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Existence of positive periodic solutions for a neutral Liénard equation with a singularity of repulsive type

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Abstract. The periodic problem is studied in this paper for the neutral Liénard equation with a singularity of repulsive type

$$(x(t) - cx(t-\sigma))'' + f(x(t))x'(t) + \varphi(t)x(t-\tau) - \frac{r(t)}{x^{\mu}(t)} = h(t),$$

where $f : [0, +\infty) \to R$ is continuous, $r : R \to (0, +\infty)$ and $\varphi : R \to R$ are continuous with T-periodicity in the t variable, c, μ, σ, τ are constants with $|c| > 1, \mu > 1, 0 < \sigma, \tau < T$. Many authors obtained the existence of periodic solutions under the condition $|c| < 1$, and we extend their results to the case of $|c| > 1$. The proof of the main result relies on a continuation theorem of coincidence degree theory established by Mawhin.

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1. Introduction

As we all know, differential equations with singularities have a wide range of applications in physics, mechanics and biology (see [1–5]). For example, the positive periodic solutions for the singular equation [6]

$$x''(t) + cx'(t) - \frac{1}{x(t)} = e(t)$$

can be used to describe the movement of the piston at the bottom of the enclosing cylinder which under the effect of restoring forces is caused by the compressed gas. It has been recognized that the paper [7] is a major milestone in the study of problem of periodic solutions for second-order differential

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equation with singularities. In [7], Lazer and Solimini studied the following equation with a singularity of repulsive type

$$x''(t) - \frac{1}{x^{\alpha}(t)} = h(t), \tag{1.1}$$

where $h : R \to R$ is a T-periodic continuous function. Using the topological degree methods as well as the lower and upper function method, they obtained that $\frac{1}{T}\int_0^T h(s)\mathrm{d}s < 0$ is the necessary and sufficient condition for the existence of positive periodic solutions to (1.1) under the condition $\alpha \geq 1$. Since then, many authors have focused their attention on the equations with singularities of repulsive type [7–16].

In the past few years, the problem of existence of periodic solutions for neutral differential equations was studied in many papers (see [17–19] and the references therein). For example, Peng [17] studied the existence of periodic solutions for the second-order neutral differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_p(u(t) - cu(t-\sigma))'\right) + f(u'(t)) + g(u(t-\tau)) = e(t),$$

where $f, g \in C(R; R)$, c, σ and τ are constants with $|c| \neq 1$. Using the continuous theorem of coincidence degree theory and some new analysis techniques, they obtained some new results on the existence of periodic solutions. However, the study of positive periodic solutions for delay differential equations or neutral differential equations with singularities is relatively infrequent [20–22]. In [21,22], the authors investigated the periodic problem for a neutral Liénard equation with a singularity of repulsive type

$$(x(t) - cx(t-\sigma))'' + f(x(t))x'(t) + \varphi(t)x(t-\tau) - \frac{r(t)}{x^{\mu}(t)} = h(t), \tag{1.2}$$

where $f : [0, +\infty) \to R$ is continuous, $r : R \to (0, +\infty)$ and $\varphi : R \to R$ are continuous with T-periodicity in the t variable, $\mu > 1$ and $c \in R$ are constants. Under the conditions of $\varphi(t) \geq 0$ for $t \in [0, T]$, $|c| < 1$ and other suitable assumptions, some results on the existence of positive T-periodic solutions are obtained.

The parameter c in neutral functional differential equations has rich physical significance. For example, the vertical motion of the pendulum–MSD (mass–spring–damper) system can be described in [23] by the following second-order neutral differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(y(t) + \frac{M_1}{M_2}y(t-\tau)\right) + A\frac{\mathrm{d}y}{\mathrm{d}t} + By = 0, \tag{1.3}$$

where M_2 is the mass of a body which is mounted on a linear spring, to which a pendulum of mass M_1 is attached via a hinged massless rod of length. The physical meaning of the parameters of τ, A and B can be found in [23]. Corresponding to Eq. (1.2), $c = \frac{M_1}{M_2}$. Clearly, c is allowed to satisfy $c > 1$. In this case, one can see from the results in [23] that the steady state $y = 0$ of (1.3) is always unstable for any positive delay τ. For the other detailed explanation of physical meaning of the parameter c, we refer reader to see [24].

In this paper, we continue to study the periodic problem of equation (1.2) in the case of $|c| > 1$. Using a continuation theorem of coincidence degree theory established by Mawhin, we obtain the following result.

Theorem 1.1. *Assume that the following conditions hold:*

[H1] $\bar{\varphi} := \frac{1}{T}\int_0^T \varphi(s)\mathrm{d}s > 0$;

[H2] $|f(x)| \leq L$ *for all* $x \in (0, +\infty)$;

[H3] $|c| - 1 - \sigma_0 > 0$, *and* $\delta := 1 - \frac{T\overline{\varphi_+}}{2\overline{\varphi}}\left(\frac{\overline{|\varphi|}}{|c|-1-\sigma_0}\right)^{\frac{1}{2}} > 0$, *where* $\sigma_0 = \min\{\sigma L, \frac{LT}{\pi}\}$, L *is determined in [H2] and* σ *in* (1.2).

Then there exists at least one positive T*-periodic solution to* (1.2).

From Theorem 1.1, one can find that the constant c associated to the difference operator $D : C([-\sigma, 0], R) \to R, D\varphi = \varphi(0) - c\varphi(-\sigma)$ required $|c| > 1$. This is essentially different from the corresponding condition $|c| < 1$ assumed by Kong et al. [21,22]. Furthermore, the sign of $\varphi(t)$ is allowed to change. Just because of these two factors, there are more difficulties in the present paper than in [21,22] for estimating a priori bounds of all the possible positive T-periodic solutions to Eq. (1.2) with a parameter λ

$$\begin{aligned}(x(t) - cx(t-\sigma))'' + \lambda f(x(t))x'(t) + \lambda\varphi(t)x(t-\tau) - \frac{\lambda r(t)}{x^{\mu}(t)} \\ = \lambda h(t), \lambda \in (0,1).\end{aligned} \tag{1.4}$$

If $\varphi(t) \geq 0$ for $t \in [0, T]$, then using Theorem 1.1, we can get the following Corollary.

Corollary 1.2. *Assume that* [$H2$] *holds and* $\varphi(t) \geq 0$ *for* $t \in [0, T]$ *with* $\bar{\varphi} > 0$. *If*

$$|c| > 1 + \sigma_0 + \frac{T^2}{4}\bar{\varphi},$$

then there exists at least one positive T*-periodic solution to* (1.2).

For illustrating the application of Theorem 1.1, we give the following example.

Example 1.3. Consider the neutral Liénard equation with a singularity of repulsive type

$$(x(t) - 50x(t-\pi))'' + x'(t)\cdot\sin x(t) + (1 + 2\sin t)x(t-\tau) - \frac{r(t)}{x^{\mu}(t)} = h(t),$$

where $\tau \in R$, $\mu > 1$ are constants, r, h are 2π-periodic continuous functions with $r(t) > 0$ for $t \in [0, T]$.

Corresponding to (1.2), we have

$$T = 2\pi,\ f(x) = \sin x, \quad \varphi(t) = 1 + 2\sin t.$$

Clearly,

$$|f(x)| \leq 1, \quad \overline{\varphi} = 1 > 0, \quad \overline{\varphi_+} = \frac{2\pi + 3\sqrt{3}}{3\pi}, \quad \overline{\varphi_-} = \frac{-\pi + 3\sqrt{3}}{3\pi}.$$

By a direct calculation, we arrive at

$$\sigma_0 = 2, \quad |c| - 1 - \sigma_0 = 47 > 0, \quad \delta \approx 0.2 > 0.$$

Thus, assumptions of [H1]–[H3] hold, and using Theorem 1.1, we know that this equation has at least one positive 2π-periodic solution.

The rest of this paper is organized as follows. In the second section, we present some necessary lemmas. In the last section, we prove our main result (Theorem 1.1).

2. Essential definitions and lemmas

In this section, we will introduce four lemmas. The first is a continuation theorem of coincidence degree theory which was established by Mawhin in [25], and this lemma is the theoretic basis of this paper. The rest of the lemmas are used for estimating a priori bounds of periodic solutions to Eq. (1.4).

Now, we give some notations and definitions which will be used throughout this paper. For any T-periodic continuous function $y(t)$, let

$$y_+(t) = \max\{y(t), 0\}, y_-(t) = -\min\{y(t), 0\},$$
$$\bar{y} = \frac{1}{T}\int_0^T y(s)\mathrm{d}s \text{ and } ||y||_\infty = \max_{t\in[0,T]} |y(t)|.$$

Clearly, $y(t) = y_+(t) - y_-(t)$ for all $t \in R$, and $\bar{y} = \overline{y_+} - \overline{y_-}$. Let $X = C_T^1 := \{x \in C^1(R, R) : x(t+T) \equiv x(t)\}$ with the norm $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$, $Y = C_T := \{y \in C(R, R) : y(t + T) \equiv y(t)\}$ with the norm $\|y\|_Y = \|y\|_\infty$. It is easy to see that X and Y are Banach spaces.

Next, we define the linear operator L as

$$Lx = (Ax)'', \qquad L : D(L) \subset X \to Y,$$

where $A : C_T \to C_T$, $(Ax)(t) = x(t) - cx(t - \sigma)$, $D(L) = \{x \in X : Ax \in C^2(R, R)\}$, and

$$N : \Delta \to Y,\ (Nx)(t) = -f(x(t))x'(t) - \varphi(t)x(t - \tau) + \frac{r(t)}{x^\mu(t)} + h(t),$$

where $\Delta = \{x \in X : x(t) > 0,\ t \in [0, T]\}$. It is easy to see that

$$KerL = R \quad \text{and} \quad \mathrm{Im}L = \left\{y \in Y : \int_0^T y(t)\mathrm{d}t = 0\right\}.$$

This implies that L is a Fredholm operator of index zero.

Let us define two continuous projectors $P : X \to \mathrm{Ker}L$ and $Q : Y \to Y$ by setting

$$Px = \frac{1}{T}\int_0^T x(t)\mathrm{d}t \text{ and } \qquad Qy = \frac{1}{T}\int_0^T y(t)\mathrm{d}t,$$

respectively. Meanwhile, we can know that $\mathrm{ker}L = \mathrm{Im}P,\ \mathrm{ker}Q = \mathrm{Im}L$.

Let $L_p = L|_{D(L)\cap \mathrm{ker}P} \to \mathrm{Im}L$, then L_p has its inverse $L_p^{-1} : \mathrm{Im}L \to D(L) \cap KerP$ and we define $K_p : \mathrm{Im}L \to D(L) \cap KerP$ by

$$K_p = L_p^{-1}.$$

Clearly, for arbitrary $y \in \mathrm{Im}L$, we have

$$(K_p y)(t) = (A^{-1}Fy)(t),$$

where

$$(Fy)(t) = \int_0^T G(t,s)y(s)\mathrm{d}s, \quad G(t,x) = \begin{cases} \frac{(T-s)s}{2T}, \ 0 \le s < t \le T; \\ \frac{(s-2t)(T-s)}{2T}, \ 0 \le t \le s \le T. \end{cases}$$

For any bounded set $\Omega \subset \Delta$, we can prove by standard arguments that $K_p(I-Q)N$ and QN are relatively compact on the closure $\bar{\Omega}$. Therefore, N is L-compact on $\bar{\Omega}$.

Lemma 2.1. [25] *Let X and Y be two real Banach spaces. Suppose that $L: D(L) \subset X \to Y$ is a Fredholm operator with index zero and $N : \bar{\Omega} \to Y$ is L-compact on $\bar{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied.*

(S1) $Lx \neq \lambda Nx$, *for all* $x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;
(S2) $Nx \notin \mathrm{Im}L$, *for all* $x \in \partial\Omega \cap KerL$;
(S3) *The Brouwer degree* $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$, *where* $J : \mathrm{Im}Q \to KerL$ *is an isomorphism.*

Then equation $Lx = Nx$ has at least one solution on $\bar{\Omega}$.

Remark 2.2. If $\bar{r} > 0, \bar{\varphi} > 0$, then there exist two constants of D_1 and D_2 with $0 < D_1 < D_2 < +\infty$, such that

$$\frac{\bar{r}}{x^\mu} - \bar{\varphi}x + \bar{h} > 0, \ \forall x \in (0, D_1)$$

and

$$\frac{\bar{r}}{x^\mu} - \bar{\varphi}x + \bar{h} < 0, \ \forall x \in (D_2, \infty).$$

Lemma 2.3. [26] *If $|c| \neq 1$, then A has a continuous bounded inverse on C_T and the following hold:*

1. $\|A^{-1}x\| \le \frac{\|x\|}{|1-|c||}$, *for every* $t \in [0,T]$.
2. $\int_0^T |(A^{-1}f)(t)|\mathrm{d}t \le \frac{1}{|1-|c||}\int_0^T |f(t)|\mathrm{d}t$, *for every* $f \in C_T$.
3. *If* $Af \in C_T^1$, *then* $f \in C_T^1$ *and* $(Af)'(t) = (Af')(t)$, *for every* $t \in [0,T]$.

Lemma 2.4. [27] *Let $u : [0,T] \to R$ be a absolute continuous function, and $u(0) = u(T)$, then*

$$\left(\max_{t\in[0,T]} u(t) - \min_{t\in[0,T]} u(t)\right)^2 \le \frac{T}{4}\int_0^T |u'(s)|^2\mathrm{d}s.$$

Lemma 2.5. [28] *Let x be a continuously differentiable T-periodic function. Then, for any $\tau \in [0,T]$,*

$$\left(\int_0^T |x(t) - x(t-\tau)|^2\mathrm{d}t\right)^{\frac{1}{2}} \le \tau\left(\int_0^T |x'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}. \tag{2.1}$$

3. Main results

Let us define

$$D = \{x \in C_T^1 : Lx = \lambda Nx, \lambda \in (0,1); x(t) > 0, \forall t \in [0,T]\} \tag{3.1}$$

and

$$M_0 = \max\left\{\frac{T\overline{\varphi_+}}{2\delta\overline{\varphi}}\left(\frac{\overline{h_-}}{|c|-1-\sigma_0}\right)^{\frac{1}{2}} + \left(\frac{\bar{r}+|\bar{h}|}{\delta\overline{\varphi}}\right)^{\frac{1}{2}}, 1\right\}, \tag{3.2}$$

where σ_0 and δ are determined in assumption $[H3]$ of Theorem 1.1. Notice that for every $u \in D$, $u(t)$ is a positive T-periodic solution to (1.4), i.e.,

$$\begin{aligned}(u(t)-cu(t-\sigma))'' + \lambda f(u(t))u'(t) + \lambda\varphi(t)u(t-\tau) - \lambda\frac{r(t)}{u^{\mu}(t)}\\ = \lambda h(t),\ \lambda \in (0,1).\end{aligned} \tag{3.3}$$

Lemma 3.1. *Assume that* $[H2]$ *holds, then*

$$\begin{aligned}&\left|\int_0^T f(x(t))x'(t)x(t-\sigma)\mathrm{d}t\right|\\ &\le \sigma_0\int_0^T |x'(t)|^2\mathrm{d}t \text{ for all } x \in C_T^1.\end{aligned}$$

Proof. For each $x \in C_T^1$, let $\tilde{x}(t) = x(t)-\bar{x}$. Then using Wirtinger's inequality, we get

$$\left(\int_0^T |\tilde{x}(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \le \frac{T}{\pi}\left(\int_0^T |x'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}},$$

which together $[H2]$ yields

$$\begin{aligned}&\left|\int_0^T f(x(t))x'(t)x(t-\sigma)\mathrm{d}t\right|\\ &= \left|\int_0^T f(x(t))x'(t)[x(t-\sigma)-\bar{x}]\mathrm{d}t\right|\\ &\le L\int_0^T |x'(t)||x(t-\sigma)-\bar{x}|\mathrm{d}t\\ &\le L\left(\int_0^T |x'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T |x(t-\sigma)-\bar{x}|^2\mathrm{d}t\right)^{\frac{1}{2}}\\ &\le \frac{TL}{\pi}\int_0^T |x'(t)|^2\mathrm{d}t.\end{aligned} \tag{3.4}$$

On the other hand, using (2.1) in Lemma 2.5, we obtain

$$\begin{aligned}
&\left|\int_0^T f(x(t))x'(t)x(t-\sigma)\mathrm{d}t\right| \\
&= \left|\int_0^T f(x(t))x'(t)[x(t-\sigma)-x(t)]\mathrm{d}t + \int_0^T f(x(t))x(t)x'(t)\mathrm{d}t\right| \\
&= \left|\int_0^T f(x(t))x'(t)[x(t-\sigma)-x(t)]\mathrm{d}t\right| \\
&\le L\int_0^T |x'(t)||x(t)-x(t-\sigma)|\mathrm{d}t \qquad (3.5)\\
&\le L\left(\int_0^T |x(t)-x(t-\sigma)|^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T |x'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le L\sigma\left(\int_0^T |x'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T |x'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \\
&= L\sigma\int_0^T |x'(t)|^2\mathrm{d}t.
\end{aligned}$$

The conclusion follows from (3.4) and (3.5) directly. □

Lemma 3.2. *Suppose that assumption* $[H2]$ *holds, and* $|c|-1-\sigma_0>0$*, then for each* $u\in D$*,* u *satisfies*

$$\left(\int_0^T |u'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \le \left(\frac{T\overline{|\varphi|}}{|c|-1-\sigma_0}\right)^{\frac{1}{2}}||u||_\infty + \left(\frac{T\overline{|h|}}{|c|-1-\sigma_0}\right)^{\frac{1}{2}}||u||_\infty^{\frac{1}{2}}, \qquad (3.6)$$

where D *is defined by* (3.1).

Proof. Suppose that $u\in D$, then $u(t)$ satisfies (3.3). Multiplying both sides of (3.3) by $u(t-\sigma)$ and integrate it on $[0,T]$, we have

$$\begin{aligned}
&\int_0^T (u(t)-cu(t-\sigma))''u(t-\sigma)\mathrm{d}t + \lambda\int_0^T f(u(t))u'(t)u(t-\sigma)\mathrm{d}t \\
&+\lambda\int_0^T \varphi(t)u(t-\tau)u(t-\sigma)\mathrm{d}t - \lambda\int_0^T \frac{r(t)u(t-\sigma)}{u^\mu(t)}\mathrm{d}t = \lambda\int_0^T h(t)u(t-\sigma)\mathrm{d}t.
\end{aligned}$$

By a direct calculation and using Conclusion 3 of Lemma 2.3, we arrive at

$$\begin{aligned}
c\int_0^T |u'(t)|^2\mathrm{d}t &= \int_0^T u'(t)u'(t-\sigma)\mathrm{d}t - \lambda\int_0^T f(u(t))u'(t)u(t-\sigma)\mathrm{d}t \\
&- \lambda\int_0^T \varphi(t)u(t-\tau)u(t-\sigma)\mathrm{d}t + \lambda\int_0^T \frac{r(t)u(t-\sigma)}{u^\mu(t)}\mathrm{d}t + \lambda\int_0^T h(t)u(t-\sigma)\mathrm{d}t.
\end{aligned} \qquad (3.7)$$

Using Lemma 3.1, we have

$$\begin{aligned}&\left|\int_0^T f(u(t))u'(t)u(t-\sigma)\mathrm{d}t\right|\\&\le \sigma_0\int_0^T |u'(t)|^2\mathrm{d}t,\end{aligned} \tag{3.8}$$

and using Hölder inequality, we get

$$\begin{aligned}&-\int_0^T u'(t)u'(t-\sigma)\mathrm{d}t\\&\le \Big|\int_0^T u'(t)u'(t-\sigma)\mathrm{d}t\Big| \le \int_0^T |u'(t)||u'(t-\sigma)|\mathrm{d}t\\&\le \Big(\int_0^T |u'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_0^T |u'(t-\sigma)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\\&= \Big(\int_0^T |u'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_{-\sigma}^{T-\sigma} |u'(s)|^2\mathrm{d}s\Big)^{\frac{1}{2}}\\&= \Big(\int_0^T |u'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_0^T |u'(s)|^2\mathrm{d}s\Big)^{\frac{1}{2}}\\&= \int_0^T |u'(t)|^2\mathrm{d}t.\end{aligned} \tag{3.9}$$

If $c<-1$, substituting (3.8) and (3.9) into (3.7), we get

$$\begin{aligned}&|c|\int_0^T |u'(t)|^2\mathrm{d}t = -c\int_0^T |u'(t)|^2\mathrm{d}t\\&= -\int_0^T u'(t)u'(t-\sigma)\mathrm{d}t + \lambda\int_0^T f(u(t))u'(t)u(t-\sigma)\mathrm{d}t\\&\quad + \lambda\int_0^T \varphi(t)u(t-\tau)u(t-\sigma)\mathrm{d}t - \lambda\int_0^T \frac{r(t)u(t-\sigma)}{u^{\mu}(t)}\mathrm{d}t - \lambda\int_0^T h(t)u(t-\sigma)\mathrm{d}t\\&\le (1+\sigma_0)\int_0^T |u'(t)|^2\mathrm{d}t + ||u||_\infty^2 T\overline{\varphi_+} + ||u||_\infty T\overline{h_-}.\end{aligned}$$

This gives us that

$$(|c|-1-\sigma_0)\int_0^T |u'(t)|^2\mathrm{d}t \le ||u||_\infty^2 T\overline{\varphi_+} + ||u||_\infty T\overline{h_-}. \tag{3.10}$$

If $c>1$, substituting (3.8) and (3.9) into (3.7) again, we have

$$\begin{aligned}&|c|\int_0^T |u'(t)|^2\mathrm{d}t = c\int_0^T |u'(t)|^2\mathrm{d}t\\&= -\int_0^T u'(t)u'(t-\sigma)\mathrm{d}t - \lambda\int_0^T f(u(t))u'(t)u(t-\sigma)\mathrm{d}t\\&\quad -\lambda\int_0^T \varphi(t)u(t-\tau)u(t-\sigma)\mathrm{d}t + \lambda\int_0^T \frac{r(t)u(t-\sigma)}{u^{\mu}(t)}\mathrm{d}t + \lambda\int_0^T h(t)u(t-\sigma)\mathrm{d}t\\&\le (1+\sigma_0)\int_0^T |u'(t)|^2\mathrm{d}t + ||u||_\infty^2 T\overline{\varphi_-} + ||u||_\infty\Big[\int_0^T \frac{r(t)}{u^{\mu}(t)}\mathrm{d}t + T\overline{h_+}\Big].\end{aligned} \tag{3.11}$$

Integrating (3.3) on $[0,T]$, we arrive at

$$\int_0^T \varphi(t)u(t-\tau)\mathrm{d}t = \int_0^T \frac{r(t)}{u^\mu(t)}\mathrm{d}t + \int_0^T h(t)\mathrm{d}t, \tag{3.12}$$

i.e.,

$$\int_0^T \frac{r(t)}{u^\mu(t)}\mathrm{d}t = \int_0^T \varphi(t)u(t-\tau)\mathrm{d}t - T\bar{h}, \tag{3.13}$$

which together with (3.11) leads to

$$\begin{aligned}
&\big(|c|-1-\sigma_0\big)\int_0^T |u'(t)|^2\mathrm{d}t \\
&\le ||u||_\infty^2 T\overline{\varphi_-} + ||u||_\infty\Big[T\overline{h_+} + \int_0^T \varphi(t)u(t-\tau)\mathrm{d}t - T\bar{h}\Big] \\
&\le ||u||_\infty^2 T\overline{\varphi_-} + ||u||_\infty\Big[T\overline{|h|} + T\overline{\varphi_+}||u||_\infty\Big] \\
&= ||u||_\infty^2 T\overline{|\varphi|} + ||u||_\infty T\overline{|h|}.
\end{aligned} \tag{3.14}$$

Thus, from (3.10) and (3.14), we see that in either case $c<-1$ or the case $c>1$, we always have

$$\Big(|c|-1-\sigma_0\Big)\int_0^T |u'(t)|^2\mathrm{d}t \le ||u||_\infty^2 T\overline{|\varphi|} + ||u||_\infty T\overline{|h|},$$

i.e.,

$$\Big(\int_0^T |u'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \le \Big(\frac{T\overline{|\varphi|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u||_\infty + \Big(\frac{T\overline{|h|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u||_\infty^{\frac{1}{2}}. \tag{3.15}$$

The proof is complete. □

Lemma 3.3. *Suppose assumptions of [H1]–[H3] hold, then for arbitrary* $u \in D$*, there exists a* $t_0 \in [0,T]$ *such that* $u(t_0) \le M_0$*, where* M_0 *is defined by* (3.2).

Proof. Assume that the conclusion does not hold, then there exists an $u_0 \in D$ such that

$$u_0(t) > M_0, \quad \textit{for every } t \in [0,T]. \tag{3.16}$$

From the definition of D, we have

$$(u_0(t)-cu_0(t-\sigma))'' + \lambda f(u_0(t))u_0'(t) + \lambda\varphi(t)u_0(t-\tau) - \lambda\frac{r(t)}{u_0^\mu(t)} = \lambda h(t), \tag{3.17}$$

and it follows from (3.12) that

$$\int_0^T \varphi(t)u_0(t-\tau)\mathrm{d}t = \int_0^T \frac{r(t)}{u_0^\mu(t)}\mathrm{d}t + \int_0^T h(t)\mathrm{d}t,$$

which means

$$\int_0^T \varphi_+(t)u_0(t-\tau)\mathrm{d}t = \int_0^T \varphi_-(t)u_0(t-\tau)\mathrm{d}t + \int_0^T \frac{r(t)}{u_0^\mu(t)}\mathrm{d}t + \int_0^T h(t)\mathrm{d}t.$$

Using the mean value theorem of integrals, we know that there exist two constants $\zeta, \xi \in R$ such that

$$T\overline{\varphi_+}u_0(\zeta) \le T\overline{\varphi_-}u_0(\xi) + T\bar{r}M_0^{-\mu} + T\bar{h}.$$

Notice that $M_0 \ge 1$, we have

$$T\overline{\varphi_+}u_0(\zeta) \le T\overline{\varphi_-}u_0(\xi) + T\bar{r} + T\bar{h},$$

namely,

$$u_0(\zeta) \le \frac{\overline{\varphi_-}}{\overline{\varphi_+}}\|u_0\|_\infty + \frac{\bar{r} + |\bar{h}|}{\overline{\varphi_+}}. \tag{3.18}$$

By means of Lemma 2.4, we get the following inequality

$$\|u_0\|_\infty \le u_0(\zeta) + \frac{T^{\frac{1}{2}}}{2}\left(\int_0^T |u_0'(s)|\mathrm{d}s\right)^{\frac{1}{2}}. \tag{3.19}$$

Substituting (3.18) into (3.19), and from $[H1]$, we have

$$\|u_0\|_\infty \le \frac{T^{\frac{1}{2}}\overline{\varphi_+}}{2\overline{\varphi}}\Big(\int_0^T |u_0'(s)|\mathrm{d}s\Big)^{\frac{1}{2}} + \frac{\bar{r} + |\bar{h}|}{\overline{\varphi}}. \tag{3.20}$$

On the other hand, using Lemma 3.2, we see that

$$\Big(\int_0^T |u_0'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \le \Big(\frac{T\overline{|\varphi|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u_0||_\infty + \Big(\frac{T\overline{|h|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u_0||_\infty^{\frac{1}{2}}. \tag{3.21}$$

Substituting (3.21) into (3.20), we get

$$\begin{aligned} \|u_0\|_\infty &\le \frac{T^{\frac{1}{2}}\overline{\varphi_+}}{2\overline{\varphi}}\Big[\Big(\frac{T\overline{|\varphi|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u_0||_\infty + \Big(\frac{T\overline{|h|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u_0||_\infty^{\frac{1}{2}}\Big] + \frac{\bar{r}+\bar{h}}{\overline{\varphi}} \\ . &= \frac{T\overline{\varphi_+}}{2\overline{\varphi}}\Big(\frac{\overline{|\varphi|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u_0||_\infty + \frac{T\overline{\varphi_+}}{2\overline{\varphi}}\Big(\frac{\overline{|h|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u_0||_\infty^{\frac{1}{2}} \\ &\quad + \frac{\bar{r}+|\bar{h}|}{\overline{\varphi}}. \end{aligned} \tag{3.22}$$

Since $\delta = 1 - \frac{T\overline{\varphi_+}}{2\overline{\varphi}}\Big(\frac{\overline{|\varphi|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}} > 0$, it follows from (3.22) that

$$||u_0||_\infty \le \frac{T\overline{\varphi_+}}{2\delta\overline{\varphi}}\Big(\frac{\overline{|h|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u_0||_\infty^{\frac{1}{2}} + \frac{\bar{r}+|\bar{h}|}{\delta\overline{\varphi}}.$$

By simply calculating, we have

$$||u_0||_\infty < \frac{T\overline{\varphi_+}}{2\delta\overline{\varphi}}\Big(\frac{\overline{|h|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}} + \Big(\frac{\bar{r}+|\bar{h}|}{\delta\overline{\varphi}}\Big)^{\frac{1}{2}},$$

i.e., $\|u_0\|_\infty < M_0$, which contradicts (3.16). This contradiction implies that the conclusion of Lemma 3.3 holds. □

Lemma 3.4. *Assume that* $[H1]$ *holds, then there exists a constant*

$$\gamma = \min\left\{\frac{|\bar{h}|+1}{\overline{\varphi_+}}, \left(\frac{\bar{r}}{2|\bar{h}|+1}\right)^{\frac{1}{\mu}}\right\} \tag{3.23}$$

such that, for every $u \in D$, there always exists a $t_1 \in [0,T]$ satisfies $x(t_1) \geq \gamma$.

Proof. Assume that the conclusion does not hold, then there exists an $u_1 \in D$ satisfies

$$u_1(t) < \gamma, \quad \text{for all } t \in [0,T] \tag{3.24}$$

and

$$(u_1(t) - cu_1(t-\sigma))'' + \lambda f(u_1(t))u_1'(t) + \lambda\varphi(t)u_1(t-\tau) - \frac{\lambda r(t)}{u_1^{\mu}(t)} = \lambda h(t).$$

Integrating it on $[0,T]$, we arrive at

$$\int_0^T \varphi(t)u_1(t-\tau)\mathrm{d}t - \int_0^T \frac{r(t)}{u_1^{\mu}(t)}\mathrm{d}t = \int_0^T h(t)\mathrm{d}t,$$

which results in

$$\int_0^T \varphi_+(t)u_1(t-\tau)\mathrm{d}t \geq \int_0^T \frac{r(t)}{u_1^{\mu}(t)}\mathrm{d}t + T\bar{h}.$$

Using (3.24) and (3.23), we get

$$\begin{aligned} T\gamma\overline{\varphi_+} = \gamma\int_0^T \varphi_+(t)\mathrm{d}t &> \int_0^T \varphi_+(t)u_1(t-\tau)\mathrm{d}t \\ &\geq \frac{T\bar{r}}{\gamma^{\mu}} + T\bar{h} \\ &= T(2|\bar{h}|+1) + T\bar{h} \\ &\geq T|\bar{h}| + T. \end{aligned}$$

By simply calculating, we have

$$\gamma > \frac{|\bar{h}|+1}{\overline{\varphi_+}},$$

which contradicts (3.23). So, for every $u \in D$, there always exists a $t_1 \in [0,T]$ satisfies $x(t_1) \geq \gamma$. □

Finally, we are going to prove Theorem 1.1.

Proof. For $u \in D$, according to Lemma 3.3 and Lemma 3.4, we know that there exist $t_0, t_1 \in [0,T]$ such that

$$u(t_0) \leq M_0, \quad u(t_1) \geq \gamma. \tag{3.25}$$

Lemma 3.3 gives us that

$$\Big(\int_0^T |u'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \leq \Big(\frac{T\overline{|\varphi|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u||_\infty + \Big(\frac{T\overline{|h|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u||_\infty^{\frac{1}{2}}. \tag{3.26}$$

From (3.25) and using Lemma 2.4, we get

$$\|u\|_\infty \leq M_0 + \frac{T^{\frac{1}{2}}}{2}\Big(\int_0^T |u'(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}},$$

which together with (3.26) yields

$$\|u\|_\infty \leq M_0 + \frac{T}{2}\Big(\frac{\overline{|\varphi|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u||_\infty + \frac{T}{2}\Big(\frac{\overline{|h|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u||_\infty^{\frac{1}{2}},$$

i.e.,

$$\delta_1\|u\|_\infty \le M_0 + \frac{T}{2}\Big(\frac{\overline{|h|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}||u||_\infty^{\frac{1}{2}}, \tag{3.27}$$

where $\delta_1 := 1-\frac{T}{2}\Big(\frac{\overline{|\varphi|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}}$. Since $\bar{\varphi}>0$, it follows from $[H3]$ that $\delta_1 \ge \delta > 0$, and then by (3.27), we get

$$\|u\|_\infty \le \frac{T}{2\delta_1}\Big(\frac{\overline{|h|}}{|c|-1-\sigma_0}\Big)^{\frac{1}{2}} + \Big(\frac{M_0}{\delta_1}\Big)^{\frac{1}{2}} := M_1. \tag{3.28}$$

Thanks to $Au \in C_T^2$, there exists a $t_2 \in [0,T]$ s.t. $(Au)'(t_2)=0$. Integrating (3.3) on $[t_2,t]$, we have

$$\begin{aligned}(Au)'(t) = &-\lambda\int_{t_2}^{t} f(u(t))u'(t)\mathrm{d}t - \lambda\int_{t_2}^{t}\varphi(t)u(t-\tau)\mathrm{d}t\\ &+\lambda\int_{t_2}^{t}\frac{r(t)}{u^\mu(t)}\mathrm{d}t + \lambda\int_{t_2}^{t}h(t)\mathrm{d}t,\end{aligned}$$

where $t\in[t_2,t_2+T]$. And From $F(x)=\int_0^x f(s)\mathrm{d}s$, we know

$$|(Au)'(t)| \le 2\lambda\max_{u\in[0,M_1]}|F(u)| + \lambda T\overline{|\varphi|}\|u\|_\infty + \lambda T\overline{|h|} + \lambda\int_0^T\frac{r(t)}{u^\mu(t)}\mathrm{d}t.$$

Using the conclusion (3) of Lemma 2.3, we have from (3.13) that

$$\|Au'\|_\infty = \|(Au)'\|_\infty \le 2\lambda(\max_{u\in[0,M_1]}|F(u)| + T\overline{|\varphi|}\|u\|_\infty + T\overline{|h|}). \tag{3.29}$$

It follows from conclusion (2) of Lemma 2.3 that

$$\begin{aligned}\|u'\|_\infty = \|A^{-1}Au'\|_\infty \le \frac{\|Au'\|_\infty}{|c|-1} < \frac{2(\max_{u\in[0,M_1]}|F(u)| + T\overline{|\varphi|}\|u\|_\infty + T\overline{|h|})}{|c|-1}\\ := M_2,\end{aligned}$$

namely,

$$\|u'\|_\infty < M_2. \tag{3.30}$$

In the following part, we will give a priori lower estimate over the set D. To do it, multiplying both sides of (3.3) by $\frac{u'(t)}{r(t)}$ and integrating it on $[t_1,t]$, where t_1 is determined in Lemma 3.4, we get

$$\begin{aligned}\lambda\Big|\int_{t_1}^{t}\frac{u'(s)}{u^\mu(s)}\mathrm{d}s\Big| = &\int_{t_1}^{t}\Big|(Au)''(s)\frac{u'(s)}{r(s)}\Big|\mathrm{d}s + \lambda\int_{t_1}^{t}\Big|f(u(s))u'(s)\frac{u'(s)}{r(s)}\Big|\mathrm{d}s\\ &+ \lambda\int_{t_1}^{t}\Big|\varphi(s)u(s-\tau)\frac{u'(s)}{r(s)}\Big|\mathrm{d}s - \lambda\int_{t_1}^{t}\Big|h(s)\frac{u'(s)}{r(s)}\Big|\mathrm{d}s\\ &\le \frac{||u'||_\infty}{r_l}\int_0^T|(Au)''(s)|\mathrm{d}s + \lambda TL\frac{||u'||_\infty^2}{r_l} +\\ &\lambda T\overline{|\varphi|}\frac{||u||_\infty||u'||_\infty}{r_l} + \lambda T\overline{|h|}\frac{||u'||_\infty}{r_l},\end{aligned}$$

where $t \in [t_1, t_1+T]$ and $r_l = \min_{t\in[0,T]} r(t)$, which together with (3.28) and (3.30) yields

$$\begin{aligned}\lambda\Big|\int_{u(t_1)}^{u(t)} \frac{1}{v^{\mu}}dv\Big| \leq \frac{M_2}{r_l}\int_0^T |(Au)''(s)|\mathrm{d}s + \frac{\lambda T L M_2^2}{r_l} + \\ \frac{\lambda T\overline{|\varphi|}M_1M_2}{r_l} + \frac{\lambda T\overline{|h|}M_2}{r_l}.\end{aligned} \tag{3.31}$$

From (3.3), we know

$$\int_0^T |(Au)''(s)|\mathrm{d}s \leq \lambda T(M_2L + 2M_1\overline{|\varphi|} + 2\overline{|h|}). \tag{3.32}$$

Substituting (3.32) into (3.31), we arrive at

$$\begin{aligned}\Big|\int_{u(t_1)}^{u(t)} \frac{1}{v^{\mu}}dv\Big| &\leq \frac{2TLM_2^2}{r_l} + \frac{3T\overline{|\varphi|}M_1M_2}{r_l} + \frac{3T\overline{|h|}M_2}{r_l} \\ &:= M_3.\end{aligned} \tag{3.33}$$

Since $\mu > 1$, it follows that there exists a constant $\gamma_0 \in (0,\gamma)$ such that

$$\int_{\varepsilon}^{\gamma} \frac{1}{v^{\mu}}dv > M_3, \qquad \text{for every } \varepsilon \in (0,\gamma_0],$$

where γ is defined by (3.23). If $u(t) \leq \gamma_0$, then we obtain that

$$\Big|\int_{u(t_1)}^{u(t)} \frac{1}{v^{\mu}(s)}dv\Big| \geq \int_{\gamma_0}^{\gamma} \frac{1}{v^{\mu}}dv > M_3,$$

which contradicts to (3.33), and therefore we get

$$u(t) > \gamma_0, \qquad \text{for every } t \in [0,T]. \tag{3.34}$$

From (3.28), (3.30) and (3.34), we have

$$\gamma_0 < \min_{t\in[0,T]} u(t), \ \max_{t\in[0,T]} u(t) < M_1, \ ||u'||_\infty < M_2, \quad \forall u \in D. \tag{3.35}$$

Let $m_0 = \min\{\gamma_0, D_1\}$, $m_1 = \max\{M_1, D_2\}$, where D_1 and D_2 are determined in Remark 2.2. Set

$$\Omega = \{u \in C_T : m_0 < u(t) < m_1, \ |u'(t)| < M_2, \ \text{ for every } t \in [0,T]\},$$

we can easily verify that conditions of $(S1)$ and $(S2)$ in Lemma 2.1 hold. And we also have

$$\left(\frac{\bar{r}}{m_0^{\mu}} - \bar{\varphi}m_0 + \bar{h}\right)\left(\frac{\bar{r}}{m_1^{\mu}} - \bar{\varphi}m_1 + \bar{h}\right) < 0,$$

which means $deg\{JQN, \Omega\cap kerL, 0\} \neq 0$, namely, the condition $(S3)$ holds also. Thus, using Lemma 2.1 we know that Eq. (1.2) has at least one positive T-periodic solution $u_1 \in \overline{\Omega}$. □

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