\}$ for $x \leq 0$. We leave it to the interested reader to work out the details.

Armed with the theory introduced so far, we can prove an important property of OT and CK ball spaces in a complete metric space.

Proposition 16. *Let (X,d) be a complete metric space.*

(1) *If ϕ is an OT function on X, then for every OT element x_0 for ϕ in X, there exists an element $a \in B_0$ such that $B_a^\phi = \{a\}$.*
(2) *If φ is a CK function on X, then for every $x \in X$, there exists $a \in B_x^\varphi$ such that $B_a^\varphi = \{a\}$.*

Proof. Assertion (2) follows from assertion (1) by Fact 9. To prove assertion (1) let ϕ, x_0 and B_0 be as in the assumption of the Proposition. By Proposition 14, the OT ball space $\mathcal{B}_{x_0}^\phi$ is spherically complete, and by Corollary 11 it is strongly contractive. Theorem 2 yields the result. □

Remark 17. In what follows, we present a proof of the previous result without employing the full AC. This approach is analogous to the one given in [15], and it is purely metric in the sense of [9]. Nevertheless, DC is needed. Note that when proving $(i) \Rightarrow (ii)$ in Proposition 14, the reasoning was purely metric as well. Thus, we will apply this fact.

Assertion (2) follows from assertion (1) by Fact 9. Let $x_0 \in X$ be an OT element for ϕ and set

$$\gamma_0 := \inf_{x \in B_0} \phi(x_0, x) \in \mathbb{R}.$$

Inductively define a sequence $(x_n)_{n\in\mathbb{N}}$ of points of X and a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of nonpositive reals in the following way. For any $n \in \mathbb{N}$, denote by B_n the ball $B^{\phi}_{x_n}$. Assume that $x_{n-1} \in X$ and γ_{n-1} have been constructed. Choose $x_n \in B_{n-1}$ such that

$$\phi(x_{n-1}, x_n) \leq \gamma_{n-1} + \frac{1}{n}. \tag{5}$$

Recall that elements of B_0 are OT elements by Lemma 12, so we can put

$$\gamma_n := \inf_{x \in B_n} \phi(x_n, x) \geq \inf_{x \in X} \phi(x_n, x) \in \mathbb{R}.$$

Since $x_n \in B_n$ and $\phi(x_n, x_n) = 0$, we find that $\gamma_n \leq 0$. Corollary 11 yields $B_n \subseteq B_{n-1}$ for every $n \in \mathbb{N}$. Since X is complete, by Proposition 14 there exists $z \in \bigcap_{n\in\mathbb{N}} B_n$. We are going to show that this intersection is a singleton. We have

$$\begin{aligned}
0 \geq \gamma_n &= \inf_{x \in B_n} \phi(x_n, x) \\
&\geq \inf_{x \in B_n} \big(\phi(x_{n-1}, x) - \phi(x_{n-1}, x_n)\big) \\
&= \big(\inf_{x \in B_n} \phi(x_{n-1}, x)\big) - \phi(x_{n-1}, x_n) \\
&\geq \big(\inf_{x \in B_{n-1}} \phi(x_{n-1}, x)\big) - \phi(x_{n-1}, x_n) \\
&= \gamma_{n-1} - \phi(x_{n-1}, x_n) \geq -\frac{1}{n}.
\end{aligned}$$

For any $n \in \mathbb{N}$ and any $x \in B_n$ we have that

$$d(x, x_n) \leq -\phi(x_n, x) \leq -\gamma_n \leq \frac{1}{n},$$

so the radii of the sets B_n approach 0 as $n \to \infty$. This shows that

$$\bigcap_{n\in\mathbb{N}} B_n = \{z\}.$$

From part (2) of Lemma 10, it follows that $B_z \subseteq B_n$ for every $n \in \mathbb{N}$, so B_z is a singleton ball.

2.3. Generalized Caristi–Kirk ball spaces

Consider a function $\tilde{\varphi} : X \to (-\infty, +\infty]$ which is lower semicontinuous, bounded from below and not constantly equal to $+\infty$. We will call such $\tilde{\varphi}$ a *CK*$^\infty$ *function on* X. In this setting, we may define the *CK*$^\infty$ *balls* as follows:

$$B^{\tilde{\varphi}}_x := \{x \in X \mid \tilde{\varphi}(y) + d(x, y) \leq \tilde{\varphi}(x)\}.$$

If an element $x_0 \in X$ satisfies $\tilde{\varphi}(x) < +\infty$, we will call it a *CK element for* $\tilde{\varphi}$ *in* X (or simply a *CK element*).

An easy observation is that every CK function is a CK$^\infty$ function. However, for a CK$^\infty$ function $\tilde{\varphi}$, setting

$$\phi(x, y) := \tilde{\varphi}(y) - \tilde{\varphi}(x) \tag{6}$$

as we did in the CK case (1), may not make sense. Indeed, in the case $\tilde{\varphi}(x) = +\infty = \tilde{\varphi}(y)$ there is no natural choice for the value of $\phi(x, y)$.

In this subsection, if a CK element x_0 is fixed, we will write B_0 in place of $B^{\tilde{\varphi}}_{x_0}$.

For a CK element x_0, we define the CK$^\infty$ *ball space generated by* x_0 as the ball space $(B_0, \mathcal{B}^{\tilde{\varphi}}_{x_0})$, where

$$\mathcal{B}^{\tilde{\varphi}}_{x_0} := \{B^{\tilde{\varphi}}_x \mid x \in B_0\}.$$

Note that in general the ball space $\{B^{\tilde{\varphi}}_x \mid x \in X\}$ is not strongly contractive. Indeed, if $x, y \in X$, $x \neq y$, satisfy $\tilde{\varphi}(x) = \tilde{\varphi}(y) = +\infty$, then $B^{\tilde{\varphi}}_x = X = B^{\tilde{\varphi}}_y$. However, if we work inside a CK$^\infty$ ball space, strong contractiveness holds, as stated in the following lemma.

Lemma 18. *Take a metric space (X, d) and any function $\tilde{\varphi} : X \to (-\infty, +\infty]$. The following assertions hold for every $x \in X$.*

(1) *$x \in B^{\tilde{\varphi}}_x$.*
(2) *If $y \in B^{\tilde{\varphi}}_x$, then $B^{\tilde{\varphi}}_y \subseteq B^{\tilde{\varphi}}_x$ and $\tilde{\varphi}(y) \leq \tilde{\varphi}(x)$.*
(3) *Let $x \in X$ be such that $\tilde{\varphi}(x) < +\infty$ and let $y \in B^{\tilde{\varphi}}_x \backslash \{x\}$. Then $B^{\tilde{\varphi}}_y \subsetneq B^{\tilde{\varphi}}_x$ and $\tilde{\varphi}(y) < +\infty$.*

In particular, if x_0 is a CK element for $\tilde{\varphi}$, then for every $y \in B_0$, y is also a CK element for $\tilde{\varphi}$. Further, $\tilde{\varphi}|_{B_0}$ is a CK function and $(B_0, \mathcal{B}^{\tilde{\varphi}}_{x_0})$ is a CK ball space in the sense of Sect. 2.1.

Proof. (1): Indeed, $\tilde{\varphi}(x) + d(x, x) = \tilde{\varphi}(x)$.
(2): If $\varphi(x) = +\infty$, then $B^{\tilde{\varphi}}_x = X$ and $B^{\tilde{\varphi}}_y \subseteq B^{\tilde{\varphi}}_x$ as well as $\tilde{\varphi}(y) \leq \tilde{\varphi}(x)$ trivially. Now assume that $\tilde{\varphi}(x) < +\infty$ and $y \in B^{\tilde{\varphi}}_x$. Then also $\tilde{\varphi}(y) < +\infty$ because

$$\tilde{\varphi}(y) \leq \tilde{\varphi}(x) - d(x, y) \leq \tilde{\varphi}(x) < +\infty.$$

This implies the last assertions of the statement. Hence, (3), as well as the rest of (2), follow from Lemma 4. $\square$

Lemma 19. *For every $x \in X$ and every CK$^\infty$ function $\tilde{\varphi}$, the ball $B^{\tilde{\varphi}}_x$ is closed in the topology induced by the metric.*

Proof. The complement $\{y \in X \mid d(x, y)+\tilde{\varphi}(y) > \tilde{\varphi}(x)\}$ of $B^{\tilde{\varphi}}_x$ is the preimage of the final segment $(\tilde{\varphi}(x), +\infty]$ of $(-\infty, +\infty]$, which is open in the Scott topology, under the function

$$X \ni y \xmapsto{\psi} d(x, y) + \tilde{\varphi}(y).$$

Whenever $\tilde{\varphi}$ is lower semicontinuous, then so is ψ and this preimage is open in X. $\square$

We are now ready to prove a result analogous to Propositions 6 and 16.

Proposition 20. *Let (X, d) be a complete metric space and $\tilde{\varphi}$ be a CK$^\infty$ function on X. If $x_0 \in X$, then there exists $a \in B^{\tilde{\varphi}}_{x_0}$ such that $B^{\tilde{\varphi}}_a = \{a\}$.*

Proof. Consider a complete metric space (X, d), fix any element $x_0 \in X$ and consider the ball $B_0 := B^{\tilde{\varphi}}_{x_0}$.

Assume first that x_0 is a CK element for $\tilde{\varphi}$ in X. As we know from Lemma 19, B_0 is closed; hence, complete. Moreover, the function $\varphi := \tilde{\varphi}|_{B_0}$ is a CK function. Note that for every $x \in B_0$ we have

$$\begin{aligned} B_x^{\varphi} &= \{y \in B_0 \mid d(x,y) \leq \varphi(x) - \varphi(y)\} \\ &\subseteq B_x^{\tilde{\varphi}} = \{y \in X \mid \tilde{\varphi}(y) + d(x,y) \leq \tilde{\varphi}(x)\}. \end{aligned}$$

We wish to show that the above sets are equal. By assertion (2) of Lemma 18 we know that $B_x^{\tilde{\varphi}} \subseteq B_0$. On B_0 we have $\varphi = \tilde{\varphi}$ so that the values of $\tilde{\varphi}$ are finite and $\tilde{\varphi}(y) + d(x,y) \leq \tilde{\varphi}(x)$ is equivalent to $d(x,y) \leq \varphi(x) - \varphi(y)$. This yields $B_x^{\tilde{\varphi}} \subseteq B_x^{\varphi}$.

Since φ is a CK function on a complete metric space B_0, we may apply assertion (2) of Proposition 16 to the CK ball space $(B_0, \mathcal{B}_{x_0}^{\varphi})$, to acquire $a \in B_0$ such that

$$\{a\} = B_a^{\varphi} = B_a^{\tilde{\varphi}}.$$

Assume now that $x_0 \in X$ is not a CK element for $\tilde{\varphi}$. Then we obtain that $B_0 = X$. Inside the ball B_0 we may thus find a CK element x_1 for $\tilde{\varphi}$ since by definition $\tilde{\varphi}$ is not constantly ∞. From what we have proved above, there exists $a \in B_{x_1}^{\tilde{\varphi}} \subseteq X = B_0$ such that $B_a^{\tilde{\varphi}} = \{a\}$. □

3. Applications of Proposition 16

In this section, we give simple proofs for a number of known theorems, in versions that involve OT functions, by applying Proposition 16. Note that the multivalued Caristi–Kirk FPT, Ekeland's principle and Takahashi's theorem have already been proved in the OT form in [15] using the Oettli–Théra theorem. The original versions of these theorems are listed in Sect. 4.

Theorem 21 (Caristi–Kirk FPT, OT form). *Let (X,d) be a complete metric space and ϕ an OT function on X. If a function $f : X \to X$ satisfies*

$$\underset{x \in X}{\forall} \; d(x, f(x)) \leq -\phi(x, f(x)), \tag{7}$$

then f has a fixed point on X, i.e., there exists an element $a \in X$ such that $f(a) = a$.

Proof. Condition (7) implies that for every $x \in X$ we have

$$f(x) \in B_x.$$

Proposition 16 gives us the existence of $a \in X$ such that $B_a = \{a\}$. In particular, since $f(a) \in B_a$, we have $f(a) = a$. □

Theorem 22 (Caristi–Kirk FPT, multivalued version, OT form). *Let (X,d) be a complete metric space and ϕ an OT function on X.*

If a function $F : X \to \mathcal{P}(X)$ satisfies:

$$\underset{x \in X}{\forall} \; \underset{y \in F(x)}{\exists} \; d(x,y) \leq -\phi(x,y), \tag{8}$$

then F has a fixed point on X, i.e., there exists $a \in X$ such that $a \in F(a)$.

Proof. Condition (8) means that for every $x \in X$ there exists $y \in F(x) \cap B_x$. In particular, for $x = a$ with $a \in X$ given by Proposition 16 we obtain

$$y \in F(a) \cap B_a \subseteq \{a\},$$

whence $a = y \in F(a)$. □

Theorem 23 (Basic Ekeland's principle, OT form). *Let* (X, d) *be a complete metric space and* ϕ *an OT function on* X*. There exists* $a \in X$ *such that*

$$\underset{x \in X \setminus \{a\}}{\forall} \quad -\phi(a, x) < d(a, x). \tag{9}$$

Proof. Property (9) is equivalent to $B_a = \{a\}$ and the existence of such $a \in X$ follows from Proposition 16. □

Theorem 24 (Altered Ekeland's principle, OT form). *Let* (X, d) *be a complete metric space and* ϕ *an OT function on* X*. For any* $\gamma > 0$ *and any OT element* x_0 *for* ϕ *in* X *there exists* $a \in X$ *such that*

$$\underset{x \in X \setminus \{a\}}{\forall} \quad -\phi(a, x) < \gamma d(a, x) \tag{10}$$

and

$$-\phi(x_0, a) \geq \gamma d(x_0, a). \tag{11}$$

Proof. Since $\gamma > 0$, the function $\psi := \gamma^{-1}\phi$ is an OT function on X, so we can work with ψ and the respective ball space $\mathcal{B}^\psi$.

We apply Proposition 16 to the given complete metric space X, the function ψ and the ball $B_0 := B^\psi_{x_0}$. This gives us the existence of an element $a \in B_0$ such that $B^\psi_a = \{a\}$. Now, the assertion $a \in B_0$ means that

$$d(x_0, a) \leq -\psi(x_0, a) = -\gamma^{-1}\phi(x_0, a),$$

which is equivalent to property (11). Similarly, $B^\psi_a = \{a\}$ implies

$$\underset{x \in X \setminus \{a\}}{\forall} d(a, x) > -\psi(a, x) = -\gamma^{-1}\phi(a, x),$$

which is equivalent to property (10). □

Theorem 25 (Ekeland's usual variational theorem, OT form). *Let* (X, d) *be a complete metric space and* ϕ *an OT function on* X*. Fix* $\varepsilon \geq 0$ *and* $x_0 \in X$ *such that* $-\varepsilon \leq \inf_{x \in X} \phi(x_0, x)$*. Then for any* $\gamma > 0$ *and* $\delta \geq 0$ *with* $\gamma\delta \geq \varepsilon$ *there exists* $a \in X$ *such that* $d(a, x_0) \leq \delta$ *and* a *is the strict minimum point of the function* $\phi_\gamma : X \to (-\infty, +\infty]$ *defined as*

$$\phi_\gamma(x) = \phi(a, x) + \gamma d(x, a).$$

Proof. Take $\varepsilon \geq 0$ and x_0 as in the assumptions of the theorem, and fix arbitrary real numbers $\gamma > 0$ and $\delta \geq 0$ such that $\gamma\delta \geq \varepsilon$. The function $\psi := \gamma^{-1}\phi : X \times X \to (-\infty, +\infty]$ is an OT function on X, so we can apply Proposition 16 with the function ψ and $B_0 := B^\psi_{x_0}$ (note that x_0 is an OT element for ψ in X). We deduce the existence of $a \in B_0$ such that $B^\psi_a = \{a\}$. Now, the property $a \in B_0$ means that

$$d(x_0, a) \leq -\psi(x_0, a),$$

or in other words,

$$\gamma d(x_0, a) \leq -\phi(x_0, a) \leq -\inf_{x \in X} \phi(x_0, x) \leq \varepsilon \leq \gamma\delta.$$

Thus,

$$d(a, x_0) = d(x_0, a) \leq \delta.$$

The property $B_a = \{a\}$ means that for every $x \in X \backslash \{a\}$ we have that

$$d(x, a) > -\psi(a, x) = -\gamma^{-1}\phi(a, x).$$

From this, we obtain that

$$\phi_\gamma(x) = \phi(a, x) + \gamma d(x, a) > 0 = \phi_\gamma(a),$$

which means that a is the strict minimum point of the function ϕ_γ. $\square$

Definition 26. Let (X, d) be a metric space. Take $\gamma \in (0, \infty)$ and $a, b \in X$. The *petal associated with* γ *and* a, b is the subset $P_\gamma(a, b)$ of X defined as follows:

$$P_\gamma(a, b) = \{y \in X \mid \gamma d(y, a) + d(y, b) \leq d(a, b)\}.$$

Theorem 27 (Flower petal theorem). *Let M be a complete subset of a metric space (X, d). Take $x_0 \in M$ and $b \in X \backslash M$. Then for each $\gamma > 0$ there exists $a \in P_\gamma(x_0, b) \cap M$ such that*

$$P_\gamma(a, b) \cap M = \{a\}.$$

Proof. We use the notation from the assertion of the theorem. As $\gamma > 0$, the function $\varphi : M \to \mathbb{R}$ given by

$$\varphi(x) := \gamma^{-1} d(x, b)$$

is a CK function on M. In this setting we have, for every $x \in M$,

$$P_\gamma(x, b) \cap M = \{y \in M \mid d(x, y) \leq \varphi(x) - \varphi(y)\} = B_x^\varphi.$$

To conclude we use assertion (2) of Proposition 16 with M in place of X and $x := x_0$, which yields the existence of $a \in B_{x_0}^\varphi = P_\gamma(x_0, b) \cap M$ such that

$$\{a\} = B_a^\varphi = P_\gamma(a, b) \cap M.$$

$\square$

Theorem 28 (Takahashi, OT form). *Let (X, d) be a complete metric space, ϕ an OT function on X and $x_0 \in X$ an OT element for ϕ in X. Assume that for every $u \in B_{x_0}$ with $\inf_{x \in X} \phi(u, x) < 0$ there exists $v \in X$ such that $v \neq u$ and $d(u, v) \leq -\phi(u, v)$. Then there exists $a \in B_{x_0}$ such that $\inf_{x \in X} \phi(a, x) = 0$.*

Proof. Proposition 16 gives us the existence of $a \in B_{x_0}$ such that $B_a = \{a\}$. If $\inf_{x \in X} \phi(a, x) < 0$, then by assumption there would exist $v \in X \backslash \{a\}$ such that $d(a, v) \leq -\phi(a, v)$, which would mean that B_a is not a singleton, contradiction. So $\inf_{x \in X} \phi(a, x) \geq 0$, but $\phi(a, a) = 0$, which proves the claim. $\square$

Theorem 29 (Oettli–Théra). *Let (X, d) be a complete metric space, ϕ an OT function on X and $x_0 \in X$ an OT element for ϕ in X. Let $\Psi \subseteq X$ have the property that*

$$\underset{x \in B_{x_0} \backslash \Psi}{\forall} \; \underset{y \in X \backslash \{x\}}{\exists} \; d(x, y) \leq -\phi(x, y). \tag{12}$$

Then there exists $a \in B_{x_0} \cap \Psi$.

Proof. From Proposition 16 there exists $a \in B_{x_0}$ such that $B_a = \{a\}$. If $a \notin \Psi$, then by assumption B_a would contain another element $y \neq a$, which would mean that B_a is not a singleton, contradiction. $\square$

4. Applications of Proposition 20

Many of the theorems mentioned in the previous section have been originally stated and proved using the CK function φ. By Fact 9, proving the version involving ϕ, through (1), will also automatically prove the version involving φ. However, many sources (e.g., [1,16]) cite the theorems in a CK$^\infty$ form. As already remarked, we cannot directly define an OT function from a CK$^\infty$ function. Nevertheless, we can use Proposition 20 to prove these versions in the same way we did in the previous section using Proposition 16. Therefore, we leave the proofs to the reader.

Here we do not include the Oettli–Théra Theorem (since it has originally been stated only in the OT form) nor the Flower Petal Theorem (since it does not include any CK$^\infty$ function in its statement and proof).

For the following theorems, fix a complete metric space (X, d) and a CK$^\infty$ function $\tilde{\varphi}$ on X.

Theorem 30 (Caristi–Kirk FPT, *CK*$^\infty$ form). *If a function $f : X \to X$ satisfies*

$$\forall_{x\in X}\ \tilde{\varphi}(f(x)) + d(x, f(x)) \leq \tilde{\varphi}(x),$$

then f has a fixed point on X, i.e., there exists an element $a \in X$ such that $f(a) = a$.

Theorem 31 (Caristi–Kirk FPT, multivalued version, *CK*$^\infty$ form). *If a function $F : X \to \mathcal{P}(X)$ satisfies*

$$\forall_{x\in X}\ \exists_{y\in F(x)}\ \tilde{\varphi}(y) + d(x, y) \leq \tilde{\varphi}(x), \tag{13}$$

then F has a fixed point on X, i.e., there exists $a \in X$ such that $a \in F(a)$.

Theorem 32 (Basic Ekeland's principle, *CK*$^\infty$ form). *There exists $a \in X$ such that*

$$\forall_{x\in X\setminus\{a\}}\ \tilde{\varphi}(a) < \tilde{\varphi}(x) + d(a, x).$$

Theorem 33 (Altered Ekeland's principle, *CK*$^\infty$ form). *For all $\gamma > 0$ and any $x_0 \in X$ there exists $a \in X$ such that*

$$\forall_{x\in X\setminus\{a\}}\ \tilde{\varphi}(a) < \tilde{\varphi}(x) + \gamma d(a, x)$$

and

$$\tilde{\varphi}(a) \leq \tilde{\varphi}(x_0) - \gamma d(a, x_0).$$

Theorem 34 (Ekeland's usual variational theorem, *CK*$^\infty$ form). *Let $\varepsilon \geq 0$ and $x_0 \in X$ be such that $\tilde{\varphi}(x_0) \leq \inf \tilde{\varphi}(X) + \varepsilon$. Then for any $\gamma > 0$ and $\delta \geq 0$ with $\gamma\delta \geq \varepsilon$ there exists $a \in X$ such that $d(a, x_0) \leq \delta$ and a is the strict minimum point of the function*

$$\tilde{\varphi}_\gamma(x) = \tilde{\varphi}(x) + \gamma d(x, a).$$

Theorem 35 (Takahashi, CK^∞ form). *Suppose that for each $u \in X$ with $\inf_{x\in X} \tilde{\varphi}(x) < \tilde{\varphi}(u)$ there exists $v \in X$ such that $v \neq u$ and $\tilde{\varphi}(v) + d(u,v) \leq \tilde{\varphi}(u)$. Then there exists $a \in X$ such that $\inf_{x\in X} \tilde{\varphi}(x) = \tilde{\varphi}(a)$.*

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