J. Fixed Point Theory Appl. (2019) 21:20
https://doi.org/10.1007/s11784-018-0656-9
Published online January 3, 2019

Journal of Fixed Point Theory and Applications

Extragradient methods for solving non-Lipschitzian pseudo-monotone variational inequalities

Duong Viet Thong and Aviv Gibali

Abstract. The purpose of this paper is to study and analyze two new extragradient methods for solving non-Lipschitzian and pseudo-monotone variational inequalities in real Hilbert spaces. Under suitable conditions, weak and strong convergence theorems of the proposed methods are established. We present academic and numerical examples for illustrating the behavior of the proposed algorithms.

Mathematics Subject Classification. 65J15, 65Y05, 47H05, 47J25.

Keywords. Extragradient method, Halpern method, variational inequality, pseudo-monotone operator.

1. Introduction

We focus on the following classical variational inequality (VI) ([9,10]) which consists in finding a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \ \ \forall x \in C, \tag{1.1}$$

where C is a nonempty closed convex subset in a real Hilbert space H, and is a single-valued mapping $A : H \to H$. As commonly done, we denote by $\mathrm{VI}(C, A)$ the solution set of VI (1.1). Variational inequalities are fundamental in a broad range of mathematical and applied sciences; the theoretical and algorithmic foundations as well as the applications of variational inequalities have been extensively studied in the literature and continue to attract intensive research. For the current state of the art in a finite-dimensional setting, see for instance [8] and the extensive list of references therein.

Many authors have proposed and analyzed several iterative methods for solving the variational inequality (1.1). The simplest one is the following projection method, which can be seen as an extension of the projected gradient method introduced for solving optimization problems.

$$x_{n+1} = P_C(x_n - \lambda Ax_n), \tag{1.2}$$

for each $n \geq 1$, where P_C denotes the metric projection from H onto C. It is known that the assumptions which imply convergence of this method are quite restrictive and require A to be L-Lipschitz continuous and α-strongly monotone (see Chapter 13 in [19]) and $\lambda \in \left(0, \frac{2\alpha}{L^2}\right)$.

One way to weaken the (1.2) convergence assumptions, Korpelevich [20] (also independently by [1]) proposed a double projection method known as the extragradient method in Euclidean space when A is monotone and L-Lipschitz continuous. The iterative step of the method is as follows:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = P_C(x_n - \lambda_n A y_n), \end{cases}$$

where $\lambda_n \in \left(0, \frac{1}{L}\right)$.

In recent years, the extragradient method was further extended to infinite-dimensional spaces in various ways, see, e.g. [2–5,16,23,24,30–32,34] and the references therein.

Obviously, when A is not Lipschitz continuous or its Lipschitz constant L is difficult to compute/approximate, Korpelevich's method fails since the step-size λ_n depends on this. So, Iusem [14] proposed the following algorithm in a way to avoid this obstacle.

Algorithm 1.1.

Initialization: Given $\gamma > 0, l \in (0,1), \mu \in (0,1)$. Let $x_0 \in C$ be arbitrary

---

Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma l^m \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|. \tag{1.3}$$

If $x_n = y_n$ then stop and x_n is a solution of $VI(C, A)$. Otherwise
Step 2. Compute

$$x_{n+1} = P_C(x_n - \beta_n A y_n),$$

where

$$\beta_n := \frac{\langle Ay_n, x_n - y_n \rangle}{\|Ay_n\|^2}.$$

Set $n := n + 1$ and go to **Step 1**.

Similar extensions have been developed by many authors, for example Khobotov [18] and Marcotte [25]. These algorithms assume that A is monotone and continuous on C. Thus, in this spirit, we wish to construct an extragradient modification which converges under a weaker condition in

Hilbert spaces. To be more specific, we assume that A is a uniformly continuous pseudo-monotone operator. Our scheme also make use of a different Armijo-type line-search and then A is only assumed to be pseudo-monotone on C in the sense of Karamardian [17]. Weak and strong convergence of these proposed algorithms is established in real Hilbert spaces.

The paper is organized as follows. We first recall some basic definitions and results in Sect. 2. Our algorithms are presented and analyzed in Sect. 3. In Sect. 4, we present some numerical experiments which demonstrate the algorithms performances as well as provide a preliminary computational overview by comparing it with some related algorithms.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$. For each $x, y \in H$ and $\alpha \in \mathbb{R}$, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$
$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Definition 2.1. Let $T : H \to H$ be an operator.

1. The operator T is called L-`Lipschitz continuous` with $L > 0$ if
$$\|Tx - Ty\| \leq L\|x-y\| \quad \forall x, y \in H. \tag{2.1}$$
if $L = 1$ then the operator T is called `nonexpansive` and if $L \in (0,1)$, T is called `a contraction`.
2. The operator T is called `monotone` if
$$\langle Tx - Ty, x-y\rangle \geq 0 \quad \forall x, y \in H. \tag{2.2}$$
3. The operator T is called `pseudo-monotone` if
$$\langle Tx, y-x\rangle \geq 0 \Longrightarrow \langle Ty, y-x\rangle \geq 0 \quad \forall x, y \in H. \tag{2.3}$$
4. The operator T is called α-`strongly monotone` if there exists a constant $\alpha > 0$ such that
$$\langle Tx - Ty, x-y\rangle \geq \alpha\|x-y\|^2 \ \ \forall x, y \in H.$$
5. The operator T is called `sequentially weakly continuous` if for each sequence $\{x_n\}$ we have: $\{x_n\}$ converges weakly to x implies Tx_n converges weakly to Tx.

It is easy to see that every monotone operator is pseudo-monotone, but the converse is not true.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\| \ \forall y \in C$. P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive.

Lemma 2.2. [12] *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x \Longleftrightarrow \langle x-z, z-y\rangle \geq 0 \ \ \forall y \in C$.*

Lemma 2.3. [12] *Let C be a closed and convex subset in a real Hilbert space H, $x \in H$. Then*

(i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \ \forall y \in C$;
(ii) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2 \ \forall y \in C$;
(iii) $\langle (I - P_C)x - (I - P_C)y, x - y\rangle \geq \|(I - P_C)x - (I - P_C)y\|^2 \ \forall y \in C$.

For properties of the metric projection, the interested reader could be referred to [12, Section 3].

The following Lemmas are useful for the convergence of our proposed methods.

Lemma 2.4. [7] *For $x \in H$ and $\alpha \geq \beta > 0$ the following inequalities hold.*

$$\frac{\|x - P_C(x - \alpha Ax)\|}{\alpha} \leq \frac{\|x - P_C(x - \beta Ax)\|}{\beta},$$
$$\|x - P_C(x - \beta Ax)\| \leq \|x - P_C(x - \alpha Ax)\|.$$

Lemma 2.5. [15] *Let H_1 and H_2 be two real Hilbert spaces. Suppose $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then $A(M)$ (the image of M under A) is bounded.*

Lemma 2.6. ([6], Lemma 2.1) *Consider the problem $VI(C, A)$ with C being a nonempty, closed, convex subset of a real Hilbert space H and $A : C \to H$ being pseudo-monotone and continuous. Then, x^* is a solution of $VI(C, A)$ if and only if*

$$\langle Ax, x - x^*\rangle \geq 0 \quad \forall x \in C.$$

Lemma 2.7. [26] *Let C be a nonempty set of H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

(i) *for every $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*
(ii) *every sequential weak cluster point of $\{x_n\}$ is in C.*

Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.8. [22] *Let $\{a_n\}$ be a sequence of non-negative real numbers such that there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \dots, k\}$ such that $a_n < a_{n+1}$.

The next technical lemma is very useful and used by many authors, for example Liu [21] and Xu [35]. Furthermore, a variant of Lemma 2.9 has already been used by Reich in [29].

Lemma 2.9. *Let $\{a_n\}$ be sequence of non-negative real numbers such that:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{b_n\}$ is a sequence such that

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $\limsup_{n\to\infty} b_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we introduce the two new extragradient modifications for solving the VI (1.1). We present the weak and strong convergence of the schemes under the assumptions.

Condition 3.1. The feasible set C of the VI (1.1) is a nonempty, closed, and convex subset of the real Hilbert space H.

Condition 3.2. The VI (1.1) associated operator $A : C \to H$ is a pseudo-monotone, sequentially weakly continuous on C and uniformly continuous on bounded subsets of C.

Condition 3.3. The solution set of the VI (1.1) is nonempty, that is VI(C, A) $\neq \emptyset$.

3.1. Weak convergence

Algorithm 3.1.

Initialization: Given $\gamma > 0, l \in (0,1), \mu \in (0,1)$. Let $x_1 \in C$ be arbitrary

---

Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma l^m \langle Ax_n - Ay_n, x_n - y_n \rangle \leq \mu \|x_n - y_n\|^2. \tag{3.1}$$

If $x_n = y_n$ or $Ay_n = 0$ then stop and x_n is a solution of $VI(C, A)$. Otherwise
Step 2. Compute

$$x_{n+1} = P_C(x_n - \beta_n A y_n),$$

where

$$\beta_n := \frac{1-\mu}{\gamma} \frac{\|x_n - y_n\|^2}{\|Ay_n\|^2}.$$

Set $n := n + 1$ and go to **Step 1**.

We start the algorithm's convergence analysis by proving that (3.1) terminates after finite steps.

Lemma 3.2. *Assume that Conditions 3.1–3.3 hold. The Armijo line-search rule (3.1) is well defined. In addition, we have* $\lambda_n \leq \gamma$.

Proof. If $x_n \in VI(C,A)$ then $x_n = P_C(x_n - \gamma A x_n)$ and $m_n = 0$. We consider the situation $x_n \notin VI(C,A)$ and assume the contrary that for all m we have

$$\begin{aligned} &\gamma l^m \langle A x_n - A P_C(x_n - \gamma l^m A x_n), x_n - P_C(x_n - \gamma l^m A x_n) \rangle \\ &\quad > \mu \|x_n - P_C(x_n - \gamma l^m A x_n)\|^2 \end{aligned} \tag{3.2}$$

By Cauchy–Schwartz inequality, we have

$$\begin{aligned} &\gamma l^m \langle A x_n - A P_C(x_n - \gamma l^m A x_n), x_n - P_C(x_n - \gamma l^m A x_n) \rangle \\ &\quad \le \gamma l^m \|A x_n - A P_C(x_n - \gamma l^m A x_n)\| \|x_n - P_C(x_n - \gamma l^m A x_n)\|. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) we find

$$\gamma l^m \|A x_n - A P_C(x_n - \gamma l^m A x_n)\| > \mu \|x_n - P_C(x_n - \gamma l^m A x_n)\|.$$

This implies that

$$\|A x_n - A P_C(x_n - \gamma l^m A x_n)\| > \mu \frac{\|x_n - P_C(x_n - \gamma l^m A x_n)\|}{\gamma l^m}. \tag{3.4}$$

Since $x_n \in C$ for all n and P_C is continuous, we have $\lim_{m\to\infty} \|x_n - P_C(x_n - \gamma l^m A x_n)\| = 0$. Since A is uniformly continuous on bounded subsets of C (Condition 3.2), we get that

$$\lim_{m\to\infty} \|A x_n - A P_C(x_n - \gamma l^m A x_n)\| = 0. \tag{3.5}$$

Combining (3.4) and (3.5) we get

$$\lim_{m\to\infty} \frac{\|x_n - P_C(x_n - \gamma l^m A x_n)\|}{\gamma l^m} = 0. \tag{3.6}$$

Assume that $z_m = P_C(x_n - \gamma l^m A x_n)$ we have

$$\langle z_m - x_n + \gamma l^m A x_n, x - z_m \rangle \ge 0 \quad \forall x \in C.$$

This implies that

$$\langle \frac{z_m - x_n}{\gamma l^m}, x - z_m \rangle + \langle A x_n, x - z_m \rangle \ge 0 \quad \forall x \in C. \tag{3.7}$$

Taking the limit $m \to \infty$ in (3.7) and using (3.6), we obtain

$$\langle A x_n, x - x_n \rangle \ge 0 \quad \forall x \in C,$$

which implies that $x_n \in VI(C,A)$ this is a contraction. □

Remark 3.3.

1. In the proof of Lemma 3.2 we do not use the pseudomonotonicity of A.
2. Now we show that if $x_n = y_n$ then stop and x_n is a solution of $VI(C,A)$. Indeed, we have $0 < \lambda_n \le \gamma$, which together with Lemma 2.4 we get

$$0 = \frac{\|x_n - y_n\|}{\lambda_n} = \frac{\|x_n - P_C(x_n - \lambda_n A x_n)\|}{\lambda_n} \ge \frac{\|x_n - P_C(x_n - \gamma A x_n)\|}{\gamma}.$$

This implies that x_n is a solution of $VI(C,A)$.

Lemma 3.4. *Assume that Conditions 3.1–3.3 hold and let $\{x_n\}$ be any sequence generated by Algorithm 3.1. Then we have*

$$\langle A x_n, x_n - y_n \rangle \ge \frac{1}{\gamma} \|x_n - y_n\|^2.$$

Proof. Recall one of the metric projection property

$$\|x - P_C y\|^2 \leq \langle x - y, x - P_C y\rangle \text{ for all } x \in C \text{ and } y \in H.$$

By denoting $y = x_n - \lambda_n Ax_n$ and $x = x_n$, we get

$$\|x_n - P_C(x_n - \lambda_n Ax_n)\|^2 \leq \lambda_n \langle Ax_n, x_n - P_C(x_n - \lambda_n Ax_n)\rangle,$$

thus

$$\langle Ax_n, x_n - y_n\rangle \geq \lambda_n^{-1}\|x_n - y_n\|^2,$$

which, together with $\lambda_n \leq \gamma$ we find

$$\langle Ax_n, x_n - y_n\rangle \geq \frac{1}{\gamma}\|x_n - y_n\|^2.$$

□

Lemma 3.5. *Assume that Conditions 3.1–3.3 hold and let $\{x_n\}$ be any sequence generated by Algorithm 3.1. Then we have*

$$\langle Ay_n, x_n - p\rangle \geq \frac{1-\mu}{\gamma}\|x_n - y_n\|^2.$$

Proof. Indeed, let $p \in VI(C, A)$, since $y_n \in C$, we have $\langle Ap, y_n - p\rangle \geq 0$. Due to the pseudomonotonicity of A, we get

$$\langle Ay_n, y_n - p\rangle \geq 0. \tag{3.8}$$

On the other hand, according to Lemma 3.4 we have

$$\langle Ax_n, x_n - y_n\rangle \geq \frac{1}{\lambda_n}\|x_n - y_n\|^2. \tag{3.9}$$

Now, using (3.1), (3.8) and (3.9), we get

$$\begin{aligned}
\langle Ay_n, x_n - p\rangle &= \langle Ay_n, x_n - y_n\rangle + \langle Ay_n, y_n - p\rangle \\
&\geq \langle Ay_n, x_n - y_n\rangle \\
&= \langle Ax_n, x_n - y_n\rangle - \langle Ax_n - Ay_n, x_n - y_n\rangle \\
&\geq \frac{1}{\lambda_n}\|x_n - y_n\|^2 - \frac{\mu}{\lambda_n}\|x_n - y_n\|^2 \\
&= \frac{1-\mu}{\lambda_n}\|x_n - y_n\|^2.
\end{aligned}$$

Since $\lambda_n \leq \gamma$ we get

$$\langle Ay_n, x_n - p\rangle \geq \frac{1-\mu}{\gamma}\|x_n - y_n\|^2.$$

□

Remark 3.6. From Lemma 3.5 we see that if $Ay_n = 0$ then $x_n = y_n$, this implies that x_n is a solution of $VI(C, A)$.

Lemma 3.7. *Assume that Conditions 3.1–3.3 hold and let $\{x_n\}$ be any sequence generated by Algorithm 3.1. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z \in C$ and $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$, then $z \in VI(C, A)$.*

Proof. We have $y_{n_k} = P_C(x_{n_k} - \lambda_{n_k} Ax_{n_k})$ thus,

$$\langle x_{n_k} - \lambda_{n_k} Ax_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0 \ \ \forall x \in C,$$

or equivalently

$$\frac{1}{\lambda_{n_k}}\langle x_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq \langle Ax_{n_k}, x - y_{n_k}\rangle \ \ \forall x \in C.$$

This implies that

$$\frac{1}{\lambda_{n_k}}\langle x_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle Ax_{n_k}, y_{n_k} - x_{n_k}\rangle \leq \langle Ax_{n_k}, x - x_{n_k}\rangle \ \forall x \in C. \quad (3.10)$$

Now, we show that

$$\liminf_{k\to\infty}\langle Ax_{n_k}, x - x_{n_k}\rangle \geq 0. \quad (3.11)$$

For showing this, we consider two possible cases. Suppose first that $\liminf_{k\to\infty}\lambda_{n_k} > 0$. We have $\{x_{n_k}\}$ is a bounded sequence, A is uniformly continuous on bounded subsets of C. By Lemma 2.6, we get that $\{Ax_{n_k}\}$ is bounded. Taking $k \to \infty$ in (3.10) since $\|x_{n_k} - y_{n_k}\| \to 0$, we get

$$\liminf_{k\to\infty}\langle Ax_{n_k}, x - x_{n_k}\rangle \geq 0.$$

Now, we assume that $\liminf_{k\to\infty}\lambda_{n_k} = 0$. Assume $z_{n_k} = P_C(x_{n_k} - \lambda_{n_k}.l^{-1} Ax_{n_k})$, we have $\lambda_{n_k} l^{-1} > \lambda_{n_k}$. Applying Lemma 3.5, we obtain

$$\|x_{n_k} - z_{n_k}\| \leq \frac{1}{l}\|x_{n_k} - y_{n_k}\| \to 0 \text{ as } k \to \infty.$$

Consequently, $z_{n_k} \rightharpoonup z \in C$, this implies that $\{z_{n_k}\}$ is bounded, and due to Condition 3.2, we get that

$$\|Ax_{n_k} - Az_{n_k}\| \to 0 \text{ as } k \to \infty. \quad (3.12)$$

By the Armijo line-search rule (3.1), we have

$$\lambda_{n_k}.l^{-1}\|Ax_{n_k} - AP_C(x_{n_k} - \lambda_{n_k} l^{-1}Ax_{n_k})\| > \mu\|x_{n_k} - P_C(x_{n_k} - \lambda_{n_k} l^{-1} Ax_{n_k})\|.$$

That is,

$$\frac{1}{\mu}\|Ax_{n_k} - Az_{n_k}\| > \frac{\|x_{n_k} - z_{n_k}\|}{\lambda_{n_k} l^{-1}}. \quad (3.13)$$

Combining (3.12) and (3.13), we obtain

$$\lim_{k\to\infty}\frac{\|x_{n_k} - z_{n_k}\|}{\lambda_{n_k} l^{-1}} = 0.$$

Furthermore, we have

$$\langle x_{n_k} - \lambda_{n_k} l^{-1} Ax_{n_k} - z_{n_k}, x - z_{n_k}\rangle \leq 0 \ \ \forall x \in C.$$

This implies that

$$\frac{1}{\lambda_{n_k} l^{-1}}\langle x_{n_k} - z_{n_k}, x - z_{n_k}\rangle + \langle Ax_{n_k}, z_{n_k} - x_{n_k}\rangle \leq \langle Ax_{n_k}, x - x_{n_k}\rangle \ \ \forall x \in C. \quad (3.14)$$

Taking the limit $k \to \infty$ in (3.14), we get

$$\liminf_{k\to\infty}\langle Ax_{n_k}, x - x_{n_k}\rangle \geq 0.$$

Therefore, the inequality (3.11) is proved. Next, we show that $z \in \mathrm{VI}(C, A)$.

Now we choose a sequence $\{\epsilon_k\}$ of positive numbers decreasing and tending to 0. For each k, we denote by N_k the smallest positive integer such that

$$\langle Ax_{n_j}, x - x_{n_j}\rangle + \epsilon_k \geq 0 \quad \forall j \geq N_k, \tag{3.15}$$

where the existence of N_k follows from (3.11). Since $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Furthermore, for each k, since $\{x_{N_k}\} \subset C$ we have $Ax_{N_k} \neq 0$ and, setting

$$v_{N_k} = \frac{Ax_{N_k}}{\|Ax_{N_k}\|^2},$$

we have $\langle Ax_{N_k}, x_{N_k}\rangle = 1$ for each k. Now, we can deduce from (3.15) that for each k

$$\langle Ax_{N_k}, x + \epsilon_k v_{N_k} - x_{N_k}\rangle \geq 0.$$

Since the fact that A is pseudo-monotone, we get

$$\langle A(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - x_{N_k}\rangle \geq 0.$$

This implies that

$$\langle Ax, x - x_{N_k}\rangle \geq \langle Ax - A(x+\epsilon_k v_{N_k}), x+\epsilon_k v_{N_k} - x_{N_k}\rangle - \epsilon_k\langle Ax, v_{N_k}\rangle. \tag{3.16}$$

Now, we show that $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$. Indeed, we have $x_{n_k} \rightharpoonup z$ as $k \to \infty$. Since A is sequentially weakly continuous on C, $\{Ax_{n_k}\}$ converges weakly to Az. We have that $Az \neq 0$ (otherwise, z is a solution). Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$0 < \|Az\| \leq \liminf_{k\to\infty} \|Ax_{n_k}\|.$$

Since $\{x_{N_k}\} \subset \{x_{n_k}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, we obtain

$$0 \leq \limsup_{k\to\infty} \|\epsilon_k v_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\epsilon_k}{\|Ax_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|Ax_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$.

Now, letting $k \to \infty$, then the right-hand side of (3.16) tends to zero by A is uniformly continuous, $\{x_{N_k}\}, \{v_{N_k}\}$ are bounded and $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$. Thus, we get

$$\liminf_{k\to\infty} \langle Ax, x - x_{N_k}\rangle \geq 0.$$

Hence, for all $x \in C$ we have

$$\langle Ax, x - z\rangle = \lim_{k\to\infty} \langle Ax, x - x_{N_k}\rangle = \liminf_{k\to\infty} \langle Ax, x - x_{N_k}\rangle \geq 0.$$

By Lemma 2.6, we obtain $z \in VI(C, A)$ and the proof is complete. □

Remark 3.8. When the mapping A is monotone, it is not necessary to impose the sequential weak continuity on A.

Theorem 3.9. *Assume that Conditions 3.1–3.3 hold. Then any sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to an element of $VI(C, A)$.*

Proof. We divide the proof into two claims.

Claim 1. We show that

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2 - \frac{(1-\mu)^2}{\gamma^2}\frac{\|x_n-y_n\|^4}{\|Ay_n\|^2}.$$

Indeed, since P_C is nonexpansive we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|P_C(x_n-\beta_n Ay_n)-P_C p\|^2 \leq \|x_n-\beta_n Ay_n-p\|^2\\ &= \|x_n-p\|^2 - 2\beta_n\langle Ay_n, x_n-p\rangle + \beta_n^2\|Ay_n\|^2,\end{aligned}$$

which, together with Lemma 3.5 we obtain

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2 - 2\beta_n\frac{1-\mu}{\gamma}\|x_n-y_n\|^2 + \beta_n^2\|Ay_n\|^2. \tag{3.17}$$

Substituting $\beta_n = \frac{1-\mu}{\gamma}\frac{\|x_n-y_n\|^2}{\|Ay_n\|^2}$ into (3.17), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|x_n-p\|^2 - 2\frac{(1-\mu)^2}{\gamma^2}\frac{\|x_n-y_n\|^4}{\|Ay_n\|^2} + \frac{(1-\mu)^2}{\gamma^2}\frac{\|x_n-y_n\|^4}{\|Ay_n\|^2}\\ &= \|x_n-p\|^2 - \frac{(1-\mu)^2}{\gamma^2}\frac{\|x_n-y_n\|^4}{\|Ay_n\|^2}.\end{aligned}$$

Claim 2. Now, we show that $\{x_n\}$ converges weakly to an element of $VI(C,A)$. Thanks to **Claim 1**, we have

$$\|x_{n+1}-p\| \leq \|x_n-p\| \quad \forall p \in VI(C,A).$$

This implies that for all $p \in VI(C,A)$ then $\lim_{n\to\infty}\|x_n-p\|$ exists, thus the sequence $\{x_n\}$ is bounded. Consequently, $\{y_n\}$ is bounded.

On the other hand, according to **Claim 1**, we get

$$\frac{(1-\mu)^2}{\gamma^2}\frac{\|x_n-y_n\|^4}{\|Ay_n\|^2} \leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2$$

which implies that

$$\lim_{n\to\infty}\frac{\|x_n-y_n\|^4}{\|Ay_n\|^2} = 0. \tag{3.18}$$

Since $\{y_n\} \subset C$ is bounded, A is uniformly continuous on bounded subsets of C, according to Lemma 2.5 we get $\{Ay_n\}$ is bounded, which together with (3.18) we get

$$\lim_{n\to\infty}\|x_n-y_n\| = 0. \tag{3.19}$$

Since $\{x_n\}$ is a bounded sequence, there exists the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z \in C$. It implies from Lemma 3.7 and (3.19) that $z \in \mathrm{VI}(C,A)$.

Therefore, we showed that:

(i) For every $p \in \mathrm{VI}(C,A)$, then $\lim_{n\to\infty}\|x_n-p\|$ exists;
(ii) Every sequential weak cluster point of the sequence $\{x_n\}$ is in $\mathrm{VI}(C,A)$.

By Lemma 2.7 the sequence $\{x_n\}$ converges weakly to an element of $\mathrm{VI}(C,A)$. □

Remark 3.10. Our result is more general than related results in the literature and hence might be applied for a wider class of mappings. For example, we next present the advantage of our method compared with the recent result [33, Theorem 3.1].

As in Theorem 3.9, $A : C \to H$ is assumed to be uniformly continuous on bounded subsets instead of Lipschitz continuous in [33].

3.2. Strong convergence

In this subsection, we introduce our second extragradient modification which is based on Halpern method [13] (see also [28]) and hence has a strong convergence property. An additional assumption for the analysis of our method is the following.

Condition 3.4. Let $\{\alpha_n\}$ be a real sequences in $(0, 1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

The proposed algorithm is of the form

Algorithm 3.11.

Initialization: Given $\gamma > 0, l \in (0, 1), \mu \in (0, 1)$. Let $x_0 \in C$ be arbitrary

---

Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute

$$y_n = P_C(x_n - \lambda_n A x_n),$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma l^m \langle Ax_n - Ay_n, x_n - y_n\rangle \le \mu \|x_n - y_n\|^2.$$

If $x_n = y_n$ then stop and x_n is a solution of $VI(C, A)$. Otherwise
Step 2. Compute

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) z_n,$$

where

$$z_n = P_C(x_n - \beta_n A y_n),$$

and

$$\beta_n := \frac{1-\mu}{\gamma} \frac{\|x_n - y_n\|^2}{\|Ay_n\|^2}.$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 3.12. *Assume that Conditions 3.1–3.4 hold. Then any sequence* $\{x_n\}$ *generated by Algorithm 3.11 converges strongly to* $p \in \mathrm{VI}(C, A)$*, where* $p = P_{\mathrm{VI}(C,A)} x_0$.

Proof. Similar to the proof of Theorem 3.9, and in order to keep it simple, we divide the proof into four claims.

Claim 1. We prove that $\{x_n\}$ is bounded. Indeed, thanks to **Claim 1** in the proof of Theorem 3.9, we have

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \frac{(1-\mu)^2}{\gamma^2}\frac{\|x_n - y_n\|^4}{\|Ay_n\|^2}. \tag{3.20}$$

This implies that

$$\|z_n - p\| \leq \|x_n - p\|. \tag{3.21}$$

Using (3.21), we have

$$\begin{aligned}\|x_{n+1} - p\| =& \|\alpha_n x_0 + (1-\alpha_n)z_n - p\| \\ &= \|\alpha_n(x_0 - p) + (1-\alpha_n)(z_n - p)\| \\ &\leq \alpha_n\|x_0 - p\| + (1-\alpha_n)\|z_n - p\| \\ &\leq \alpha_n\|x_0 - p\| + (1-\alpha_n)\|x_n - p\| \\ &\leq \max\{\|x_0 - p\|, \|x_n - p\|\} \\ &\leq \cdots \leq \|x_0 - p\|.\end{aligned}$$

Thus, the sequence $\{x_n\} \subset C$ is bounded. Consequently, the sequences $\{y_n\}, \{z_n\}, \{Ay_n\}$ are bounded.

Claim 2. We prove that

$$\frac{(1-\mu)^2}{\gamma^2}\frac{\|x_n - y_n\|^4}{\|Ay_n\|^2} \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M,$$

for some $M > 0$. Indeed, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(x_0 - p) + (1-\alpha_n)(z_n - p)\|^2 \\ &\leq (1-\alpha_n)\|z_n - p\|^2 + 2\alpha_n\langle x_0 - p, x_{n+1} - p\rangle \\ &\leq \|z_n - p\|^2 + 2\alpha_n\langle x_0 - p, x_{n+1} - p\rangle. \end{aligned} \tag{3.22}$$

Substituting (3.20) into (3.23), we get

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \frac{(1-\mu)^2}{\gamma^2}\frac{\|x_n - y_n\|^4}{\|Ay_n\|^2} + 2\alpha_n\langle x_0 - p, x_{n+1} - p\rangle.$$

This implies that

$$\begin{aligned}\frac{(1-\mu)^2}{\gamma^2}\frac{\|x_n - y_n\|^4}{\|Ay_n\|^2} &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\langle x_0 - p, x_{n+1} - p\rangle \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\|x_0 - p\|\|x_{n+1} - p\| \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n D,\end{aligned}$$

where $D := \sup\{2\|x_0 - p\|\|x_{n+1} - p\| : n \in \mathbb{N}\}$.

Claim 3. We prove that

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle x_0 - p, x_{n+1} - p\rangle.$$

Using (3.21) we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(x_0 - p) + (1-\alpha_n)(z_n - p)\|^2 \\ &\leq (1-\alpha_n)\|z_n - p\|^2 + 2\alpha_n\langle x_0 - p, x_{n+1} - p\rangle \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle x_0 - p, x_{n+1} - p\rangle. \end{aligned} \tag{3.23}$$

Claim 4. Now, we will show that the sequence $\{\|x_n - p\|^2\}$ converges to zero by considering two possible cases on the sequence $\{\|x_n - p\|^2\}$.

Case 1: There exists an $N \in \mathbb{N}$ such that $\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2$ for all $n \geq N$. This implies that $\lim_{n\to\infty} \|x_n - p\|^2$ exists. It implies from **Claim 2** that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.24}$$

On the other hand,

$$\begin{aligned}\|z_n - x_n\| &= \|P_C(x_n - \beta_n Ay_n) - P_C x_n\| \\ &\leq \|x_n - \beta_n Ay_n - x_n\| = \beta_n\|Ay_n\| = \frac{1-\mu}{\gamma}\frac{\|x_n - y_n\|^2}{\|Ay_n\|}.\end{aligned}$$

Using (3.24), it implies that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Since $\{x_n\} \subset C$ is a bounded sequence, we assume that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z \in C$ and

$$\limsup_{n\to\infty}\langle x_0 - p, x_n - p\rangle = \lim_{j\to\infty}\langle x_0 - p, x_{n_j} - p\rangle = \langle x_0 - p, z - p\rangle. \tag{3.25}$$

Since $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and $x_{n_j} \rightharpoonup z \in C$, according Lemma 3.7, we get $z \in VI(C, A)$. On the other hand,

$$\|x_{n+1} - z_n\| = \alpha_n\|x_0 - z_n\| \to 0 \text{ as } n \to \infty.$$

Thus

$$\|x_{n+1} - x_n\| = \|x_{n+1} - z_n\| + \|z_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.26}$$

Using (3.26), $p = P_{VI(C,A)}x_0$ and $x_{n_k} \rightharpoonup z \in VI(C, A)$, we get

$$\limsup_{n\to\infty}\langle x_0 - p, x_{n+1} - p\rangle \leq \langle x_0 - p, z - p\rangle \leq 0,$$

which, together with **Claim 3**, it implies from Lemma 2.9 that

$$x_n \to p \text{ as } n \to \infty.$$

Case 2: There exists a subsequence $\{\|x_{n_j} - p\|^2\}$ of $\{\|x_n - p\|^2\}$ such that $\|x_{n_j} - p\|^2 < \|x_{n_j+1} - p\|^2$ for all $j \in \mathbb{N}$. In this case, it follows from Lemma 2.8 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\|x_{m_k} - p\|^2 \leq \|x_{m_k+1} - p\|^2 \text{ and } \|x_k - p\|^2 \leq \|x_{m_k+1} - p\|^2. \tag{3.27}$$

According to **Claim 2**, we have

$$\frac{(1-\mu)^2}{\gamma^2}\frac{\|x_{m_k} - y_{m_k}\|^4}{\|Ay_{m_k}\|^2} \leq \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + \alpha_{m_k}M, \leq \alpha_{m_k}M.$$

This implies that

$$\lim_{k\to\infty} \|x_{m_k} - y_{m_k}\| = 0.$$

As proved in the first case, we obtain

$$\lim_{k\to\infty} \|z_{m_k} - x_{m_k}\| = 0$$

and

$$\lim_{k\to\infty} \|x_{m_k+1} - x_{m_k}\| = 0$$

and

$$\limsup_{k\to\infty} \langle x_0 - p, x_{m_k+1} - p\rangle \le 0. \tag{3.28}$$

Combining (3.27) and **Claim 3**, we obtain

$$\begin{aligned}\|x_{m_k+1} - p\|^2 &\le (1-\alpha_{m_k})\|x_{m_k} - p\|^2 + 2\alpha_{m_k}\langle x_0 - p, x_{m_k+1}\rangle \\ &\le (1-\alpha_{m_k})\|x_{m_k+1} - p\|^2 + 2\alpha_{m_k}\langle x_0 - p, x_{m_k+1} - p\rangle.\end{aligned}$$

This implies that

$$\|x_{m_k+1} - p\|^2 \le 2\langle x_0 - p, x_{m_k+1} - p\rangle,$$

which, together with (3.27) we get

$$\|x_k - p\|^2 \le \|x_{m_k+1} - p\|^2 \le 2\langle x_0 - p, x_{m_k+1} - p\rangle. \tag{3.29}$$

It implies from (3.28) and (3.29) that $\limsup_{k\to\infty} \|x_k - p\|^2 = 0$, that is $x_k \to p$ as $k \to \infty$. □

To end this section, we next present an academic example of variational inequality problem in an infinite dimensional space, where the cost function A is pseudo-monotone, L-Lipschitz continuous and sequentially weakly continuous on C but A fails to be a monotone mapping on H.

Example. Consider the Hilbert space

$$H = l_2 := \left\{ u = (u_1, u_2, \ldots, u_n, \ldots) \mid \sum_{n=1}^{\infty} |u_n|^2 < +\infty \right\}$$

equipped with the inner product and induced norm on H:

$$\langle u, v\rangle = \sum_{n=1}^{\infty} u_n v_n \text{ and } \|u\| = \sqrt{\langle u, u\rangle}$$

for any $u = (u_1, u_2, \ldots, u_n, \ldots), v = (v_1, v_2, \ldots, v_n, \ldots) \in H$.

Consider the set and the mapping:

$$C = \{u = (u_1, u_2, \ldots, u_i, \ldots) \in H \mid |u_i| \le \frac{1}{i}, i = 1, 2, \ldots, n, \ldots\},$$

$$Au = \left((\|u\| + \alpha) - \frac{1}{\|u\| + \alpha} \right) u,$$

where $\alpha > 1$ is a positive real number.

With this C and A, it is easy to see that $\mathrm{VI}(C, A) = \{0\}$ and moreover, A is pseudo-monotone, sequentially weakly continuous and uniformly continuous on C but A fails to be Lipschitz continuous on H.

First observe that since $\alpha > 1$, we get that

$$\left((\|u\| + \alpha) - \frac{1}{\|u\| + \alpha} \right) > 0 \quad \forall u \in C.$$

Now let $u, v \in C$ be such that $\langle Au, v - u\rangle \ge 0$. This implies that $\langle u, v - u\rangle \ge 0$.

Consequently,

$$\begin{aligned}\langle Av, v-u\rangle &= \left((\|u\|+\alpha) - \frac{1}{\|u\|+\alpha}\right)\langle v, v-u\rangle \\ &\geq \left((\|u\|+\alpha) - \frac{1}{\|u\|+\alpha}\right)(\langle v, v-u\rangle - \langle u, v-u\rangle) \\ &= \left((\|u\|+\alpha) - \frac{1}{\|u\|+\alpha}\right)\|v-u\|^2 \geq 0,\end{aligned}$$

meaning that A is pseudo-monotone.

Now, since C is compact, the mapping A is uniformly continuous and sequentially weakly continuous on C.

Finally, we show that A is not Lipschitz continuous on H. Assume to the contrary that A is Lipschitz continuous on H, i.e., there exists $L > 0$ such that

$$\|Au - Av\| \leq L\|u - v\| \quad \forall u, v \in H.$$

Let $u = (L, 0, ..., 0, ...)$ and $v = (0, 0, ..., 0, ...)$, then

$$\|Au - Av\| = \|Au\| = \left((\|u\|+\alpha) - \frac{1}{\|u\|+\alpha}\right)\|u\| = \left((L+\alpha) - \frac{1}{L+\alpha}\right)L.$$

Thus, $\|Au - Av\| \leq L\|u - v\|$ is equivalent to

$$\left((L+\alpha) - \frac{1}{L+\alpha}\right)L \leq L^2,$$

equivalently

$$L + \alpha \leq L + \frac{1}{L+\alpha} < L + 1,$$

which implies that $\alpha < 1$, and this leads to a contraction and thus A is not Lipschitz continuous on H.

Remark 3.13. It should be emphasized here that the example established in Section 4 in [33] is not sequentially weakly continuous.

Remark 3.14. Thank to the referee's comment, we wish to point out that since the proximity operator of a proper, lower semicontinuous and convex function is a generalization of the metric projection and our convergence analysis mainly use the firmly nonexpansiveness of the metric projection, and it is known that the proximity operator is indeed firmly nonexpansive, our proposed method can be modified to solve a more general variational inequalities.

4. Numerical illustrations

In this section, we present an example illustrating the behavior and advantages of our proposed schemes. The numerical example which is the Kojima–Shindo Nonlinear Complementarity Problem (NCP), see e.g., [27].

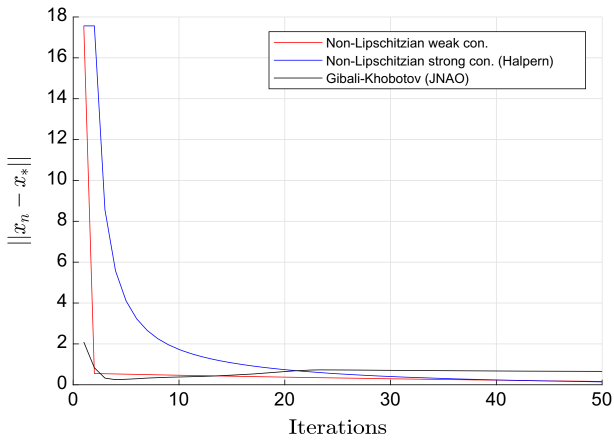

FIGURE 1. Illustration of Algorithms 3.1, 3.11 and [11, Algorithm 3.1]

TABLE 1. Algorithms 3.1, 3.11 and [11, Algorithm 3.1]

Algorithms	Total line-search iterations	CPU time
Algorithms 3.1	7	17.37
Algorithm 3.11	34	33.82
[11, Algorithm 3.1]	11	26.25

Example. In this example, we test our algorithms behavior for solving The Kojima–Shindo Nonlinear Complementarity Problem (NCP) with $n = 4$, see e.g., [27]. The cpu time is measured in seconds using the intrinsic MATLAB function cpu time. The VI feasible set is $C := \{x \in \mathbb{R}^4_+ \mid x_1+x_2+x_3+x_4 = 4\}$ and A is given as follows:

$$A(x_1, x_2, x_3, x_4) := \begin{bmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{bmatrix}. \tag{4.1}$$

The solution if the problem is $(\sqrt{6}/2, 0, 0, 0.5)$. In our experiments, we choose the stopping criteria as $\|x_n - y_n\| \leq 10^{-3}$. The projection onto the feasible set C is performed using CVX version 1.22. Other parameters are: $\gamma = 0.2, l = 0.3, \mu = 0.6$, we choose the stopping criterion $||x_n - y_n|| < 10^{-5}$. In [11, Algorithm 3.1], we choose $\varepsilon = 0.2, \beta = 0.5$ and $\alpha_{-1} = 0.7$.

The starting point for all experiments is $x_0 = (1, 1, 1, 1)$. All computations were performed using MATLAB R2017a on an Intel Core i5-4200U 2.3 GHz running 64-bit Windows. In Fig. 1, the performances of Algorithms 3.1, 3.11 and [11, Algorithm 3.1] are presented.

In Table 1, the complementary data of Fig. 1 is presented.

5. Conclusions

In this paper, we proposed two extragradient extensions for solving non-Lipschitzian pseudo-monotone variational inequalities in real Hilbert spaces. Under suitable and standard conditions, we establish weak and strong convergence theorems of the proposed schemes. Our work extends and generalizes some existing results in the literature and academic and numerical experiments demonstrate the behavior and potential applicability of the methods.

Acknowledgements

The authors would like to thank Professor Simeon Reich and the referees for their comments on the manuscript which helped in improving earlier version of this paper.

References

[1] Antipin, A.S.: On a method for convex programs using a symmetrical modification of the Lagrange function. Ekon. i Mat. Metody. **12**, 1164–1173 (1976)

[2] Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**, 318–335 (2011)

[3] Censor, Y., Gibali, A., Reich, S.: Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. Optim. Meth. Softw. **26**, 827–845 (2011)

[4] Censor, Y., Gibali, A., Reich, S.: Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space. Optimization **61**, 1119–1132 (2011)

[5] Censor, Y., Gibali, A., Reich, S.: Algorithms for the split variational inequality problem. Numer. Algorithms. **56**, 301–323 (2012)

[6] Cottle, R.W., Yao, J.C.: Pseudo-monotone complementarity problems in Hilbert space. J. Optim. Theory Appl. **75**, 281–295 (1992)

[7] Denisov, S.V., Semenov, V.V., Chabak, L.M.: Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators. Cybern. Syst. Anal. **51**, 757–765 (2015)

[8] Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer Series in Operations Research, vols. I and II. Springer, New York (2003)

[9] Fichera, G.: Sul problema elastostatico di Signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincei VIII. Ser. Rend. Cl. Sci. Fis. Mat. Nat **34**, 138–142 (1963)

[10] Fichera, G.: Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincei, Mem., Cl. Sci. Fis. Mat. Nat. Sez. I, VIII. Ser **7**, 91–140 (1964)

[11] Gibali, A.: A new non-Lipschitzian projection method for solving variational inequalities in Euclidean spaces. J. Nonlinear Anal. Optim. Theory Appl. **6**, 41–51 (2015)

[12] Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker, New York (1984)

[13] Halpern, B.: Fixed points of nonexpanding maps. Bull. Am. Math. Soc. **73**, 957–961 (1967)

[14] Iusem, A.N.: An iterative algorithm for the variational inequality problem. Comput. Appl. Math. **13**, 103–114 (1994)

[15] Iusem, A.N., Gárciga Otero, R.: Inexact versions of proximal point and augmented Lagrangian algorithms in Banach spaces. Numer. Funct. Anal. Optim. **22**, 609–640 (2001)

[16] Kassay, G., Reich, S., Sabach, S.: Iterative methods for solving systems of variational inequalities in refelexive Banach spaces. SIAM J. Optim. **21**, 1319–1344 (2011)

[17] Karamardian, S.: Complementarity problems over cones with monotone and pseudo-monotone maps. J. Optim. Theory Appl. **18**, 445–454 (1976)

[18] Khobotov, E.N.: Modifications of the extragradient method for solving variational inequalities and certain optimization problems. USSR Comput. Math. Math. Phys. **27**, 120–127 (1987)

[19] Konnov, I.V.: Equilibrium Models and Variational Inequalities. Mathematics in Science and Engineering. Elsevier, Amsterdam (2007)

[20] Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Ekon. i Mat. Metody. **12**, 747–756 (1976)

[21] Liu, L.S.: Ishikawa and Mann iteration process with errors for nonlinear strongly accretive mappings in Banach space. J. Math. Anal. Appl. **194**, 114–125 (1995)

[22] Maingé, P.E.: A hybrid extragradient-viscosity method for monotone operators and fixed point problems. SIAM J. Control Optim. **47**, 1499–1515 (2008)

[23] Malitsky, Y.V.: Projected reflected gradient methods for monotone variational inequalities. SIAM J. Optim. **25**, 502–520 (2015)

[24] Malitsky, Y.V., Semenov, V.V.: A hybrid method without extrapolation step for solving variational inequality problems. J. Glob. Optim. **61**, 193–202 (2015)

[25] Marcotte, P.: Application of Khobotov's algorithm to variational inequalities and network equilibrium problems. Inf. Syst. Oper. Res. **29**, 258–270 (1991)

[26] Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 591–597 (1967)

[27] Pang, J.S., Gabriel, S.A.: NE/SQP: a robust algorithm for the nonlinear complementarity problem. Math. Program. **60**, 295–337 (1993)

[28] Reich, S.: Strong convergence theorems for resolvents of accretive operators in Banach spaces. J. Math. Anal. Appl. **75**, 287–292 (1980)

[29] Reich, S.: Constructive Techniques for Accretive and Monotone Operators. Applied Nonlinear Analysis, pp. 335–345. Academic Press, New York (1979)

[30] Solodov, M.V., Svaiter, B.F.: A new projection method for variational inequality problems. SIAM J. Control Optim. **37**, 765–776 (1999)

[31] Thong, D.V., Hieu, D.V.: Modified subgradient extragradient method for variational inequality problems. Numer. Algorithms. **79**, 597–610 (2018)

[32] Thong, D.V., Hieu, D.V.: Inertial extragradient algorithms for strongly pseudomonotone variational inequalities. J. Comput. Appl. Math. **341**, 80–98 (2018)

[33] Vuong, P.T.: On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities. J. Optim. Theory Appl. **176**, 399–409 (2018)

[34] Vuong, P.T., Shehu, Y.: Convergence of an extragradient-type method for variational inequality with applications to optimal control problems. Numer. Algorithms. https://doi.org/10.1007/s11075-018-0547-6 (2018)

[35] Xu, H.K.: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**, 240–256 (2002)

Duong Viet Thong
Applied Analysis Research Group
Faculty of Mathematics and Statistics
Ton Duc Thang University
Ho Chi Minh City
Vietnam
e-mail: duongvietthong@tdtu.edu.vn

Aviv Gibali
Department of Mathematics
ORT Braude College
2161002 Karmiel
Israel
e-mail: avivg@braude.ac.il

and

The Center for Mathematics and Scientific Computation
University of Haifa
Mt. Carmel
3498838 Haifa
Israel