J. Fixed Point Theory Appl. (2019) 21:54
https://doi.org/10.1007/s11784-019-0695-x
Published online April 24, 2019

Journal of Fixed Point Theory and Applications

A note on the fixed point theorem of Górnicki

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Abstract. In this note, we show that the main result (Theorem 2.6) due to Górnicki (J Fixed Point Theory Appl 21:29, 2019. https://doi.org/10.1007/s11784-019-0668-0) is still valid if we replace the assumption of continuity of the mapping by some weaker versions of continuity conditions. As a by-product, we provide few more new answers to the open question of Rhoades (Contemp Math 72:233–245, 1988).

Mathematics Subject Classification. Primary 47H09, Secondary 47H10.

Keywords. Fixed point, asymptotic regularity.

1. Introduction

The following theorem is the key result of [3].

Theorem 1.1. *If* (X, d) *is a complete metric space and* $T\colon X \to X$ *is a continuous asymptotically regular mapping and if there exists* $0 \leqslant M < 1$ *and* $0 \leqslant K < +\infty$ *satisfying*

$$d(Tx, Ty) \leqslant Md(x, y) + K\{d(x, Tx) + d(y, Ty)\} \tag{1.1}$$

for all $x, y \in X$*, then* T *has a unique fixed point* $p \in X$ *and* $T^n x \to p$ *for each* $x \in X$*.*

Recall that the set $O(x; T) = \{T^n x\colon n = 0, 1, 2, \ldots\}$ is called the orbit of the self-mapping T at the point $x \in X$.

Definition 1.1. A self-mapping T of a metric space (X, d) is said to be orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O(x; T)$ for some $x \in X, x_n \to z$ implies $Tx_n \to Tz$ as $n \to \infty$.

Remark 1.1. Every continuous self-mapping of a metric space is orbitally continuous, but the converse need not be true (see Example 1.1 below).

Definition 1.2. [5] A self-mapping T of a metric space (X, d) is called k-continuous, $k = 1, 2, 3, \ldots,$ if $T^k x_n \to Tz$, whenever $\{x_n\}$ is a sequence in X such that $T^{k-1} x_n \to z$.

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Remark 1.2. It is important to note that for a self-mapping T of a metric space (X, d), the notion of 1-continuity coincides with continuity. However,

$$\text{1-continuity} \Rightarrow \text{2-continuity} \Rightarrow \text{3-continuity} \Rightarrow \cdots,$$

but not conversely. The following example illustrates this fact [5].

Example 1.1. Let $X = [0, 4]$ and d be the usual metric on X. Define $T\colon X \to X$ by

$$T(x) = 2 \text{ if } x \in [0, 2], T(x) = 0 \text{ if } x \in (2, 4].$$

Then, $Tx_n \to t \Rightarrow T^2x_n \to t$, since $Tx_n \to t$ implies $t = 0$ or $t = 2$ and $T^2x_n \to 2 = T2$ for all n. Hence, T is 2-continuous. However, T is discontinuous at $x = 2$.

In 1988, Rhoades [7] posed an open problem regarding existence of contractive definitions which yield a fixed point but the mapping need not be continuous at the fixed point. This problem was settled in the affirmative by Pant [6]. In a recent past, several new situations have been established where the existence of the fixed point is guaranteed but the mappings are discontinuous at the fixed point [1,2,4].

In this paper, we show that the assumption of continuity considered in Theorem 2.6 of [3] can be relaxed to some weaker notions of continuity, (orbital continuity or k-continuity) which thereby extends the scope of the study of fixed point theorems from the class of continuous mappings to a wider class of mappings which also include discontinuous mappings. As a by-product, we provide new answers to the open problem posed by Rhoades [7].

2. Main results

Theorem 2.1. *If (X, d) is a complete metric space and $T\colon X \to X$ is an asymptotically regular mapping and if there exists $0 \leqslant M < 1$ and $0 \leqslant K < +\infty$ satisfying*

$$d(Tx, Ty) \leqslant Md(x, y) + K\{d(x, Tx) + d(y, Ty)\} \tag{2.1}$$

for all $x, y \in X$, then T has a unique fixed point $p \in X$ provided T is either k-continuous for $k \geqslant 1$ or orbitally continuous.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = Tx_n = T^n x$. Then, following Theorem 2.6 of [3] we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $u \in X$ such that $x_n \to u$ as $n \to \infty$. Also, $Tx_n \to u$. Furthermore, for each $k \geqslant 1$ we have $T^k x_n \to u$ as $n \to \infty$. Suppose that T is k-continuous. Since $T^{k-1}x_n \to u$, k-continuity of T implies that $\lim_{n\to\infty} T^k x_n = Tu$. This yields $u = Tu$, that is, u is a fixed point of T.

Finally, suppose that T is orbitally continuous. Since $x_n \to u$, orbital continuity implies that $\lim_{n\to\infty} Tx_n = Tu$. This yields $Tu = u$, that is, u is a fixed point of T.

We now give an example to show that the condition (2.1) is strong enough to generate a fixed point but does not force the mapping to be continuous at the fixed point [6].

Example 2.1. Let $X = [0, 2]$ and d be the usual metric on X. Define $T\colon X \to X$ by

$$T(x) = \begin{cases} 1, & \text{if } 0 \leqslant x \leqslant 1; \\ 0, & \text{if } 1 < x \leqslant 2. \end{cases}$$

Then, T satisfies all the conditions of Theorem 2.1 and has a unique fixed point $x = 1$ at which T is discontinuous.

Acknowledgements

The author is thankful to the learned referee for suggesting some improvements and thereby removing certain obscurities in the presentation.

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