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Existence and uniqueness of solution for abstract differential equations with state-dependent delayed impulses

Eduardo Hernández, Katia A. G. Azevedo and Marta C. Gadotti

Abstract. We study the existence and uniqueness of mild and classical solutions for a general class of abstract impulsive differential equations with state-dependent impulses. Some examples on partial differential equations are presented.

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1. Introduction

In this paper, we study the existence and uniqueness of mild and classical solutions for a class of abstract impulsive differential equations of the form

$$u'(t) = Au(t) + f(t, u(\zeta(t, u(t)))), \qquad t \in I_i = (t_i, t_{i+1}],\ i = 0, \ldots, N, \tag{1.1}$$

$$u(t_j^+) = g_j(u(\sigma_j(u(t_j^+)))), \quad j = 1, \ldots, N, \tag{1.2}$$

$$u_0 = \varphi \in \mathcal{B} = C(I_{-1}; X), \ \ I_{-1} = [-p, 0], \tag{1.3}$$

where $A : D(A) \subset X \mapsto X$ is the generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space $(X, \| \cdot \|)$, $0 = t_0 < t_1 < t_2 < \cdots < t_N < t_{N+1} = a$ are pre-fixed numbers and $f(\cdot)$, $g_i(\cdot)$, $\sigma_i(\cdot)$, $i = 1, \ldots, N$, are functions specified be later.

The study of state-dependent delay equations is motivated by applications and theory. Related ODEs on finite dimensional spaces we cite the early works by Driver [9,10] and Aiello et al. [1], the survey by Hartung, Krisztin et al. [15], the papers by Hartung et.al. [16–18] and the references in these works. For the case PDEs and abstract differential equations with state-dependent delay, we mention [19,20,26,36–38] and the recent interesting works by Krisztin and Rezounenko [25], Yunfei et al. [33], Kosovalic et al. [26,27] and Hernandez et al. [24].

Concerning the theory of impulsive differential equations, their motivations and relevant developments, we cite the books by Bainov and Covachev [2], Lakshmikantham et al. [28], Samoilenko and Perestyuk [40] for the case of ordinary differential equations on finite dimensional space and Benchohra et al. [7] for abstract differential equations and partial differential equations. In addition, we cite the interesting papers [8,11,20–23,29,31,34,39,43] and the references therein. Related differential equations with impulse at state-dependent moments and state-dependent delayed impulses, we refer the reader to [3–6,13,14,30,41].

Our work is motivated by the papers Hakl et al. [14] related partial differential equations with impulse at state-dependent moments and Li and Wu [30] on differential equations with state-dependent delayed impulses. Specifically, we study the existence and "uniqueness" of solutions for the problem (1.1)–(1.3) which is a highly not trivial problem since functions of the form $u \mapsto u(\zeta(\cdot, u(\cdot)))$ are (in general) nonlinear and not Lipschitz on space of continuous or sectionally continuous functions. By noting that

$$\begin{aligned}
&\| u(\zeta(\cdot, u(\cdot))) - v(\zeta(\cdot, v(\cdot))) \|_{C([-p,a];X)} \\
&\quad \leq (1 + [v]_{C_{\mathrm{Lip}}([-p,a];X)}[\zeta]_{C_{\mathrm{Lip}}([0,a]\times X;[-p,a])}) \| u - v \|_{C_{\mathrm{Lip}}([-p,a];X)}, \\
&\| u(\sigma_i(u(t_i^+))) - v(\sigma_i(v(t_i^+))) \| \\
&\quad \leq (1 + [v]_{C_{\mathrm{Lip}}([-p,a];X)}[\sigma_i]_{C_{\mathrm{Lip}}(X;[-p,a]}) \| u - v \|_{C([-p,a];X)},
\end{aligned}$$

when the involved functions are Lipschitz, we study the existence of solutions on spaces of sectionally Lipschitz functions, a hard problem in the semigroup framework and in the general field of partial differential equations. In addition, we note that the Lipschizianity of $T(\cdot)g_i(u(\sigma_i(u(t_i^+))))$ not depend on the Lipschizianity of $g_i(\cdot)$ and $u(\cdot)$, which introduce a extra difficulty in our studies.

This paper has four sections. The existence and uniqueness of a classical solution via the contraction mapping principle is proved in Theorems 2.1, 2.2 and Proposition 2.3. In Theorem 2.3 we prove the existence of a mild solution using the Schauder's fixed point Theorem. The particular case in which $\sigma_i(\cdot)$ and (or) $\zeta(\cdot)$ have values in $[-p, 0]$, is studied in Propositions 2.1 and 2.2. In the last section some examples on partial differential equations are presented.

We include now some notations and results used in this work. Let $(Z, \| \cdot \|_Z)$ and $(W, \| \cdot \|_W)$ be Banach spaces. We denote by $\mathcal{L}(Z, W)$ the space of bounded linear operators from Z into W endowed with operator norm denoted by $\| \cdot \|_{\mathcal{L}(Z,W)}$ and we write $\mathcal{L}(Z)$ and $\| \cdot \|_{\mathcal{L}(Z)}$ if $Z = W$. Moreover, if $X = Z = W$ we write simply $\| \cdot \|$ for the norms $\| \cdot \|_X$ and $\| \cdot \|_{\mathcal{L}(X)}$. In addition, $B_r(z, Z) = \{y \in Z : \| y - z \|_Z \leq r\}$.

Let $J \subset \mathbb{R}$ be a bounded interval. The spaces $C(J, Z)$ and $C_{\mathrm{Lip}}(J, Z)$ and their norms denoted by $\| \cdot \|_{C(J,Z)}$ and $\| \cdot \|_{C_{\mathrm{Lip}}(J,Z)}$ are the usual. We only note that $\| \cdot \|_{C_{\mathrm{Lip}}(J;Z)}$ is given by $\| \cdot \|_{C_{\mathrm{Lip}}(J;Z)} = \| \cdot \|_{C(J;Z)} + [\cdot]_{C_{\mathrm{Lip}}(J;Z)}$ where $[\zeta]_{C_{\mathrm{Lip}}(J;Z)} = \sup_{t,s\in J, t\neq s} \frac{\|\zeta(s)-\zeta(t)\|_Z}{|t-s|}$.

The notation $\mathcal{PC}(Z)$ is used for the space formed by all the bounded functions $u : [0,a] \mapsto Z$ such that $u(\cdot)$ is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for all $i = 1, \ldots, N$, provided with the norm $\| u \|_{\mathcal{PC}(Z)} = \max_{i=0,1,\ldots,N} \| u \|_{C((t_i,t_{i+1}];Z)}$. In addition, $\mathcal{PC}_{\text{Lip}}(Z)$ represents the space of functions $u \in \mathcal{PC}(Z)$ such that $u_{|_{(t_i,_{i+1}]}} \in C_{\text{Lip}}((t_i, t_{i+1}]; Z)$ for all $i = 0, 1, \ldots t_{N+1}$, endowed with the norm $\| u \|_{\mathcal{PC}_{\text{Lip}}(Z)} = \max_{i=0,\ldots,N} \| u_{|_{(t_i,t_{i+1}]}} \|_{C_{\text{Lip}}((t_i,_{i+1}];Z)}$.

We use the symbol $\mathcal{BPC}(Z)$ for the set of all the functions $u : [-p, a] \mapsto Z$ such that $u_{|_{[-p,t_1]}} \in C([-p, t_1]; Z)$ and $u_{|_{[0,a]}} \in \mathcal{PC}(Z)$. In addition, $\mathcal{BPC}_{\text{Lip}}(Z)$ is the space formed by all the functions $u : [-p, a] \mapsto Z$ such that $u \in \mathcal{BPC}(Z)$, $u_{|_{[-p,0]}} \in C_{\text{Lip}}([-p, 0]; Z)$ and $u_{|_{[0,a]}} \in \mathcal{PC}_{\text{Lip}}(Z)$, endowed with the norm $\| u \|_{\mathcal{BPC}_{\text{Lip}}(Z)} = \max\{\| u_{|_{I_i}} \|_{C_{\text{Lip}}(I_i;Z)} : i = -1, 0, \ldots, N\}$.

For $u \in \mathcal{BPC}(Z)$ and $i \in \{-1, 0, 1, \cdots, N\}$, we use the notation $\tilde{u}_i$ for the function $\tilde{u}_i \in C([t_i, t_{i+1}]; Z)$ given by $\widetilde{u}_i(t) = u(t)$ for $t \in (t_i, t_{i+1}]$ and $\widetilde{u}_i(t) = u(t_i^+)$ for $t = t_i$. For $B \subseteq \mathcal{BPC}(Z)$ and $i \in \{-1, 0, 1, \cdots, N\}$, $\widetilde{B}_i$ is the set $\widetilde{B}_i = \{\tilde{u}_i : u \in B\}$. We note the following Ascoli–Arzela type criteria.

Lemma 1.1. *A set $B \subseteq \mathcal{BPC}(Z)$ is relatively compact in $\mathcal{BPC}(Z)$ if and only if each set $\widetilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}], Z)$.*

In this paper, X_1 is the domain of A endowed with the norm $\| x \|_{X_1} = \| x \| + \| Ax \|$ and C_0, C_1 are positive constants such that $\| AT(s) \|_{\mathcal{L}(X_1,X)} \leq C_1$, $\| T(s) \| \leq C_0$ and $\| AT(t) \| \leq \frac{C_1}{t}$ for all $s \in [0, a]$ and $t \in (0, a]$.

Related the abstract Cauchy problem

$$u'(t) = Au(t) + \xi(t), \qquad t \in [a, b], \qquad u(c) = x \in X, \tag{1.4}$$

we note that the function $u \in C([c, d]; X)$ given by $u(t) = T(t - c)x + \int_c^t T(t - s)\xi(s)\mathrm{d}s$, is called mild solution of (1.4). In addition, a function $v \in C([c, d]; X)$ is said to be a classical solution of (1.4) if $v \in C^1((c, d]; X) \cap C((c, d]; X_1)$ and $v(\cdot)$ satisfies (1.4) on $(c, d]$.

2. Existence of solutions

In this section we present some results on the existence of solution for (1.1)–(1.3). To begin, we introduce the followings concepts of solution.

Definition 2.1. A function $u \in \mathcal{BPC}(X)$ is called a mild solution of the problem (1.1)–(1.3) if $u_0 = \varphi$, $u(t_i^+) = g_i(u(\sigma_i(u(t_i^+))))$ for all $i = 1, \ldots, N$ and

$$u(t) = T(t)\varphi(0) + \int_0^t T(t - \tau)f(\tau, u(\zeta(\tau, u(\tau))))\mathrm{d}\tau, \quad t \in [0, t_1],$$

$$u(t) = T(t - t_i)g_i(u(\sigma_i(u(t_i^+)))) + \int_{t_i}^t T(t - \tau)f(\tau, u(\zeta(\tau, u(\tau))))\mathrm{d}\tau,$$

for all $t \in (t_i, t_{i+1}]$ and $i = 1, \ldots, N$.

Definition 2.2. A function $u \in \mathcal{BPC}(X)$ is called a classical solution of (1.1)–(1.3) if $u_0 = \varphi$, $u(t_i^+) = g_i(u(\sigma_i(u(t_i^+))))$ for all $i = 1, \ldots, N$ and $u(\cdot)$ satisfy (1.1).

In the remainder of this work, we assume that $(W, \|\cdot\|_W)$ is Banach continuously embedded in $(X, \|\cdot\|)$ such that $AT(\cdot) \in L^\infty([0,a]; \mathcal{L}(W,X))$. To prove our results, we introduce the following conditions.

- $\mathbf{H}_\zeta$ $\zeta \in C_{\rm Lip}([0,a] \times X; [-p,a])$ and there is a function $j : \{1,\ldots,N\} \mapsto \{-1,0,1,\ldots,N\}$ such that $\zeta \in C_{\rm Lip}(I_i \times X; I_{j(i)})$ and $j(i) \leq i$ for all $i \in \{1,\ldots,N\}$.
- $\mathbf{H}_{\sigma_\mathbf{i}}$ There is a function $q : \{1,\ldots,N\} \mapsto \{-1,0,1,\ldots,N\}$ such that $q(i) \leq i$ and $\sigma_i \in C(X, I_{q(i)})$ for all $i \in \{1,\ldots,N\}$. Next we write $[\sigma_i]_{C_{\rm Lip}}$ in place $[\sigma_i]_{C_{\rm Lip}(X;I_{q(i)})}$.
- $\mathbf{H}^{\mathbf{W}}_{\mathbf{g},\mathbf{X}}$ $g_i \in C_{\rm Lip}(X;W)$ and $\mathcal{C}_{X,W}(g_i) = \| g_i \|_{C(X;W)} < \infty$ for every $i \in \{1,\ldots,N\}$. Next, $L_{Z,W}(g_i)$ denotes the Lipschitz constant of $g_i(\cdot)$, $L_{Z,W}(g) = \max_{i=1,\ldots,N} L_{Z,W}(g_i)$ and $\mathcal{C}_{Z,W}(g) = \max_{i=1,\ldots,N} \mathcal{C}_{Z,W}(g_i)$.
- $\mathbf{H}_\mathbf{g}$ $g_i \in C_{\rm Lip}(X;X)$ and $\mathcal{C}_X(g_i) = \| g_i \|_{C(X;X)} < \infty$ for all $i \in \{1,\ldots,N\}$. Next, L_{g_i} is the Lipschitz constant of $g_i(\cdot)$, $L_g = \max_{i=1,\ldots,N} L_{g_i}$ and $\mathcal{C}_X(g) = \max_{i=1,\ldots,N} \mathcal{C}_X(g_i)$.
- $\mathbf{H}_\mathbf{f}$ $f \in C_{\rm Lip}([0,a] \times X; X)$ and $C_X(f) = \| f \|_{C([0,a]\times X;X)} < \infty$. Next, L_f denotes the Lipschitz constant of $f(\cdot)$.

Notations 1. Next, for convenience, we write $[\zeta]_{C_{\rm Lip}}$ in place $[\zeta]_{C_{\rm Lip}([0,a]\times X;[-p,a])}$, $b_i = t_{i+1} - t_i$, $b = \max_{i=1,\ldots,N} b_i$, $i_c : W \mapsto X$ is the inclusion map and

$$\Lambda_{X,W} = \max\{\| AT(\cdot) \|_{L^\infty([0,b],\mathcal{L}(W,X))}, C_0 \| i_c \|_{\mathcal{L}(W,X)}\}$$
$$\Phi_{X,W} = \Lambda_{X,W}\mathcal{C}_{X,W}(g) + C_0(C_X(f) + bL_f) + [T(\cdot)\varphi(0)]_{C_{\rm Lip}([0,a];X)} + [\varphi]_{C_{\rm Lip}([-p,0];X)}.$$

The next useful result follows from the proof of [24, Lemma 1]. The proof is omitted.

Lemma 2.2. *Assume that the conditions $\mathbf{H}_\zeta$, $\mathbf{H}_{\sigma_\mathbf{i}}$ are satisfied, $u, v \in \mathcal{BPC}_{\rm Lip}(X)$ and $u_0 = v_0$. Then $u(\zeta(\cdot, u(\cdot))) \in \mathcal{PC}_{\rm Lip}(X)$ and*

$$[u(\zeta(\cdot,u(\cdot)))]_{\mathcal{PC}_{\rm Lip}(X)} \leq [u]_{\mathcal{BPC}_{\rm Lip}(X)}[\zeta]_{C_{\rm Lip}}(1 + [u_{|_{[0,a]}}]_{\mathcal{PC}_{\rm Lip}(X)}), \tag{2.1}$$

$$\| u(\zeta(\cdot,u(\cdot))) - v(\zeta(\cdot,v(\cdot))) \|_{\mathcal{PC}(X)} \leq (1 + [v]_{\mathcal{BPC}_{\rm Lip}(X)}[\zeta]_{C_{\rm Lip}}) \| u - v \|_{\mathcal{PC}(X)}, \tag{2.2}$$

$$\| u(\sigma_i(u(t_i^+))) - v(\sigma_i(v(t_i^+))) \| \leq (1 + [v]_{\mathcal{BPC}_{\rm Lip}(X)}[\sigma_i]_{C_{\rm Lip}}) \| u - v \|_{\mathcal{PC}(X)}. \tag{2.3}$$

We can prove now our first result.

Theorem 2.1. *Assume that the conditions $\mathbf{H}_\zeta, \mathbf{H}_{\sigma_\mathbf{i}}$, $\mathbf{H}^{\mathbf{W}}_{\mathbf{g},\mathbf{X}}$ and $\mathbf{H}_\mathbf{f}$ are satisfied, $T(\cdot)\varphi(0) \in C_{\rm Lip}([0,a];X)$, $\varphi \in C_{\rm Lip}([-p,0];X)$ and*

$$2C_0bL_f(1 + [\zeta]_{C_{\rm Lip}}(1 + 2\Phi_{X,W})) + 2\Lambda_{X,W}L_{X,W}(g)(1 + 2\max_{i=1,\ldots,N}[\sigma_i]_{C_{\rm Lip}}\Phi_{X,W}) < 1. \tag{2.4}$$

Then there exists a unique classical solution $u \in \mathcal{BPC}_{\rm Lip}(X)$ of the problem (1.1)–(1.3).

Proof. Let $P : \mathbb{R} \mapsto \mathbb{R}$ be the polynomial given by

$$\begin{aligned} P(x) &= \Phi_{X,W} + (C_0 b L_f(1 + [\zeta]_{C_{\mathrm{Lip}}}) + \Lambda_{X,W} L_{X,W}(g) - 1)x \\ &\quad (\Lambda_{X,W} L_{X,W}(g) \max_{i=1,\dots,N} [\sigma_i]_{C_{\mathrm{Lip}}} + C_0 b L_f [\zeta]_{C_{\mathrm{Lip}}}) x^2. \end{aligned} \tag{2.5}$$

From (2.4) and noting that $C_0 b L_f(1+[\zeta]_{C_{\mathrm{Lip}}}) + \Lambda_{X,W} L_{X,W}(g) < 1$, we infer that $P(\cdot)$ has a root $R_1 > 0$ and there exists $R > 0$ such that $P(R) < 0$. From the definition of $P(\cdot)$, we get

$$\Phi_{X,W} + C_0 b L_f [\zeta]_{C_{\mathrm{Lip}}} (R + R^2) < R, \tag{2.6}$$

$$\Lambda_{X,W} L_{X,W}(g)(1 + \max_{i=1,\dots,N} [\sigma_i]_{C_{\mathrm{Lip}}} R) + C_0 b L_f (1 + R[\zeta]_{C_{\mathrm{Lip}}}) < 1. \tag{2.7}$$

Let $\mathcal{S}(R)$ be the space $\mathcal{S}(R) = \{u \in \mathcal{BPC}_{\mathrm{Lip}}(X)\,;\, u_0 = \varphi,\, [u_{|_{[0,a]}}]_{\mathcal{PC}_{\mathrm{Lip}}(X)} \leq R\}$, endowed with the metric $d(u, v) = \parallel u - v \parallel_{\mathcal{BPC}(X)}$ and $\Gamma : \mathcal{S}(R) \mapsto \mathcal{BPC}(X)$ be the map defined by $(\Gamma u)_0 = \varphi$ and

$$\Gamma u(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(s, u(\zeta(s, u(s))))\mathrm{d}s, \quad t \in [0, t_1],$$

$$\Gamma u(t) = T(t - t_i) g_i(u(\sigma_i(u(t_i^+)))) + \int_{t_i}^t T(t-s) f(s, u(\zeta(s, u(s))))\mathrm{d}s,$$

for $t \in (t_i, t_{i+1}]$ and $i = 1, \dots, N$.

It's easy to see that $\mathcal{S}(R)$ is closed in $\mathcal{BPC}(X)$ and that $\Gamma(\cdot)$ is well defined. Moreover, from Lemma 2.2, for $i \in \{1, \dots, i\}$, $t \in (t_i, t_{i+1})$ and $h > 0$ such that $t + h \in (t_i, t_{i+1}]$, we get

$$\begin{aligned} &\parallel \Gamma u(t+h) - \Gamma u(t) \parallel \\ &\quad \leq \int_{t-t_i}^{t+h-t_i} \parallel AT(s) g_i(u(\sigma_i(u(t_i^+)))) \parallel \mathrm{d}s \\ &\qquad + \int_{t_i}^{t_i+h} \parallel T(t+h-s) f(s, u(\zeta(s, u(s)))) \parallel \mathrm{d}s \\ &\qquad + \int_{t_i}^{t} \parallel T(t-s) \parallel \parallel f(s+h, u(\zeta(s+h, u(s+h)))) - f(s, u(\zeta(s, u(s)))) \parallel \mathrm{d}s \\ &\quad \leq \parallel AT(\cdot) \parallel_{L^\infty([0,b_i];\mathcal{L}(W,X))} h\mathcal{C}_{X,W}(g) + C_0 \mathcal{C}_X(f) h \\ &\qquad + \int_{t_i}^{t} \parallel T(t-s) \parallel L_f(1 + [u(\zeta(\cdot, u(\cdot)))]_{C_{\mathrm{Lip}}(I_i;X)}) h\mathrm{d}s \\ &\quad \leq \parallel AT(\cdot) \parallel_{L^\infty([0,b_i];\mathcal{L}(W,X))} h\mathcal{C}_{X,W}(g) + C_0(\mathcal{C}_X(f) + L_f b) h \\ &\qquad + C_0 b L_f [u]_{\mathcal{BPC}_{\mathrm{Lip}}(X)} [\zeta]_{C_{\mathrm{Lip}}} (1 + [u]_{\mathcal{PC}_{\mathrm{Lip}}(X)}) h, \end{aligned}$$

which implies that $[(\Gamma u)_{|_{I_i}}]_{C_{\mathrm{Lip}}(I_i;X)} \leq \Phi_{X,W} + C_0 b L_f [\zeta]_{C_{\mathrm{Lip}}} (R + R^2) < R$. In a similar way, we obtain that

$$\begin{aligned} [(\Gamma u)_{|_{[0,t_1]}}]_{C_{\mathrm{Lip}}([0,t_1];X)} &\leq [T(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];X)} + C_0(\mathcal{C}_X(f) + bL_f) \\ &\quad + C_0 b L_f [\zeta]_{C_{\mathrm{Lip}}} (R + R^2)) \leq R. \end{aligned}$$

From the above and noting that $[\varphi]_{C_{\mathrm{Lip}}([-p,0];X)} \leq R$, we obtain that $[\Gamma u]_{\mathcal{BPC}_{\mathrm{Lip}}(X)} \leq R$, which implies that Γ is a $\mathcal{S}(R)$-valued function.

On the other hand, using (2.2), for $u, v \in \mathcal{S}(R)$, $i = 1, \ldots, N$ and $t \in (t_i, t_{i+1}]$ we have that

$$\begin{aligned}\| \Gamma u(t) - \Gamma v(t) \| &\leq C_0 \| i_c \|_{\mathcal{L}(W,X)} L_{X,W}(g) \| u(\sigma_i(u(t_i^+))) - v(\sigma_i(v(t_i^+))) \| \\ &\quad + C_0 L_f \int_{t_i}^{t} \| u(\zeta(\cdot, u(\cdot))) - v(\zeta(\cdot, v(\cdot))) \|_{C(I_i;X)} \, \mathrm{d}s \\ &\leq \left(\Lambda_{X,W} L_{X,W}(g)(1 + R[\sigma_i]_{C_{\mathrm{Lip}}}) + C_0 b L_f (1 + R[\zeta]_{C_{\mathrm{Lip}}}) \right) d(u,v).\end{aligned}$$

In addition, for $t \in [0, t_1]$ we note that $\| \Gamma u(t) - \Gamma v(t) \| \leq C_0 b L_f (1 + R[\zeta]_{C_{\mathrm{Lip}}}) d(u,v)$. From the above estimates we infer that

$$d(\Gamma u, \Gamma v) \leq (\Lambda_{X,W} L_{X,W}(g)(1 + R \max_{i=1,\ldots,N} [\sigma_i]_{C_{\mathrm{Lip}}}) + C_0 b L_f (1 + R[\zeta]_{C_{\mathrm{Lip}}})) d(u,v).$$

Thus, $\Gamma(\cdot)$ is a contraction and there exists a unique mild solution $u \in \mathcal{S}(R)$ of (1.1)–(1.3).

We prove now that $u(\cdot)$ is a classical solution. Let $\widetilde{u}_i$, $i = 1, \ldots N$, be defined as in the introduction. It is easy to see that $\widetilde{u}_i(\cdot)$ is the mild solution of the problem

$$w'(t) = Aw(t) + f(t, u(\zeta(t, u(t)))), \qquad t \in I_i = [t_i, t_{i+1}], \tag{2.8}$$

$$w(t_i) = g_i(u(\sigma_i(u(t_i^+)))). \tag{2.9}$$

Since $f(\cdot, u(\zeta(\cdot, u(\cdot))))$ is Lipschitz on I_i and the semigroup is analytic, from [35, Theorem 4.3.2] it follows that $\widetilde{u}_i$ is a classical solution of (2.8)–(2.9). The same argument prove that $\widetilde{u}_0$ is a classical solution of (2.8) on $[0, t_1]$ with initial condition $u(0) = \varphi(0)$. From the above, we obtain that $u(\cdot)$ is a classical solution of (1.1)–(1.3). □

In the next result we establish the existence and uniqueness of a classical solution without to use condition $\mathbf{H}^{\mathbf{W}}_{\mathbf{g},\mathbf{X}}$. In place of this condition, we introduce the following one:

$\mathbf{H}_{g_i,\sigma_j}$ $\sigma_i \in C_{\mathrm{Lip}}(X, [-p, a])$ for all $i \in \{1, \ldots, N\}$, $\overline{\cup_{i=1}^N \sigma_i(X)} \subset \cup_{i=0}^N I_i \cup [-p, 0]$, $g_i \in C(X_1; X_1) \cap C_{\mathrm{Lip}}(X; X)$ and there are constants l_{g_i}, k_{g_i} such that $\| A g_i(x) \| \leq l_{g_i} r + k_{g_i}$ for all $x \in B_r(0, X_1)$, $i \in \{1, \ldots, N\}$ and every $r > 0$.

Notations 2. If condition $\mathbf{H}_{g_i,\sigma_j}$ is verified, we use the notations $l_g = \max_{i=1,\ldots} l_{g_i}$ and

$$\begin{aligned}\Upsilon &= C_0 \max_{i=1,\ldots,N} k_{g_i} + 2C_0 C_X(f) + b(C_0 + C_1) L_f + [T(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([-p,0];X)} \\ &\quad + \| \varphi \|_{C([-p,0];X_1)} + [\varphi]_{C_{\mathrm{Lip}}([-p,0];X)}.\end{aligned}$$

Theorem 2.2. *Assume that the conditions $\mathbf{H}_\zeta$, $\mathbf{H}_{g_i,\sigma_j}$, $\mathbf{H}_\mathbf{g}$ and $\mathbf{H}_\mathbf{f}$ are satisfied, X is a Hilbert space, A is self-adjoint, $T(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,a]; X)$, $\varphi \in C_{\mathrm{Lip}}([-p,0]; X) \cap C([-p,0]; X_1)$ and*

$$2bL_f((C_0 + C_1)[\zeta]_{C_{\mathrm{Lip}}}(1 + 2\Upsilon) + C_0) + 2C_0(l_g + L_g(1 + 2 \max_{i=1,\ldots,N} [\sigma_i]_{C_{\mathrm{Lip}}} \Upsilon)) < 1. \tag{2.10}$$

Then there exists a unique classical solution $u \in \mathcal{BPC}_{\mathrm{Lip}}(X)$ of the problem (1.1)–(1.3) such that $A\widetilde{u}_i \in C([t_i, t_{i+1}]; X)$ for all $i = 1, \ldots, N$.

Proof. Let $P : \mathbb{R} \mapsto \mathbb{R}$ be the polynomial given by

$$P(x) = \Upsilon + (bL_f((C_0 + C_1)[\zeta]_{C_{\mathrm{Lip}}} + C_0) + C_0(L_g + L_G) - 1)x \\ (C_0 L_g \max_{i=1,\ldots,N} [\sigma_i]_{C_{\mathrm{Lip}}} + bL_f(C_0 + C_1)[\zeta]_{C_{\mathrm{Lip}}})x^2. \quad (2.11)$$

From (2.10) there exists $R > 0$ such that $P(R) < 0$ and

$$\Upsilon + C_0 l_g R + (C_0 + C_1)bL_f[\zeta]_{C_{\mathrm{Lip}}}(R + R^2) < R, \quad (2.12)$$

$$C_0 L_{X,X}(g)(1 + R \max_{i=1,\ldots,N} [\sigma_i]_{C_{\mathrm{Lip}}}) + C_0 bL_f(1 + R[\zeta]_{C_{\mathrm{Lip}}}) < 1. \quad (2.13)$$

Let $\mathcal{S}(R)$ the space in the proof of Theorem 2.1 and $\mathcal{S}(\sigma_i, R)$ be the space

$$\mathcal{S}(\sigma_i, R) = \{u \in \mathcal{S}(R) : u(t) \in D(A) \text{ and } \| Au(t) \| \leq R, \ \forall\, t \in \cup_{i=1}^{N} \sigma_i(X)\}, \quad (2.14)$$

endowed with the metric $d(u, v) = \| u - v \|_{\mathcal{PC}(X)}$. Let $\Gamma : \mathcal{S}(\sigma_i, R) \mapsto \mathcal{BPC}(X)$ be defined as in the proof of Theorem 2.1. Next we prove that Γ is a contraction on $\mathcal{S}(\sigma_i, R)$.

Let $u \in \mathcal{S}(\sigma_i, R)$, $i \in \{1, \ldots, N\}$, $t \in (t_i, t_{i+1})$ and $h > 0$ such that $t + h \in (t_i, t_{i+1}]$. Arguing as in the proof of Theorem 2.1 and noting that $u(\sigma(u(t_i^+))) \in X_1$, we see that

$$\begin{aligned} &\| \Gamma u(t + h) - \Gamma u(t) \| \\ &\quad \leq \int_{t-t_i}^{t+h-t_i} \| T(s)Ag_i(u(\sigma_i(u(t_i^+)))) \| \,\mathrm{d}s \ + (C_0(\mathcal{C}_X(f) + bL_f)h \\ &\qquad + C_0 bL_f[\zeta]_{C_{\mathrm{Lip}}}(R + R^2)h \leq C_0(l_{g_i} R + k_{g_i})h \\ &\qquad + C_0(\mathcal{C}_X(f) + bL_f)h \\ &\qquad + C_0 bL_f[\zeta]_{C_{\mathrm{Lip}}}(R + R^2)h, \end{aligned}$$

and hence, $[(\Gamma u)_{|I_i}]_{C_{\mathrm{Lip}}(I_i;X)} \leq \Upsilon + C_0 l_{g_i} R + C_0 bL_f[\zeta]_{C_{\mathrm{Lip}}}(R + R^2) \leq R$. In addition, it is easy to see that

$$[(\Gamma u)_{|[0,t_1]}]_{C_{\mathrm{Lip}}([0,t_1];X)} \leq [T(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([-p,0];Z)} + C_0(\mathcal{C}_X(f) + bL_f) \\ + C_0 bL_f[\zeta]_{C_{\mathrm{Lip}}}(R + R^2)) \leq R.$$

From the above remarks we have that $[(\Gamma u)_{|[0,a]}]_{\mathcal{PC}_{\mathrm{Lip}}(X)} \leq R$ which shows that $\Gamma u \in \mathcal{S}(R)$. In addition, arguing as in the proof of Theorem 2.1 it follows that

$$d(\Gamma u, \Gamma v) \leq C_0(L_g(1 + R \max_{i=1,\ldots,N} [\sigma_i]_{C_{\mathrm{Lip}}}) + bL_f(1 + R[\zeta]_{C_{\mathrm{Lip}}}))d(u, v).$$

From the above remarks, we have that Γ is a contraction on $\mathcal{S}(R)$.

Next we show that $\| A\Gamma u(t) \| \leq R$ for all $t \in \cup_{j=1}^{N} \sigma_j(X)$. Let $t \in \cup_{j=1}^{N} \sigma_j(X)$ and assume that $t \in (t_i, t_{i+1}]$ for $i \geq 1$. Using that $(T(t))_{t\geq 0}$ is analytic and that $u(\sigma(u(t_i^+))) \in X_1$ and $\| Au(\sigma(u(t_i^+))) \| \leq l_{g_i} R + k_{g_i}$, we note that

$$A\Gamma u(t) = T(t - t_i)Ag_i(u(\sigma_i(u(t_i^+)))) \\ + T(t - t_i)f(t, u(\zeta(t, u(t)))) - f(t, u(\zeta(t, u(t))))$$

$$+ \int_{t_i}^{t} AT(t-s)(f(s,u(\zeta(s,u(s)))) - f(t,u(\zeta(t,u(t))))\mathrm{d}s,$$

$$\begin{aligned}\| A\Gamma u(t) \| &\leq C_0(l_{g_i}R + k_{g_i}) + 2C_0C_X(f) \\ &\quad + \int_0^t \frac{C_1}{t-s} L_f(1 + [u(\zeta(\cdot,u(\cdot)))]_{C_{\mathrm{Lip}}(I_i;X)})(t-s)\mathrm{d}s \\ &\leq C_0(l_{g_i}R + k_{g_i}) + 2C_0C_X(f) + bC_1L_f + C_1bL_f[\zeta]_{C_{\mathrm{Lip}}}(R+R^2),\end{aligned}$$

which implies that $\| A\Gamma u(t) \| \leq \Upsilon + C_0 l_{g_i}R + C_1bL_f[\zeta]_{C_{\mathrm{Lip}}}(R+R^2) \leq R$. If $t \in I_1$ we see that

$$\| A\Gamma u(t) \| \leq C_0 \| A\varphi(0) \| + 2C_0C_X(f) + bC_1L_f + C_1bL_f[\zeta]_{C_{\mathrm{Lip}}}(R+R^2) \leq R.$$

Thus, $\| A\Gamma u(t) \| \leq R$ for all $t \in \cup_{i=1}^{N}\sigma_i(X)$ and Γ is a $\mathcal{S}(\sigma_i,R)$-valued function.

To finish the proof, we prove that $\mathcal{S}(\sigma_i,R)$ is a closet subset of $\mathcal{S}(R)$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(\sigma_i,R)$ and $u \in \mathcal{BPC}(X)$ such that $u_n \to u$ as $n \to \infty$. Let $t \in \cup_{i=1}^{N}\sigma_i(X)$. Since $(Au_n(t))_{n\in\mathbb{N}}$ is bounded, there exists $w \in X$ such that $< Au_n(t), z > \to < w, z >$ as $n \to \infty$ for all $z \in X$. In particular, for $v \in X_1$ we have that $< Au_n(t), v > = < u_n(t), Av > \to < u(t), Av >$ as $n \to \infty$, which implies that $< w, v > = < u(t), Av >$ for all $v \in X_1$. Using that A is self-adjoint, we obtain that $u(t) \in X_1$, $Au(t) = w$ and $\| Au(t) \| = \| w \| \leq \liminf_{n\to\infty} \| Au_n(t) \| \leq R$, which completes the proof that $\mathcal{S}(\sigma_i,R)$ is closed.

From the above it follows that Γ is a contraction on $\mathcal{S}(\sigma_i,R)$ and there exists a unique mild solution $u \in \mathcal{S}(\sigma_i,R)$. The fact that $u(\cdot)$ is a classical solution follows from the proof of Theorem 2.1. □

The next result consider the case where $\sigma_i(X) \subset [-p,0]$ for all $i = 1,\ldots,N$. The proof use the ideas in the proof of Theorem 2.1 and we include a short proof for completeness.

Proposition 2.1. *Let conditions* $\mathbf{H_g}$ *and* $\mathbf{H_f}$ *be holds. Assume* $\zeta \in C_{\mathrm{Lip}}([0,a]\times X;[-p,a])$, $\sigma_i \in C_{\mathrm{Lip}}(X;[-p,0])$ *for all* $i = 1,\ldots,N$, $T(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,a];X)$, $\varphi \in C_{\mathrm{Lip}}([-p,0];X)$, $g_i(\varphi(\cdot)) \in C([-p,0];W)$ *for all* $i = 1,\ldots,N$ *and*

$$2C_0bL_f(1 + [\zeta]_{C_{\mathrm{Lip}}}(1 + 2\Phi_{X,W,\varphi})) + 2L_g\Psi_{\varphi,\sigma_i,g_i} < 1. \tag{2.15}$$

where $\Phi_{X,W,\varphi} = \Phi_{X,W}\max_{i=1,\ldots,N} \| g_i(\varphi(\cdot)) \|_{C([-p,0];W)} + C_0(\mathcal{C}_X(f) + bL_f) + [\varphi]_{C_{\mathrm{Lip}}([-p,0];X)} + [T(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([-p,0];X)}$ *and* $\Psi_{\varphi,\sigma_i,g_i} = C_0[\varphi]_{C_{\mathrm{Lip}}([-p,0];X)}\max_{i=1,\ldots,N}[\sigma_i]_{C_{\mathrm{Lip}}}$. *Then there exists a unique classical solution* $u \in \mathcal{BPC}_{\mathrm{Lip}}(X)$ *of the problem* (1.1)–(1.3).

Proof. Let $P : \mathbb{R} \mapsto \mathbb{R}$ be given by $P(x) = \Phi_{X,W,\varphi} + (C_0bL_f(1 + [\zeta]_{C_{\mathrm{Lip}}}) + L_g\Psi_{\varphi,\sigma_i,g_i} - 1)x + C_0bL_f[\zeta]_{C_{\mathrm{Lip}}}x^2$. From (2.15) there exists $R > 0$ such that $P(R) < 0$. Let $\mathcal{S}(R)$ be defined as in the proof of Theorem 2.1 and $\Gamma : \mathcal{S}(R) \mapsto \mathcal{BPC}(X)$ be the map given by $\Gamma u_0 = \varphi$ and

$$\Gamma u(t) = T(t)\varphi(0) + \int_0^t T(t-\tau)f(\tau,u(\zeta(\tau,u(\tau))))\mathrm{d}\tau, \quad t \in [0,t_1],$$

$$\Gamma u(t) = T(t-t_i)g_i(\varphi(\sigma_i(u(t_i^+)))) + \int_{t_i}^{t} T(t-s)f(s,u(\zeta(s,u(s))))\mathrm{d}s, \quad t \in (t_i,t_{i+1}].$$

Arguing as in the proof of Theorem 2.1, for $i \in \{1, \dots, i\}$, $t \in (t_i, t_{i+1})$ and $h > 0$ such that $t + h \in (t_i, t_{i+1}]$, it is easy to see that

$$\| \Gamma u(t+h) - \Gamma u(t) \| \leq \Phi_{X,W} \max_{j=1,\dots,N,} \| g_j(\varphi(\cdot)) \|_{C([-p,0];W)} h$$
$$+ (C_0 \mathcal{C}_X(f) + bL_f)h + C_0 bL_f [\zeta]_{C_{\mathrm{Lip}}} (R + R^2)h,$$

which implies (from the definition of $P(\cdot)$) that $[(\Gamma u)_{|I_i}]_{C_{\mathrm{Lip}}(I_i;X)} \leq R$. Similarly, we have that

$$[(\Gamma u)_{|[0,t_1]}]_{C_{\mathrm{Lip}}([0,t_1];X)} \leq [T(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([-p,0];X)} + C_0(\mathcal{C}_X(f) + bL_f)$$
$$+ C_0 bL_f [\zeta]_{C_{\mathrm{Lip}}} (R + R^2)$$
$$\leq \Phi_{X,W,\varphi} + C_0 bL_f [\zeta]_{C_{\mathrm{Lip}}} (R + R^2) \leq R.$$

From the above, $[(\Gamma u)_{|[0,a]}]_{\mathcal{PC}_{\mathrm{Lip}}(X)} \leq R$, which proves that Γ is a $\mathcal{S}(R)$-valued function.

On the other hand, for $u, v \in \mathcal{S}(R)$, $i = 1, \dots, N$, $t \in (t_i, t_{i+1}]$ and $s \in [0, t_1]$ we get

$$\| \Gamma u(t) - \Gamma v(t) \| \leq (C_0 L_g [\varphi]_{C_{\mathrm{Lip}}([-p,0];X)} \max_{j=1,\dots,N} [\sigma_j]_{C_{\mathrm{Lip}}}$$
$$+ C_0 bL_f (1 + R[\zeta]_{C_{\mathrm{Lip}}}))d(u,v),$$
$$\| \Gamma u(s) - \Gamma v(s) \| \leq C_0 bL_f (1 + R[\zeta]_{C_{\mathrm{Lip}}})d(u,v),$$

which allows us infer that Γ is a contraction and there exists a unique mild solution $u \in \mathcal{S}(R)$ of the problem (1.1)–(1.3). The fact that $u(\cdot)$ is a classical solution follows from the proof of Theorem 2.1. □

In the next result, we assume that the functions $\zeta(\cdot)$ and $\sigma_i(\cdot)$ have values in $[-r, 0]$.

Proposition 2.2. *Suppose that the conditions* $\mathbf{H_g}$, $\mathbf{H_f}$ *are satisfied,* $\varphi \in C_{\mathrm{Lip}}([-p,0];X)$, $\sigma_i \in C_{\mathrm{Lip}}(X;[-p,0])$ *for all* $i = 1, \dots, N$, $\zeta \in C_{\mathrm{Lip}}([0,a] \times X;[-p,0])$ *and*

$$C_0[\varphi]_{C_{\mathrm{Lip}}([-p,0];X)}(L_g \max_{j=1,\dots,N} [\sigma_j]_{C_{\mathrm{Lip}}} + bL_f[\zeta]_{C_{\mathrm{Lip}}}) < 1.$$

Then there exists a unique mild solution $u \in \mathcal{PC}_{\mathrm{Lip}}(X)$ *of* (1.1)–(1.3).

Proof. Let $\Gamma : \mathcal{BPC}(X) \mapsto \mathcal{BPC}(X)$ be defined as in the proof of Theorem 2.1, but using $f(\tau, \varphi(\zeta(\tau, u(\tau))))$ in place $f(\tau, u(\zeta(\tau, u(\tau))))$. In this case, for $u, v \in \mathcal{BPC}_{\mathrm{Lip}}(X)$ we see that

$$\| \Gamma u - \Gamma v \|_{C((t_i,t_{i+1}];X)} \leq C_0 L_g [\varphi]_{C_{\mathrm{Lip}}([-p,0];X)} \max_{j=1,\dots,N} [\sigma_j]_{C_{\mathrm{Lip}}} d(u,v)$$
$$+ C_0 bL_f [\varphi]_{C_{\mathrm{Lip}}([-p,0];X)} [\zeta]_{C_{\mathrm{Lip}}} d(u,v),$$
$$\| \Gamma u - \Gamma v \|_{C([0,t_1];X)} \leq C_0 bL_f [\varphi]_{C_{\mathrm{Lip}}([-p,0];X)} [\zeta]_{C_{\mathrm{Lip}}} d(u,v),$$

which allows us to conclude that Γ is a contraction. □

Next, we discuss briefly the case in which the functions $f(\cdot)$ and $g_i(\cdot)$ are locally bounded and (or) locally Lipschitz. For sake of clarity, we include the next conditions.

$\mathcal{H}^W_{g,X}$ For all $i = 1, \ldots, N$, there is $L_{X,W}(g_i, \cdot) \in C(\mathbb{R}; \mathbb{R})$ such that $\| g_i(x) - g_i(y) \|_W \leq L_{X,W}(g_i, r) \| x - y \|$ for all $x, y \in B_r(0, X)$ and every $r > 0$. Next, $L_{X,W}(g, r) = \max_{i=1,\ldots,N} L_{X,W}(g_i, r)$ and $\mathcal{C}_{X,W}(g_i, r) = \| g_i \|_{C(B_r(0,X);W)}$.

$\mathcal{H}_f$ There is $L_f \in C(\mathbb{R}; \mathbb{R})$ such that $\| f(t, x) - f(s, y) \| \leq L_f(r)(| t - s | + \| x - y \|)$ for all $x, y \in B_r(0, X)$, $t, s \in [0, a]$ and $r > 0$. Next, for $r > 0$ we use the notation $\mathcal{C}_X(f, r) = \| f \|_{C([0,a]\times B_r(0,X);X)}$.

$\mathcal{H}_\mathbf{g}$ There are functions $L_{g_i} \in C(\mathbb{R}; \mathbb{R})$ such that $\| g_i(x) - g_i(y) \| \leq L_{g_i}(r) \| x - y \|$ for all $x, y \in B_r(0, X)$ and $r > 0$. Next, $L_g(r) = \max_{i=1,\ldots,N} L_{g_i}(r)$, $\mathcal{C}_X(g)(r) = \max_{i=1,\ldots,N} \mathcal{C}_X(g_i)(r)$ and $\mathcal{C}_X(g_i, r) = \| g_i \|_{C(B_r(0,X);X)}$.

Notations 3. For $r > 0$, we define $\Phi_{X,W}(r) = \Lambda_{X,W}\mathcal{C}_{X,W}(g, r) + C_0 (C_X(f, r) + bL_f(r)) + [T(\cdot)\varphi(0)]_{C_{\rm Lip}([-p,0];X)} + [\varphi]_{C_{\rm Lip}([-p,0];X)}$.

The proof of Proposition 2.3 follows from the proof of Theorem 2.1.

Proposition 2.3. *Let conditions* $\mathbf{H}_\zeta, \mathbf{H}_{\sigma_\mathbf{i}}, \mathcal{H}^W_{g,X}$ *and* $\mathcal{H}_f$ *be holds. Suppose that* $T(\cdot)\varphi(0) \in C_{\rm Lip}([0, a]; X)$, $\varphi \in C_{\rm Lip}([-p, 0]; X)$ *and there is* $r > 0$ *such that* (2.4) *is satisfied with* $L_f(r)$, $\Phi_{X,W}(r)$ *and* $L_{X,W}(g, r)$ *in place* $L_f, \Phi_{X,W}$ *and* $L_{X,W}(g)$, *and*

$$\max\{C_0(\max\{\| \varphi(0) \|, \| i_c \|_{\mathcal{L}(W,X)} \mathcal{C}_{X,W}(g, r)\} + b\mathcal{C}_X(f, r)), \| \varphi \|_{C([-p,0];X)}\} \leq r.$$

Then there exists a unique classical solution $u \in \mathcal{BPC}_{\rm Lip}(X)$ *of the problem* (1.1)–(1.3).

Proof. Let $P : \mathbb{R} \mapsto \mathbb{R}$ be defined as in the proof of Theorem 2.1, but using $L_f(r), \Phi_{X,W}(r)$ and $L_{X,W}(g, r)$ in place $L_f, \Phi_{X,W}$ and $L_{X,W}(g)$. Arguing as in the proof of Theorem 2.1 we infer that there exists $R > 0$ such that

$$\Phi_{X,W}(r) + C_0 bL_f(r)[\zeta]_{C_{\rm Lip}}(R + R^2) < R, \tag{2.16}$$

$$\Lambda_{X,W} L_{X,W}(g, r)(1 + R \max_{i=1,\ldots,N}[\sigma_i]_{C_{\rm Lip}}) + C_0 bL_f(r)(1 + R[\zeta]_{C_{\rm Lip}}) < 1. \tag{2.17}$$

Let $\mathcal{S}(R)$ be the space in the proof of Theorem 2.1 and $\mathcal{S}(r, R) = \{u \in \mathcal{S}(R) : \| u \|_{\mathcal{BPC}(X)} \leq r\}$, endowed with the metric $d(u, v) = \| u - v \|_{\mathcal{BPC}(X)}$. Let $\Gamma : \mathcal{S}(r, R) \mapsto \mathcal{BPC}(X)$ be defined as in the proof of Theorem 2.1.

From the proof of Theorem 2.1 we infer that Γ is a contraction on $\mathcal{S}(R)$. Moreover, for $t \in I_i$ with $i \geq 0$ it is easy to see that

$$\| \Gamma u(t) \| \leq C_0 \max\{\| \varphi(0) \|, \| i_c \|_{\mathcal{L}(W,X)} \mathcal{C}_{X,W}(g, r)\} + C_0 b\mathcal{C}_X(f, r) \leq r,$$

which implies that $\| \Gamma u \|_{\mathcal{BPC}(X)} \leq r$ since $r > \| \varphi \|_{C([-p,0];X)}$. Thus, Γ is a contraction on $\mathcal{S}(r, R)$ and there exists a unique mild solution $u \in \mathcal{S}(r, R)$ of (1.1)–(1.3). Finally, from [35, Theorem 4.3.2] we infer that $u(\cdot)$ is a classical solution. □

Corollary 2.1. *Assume that the conditions* $\mathbf{H}_\zeta, \mathbf{H}_{\sigma_\mathbf{i}}, \mathcal{H}^W_{g,X}$ *and* $\mathcal{H}_f$ *are satisfied, the functions* $L_f(\cdot)$, $\mathcal{C}_X(f, \cdot)$, $L_{X,W}(g, \cdot)$ *and* $\mathcal{C}_{X,W}(g, \cdot)$ *are non-decreasing,* $\varphi \in C_{\rm Lip}([-p, 0]; X)$, $T(\cdot)\varphi(0) \in C_{\rm Lip}([0, a]; X)$, $\limsup_{r\to\infty} \frac{1}{r} C_0(\|$

$i_c \|_{\mathcal{L}(W,X)} \mathcal{C}_{X,W}(g,r) + b\mathcal{C}_X(f,r)) < 1$ *and*

$$2C_0 b \limsup_{r\to\infty} L_f(r)\frac{1}{r}(1+[\zeta]_{C_{\mathrm{Lip}}}(1+2\Phi_{X,W}(r)) \\ +\ 2\Lambda_{X,W} \limsup_{r\to\infty} L_{X,W}(g,r)\frac{1}{r}(1+2\max_{i=1,\ldots,N}[\sigma_i]_{C_{\mathrm{Lip}}}\Phi_{X,W}(r)) < 1. \tag{2.18}$$

Then there exists a unique classical solution $u \in \mathcal{BPC}_{\mathrm{Lip}}(X) \cap B_r(0, \mathcal{BPC}(X))$ *of* (1.1)–(1.3).

We establish now, without proof, a result similar to Theorem 2.2 for the case where $f(\cdot)$ satisfy the condition $\mathcal{H}_f$.

Proposition 2.4. *Suppose the conditions* $\mathbf{H}_\zeta$, $\mathbf{H}_{g_i,\sigma_j}$, $\mathcal{H}_g$ *and* $\mathcal{H}_f$ *be holds,* X *is a Hilbert space,* A *is self-adjoint,* $T(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,a];X)$ *and* $\varphi \in C_{\mathrm{Lip}}([-p,0];X) \cap C([-p,0];X_1)$. *If there is* $r > 0$ *such that the inequality* (2.10) *is valid with* $L_f(r)$, $L_g(r)$, $\mathcal{C}_X(f,r)$ *and* $\mathcal{C}_X(g,r)$ *in place* L_f, L_g, $\mathcal{C}_X(f)$ *and* $\mathcal{C}_X(g)$, *and* $C_0(\max\{\| \varphi(0) \|, \mathcal{C}_X(g,r)\} + b\mathcal{C}_X(f,r)) \le r$, *then there exists a unique classical solution* $u \in B_r(0, \mathcal{BPC}(X)) \cap \mathcal{PC}_{\mathrm{Lip}}(X))$ *of* (1.1)–(1.3).

To complete this section, we study the existence of solution using the Schauder's fixed point Theorem. The next lemma follows from the proof of [32, Proposition 4.2.1].

Lemma 2.3. *Let* $\alpha \in (0,1)$, $\xi \in L^\infty([b,c];X)$ *and* $v : [b,c] \mapsto X$ *be the function defined by* $v(t) = \int_b^t T(t-s)\xi(s)ds$. *Then* $[v]_{C^\alpha([b,c];X)} \le \| \xi \|_{L^\infty([b,c];X)} ((c-b)^{1-\alpha}C_0 + \frac{C_1}{\alpha(1-\alpha)})$.

Theorem 2.3. *Assume that the conditions* $\mathbf{H}_\zeta$ *and* $\mathbf{H}_{\sigma_\mathrm{i}}$ *are satisfied, there is a Banach space* $(Y, \|\cdot\|_Y) \hookrightarrow (X, \|\cdot\|)$ *such that* $\| T(t) - I \|_{\mathcal{L}(Y,X)} \to 0$ *as* $t \to 0$, $g_i \in C(X;Y)$ *for all* i, $f \in C([0,a]\times X;X)$, *the functions* $g_i(\cdot), f(\cdot)$ *are bounded and* $(T(t))_{t\ge 0}$ *is compact. Then there exists a mild solution of the problem* (1.1)–(1.3).

Proof. Let $\mathcal{C}_{X,Y}(g) = \max_{i=1,\ldots,N} \| g_i \|_{C(X;Y)}$, $C_X(f) = \| f \|_{C([0,a]\times X;X)}$ and $\alpha \in (0,1)$. Let $\mathcal{BPC}_\varphi(X) = \{u \in \mathcal{BPC}(X) : u_0 = \varphi\}$ endowed with the metric $d(u,v) = \| u - v \|_{\mathcal{BPC}(X)}$ and $\Gamma : \mathcal{BPC}_\varphi(X) \mapsto \mathcal{BPC}(X)$ be defined as in the proof of Theorem 2.1.

It is easy to prove that Γ is continuous. Next, using Lemma 1.1, we show that Γ is completely continuous.

Let $i \in \{1,\ldots,N\}$. From Lemma 2.3, for $t \in (t_i, t_{i+1})$, $h > 0$ with $t+h \in (t_i, t_{i+1}]$, we get

$$\begin{aligned} &\| \Gamma u(t+h) - \Gamma u(t) \| \\ &\le \| (T(t+h-t_i) - T(t-t_i))g_i(u(\sigma_i(u(t_i^+)))) \| \\ &\quad + \| \int_{t_i}^{t+h} T(t+h-s)f(s,u(\zeta(s,u(s))))\mathrm{d}s - \int_{t_i}^{t} T(t-s)f(s,u(\zeta(s,u(s))))\mathrm{d}s \| \\ &\le \| (T(t+h-t_i) - T(t-t_i)) \|_{\mathcal{L}(Y,X)} \mathcal{C}_{X,Y}(g) + \mathcal{C}_X(f)\left(a^{1-\alpha}C_0 + \frac{C_1}{\alpha(1-\alpha)}\right)h^\alpha, \end{aligned}$$

which shows that $\{(\Gamma u)_{|_{I_i}} : u \in \mathcal{BPC}_\varphi(X)\}$ is right equicontinuous at $t \in (t_i, t_{i+1})$. A similar argument prove that $\{(\Gamma u)_{|_{I_i}} : u \in \mathcal{BPC}_\varphi(X)\}$ is left equicontinuous at $t = t_{i+1}$, which implies that $\{(\Gamma u)_{|_{I_i}} : u \in \mathcal{BPC}_\varphi(X)\}$ is equicontinuous on I_i. In addition, for $u \in \mathcal{BPC}_\varphi(X)$ and $0 < h < \delta$ we note that

$$\begin{aligned}
&\| \widetilde{\Gamma u}(t_i + h) - \widetilde{\Gamma u}(t_i) \| \\
&\quad = \| (T(h) - I) g_i(u(\sigma_i(u(t_i^+)))) \| + \int_{t_i}^{t_i+h} T(t_i + h - s) f(s, u(\zeta(s, u(s)))) \mathrm{d}s \| \\
&\quad \leq \| T(h) - I \|_{\mathcal{L}(Y,X)} \mathcal{C}_{X,Y}(g) + \mathcal{C}_X(f) \left(a^{1-\alpha} C_0 + \frac{C_1}{\alpha(1-\alpha)} \right) h^\alpha,
\end{aligned}$$

which proves that $\Gamma \widetilde{\mathcal{BPC}_\varphi(X)}_i = \{(\widetilde{\Gamma u})_i : u \in \mathcal{BPC}_\varphi(X)\}$ is right equicontinuous at t_i. From the above it follows that $\{\widetilde{(\Gamma u)}_i : u \in \mathcal{BPC}_\varphi(X)\}$ is equicontinuous on I_i.

We prove now that $\{(\widetilde{\Gamma u})_i(t) : u \in \mathcal{BPC}_\varphi(X)\}$ is relatively compact in X for all $t \in [t_i, t_{i+1}]$. Since the semigroup is compact, $(Y, \| \cdot \|_Y) \hookrightarrow (X, \| \cdot \|)$ and $g_i(\cdot)$ is bounded with values in Y, we have that $U = \{g_j(u(\sigma_j(u(t_j^+)))) : u \in \mathcal{BPC}_\varphi(X),\ j = 1, \ldots, N\}$ is relatively compact in X. For $t \in (t_i, t_{i+1}]$ and $0 < \varepsilon < t - t_i$, we note that

$$\begin{aligned}
(\widetilde{\Gamma u})_i(t) &= T(t - t_i) U + T(\varepsilon) \int_{t_i}^{t-\varepsilon} T(t - \varepsilon - s) f(s, u(\zeta(s, u(s)))) \mathrm{d}s \\
&\quad + \int_{t-\varepsilon}^{t} T(t - s) f(s, u(\zeta(s, u(s)))) \mathrm{d}s \\
&\in T(t - t_i) U + T(\varepsilon) C_0 (t - \varepsilon - t_i) \mathcal{C}_X(f) B_1(0, X) + \varepsilon C_0 \mathcal{C}_X(f) B_1(0, X),
\end{aligned}$$

and hence, $\{(\widetilde{\Gamma u})_i(t) : u \in \mathcal{BPC}_\varphi(X)\} \subset K_\varepsilon + D_\varepsilon$, where K_ε is relatively compact and the diameter of D_ε converges to zero as $\varepsilon \to 0$. This prove that the set $\Gamma \mathcal{BPC}_\varphi(X)(t)$ is relatively compact in X. Moreover, since $\Gamma \widetilde{\mathcal{BPC}_\varphi(X)}_i(t_i)$ $= \{g_i(u(\sigma_i(u(t_i^+)))) : u \in \mathcal{BPC}_\varphi(X)\} \subset \overline{U}$, we obtain that $\Gamma \widetilde{\mathcal{BPC}_\varphi(X)}(t_i)$ is relatively compact in X. From the above remarks we have that $(\Gamma \widetilde{\mathcal{BPC}_\varphi(X)})_i$ is relatively compact in $C([t_i, t_{i+1}]; X)$. Moreover, the same argument also prove that $(\Gamma \widetilde{\mathcal{BPC}_\varphi}(X))_1 = \{(\Gamma u)_{|_{[0,t_1]}} : u \in \mathcal{BPC}_\varphi(X))\}$ is relatively compact in $C([0, t_1]; X)$.

From the above and Lemma 1.1, it follows that Γ is completely continuous and noting that the functions $f(\cdot)$ and $g_i(\cdot)$ are bounded, we infer that there exists $r > 0$ such that $\Gamma(\mathcal{BPC}_\varphi(X)) \subset B_r(0, \mathcal{BPC}_\varphi(X))$. Thus, Γ is completely continuous from $B_r(0, \mathcal{BPC}_\varphi(X))$ into $B_r(0, \mathcal{BPC}_\varphi(X))$ and there exists a mild solution $u \in B_r(0, \mathcal{BPC}_\varphi(X))$ of (1.1)–(1.3). □

3. Examples

In this section, $X = L^2(\Omega; \mathbb{R})$ or $X = C(\Omega; \mathbb{R})$, $\Omega \subset \mathbb{R}^n$ is a open set with smooth boundary and $A : D(A) \subset X \mapsto X$ is the realization of an second order strongly elliptic operator. Next, we assume that $(T(t))_{t\geq}$ is the analytic semigroup generated by A, $D(A) = \{u \in L^2(\Omega) : Au \in L^2(\Omega)\}$ if $\Omega = \mathbb{R}^n$ and $D(A) = W^{2,2}(\Omega) \cap W_0^{2,1}(\Omega)$ if Ω is bounded. For sake of simplicity, we suppose that the conditions $\mathbf{H}_\zeta$ and $\mathbf{H}_{\sigma_\mathbf{i}}$ are satisfies, $0 \in \rho(A)$, $\varphi \in C_{\mathrm{Lip}}([-p,0];X)$ and $T(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,a];X)$. In addition, X_1 is the domain of A endowed with the norm $\| x \|_{X_1} = \| Ax \|$ and C_0, C_1 are the constants in the introduction.

To begin, we study the impulsive problem

$$u'(t,\xi) = Au(t)(\xi) + \beta_1(t,\xi,u(\zeta(u(t))-t,\xi)) + \beta_2(t)u(\zeta(u(t)),\xi),\ t \in I_i,\ \xi \in \Omega, \tag{3.1}$$

$$u(t_i^+,\xi) = \int_{\mathbb{R}^n} \mathcal{L}_i(\xi,y)u(\sigma(u(t_i^+)),y)\mathrm{d}y, \tag{3.2}$$

$$u(\theta,\xi) = \varphi(\theta,\xi), \qquad \theta \in [-p,0], \tag{3.3}$$

where $\Omega = \mathbb{R}^n$, $X = L^2(\Omega;\mathbb{R})$, $0 = t_0 < \cdots < t_{N+1} = a$ are pre-fixed, $I_i = (t_i, t_{i+1}]$, $\beta_1 \in C_{\mathrm{Lip}}([0,a]\times\mathbb{R};\mathbb{R})$, $\beta_1(\cdot)$ is bounded, $\beta_2 \in C_{\mathrm{Lip}}([-p,a];\mathbb{R})$ and $\mathcal{L}_i, A\mathcal{L}_i \in L^2(\Omega\times\Omega,\mathbb{R})$. In addition, we assume that there is $\gamma \in L^p(\Omega)$ such that

$$| \beta_1(t,\xi,x) - \beta_1(s,\xi,y) | \leq \gamma(\xi)(| t-s | + | x-y |), \qquad \forall\ t,s \in [0,a], \xi, x, y \in \mathbb{R}^n.$$

To represent this problem in the form (1.1)–(1.3) we define the functions $g_i(\cdot)$ and $f(\cdot)$ by $g_i(t,x)(\xi) = \int_{\mathbb{R}^n} \mathcal{L}(\xi,y)x(y)\mathrm{d}y$ and $f(t,x)(\xi) = \beta_1(t,\xi,x(\xi)) + \beta_2(t)x(\xi)$. It is easy to see that $\| Ag_i(x) \| \leq \| A\mathcal{L}_i \|_{L^2(\Omega\times\Omega;\mathbb{R})} \| x \|$ and

$$\begin{aligned} \| f(t,x) - f(s,y) \| &\leq [\gamma]_{C_{\mathrm{Lip}}} | t-s | + \| \gamma \|_{C(\Omega)} \| x-y \| \\ &\quad + [\beta_2]_{C_{\mathrm{Lip}}([0,a];\mathbb{R})} \| x \| | t-s | + \| \beta_2 \|_{C(\Omega)} \| x-y \| . \end{aligned}$$

Thus, we can apply Proposition 2.3 with $L_f(r) = \| \gamma \|_{C_{\mathrm{Lip}}(\Omega)} + [\beta_2]_{C_{\mathrm{Lip}}} r + \| \beta_2 \|_{C(\Omega)}$, $C_X(f,r) = \| \beta_1 \|_{C([0,a]\times\Omega\times\Omega;\mathbb{R})} + \| \beta_2 \|_{C(\Omega)}\ r$, $L_{X,X_1}(g_i) = \| A\mathcal{L}_i \|_{L^2(\Omega\times\Omega;\mathbb{R})}$, $C_{X,X_1}(g,r) = \max_{i=1,\dots,n} \| A\mathcal{L}_i \|_{L^2(\Omega\times\Omega;\mathbb{R})}\ r$ and $L_{X,X_1}(g) = \sup_{i=1,\dots,n} \| A\mathcal{L}_i \|_{L^2(\Omega\times\Omega;\mathbb{R})}$.

In the next result, we adopt the above notations and the notations in Remark 1. In addition, we say that $u \in \mathcal{BPC}(X)$ is a classical solution of (3.1)–(3.3) if $u(\cdot)$ is a classical solution of the associate problem (1.1)–(1.3) and we adopt a similar (for mild and classical solutions) in the following examples.

Proposition 3.5. *If* $\max\{\| \varphi \|_{C([-p,0];X)}, C_0 b\mathcal{C}_X(f,r) + C_1\mathcal{C}_{X,W}(g,r)\} \leq r$ *and*

$$\begin{aligned} &2C_0 bL_f(r)\frac{1}{r}(1 + [\zeta]_{C_{\mathrm{Lip}}}(1 + 2\Phi_{X,W}(r)) \\ &\quad + 2\Lambda_{X,W}L_{X,W}(g,r)\frac{1}{r}(1 + 2\max_{i=1,\dots,N}[\sigma_i]_{C_{\mathrm{Lip}}}\Phi_{X,W}(r)) < 1, \end{aligned}$$

for some $r > 0$, then there exists a unique classical solution $u \in \mathcal{BPC}_{\mathrm{Lip}}(X)$ of (3.1)–(3.3).

We study now the problem

$$u'(t,x) = Au(t)(x) + \int_0^t \beta_1(s,u(\zeta(u(t))-t,x))\mathrm{d}s, \quad x \in \Omega,\ t \in I_i = (t_i,t_{i+1}], \tag{3.4}$$

$$u(t_i^+, x) = \alpha_i u(\sigma_i(u(t_i^+)), x), \tag{3.5}$$
$$u(\theta, x) = \varphi(\theta, x), \qquad \theta \in [-p, 0], x \in \Omega, \tag{3.6}$$

where Ω is bounded, $\beta_1 \in C_{\text{Lip}}([0,a] \times \mathbb{R}; \mathbb{R})$ and $\beta_1(\cdot)$ is bounded.

To apply Theorem 2.2, we assume $X = L^2(\Omega)$, the condition $\mathbf{H}_{g_i,\sigma_j}$ is satisfied and we define $g_i(\cdot)$ and $f(\cdot)$ by $f(t,x)(\xi) = \int_0^t \beta_1(\tau, x(\xi))\mathrm{d}\tau$ and $g_i(t,x)(\xi) = \alpha_i x(\xi)$. From the above,

$$\| f(t,x) - f(s,y) \| \leq \| \beta_1 \|_{C([0,a]\times\mathbb{R};\mathbb{R})} | t - s | + b[\beta_1]_{C_{\text{Lip}}([0,a]\times\mathbb{R};\mathbb{R})} \| x - y \|,$$
$$\| g_i(x) - g_i(y) \| \leq | \alpha_i | \| x - y \|, \quad \| Ag_i(z) \| \leq | \alpha_i | \| Az \|,$$

for $t, s \in [0, a]$, $x, y \in X$ and $z \in D(A)$, and the conditions in Theorem 2.2 are satisfied with $L_f = (1 + b) \| \beta_1 \|_{C_{\text{Lip}}([0,a]\times\mathbb{R};\mathbb{R})}$, $C_X(f) = a \| \beta_1 \|_{C([0,a]\times\mathbb{R};\mathbb{R})}$, $l_{g_i} = | \alpha_i |$, $k_{g_i} = 0$, $L_g = \max_{i=1,\ldots,N} | \alpha_i |$ and $\Upsilon = 2C_0 C_X(f) + b(C_0 + C_1)L_f + [T(\cdot)\varphi(0)]_{C_{\text{Lip}}([-p,0];X)} + [\varphi]_{C_{\text{Lip}}([-p,0];X)}$. The next result follows from Theorem 2.2.

Proposition 3.6. *Under the above conditions and notations, if the inequality (2.10) is verified, then there exists a unique classical solution* $u \in \mathcal{BPC}_{\text{Lip}}(X)$ *of (3.4)–(3.6).*

We complete this section studying a problem motivated by equations arising in population dynamics. Consider the problem

$$u'(t,x) = Au(t)(x) + \alpha u(t,x)(1 - u(t,x)), \quad x \in \Omega,\ t \in I_i = (t_i, t_{i+1}], \tag{3.7}$$
$$u(t_i^+, x) = \alpha_i u(\sigma_i(u(t_i^+)), x), \tag{3.8}$$
$$u(\theta, x) = \varphi(\theta, x), \qquad \theta \in [-p, 0]. \tag{3.9}$$

To treat this problem, we assume $X = C(\Omega; \mathbb{R})$ and $\alpha, \alpha_i \in \mathbb{R}$ and we define $g_i(\cdot)$ and $f(\cdot)$ by $g_i(t,x)(\xi) = \alpha_i x(\xi)$ and $f(t,x)(\xi) = \alpha x(\xi)(1 - x(\xi))$. It is trivial to see that

$$\| f(t,x) - f(s,y) \| \leq | \alpha | (1 + 2r) \| x - y \|, \quad \| f(t,x) \| \leq | \alpha | r(1 + r),$$
$$\| g_i(x) - g_i(y) \| \leq | \alpha_i | \| x - y \| \ \text{ and } \ \| Ag_i(z) \| \leq | \alpha_i | \| Az \|,$$

for all $t, s \in [0, a]$, $x, y \in B_r(0; X)$ and $z \in D(A)$. From Proposition 2.4, we get.

Proposition 3.7. *Suppose that there is* $r > \| \varphi \|_{C([-p,0];X)}$ *such that the inequality* (2.10) *is verified with* $L_f(r)$ *in place* L_f *and* $C_0(\| \varphi(0) \| + b | \alpha | (1 + 2r)) + C_0(\max_{i=1,\ldots,N} l_{g_i} r + k_{g_i}) < r$. *Then there exists a unique classical solution* $u \in \mathcal{BPC}_{\text{Lip}}(X)$ *of* (3.7)–(3.9).

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Eduardo Hernández and Katia A. G. Azevedo
Departamento de Computação e Matemática, Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto
Universidade de São Paulo
Ribeirão Preto SPCEP 14040-901
Brazil
e-mail: lalohm@ffclrp.usp.br

Katia A. G. Azevedo
e-mail: kandreia@ffclrp.usp.br

Marta C. Gadotti
Departamento de Matemática, Instituto de Geociências e Ciências Exatas.
Universidade Estadual Paulista
Rio Claro SPCEP 13506-900
Brazil
e-mail: martacg@rc.unesp.br