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A variational principle and coupled fixed points

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Abstract. We generalize Ekeland's variational principle in partially ordered complete metric spaces on sets defined by functions with the mixed monotone property. We apply the result in giving an alternative proofs for coupled fixed points for mixed monotone maps in partially ordered complete metric spaces by using a variational technique. We succeed in generalizing some of the known results about coupled fixed points.

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1. Introduction and preliminaries

Ekeland proved a variational principle in [19]. In a series of articles [20,21] he enriches the results. Later he presented a more concise proof [22], which technique we will use. In the same article [22], various applications of the variational principle in different fields of mathematics are presented. Ekeland's variational principle has many applications and generalizations [4–6,9,12,16,33,35,46]. It is well known that fixed point theorems and variational principles are closely related [10,11,15,22,27,30].

Fixed point theorems, initiated by Banach's Contraction Principle has proved to be a powerful tool in nonlinear analysis. We cannot mention all kinds of generalizations of Banach's Contraction Principle. One direction for generalization of it is the notion of coupled fixed points [25], where mixed monotone maps in partially ordered by a cone Banach spaces are investigated. Later this idea was developed for mixed monotone maps in partially ordered metric spaces [8]. It is impossible to summarize all generalizations of the ideas of coupled fixed points, for mixed monotone maps, in partially ordered metric spaces. The investigation on the subject continuous as seen [1–3,13, 23,24,26,29,31,32,36,39,42–45,47], which is far from exhausting the most recent results. Another kind of maps considered in partially ordered complete metric spaces are for monotone maps without the mixed monotone property [17,18,28,40].

Let us mention that Ekeland's variational principle holds for any l.s.c maps $T : X \times X \to \mathbb{R}$, provided that X is a partially ordered complete metric space. Unfortunately, when investigating contraction type of maps $F : X \times X \to X$, satisfying the mixed monotone property in a partially ordered complete metric space $X \times X$. the contraction conditions holds only for part of the points $(x, y), (u, v) \in X \times X$. Thus we can not apply Ekeland's variational principle, as it is done in [22]. Therefore we will try to generalize Ekeland's variational principle on classes of subsets of partially ordered complete metric space $X \times X$, which need not to be compact or even closed, and then to apply it for the existence of coupled fixed points for maps that satisfy the mixed monotone property.

A similar approach was used in [37,38], where variational principles in partially ordered metric spaces were obtained and used to investigated problems, otherwise impossible to solve with the known variational principles.

Let (X, ρ) be a metric space. Following [9] an extended real valued function $T : X \to (-\infty, +\infty]$ on X is called lower semicontinuous (for short l.s.c) if $\{x \in X : f(x) > a\}$ is an open set for each $a \in (-\infty, +\infty]$. Equivalently T is l.s.c if and only if at any point $x_0 \in X$ there holds $\liminf_{x\to x_0} f(x) \geq f(x_0)$. A function T is called to be proper function, provided that $T \not\equiv +\infty$.

Following [8,25] let X be a set and let $\preceq$ be a partial order in X, then $(X, \preceq)$ is called a partially ordered set. We call two elements $x, y \in X$ comparable if either $x \preceq y$ or $y \preceq x$. We denote $x \succeq y$ if $y \preceq x$. We say that $x \prec$ if $x \preceq y$ but $x \neq y$. Let (X, ρ) be a metric space with a partial order $\preceq$, then the triple $(X, \rho, \preceq)$ is called a partially ordered metric space. Ran and Reurings in [41] initiate the fixed point theory in partially ordered metric spaces.

Definition 1.1 ([8,25]). Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$. The function F is said to have the mixed monotone property if

$$\begin{gathered}\text{for any } \ x_1, x_2, y \in X \ \text{ such that } \ x_1 \preceq x_2\\ \text{there holds } \ F(x_1, y) \preceq F(x_2, y)\end{gathered}$$

and

$$\begin{gathered}\text{for any } \ y_1, y_2, x \in X \ \text{ such that } \ y_1 \preceq y_2\\ \text{there holds } \ F(x, y_1) \succeq F(x, y_2).\end{gathered}$$

Definition 1.2 ([8,25]). Let $F : X \times X \to X$. An ordered pair $(x, y) \in X \times X$ is called coupled fixed point of F if $x = F(x, y)$ and $y = F(y, x)$.

Let $(X, \rho, \preceq)$ be a partially ordered complete metric space. We endow the product space $X \times X$ with the following partial order $(u, v) \preceq (x, y)$, provided that $x \succeq u$ and $y \preceq v$ hold simultaneously and with the following metric $d((x, y), (u, v)) = \rho(x, u) + \rho(y, v)$ for $(x, y), (u, v) \in X \times X$.

Everywhere for a partially ordered metric space $(X, \rho, \preceq)$ we will consider the product space $(X \times X, d, \preceq)$ endowed with the above-mentioned partial order and metric.

2. Main result

Just to fit some of the formulas in the text field we will use the notation $u = \big(u^{(1)}, u^{(2)}\big) \in X \times X$ and for any $u \in X \times X$ let us denote $\overline{u} = \big(u^{(2)}, u^{(1)}\big)$.

Theorem 2.1. *Let $(X, \rho, \preceq)$ be a partially ordered complete metric space, $(X \times X, d, \preceq)$ and $F : X \times X \to X$ be a continuous map with the mixed monotone property. Let*

$$V \times V = \{x = (x^{(1)}, x^{(2)}) \in X \times X : x^{(1)} \preceq F(x) \text{ and } x^{(2)} \succeq F(\overline{x})\} \neq \emptyset.$$

Let $T : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a proper, l.s.c, bounded from below function. Let $\varepsilon > 0$ be arbitrary chosen and fixed and let $u_0 \in V \times V$ be an ordered pair such that the inequality

$$T(u_0) \leq \inf_{V \times V} T(v) + \varepsilon \tag{2.1}$$

holds. Then there exists an ordered pair $x \in V \times V$, such that

(i) $T(x) \leq \inf_{u \in V \times V} T(u)$;

(ii) $d(x, u_0) \leq 1$;

(iii) *For every $w \in V \times V$ different from $x \in V \times V$ holds the inequality*

$$T(w) > T(x) - \varepsilon d(w, v).$$

Proof. Let us define inductively a sequence of ordered pairs $\{u_n\}_{n=0}^{\infty} \subset X \times X$, starting with the pair $u_0 \in V \times V$, that satisfies (2.1).

Suppose that we have already chosen $u_n \in V \times V$. There holds either:

(a) For every ordered pair $w \neq u_n$, $w \in V \times V$, holds the inequality $T(w) > T(u_n) - \varepsilon d(w, u_n)$;
or

(b) There exists $w \neq u_n$, $w \in V \times V$, so that there holds the inequality

$$T(w) \leq T(u_n) - \varepsilon d(w, u_n). \tag{2.2}$$

If case (a) holds, we choose $u_{n+1} = u_n$. In case of (b), let us denote by $S_n \subset V \times V$ the set of all ordered pairs $w \in V \times V$, which satisfy the inequality (2.2). We choose $u_{n+1} \in S_n$ so that

$$T(u_{n+1}) \leq \frac{T(u_n)}{2} + \frac{\inf_{v \in S_n} T(v)}{2}. \tag{2.3}$$

We claim that in both cases $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Indeed if case (a) ever occurs the sequence is stationary, starting from some index n. If case (a) does not occur for any index $n \in \mathbb{N}$, then it should be case (b) for all indexes $n \in \mathbb{N}$. Therefore by (2.2), we have the inequalities

$$d(u_k, u_{k+1}) \leq T(u_k) - T(u_{k+1})$$

for $k = 0, 1, 2, \ldots$. Summing up the above inequalities for k from n to $p - 1$, we get

$$\begin{aligned}\varepsilon d(u_n,u_p) &\le \sum_{k=n}^{p-1}\varepsilon d(u_k,u_{k+1})\\ &\le \sum_{k=n}^{p-1}(T(u_k)-T(u_{k+1}))=T(u_n)-T(u_p).\end{aligned} \tag{2.4}$$

From the inequality

$$T(u_{n+1})\le T(u_n)-\varepsilon d(u_n,u_{n+1})<T(u_n),$$

it follows that the sequence $\{T(u_n)\}_{n=0}^\infty$ is a decreasing one and bounded from below (by $\inf_{v\in V\times V}T(v)$). Hence it is convergent. So the right-hand side in (2.4) goes to zero, when n and p go to infinity simultaneously. Consequently, $\{u_n\}_{n=0}^\infty$ is a Cauchy sequence. Since $(X\times X,d)$ is a complete metric space (because (X,ρ) is complete), it follows that the sequence $\{u_n\}_{n=0}^\infty=\left\{(u_n^{(1)},u_n^{(2)})\right\}_{n=0}^\infty$ converges to some $x=\left(x^{(1)},x^{(2)}\right)\in X\times X$.

We claim that $\left(x^{(1)},x^{(2)}\right)\in V\times V$ and satisfies (i), (ii) and (iii).

Indeed from the continuity of F and the choice of $u_n=\left(u_n^{(1)},u_n^{(2)}\right)\in V\times V$ we have

$$x^{(1)}=\lim_{n\to\infty}u_n^{(1)}\preceq\lim_{n\to\infty}F(u_n)=F(x)$$

and

$$x^{(2)}=\lim_{n\to\infty}u_n^{(2)}\succeq\lim_{n\to\infty}F(\overline{u}_n)=F(\overline{x}).$$

(i) By construction the sequence $\{T(u_n)\}_{n=0}^\infty$ is monotonously decreasing, and consequently using the l.s.c. of T we get $T(x)\le\lim_{n\to\infty}T(u_n)\le T(u_0)$, and consequently (i) holds.

(ii) Let us put $n=0$ in (2.4), i.e.,

$$\varepsilon d(u_0,u_p)\le T(u_0)-T(u_p)\le T(u_0)-\inf_{v\in V\times V}T(v)\le\varepsilon.$$

Letting p to infinity in the last inequality we get

$$\varepsilon d(u_0,x)=\lim_{p\to\infty}\varepsilon d(u_0,u_p)\le\varepsilon,$$

i.e., $d(x,u)\le 1$.

(iii) Let us suppose that (iii) were not true for all $w\in V\times V$. Therefore we can choose $w\ne x$, $w\in V\times V$, so that

$$T(w)\le T(x)-\varepsilon d(w,x)<T(x). \tag{2.5}$$

Letting $p\to\infty$ in (2.4), we obtain

$$\varepsilon d(u_n,x)\le T(u_n)-T(x). \tag{2.6}$$

From (2.5) and (2.6) we get the chain of inequalities

$$\begin{aligned}T(w)&\le T(x)-\varepsilon d(w,x)\le T(u_n)-\varepsilon d(u_n,x)-\varepsilon d(x,w)\\ &=T(u_n)-\varepsilon(d(u_n,x)+d(x,w))\le T(u_n)-\varepsilon d(u_n,w)\end{aligned}$$

and thus $w\in S_n$ for all $n\in\mathbb{N}$. From (2.3) we have

$$2T(u_{n+1})-T(u_n)\le\inf_{S_n}F\le T(w), \tag{2.7}$$

because $w \in \cap_{n=0}^{\infty} S_n$. From the existence of $\lim_{n\to\infty} F(u_n) = l$ and (2.7) it follows that

$$\lim_{n\to\infty} (2T(u_{n+1}) - T(u_n)) = \lim_{n\to\infty} T(u_n) = l \leq T(w). \tag{2.8}$$

Since T is l.s.c, we have the inequality

$$T(x) \leq \lim_{n\to\infty} T(u_n) = l \tag{2.9}$$

and thus (2.8) and (2.9) imply that $T(x) \leq T(w)$, a contradiction with (2.5). □

3. Applications

We will need the next observation, that is used in [7,8,14], but not stated as a proposition.

Proposition 3.1. *Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a map with the mixed monotone property. Let $(x, y) \in X \times X$ satisfy the inequalities $x \preceq F(x, y)$, $y \succeq F(y, x)$ and let us put $u = F(x, y)$ and $v = F(y, x)$. Then there hold $u \preceq F(u, v)$, $v \succeq F(v, u)$, $u \succeq x$ and $v \preceq y$.*

Proof. By the definition of $(u, v) \in X \times X$ there hold $x \preceq F(x, y) = u$ and $y \succeq F(y, x) = v$. From the assumption that F satisfies the mixed monotone property, we get the inequalities

$$F(u, v) \succeq F(x, v) \succeq F(x, y) = u,$$

and

$$F(v, u) \preceq F(y, u) \preceq F(y, x) = v.$$

□

Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. Following [25], for any $(\xi_0, \eta_0) \in X \times X$ we will consider the sequence $\{\xi_n, \eta_n\}_{n=0}^{\infty}$, defined by $\xi_n = F(\xi_{n-1}, \eta_{n-1})$ and $\eta_n = F(\eta_{n-1}, \xi_{n-1})$ for $n \in \mathbb{N}$.

Proposition 3.2. *Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a map with the mixed monotone property. Let $(x, y) \in X \times X$ be a coupled fixed point, i.e., $x = F(x, y)$, $y = F(y, x)$ and let (ξ_0, η_0) be comparable with (x, y). Then (ξ_n, η_n) is comparable with $(x, y) = (F(x, y), F(y, x))$ and (η_n, ξ_n) is comparable with $(y, x) = (F(y, x), F(x, y))$.*

Proof. If (ξ_0, η_0) is comparable with (x, y), then there holds either $\xi_0 \preceq x$ and $\eta_0 \succeq y$ or $\xi_0 \succeq x$ and $\eta_0 \preceq y$. Let us assume that there holds the second one (i.e., $\xi_0 \succeq x$ and $\eta_0 \preceq y$). Using the mixed monotone property we get

$$\xi_1 = F(\xi_0, \eta_0) \succeq F(x, \eta_0) \succeq F(x, y) = x,$$

and

$$\eta_1 = F(\eta_0, \xi_0) \preceq F(y, \xi_0) \preceq F(y, x) = y.$$

Therefore, $(\xi_1, \eta_1) \succeq (x, y) = (F(x, y), F(y, x))$. We can get by induction that

$$\xi_n = F(\xi_{n-1}, \eta_{n-1}) \succeq F(x, \eta_{n-1}) \succeq F(x, y) = x$$

and

$$\eta_n = F(\eta_{n-1}, \xi_{n-1}) \preceq F(y, \xi_{n-1}) \preceq F(y, x) = y.$$

Consequently, (ξ_n, η_n) is comparable with (x, y) and

$$(\xi_n, \eta_n) \succeq (x, y) = (F(x, y), F(y, x)).$$

If there holds the first case (i.e., $\xi_0 \preceq x$ and $\eta_0 \succeq y$), we can get in a similar fashion that there hold $\xi_n \preceq x$ and $\eta_n \succeq y$ and thus

$$(F(x, y), F(y, x)) = (x, y) \succeq (\xi_n, \eta_n).$$

Therefore, (ξ_n, η_n) is comparable with $(x, y) = (F(x, y), F(y, x))$ in both cases. □

We will need the result from [34] that in a partially ordered space any element with an lower or an upper bound is equivalent to for every two elements there exists an element, which is comparable with both of them.

We will give an alternative proof of ([7], Theorem 3) for the existence of coupled fixed points using the variational principle from the previous section.

Theorem 3.3. *Let $(X, \rho, \preceq)$ be a partially ordered complete metric space, $(X \times X, d, \preceq)$ and $F : X \times X \to X$ be a continuous map with the mixed monotone property. Let there exist $\alpha \in [0, 1)$, so that the inequality*

$$\rho(F(x, y), F(u, v)) + \rho(F(y, x), F(v, u)) \leq \alpha\rho(x, u) + \alpha\rho(y, v) \quad (3.1)$$

holds for all $x \succeq u$ and $y \preceq v$. If there exists at least one ordered pair (x, y), such that $x \preceq F(x, y)$ and $y \succeq F(y, x)$, then there exists a coupled fixed points (x, y) of F.

If in addition every pair of elements in $X \times X$ has an lower or an upper bound, then the coupled fixed point is unique.

Proof. Let us consider the function $T : X \times X \to \mathbb{R}$, defined by

$$T(z) = d(z, (F(z), F(\overline{z})),$$

where $z = (x, y) \in X \times X$ and $\overline{z} = (y, x)$. The map T satisfies the conditions of Theorem 2.1, as far as T is continuous, proper function, bounded from below and the set of all $z \in X \times X$, such that $x \preceq F(z)$ and $y \succeq F(\overline{z})$ is not empty. Let us choose $\varepsilon \in (0, 1 - \alpha)$. By Theorem 2.1 there exists (x, y), satisfying $x \preceq F(x, y)$ and $y \succeq F(y, x)$, such that there holds the inequality

$$T(x, y) \leq T(u, v) + \varepsilon d((x, y), (u, v)) \quad (3.2)$$

for every $u \preceq F(u, v)$ and $v \succeq F(v, u)$.

Let us put $u = F(x, y)$, $v = F(y, x)$ and $w = (u, v)$. By Proposition 3.1 it follows that $u \preceq F(u, v)$, $v \succeq F(v, u)$, $u \succeq x$ and $v \preceq y$. From (3.1) using the symmetry of the metrics ρ, we obtain

$$\begin{aligned} T(w) &= d(w, (F(w), F(\overline{w})) \\ &= \rho(F(x, y), F(F(x, y), F(y, x))) \end{aligned}$$

$$
\begin{aligned}
&+\rho(F(y,x),F(F(y,x),F(x,y)))\\
&=\rho(F(F(x,y),F(y,x)),F(x,y))\\
&\quad+\rho(F(F(y,x),F(x,y)),F(y,x))\\
&\le \alpha\left(\rho(F(x,y),x)+\rho(F(y,x),y)\right)=\alpha T(x,y). \qquad (3.3)
\end{aligned}
$$

Consequently from (3.2) using (3.3), we get

$$T(x,y)\le T(w)+\varepsilon d((x,y),w)\le \alpha T(x,y)+\varepsilon T(x,y)=(\alpha+\varepsilon)\,T(x,y)$$

From the choice of $\varepsilon\in(0,1-\alpha)$, we obtain $T(x,y)<T(x,y)$. From the last inequality it follows that $T(x,y)=d((x,y),(F(x,y),F(y,x)))=0$, i.e., $\rho(x,F(x,y))+\rho(y,F(y,x))=0$. Therefore (x,y) is a coupled fixed points of F.

The proof of the uniqueness, provided that every pair of elements in $X\times X$ has an lower or an upper bound is done in [7]. □

The next results is a corollary of Theorem 3.3, which slightly generalizes the result from [8].

Corollary 3.4. *Let $(X,\rho,\preceq)$ be a partially ordered complete metric space, $(X\times X,d,\preceq)$ and $F:X\times X\to X$ be a continuous map with the mixed monotone property. Let there exist $\alpha,\beta\in[0,1)$, $\alpha+\beta<1$ so that the inequality*

$$\rho(F(x,y),F(u,v))\le \alpha\rho(x,u)+\beta\rho(y,v) \qquad (3.4)$$

holds for all $x\succeq u$ and $y\preceq v$. If there exists at least one ordered pair (x,y), such that $x\preceq F(x,y)$ and $y\succeq F(y,x)$, then there exists a coupled fixed points (x,y) of F.

If in addition every pair of elements in $X\times X$ has an lower or an upper bound, then the coupled fixed point is unique.

Proof. Let F satisfy (3.4). Then from (3.4) we get for $x\succeq u$ and $y\preceq v$ that there holds

$$\rho(F(x,y),F(u,v))\le \alpha\rho(x,u)+\beta\rho(y,v). \qquad (3.5)$$

and using the symmetry of the metric $\rho(\cdot,\cdot)$ we get, using $v\succeq y$ and $u\preceq x$

$$\rho(F(y,x),F(v,u))=\rho(F(v,u),F(y,x))\le \alpha\rho(y,v)+\beta\rho(x,u) \qquad (3.6)$$

Summing up (3.5) and (3.6), we obtain

$$\rho(F(x,y),F(u,v))+\rho(F(y,x),F(v,u))\le(\alpha+\beta)(\rho(x,u)+\rho(y,v)), \qquad (3.7)$$

and consequently the conditions of Theorem 3.3 are satisfied. □

It was proved in [14] the existence and uniqueness of coupled fixed points for Kannan type maps in metric space. We present a generalization in the context of mixed monotone maps in partially ordered metric spaces.

Theorem 3.5. *Let $(X,\rho,\preceq)$ be a partially ordered complete metric space, $(X\times X,d,\preceq)$ and $F:X\times X\to X$ be a continuous map with the mixed monotone property. Let there exist $\alpha\in[0,1/2)$, so that the inequality*

$$\rho(F(x,y),F(u,v))\le \alpha\rho(x,F(x,y))+\alpha\rho(u,F(u,v)) \qquad (3.8)$$

holds for all $x \succeq u$ *and* $y \preceq v$*. If there exists at least one ordered pair* (x, y)*, such that* $x \preceq F(x, y)$ *and* $y \succeq F(y, x)$*, then there exists a coupled fixed point* (x, y) *of* F*.*

Proof. It is well known that $\frac{\alpha}{1-\alpha} \in [0, 1)$ for any $\alpha \in [0, 1/2)$. Let us consider the function $T : X \times X \to \mathbb{R}$, defined by

$$T(z) = d(z, (F(z), F(\overline{z})) = \rho(x, F(x, y)) + \rho(y, F(y, x)),$$

where $z = (x, y) \in X \times X$. The map T satisfies the conditions of Theorem 2.1, as far as T is continuous, proper function, bounded from below and the set of all $z \in X \times X$, such that $x \preceq F(z)$ and $y \succeq F(\overline{z})$ is not empty. Let us choose $\varepsilon \in (0, 1 - \alpha)$. By Theorem 2.1 there exists (x, y), satisfying $x \preceq F(x, y)$ and $y \succeq F(y, x)$, such that there holds the inequality

$$T(x, y) \le T(u, v) + \varepsilon d((x, y), (u, v)) \tag{3.9}$$

for every $u \preceq F(u, v)$ and $v \succeq F(v, u)$.

Let us put $u = F(x, y)$, $v = F(y, x)$ and $w = (u, v)$. By Proposition 3.1 it follows that $u \preceq F(u, v)$, $v \succeq F(v, u)$, $u \succeq x$ and $v \preceq y$. From (3.8) using the symmetry of $\rho(\cdot, \cdot)$, we obtain

$$\begin{aligned} S_1 &= \rho(F(x, y), F(F(x, y), F(y, x))) \\ &= \rho(F(F(x, y), F(y, x)), F(x, y)) \\ &\le \alpha\rho(x, F(x, y)) + \alpha\rho(F(x, y), F(F(x, y), F(y, x))), \end{aligned}$$

because $F(x, y) \succeq x$ and $F(y, x) \preceq y$ and thus

$$\rho(F(x, y), F(F(x, y), F(y, x))) \le \frac{\alpha}{1 - \alpha}\rho(x, F(x, y)). \tag{3.10}$$

Similarly from (3.8) we get

$$\begin{aligned} S_2 &= \rho(F(y, x), F(F(y, x), F(x, y))) \\ &\le \alpha\rho(y, F(y, x)) + \alpha\rho(F(y, x), F(F(y, x), F(x, y))), \end{aligned}$$

and consequently

$$\rho(F(y, x), F(F(y, x), F(x, y))) \le \frac{\alpha}{1 - \alpha}\rho(y, F(y, x)). \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\begin{aligned} T(w) &= d(w, (F(w), F(\overline{w})) \\ &= \rho(F(x, y), F(F(x, y), F(y, x))) \\ &\quad +\rho(F(y, x), F(F(y, x), F(x, y)))) \\ &\le \frac{\alpha}{1 - \alpha}\left(\rho(x, F(x, y)) + \rho(y, F(y, x))\right) = \frac{\alpha}{1 - \alpha}T(x, y). \end{aligned} \tag{3.12}$$

Consequently from (3.10) using (3.12) we get

$$\begin{aligned} T(x, y) &\le T(u, v) + \varepsilon d((x, y), (u, v)) \\ &\le \frac{\alpha}{1 - \alpha}T(x, y) + \varepsilon T(x, y) = \left(\frac{\alpha}{1 - \alpha} + \varepsilon\right) T(x, y). \end{aligned}$$

From the choice of $\varepsilon \in \left(0, 1 - \frac{\alpha}{1-\alpha}\right)$, we obtain $T(x, y) < T(x, y)$. From the last inequality it follows that $T(x, y) = d((x, y), (F(x, y), F(y, x))) = 0$, i.e.,

$\rho(x,F(x,y))+\rho(y,F(y,x))=0$. Therefore (x,y) is a coupled fixed points of F.

Let there be two coupled fixed points $(x,y),(u,v)\in X\times X$, then $x=F(x,y)$, $y=F(y,x)$, $u=F(u,v)$ and $v=F(v,u)$. By the assumption that any element has an lower or an upper bound it follows from [34] that there exists (ξ_0,η_0) comparable with (x,y) and (u,v). From Proposition 3.2 it follows that (ξ_n,η_n) is comparable with both $(x,y)=(F(x,y),F(y,x))$ and $(u,v)=(F(u,v),F(v,u))$ and (η_n,ξ_n) is comparable with both (y,x) and (v,u).

We will apply inequality (3.8). If $(\xi_n,\eta_n)\succeq(x,y)$, then it satisfies the assumptions of the theorem.

If $(\xi_n,\eta_n)\preceq(x,y)$, using the symmetry of the metrics ρ we get

$$\begin{aligned}\rho(F(\xi_n,\eta_n),F(x,y))&=\rho(F(x,y),F(\xi_n,\eta_n))\\&\le\alpha\rho(x,F(x,y))+\alpha\rho(\xi_n,F(\xi_n,\eta_n)).\end{aligned}$$

Consequently, we can apply (3.8) when (ξ_n,η_n) is comparable with

$$(F(x,y),F(y,x)).$$

There exists $n_0\in\mathbb{N}$, such that $\left(\frac{\alpha}{1-\alpha}\right)^{n_0}<\frac{\rho(\xi_0,x)+\rho(\eta_0,y)+\rho(\xi_0,u)+\rho(\eta_0,v)}{\rho(x,u)+\rho(y,v)}$.

Let us denote $I_n=\rho(\xi_n,x)$ and $J_n=\rho(\eta_n,y)$. Using inequality (3.8), we get that

$$\begin{aligned}I_n&=\rho(\xi_n,x)=\rho(F(\xi_{n-1},\eta_{n-1}),F(x,y))\\&\le\alpha\rho(\xi_{n-1},F(\xi_{n-1},\eta_{n-1}))+\alpha\rho(x,F(x,y))\\&=\alpha\rho(\xi_{n-1},\xi_n)=\alpha\rho(\xi_{n-1},x)+\alpha\rho(x,\xi_n),\end{aligned}$$

and

$$\begin{aligned}J_n&=\rho(\eta_n,y)=\rho(F(\eta_{n-1},\xi_{n-1}),F(y,x))\\&\le\alpha\rho(\eta_{n-1},F(\eta_{n-1},\xi_{n-1}))+\alpha\rho(y,F(y,x))\\&=\alpha\rho(\eta_{n-1},\eta_n)=\alpha\rho(\eta_{n-1},y)+\alpha\rho(y,\eta_n).\end{aligned}$$

Summing the last two inequalities, we obtain

$$\begin{aligned}I_n+J_n&\le\alpha\left(\rho(\xi_{n-1},x)+\rho(\eta_{n-1},y)+\rho(x,\xi_n)+\rho(y,\eta_n)\right)\\&=\alpha(I_{n-1}+J_{n-1})+\alpha(I_n+J_n).\end{aligned}$$

Consequently, $I_n+J_n\le\frac{\alpha}{1-\alpha}(I_{n-1}+J_{n-1})$ and thus

$$I_n+J_n\le\left(\frac{\alpha}{1-\alpha}\right)^n(\rho(\xi_0,x)+\rho(\eta_0,y)).$$

Then we obtain,

$$\begin{aligned}\rho(x,u)+\rho(y,v)&\le\rho(x,\xi_{n_0})+\rho(\xi_{n_0},u)+\rho(y,\eta_{n_0})+\rho(\eta_{n_0},v)\\&\le\left(\frac{\alpha}{1-\alpha}\right)^{n_0}(\rho(\xi_0,x)+\rho(\eta_0,y)+\rho(\xi_0,u)+\rho(\eta_0,v))\\&<\rho(x,u)+\rho(y,v),\end{aligned}$$

which is a contradiction and that $(x,y)=(u,v)$. □

We would like to finish with a particular example. Let $X = \ell_1$, endowed with its classical norm $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ and the metric $\rho_1(x, y) = \|x - y\|$. Let us define a partial order in X by $x \preceq y$, if $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$. Let us define $F : X \times X \to X$ by

$$F(x,y) = \left\{ \frac{|x_i|}{2} - \frac{|y_i|}{3} + \frac{1}{2^i} \right\}_{i=1}^{\infty}.$$

Let us consider the ordered pair $(x_0, y_0) = \left(\left\{ \frac{2}{5.2^i} \right\}_{i=1}^{\infty}, \left\{ \frac{2}{2^i} \right\}_{i=1}^{\infty} \right)$. Then

$$F(x_0, y_0) = \left\{ \frac{1}{5.2^i} - \frac{2}{3.2^i} + \frac{1}{2^i} \right\}_{i=1}^{\infty} = \left\{ \frac{8}{15.2^i} \right\}_{i=1}^{\infty},$$

and

$$F(y_0, x_0) = \left\{ \frac{1}{2^i} - \frac{2}{15.2^i} + \frac{1}{2^i} \right\}_{i=1}^{\infty} = \left\{ \frac{28}{15.2^i} \right\}_{i=1}^{\infty}$$

The map $F : X \times X \to X$ is a continuous map. From the inequality $\left| \frac{2}{5.2^i} \right| < \left| \frac{8}{15.2^i} \right|$ it follows that $x_0 \preceq F(x_0, y_0)$, and from the inequality $\left| \frac{2}{2^i} \right| < \left| \frac{28}{15.2^i} \right|$ it follows that $y \succeq F(y_0, x_0)$.

The map F has the mixed monotone property. Indeed it $x \preceq z$, then

$$F(x,y) = \left\{ \frac{|x_i|}{2} - \frac{|y_i|}{3} + \frac{1}{2^i} \right\}_{i=1}^{\infty} \preceq \left\{ \frac{|z_i|}{2} - \frac{|y_i|}{3} + \frac{1}{2^i} \right\}_{i=1}^{\infty} = F(z,y).$$

Let $z \preceq y$ hold, then $F(x,z) \succeq F(x,y)$.

$$F(x,z) = \left\{ \frac{|x_i|}{2} - \frac{|z_i|}{3} + \frac{1}{2^i} \right\}_{i=1}^{\infty} \succeq \left\{ \frac{|x_i|}{2} - \frac{|y_i|}{3} + \frac{1}{2^i} \right\}_{i=1}^{\infty} = F(x,y).$$

Let now $x \succeq u$ and $y \preceq v$ hold.Then using that $|x_i| - |u_i| \geq 0$ and $|v_i| - |y_i| \geq 0$ we get

$$\begin{aligned}
\rho_1(F(x,y), F(u,v)) &= \|F(x,y) - F(u,v)\|_1 \\
&= \sum_{i=1}^{\infty} \left| \frac{|x_i|}{2} - \frac{|y_i|}{3} + \frac{1}{2^i} - \left(\frac{|u_i|}{2} - \frac{|v_i|}{3} + \frac{1}{2^i} \right) \right| \\
&= \sum_{i=1}^{\infty} \left| \frac{|x_i|}{2} - \frac{|u_i|}{2} + \frac{|v_i|}{3} - \frac{|y_i|}{3} \right| \\
&= \sum_{i=1}^{\infty} \left| \frac{|x_i|}{2} - \frac{|u_i|}{2} \right| + \sum_{i=1}^{\infty} \left| \frac{|v_i|}{3} + \frac{y_i}{3} \right| \\
&\leq \frac{1}{2} \sum_{i=1}^{\infty} |x_i - u_i| + \frac{1}{3} \sum_{i=1}^{\infty} |v_i - y_i| \\
&= \frac{1}{2} \rho_1(x,u) + \frac{1}{3} \rho_1(y,v).
\end{aligned}$$

Therefore the map F satisfies the conditions of Corollary 3.4, and consequently F has a coupled fixed point.

It is easy to observe that for any two elements $x, y \in (X, \rho_1, \preceq)$ there exists an element z, which is comparable with both of them (we can choose $z_i \geq \max\{|x_i|, |y_i|\}$). Thus the coupled fixed point is unique.

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