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Coupled fixed point theorems in partially ordered metric spaces via mixed g-monotone property

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Abstract. In this paper we study some coupled fixed point theorems and coupled coincidence fixed point theorems for infinite family mappings satisfying different contractive conditions on the complete partially ordered metric space with the help of concept of mixed g-monotone property. Further we used generalized Darbo type coupled fixed point theorem to find the existence of solutions for a system of nonlinear functional integral equations in Banach space with the help of measure of noncompactness.

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Keywords. Coupled coincidence point, coupled common fixed point, mixed g-monotone property, partially ordered set, measure of noncompactness.

1. Introduction

In [9], Bhaskar and Lakshmikantham introduced the notion of a coupled contraction mapping principle and proved coupled fixed point results for the mixed monotone property in partially ordered metric spaces. After that, many authors have carried out further studies on the coupled fixed point, the coupled coincidence point and the coupled common fixed point (see, e.g., [2]–[17]). Common fixed point theorems for generalized contractions invariably require a commutativity condition and the continuity of one of the mappings $\{T_\alpha : X \times X \longrightarrow X : \alpha \in \Lambda\}$ or X the property be regular. The purpose of this work was to prove some coupled common fixed point and common fixed point theorems for infinite family mappings satisfying different contractive conditions on the complete partially ordered metric space.

In [15], Lakshmikantham and Ćirić introduced the concept of mixed g-monotone property as follows:

Definition 1.1. [15] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. We say F has the mixed g-monotone property if F is a non-decreasing g-monotone in its first argument and is a non-increasing g-monotone in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X, gx_1 \leq gx_2 \Longrightarrow F(x_1, y) \leq F(x_2, y),$$

and

$$y_1, y_2 \in X, gy_1 \leq gy_2 \Longrightarrow F(x, y_1) \geq F(x, y_2).$$

Note that if g is the identity mapping, then F is said to have the mixed monotone property (see [9]).

Definition 1.2 [15]. An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \longrightarrow X$ and a mapping $g : X \longrightarrow X$ if

$$F(x, y) = gx, \ F(y, x) = gy.$$

Similarly, note that if g is the identity mapping, then (x, y) is called a coupled fixed point of the mapping F (see [9]).

Definition 1.3 [1]. An element $x \in X$ is called a common fixed point of a mapping $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if

$$F(x, x) = gx = x. \tag{1.1}$$

Definition 1.4 [15]. Let X be a nonempty set and $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. One says F and g are commutative if for all $x, y \in X$,

$$F(gx, gy) = g(F(x, y)).$$

Abbas et al. [1] introduced the concept of w-compatibility for a pair of mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$.

Definition 1.5. The mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are called w-compatible if $g(F(x, y)) = F(gx, gy))$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.

2. Main results

Throughout the paper, let Ψ be the family of all functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ satisfying the following conditions:

(a) ψ is continuous,
(b) ψ nondecreasing,
(c) $\psi(t) = 0$ if and only if $t = 0$.

We denote by Φ the set of all functions $\phi : [0, \infty) \longrightarrow [0, \infty)$ satisfying the following conditions:

(a) ϕ is lower semi-continuous,
(b) $\phi(t) = 0$ if and only if $t = 0$,

and Θ the set of all continuous functions $\theta : [0,\infty) \longrightarrow [0,\infty)$ with $\theta(t) = 0$ if and only if $t = 0$.

Now, we establish some results for the existence of coupled coincidence point and coupled common fixed point of mappings in the setup of partially ordered metric spaces. The first result in this paper is the following coupled coincidence theorem:

Theorem 2.1. *Suppose that* $(X, d, \leq)$ *is a partially ordered complete metric space. Suppose* $g : X \longrightarrow X$ *and* $\{T_\alpha : X \times X \longrightarrow X : \alpha \in \Lambda\}$ *are such that* T_{α_0} *has the mixed* g*-monotone property and commutes with* g *on* X *such that there exist two elements* $x_0, y_0 \in X$ *with* $gx_0 \leq T_{\alpha_0}(x_0, y_0)$ *and* $gy_0 \geq T_{\alpha_0}(y_0, x_0)$. *Suppose there exist* $L \geq 0$, $\psi \in \Psi$, $\phi \in \Phi$ *and* $\theta \in \Theta$ *such that*

$$\psi(d(T_{\alpha_0}(x,y), T_\alpha(u,v))) \leq \psi(M(x,y,u,v)) - \phi(M(x,y,u,v)) + L\theta(N(x,y,u,v)), \tag{2.1}$$

where

$$\begin{aligned} M(x,y,u,v) = \max\{ & d(gx,gu), d(gy,gv), d(gx,T_\alpha(x,y)), d(gu,T_{\alpha_0}(u,v)), \\ & d(gy,T_\alpha(y,x)), d(gv,T_{\alpha_0}(v,u)), \\ & \frac{d(gx,T_\alpha(u,v)) + d(gu,T_{\alpha_0}(x,y))}{2}, \\ & \frac{d(gy,T_\alpha(v,u)) + d(gv,T_{\alpha_0}(y,x))}{2}\}, \end{aligned}$$

and

$$N(x,y,u,v) = \min\{d(gx,T_{\alpha_0}(x,y)), d(gu,T_\alpha(u,v)), d(gu,T_{\alpha_0}(x,y)), d(gx,T_\alpha(u,v))\}.$$

for all $x, y, u, v \in X$, $\alpha \in \Lambda$ *for which* $gx \leq gu$ *and* $gy \geq gv$. *Suppose* $T_{\alpha_0}(X \times X) \subseteq g(X)$, g *is continuous and also suppose either*

(i) T_{α_0} *is continuous or*
(ii) X *has the following property(regular):*
  (a) *if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* n,
  (b) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all* n.

Then $\{T_\alpha : \alpha \in \Lambda\}$ *and* g *have coupled coincidence point in* X.

Proof. By the given assumptions, there exists $(x_0, y_0) \in X \times X$ such that $gx_0 \leq T_{\alpha_0}(x_0, y_0)$ and $gy_0 \geq T_{\alpha_0}(y_0, x_0)$. Since $T_{\alpha_0}(X \times X) \subseteq g(X)$, we can define $(x_1, y_1) \in X \times X$ such that $gx_1 = T_{\alpha_0}(x_0, y_0)$ and $gy_1 = T_{\alpha_0}(y_0, x_0)$, then $gx_0 \leq T_{\alpha_0}(x_0, y_0) = gx_1$ and $gy_0 \geq T_{\alpha_0}(y_0, x_0) = gy_1$. Also there exists $(x_2, y_2) \in X \times X$ such that $gx_2 = T_{\alpha_0}(x_1, y_1)$ and $gy_2 = T_{\alpha_0}(y_1, x_1)$. Since T_{α_0} has the mixed g-monotone property, we have

$$gx_1 = T_{\alpha_0}(x_0, y_0) \leq T_{\alpha_0}(x_0, y_1) \leq T_{\alpha_0}(x_1, y_1) = gx_2,$$

and

$$gy_2 = T_{\alpha_0}(y_1, x_1) \leq T_{\alpha_0}(y_0, x_1) \leq T_{\alpha_0}(y_0, x_0) = gy_1.$$

Continuing in this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = T_{\alpha_0}(x_n, y_n) \text{ and } gy_{n+1} = T_{\alpha_0}(y_n, x_n) \text{ for all } n = 0, 1, 2, \dots \tag{2.2}$$

for which

$$\begin{aligned} &gx_0 \leq gx_1 \leq gx_2 \leq \cdots \leq gx_n \leq gx_{n+1} \leq \dots, \\ &gy_0 \geq gy_1 \geq gy_2 \geq \cdots \geq gy_n \geq gy_{n+1} \geq \dots. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3) and inequality (2.1) with $(x, y) = (x_n, y_n)$ and $(u, v) = (x_{n+1}, y_{n+1})$, we obtain

$$\begin{aligned} \psi(d(gx_{n+1}, gx_{n+2})) &= \psi(d(T_{\alpha_0}(x_n, y_n), T_{\alpha_0}(x_{n+1}, y_{n+1}))) \\ &\leq \psi(M(x_n, y_n, x_{n+1}, y_{n+1})) - \phi(M(x_n, y_n, x_{n+1}, y_{n+1})) \\ &\quad + L\theta(N(x_n, y_n, x_{n+1}, y_{n+1})), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} M(x_n, y_n, x_{n+1}, y_{n+1}) &= \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gx_n, T_{\alpha_0}(x_n, y_n)), \\ &\quad d(gx_{n+1}, T_{\alpha_0}(x_{n+1}, y_{n+1})), d(gy_n, T_{\alpha_0}(y_n, x_n)), \\ &\quad d(gy_{n+1}, T_{\alpha_0}(y_{n+1}, x_{n+1})), \\ &\quad \frac{d(gx_n, T_{\alpha_0}(x_{n+1}, y_{n+1})) + d(gx_{n+1}, T_{\alpha_0}(x_n, y_n))}{2}, \\ &\quad \frac{d(gy_n, T_{\alpha_0}(y_{n+1}, x_{n+1})) + d(gy_{n+1}, T_{\alpha_0}(y_n, x_n))}{2}\} \\ &= \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), \\ &\quad d(gx_{n+1}, gx_{n+2}), d(gy_{n+1}, gy_{n+2}), \\ &\quad \frac{d(gx_n, gx_{n+2})}{2}, \frac{d(gy_n, gy_{n+2})}{2}\}, \end{aligned}$$

and

$$\begin{aligned} N(x_n, y_n, x_{n+1}, y_{n+1}) &= \min\{d(gx_n, T_{\alpha_0}(x_n, y_n)), d(gx_{n+1}, T_{\alpha_0}(x_{n+1}, y_{n+1})), \\ &\quad d(gx_{n+1}, T_{\alpha_0}(x_n, y_n)), d(gx_{n+1}, T_{\alpha_0}(x_{n+1}, y_{n+1}))\} = 0. \end{aligned}$$

Since

$$\begin{aligned} \frac{d(gx_n, gx_{n+2})}{2} &\leq \frac{d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2})}{2} \\ &\leq \max\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})\}, \end{aligned}$$

and

$$\begin{aligned} \frac{d(gy_n, gy_{n+2})}{2} &\leq \frac{d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_{n+2})}{2} \\ &\leq \max\{d(gy_n, gy_{n+1}), d(gy_{n+1}, gy_{n+2})\}, \end{aligned}$$

then we get

$$\begin{aligned} M(x_n, y_n, x_{n+1}, y_{n+1}) &= \max\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}), \\ &\quad d(gy_n, gy_{n+1}), d(gy_{n+1}, gy_{n+2})\}, \\ &\quad N(x_n, y_n, x_{n+1}, y_{n+1}) = 0. \end{aligned} \tag{2.5}$$

By (2.4) and (2.5), we have

$$\begin{aligned}\psi(d(gx_{n+1},gx_{n+2})) \leq \psi(&\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2}),\\ &d(gy_n,gy_{n+1}),d(gy_{n+1},gy_{n+2})\})\\ &-\phi(\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2}),\\ &d(gy_n,gy_{n+1}),d(gy_{n+1},gy_{n+2})\}).\end{aligned} \tag{2.6}$$

Similarly, we can show that

$$\begin{aligned}\psi(d(gy_{n+1},gy_{n+2})) \leq \psi(&\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2}),\\ &d(gy_n,gy_{n+1}),d(gy_{n+1},gy_{n+2})\})\\ &-\phi(\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2}),\\ &d(gy_n,gy_{n+1}),d(gy_{n+1},gy_{n+2})\}).\end{aligned} \tag{2.7}$$

Now denote

$$\delta_n = \max\{d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1})\}. \tag{2.8}$$

Combining (2.6),(2.7) and the fact that $\max\{\psi(a),\psi(b)\} = \psi(\max\{a,b\})$ for $a,b \in [0,+\infty)$, we have

$$\psi(\delta_{n+1}) = \max\{\psi(d(gx_{n+1},gx_{n+2})),\psi(d(gy_{n+1},gy_{n+2}))\}. \tag{2.9}$$

So, using (2.6),(2.7),(2.8) together (2.9), we obtain

$$\begin{aligned}\psi(\delta_{n+1}) \leq \psi(&\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2}),\\ &d(gy_n,gy_{n+1}),d(gy_{n+1},gy_{n+2})\})\\ &-\phi(\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2}),\\ &d(gy_n,gy_{n+1}),d(gy_{n+1},gy_{n+2})\}).\end{aligned} \tag{2.10}$$

Now we prove that for all $n \in \mathbb{N}$,

$$\begin{aligned}&\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2}),d(gy_n,gy_{n+1}),d(gy_{n+1},gy_{n+2})\} = \delta_n,\\ &\delta_{n+1} \leq \delta_n.\end{aligned} \tag{2.11}$$

For this purpose consider the following three cases:

Case 1. If $\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2}),d(gy_n,gy_{n+1}),d(gy_{n+1},gy_{n+2})\} = \delta_n$, then by (2.10), we have

$$\psi(\delta_{n+1}) \leq \psi(\delta_n) - \phi(\delta_n) < \psi(\delta_n), \tag{2.12}$$

so (2.11) obviously holds.

Case 2. If $\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2}),d(gy_n,gy_{n+1}),d(gy_{n+1},gy_{n+2})\} = d(gx_{n+1},gx_{n+2}) > 0$, then by (2.6),

$$\begin{aligned}\psi(d(gx_{n+1},gx_{n+2})) &\leq \psi(d(gx_{n+1},gx_{n+2})) - \phi(d(gx_{n+1},gx_{n+2}))\\ &< \psi(d(gx_{n+1},gx_{n+2})),\end{aligned}$$

which is a contradiction.

Case 3. If $\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2}),d(gy_n,gy_{n+1}),d(gy_{n+1},gy_{n+2})\} = d(gy_{n+1},gy_{n+2}) > 0$,
then from (2.7),

$$\psi(d(gy_{n+1},gy_{n+2})) \leq \psi(d(gy_{n+1},gy_{n+2})) - \phi(d(gy_{n+1},gy_{n+2}))$$

$$< \psi(d(gy_{n+1}, gy_{n+2})),$$

which is again a contradiction.

Thus in all cases (2.11) holds for each $n \in \mathbb{N}$. It follows that the sequence $\{\delta_n\}$ is a monotone decreasing sequence of non-negative real numbers and consequently there exists s$\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \delta. \tag{2.13}$$

We show that $\delta = 0$. Suppose, on the contrary, that $\delta > 0$. Taking the limit as $n \to \infty$ in (2.12) and using the properties of the function ϕ, we get

$$\psi(\delta) \leq \psi(\delta) - \phi(\delta) < \psi(\delta),$$

which is a contradiction. Therefore $\delta = 0$, that is,

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} = 0$$

which implies that

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d(gy_n, gy_{n+1}) = 0. \tag{2.14}$$

Now, we claim that

$$\lim_{n,m\to\infty} \max\{d(gx_n, gx_m), d(gy_n, gy_m)\} = 0. \tag{2.15}$$

Assume on the contrary that there exists $\epsilon > 0$ and subsequences $\{gx_{m(k)}\}$, $\{gx_{n(k)}\}$ of $\{gx_n\}$ and $\{gy_{m(k)}\}$, $\{gy_{n(k)}\}$ of $\{gy_n\}$ with $m(k) > n(k) \geq k$ such that

$$\max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} \geq \epsilon. \tag{2.16}$$

Additionally, corresponding to $n(k)$, we may choose $m(k)$ such that it is the smallest integer satisfying (2.16) and $m(k) > n(k) \geq k$. Thus

$$\max\{d(gx_{n(k)}, gx_{m(k)-1}), d(gy_{n(k)}, gy_{m(k)-1})\} < \epsilon. \tag{2.17}$$

Using the triangle inequality and (2.16) and (2.17) we obtain that

$$\begin{aligned}\epsilon \leq d(gx_{m(k)}, gx_{n(k)}) &\leq d(gx_{m(k)}, gx_{m(k)-1}) + d(gx_{m(k)-1}, gx_{n(k)})\\ &< d(gx_{m(k)}, gx_{m(k)-1}) + \epsilon.\end{aligned}$$

Taking the limit as $k \to \infty$ and using (2.14) we obtain

$$\lim_{k\to\infty} d(gx_{n(k)}, gx_{m(k)}) = \epsilon. \tag{2.18}$$

Similarly, we obtain

$$\lim_{k\to\infty} d(gy_{n(k)}, gy_{m(k)}) = \epsilon. \tag{2.19}$$

Also

$$\begin{aligned}\epsilon \leq d(gx_{n(k)}, gx_{m(k)}) &\leq d(gx_{n(k)}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)})\\ &\leq d(gx_{n(k)}, gx_{m(k)}) + d(gx_{m(k)}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)})\\ &\leq d(gx_{n(k)}, gx_{m(k)}) + 2\ d(gx_{m(k)}, gx_{m(k)+1}).\end{aligned}$$

So from (2.14) and (2.18), we have

$$\limsup_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{m(k)+1}) = \epsilon. \tag{2.20}$$

Similarly, we obtain

$$\limsup_{k\longrightarrow\infty} d(gy_{n(k)}, gy_{m(k)+1}) = \epsilon. \tag{2.21}$$

Also

$$\begin{aligned}\epsilon \leq d(gx_{m(k)}, gx_{n(k)}) &\leq d(gx_{m(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{n(k)})\\ &\leq d(gx_{m(k)}, gx_{n(k)}) + d(gx_{n(k)}, gx_{n(k)+1})\\ &\quad + d(gx_{n(k)+1}, gx_{n(k)})\\ &\leq d(gx_{m(k)}, gx_{n(k)}) + 2\ d(gx_{n(k)}, gx_{n(k)+1}).\end{aligned}$$

So from (2.14) and (2.18), we have

$$\limsup_{k\longrightarrow\infty} d(gx_{m(k)}, gx_{n(k)+1}) = \epsilon. \tag{2.22}$$

In a similar way, we obtain

$$\limsup_{k\longrightarrow\infty} d(gy_{m(k)}, gy_{n(k)+1}) = \epsilon. \tag{2.23}$$

Also

$$\begin{aligned}d(gx_{n(k)+1}, gx_{m(k)}) &\leq d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)})\\ &\leq d(gx_{n(k)+1}, gx_{n(k)}) + d(gx_{n(k)}, gx_{m(k)+1})\\ &\quad + d(gx_{m(k)+1}, gx_{m(k)})\end{aligned}$$

so from (2.14), (2.20) and (2.22), we have

$$\limsup_{k\longrightarrow\infty} d(gx_{n(k)+1}, gx_{m(k)+1}) = \epsilon. \tag{2.24}$$

Similarly, we obtain

$$\limsup_{k\longrightarrow\infty} d(gy_{n(k)+1}, gy_{m(k)+1}) = \epsilon. \tag{2.25}$$

$$\begin{aligned}&\limsup_{k\longrightarrow\infty} M(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}) = \max\{\limsup_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{m(k)}),\\ &\quad \limsup_{k\longrightarrow\infty} d(gy_{n(k)}, gy_{m(k)}), \limsup_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{n(k)+1}), \limsup_{k\longrightarrow\infty} d(gx_{m(k)}, gx_{m(k)+1}),\\ &\quad \limsup_{k\longrightarrow\infty} d(gy_{n(k)}, gy_{n(k)+1}), \limsup_{k\longrightarrow\infty} d(gy_{m(k)}, gy_{m(k)+1}),\\ &\quad \frac{1}{2}[\limsup_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{m(k)+1}) + \limsup_{k\longrightarrow\infty} d(gx_{m(k)}, gx_{n(k)+1})],\\ &\quad \frac{1}{2}[\limsup_{k\longrightarrow\infty} d(gy_{n(k)}, gy_{m(k)+1}) + \limsup_{k\longrightarrow\infty} d(gy_{m(k)}, gy_{n(k)+1})].\end{aligned}$$

So,

$$\limsup_{k\longrightarrow\infty} M(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}) = \epsilon. \tag{2.26}$$

Similarly, we have

$$\liminf_{k\longrightarrow\infty} M(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}) = \epsilon, \tag{2.27}$$

and

$$\lim_{k\to\infty} N(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}) = 0. \tag{2.28}$$

Since $m(k) > n(k)$ from (2.2), we have

$$gx_{n(k)} \leq gx_{m(k)},\ gy_{n(k)} \geq gy_{m(k)}.$$

Thus

$$\begin{aligned}\psi(d(gx_{n(k)+1}, gx_{m(k)+1})) &= \psi(d(T_{\alpha_0}(x_{n(k)}, y_{n(k)}), T_{\alpha_0}(x_{m(k)}, y_{m(k)})))\\ &\leq \psi(M(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}))\\ &\quad -\phi(M(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}))\\ &\quad +L\theta(N(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})),\\ \psi(d(gy_{n(k)+1}, gy_{m(k)+1})) &= \psi(d(T_{\alpha_0}(y_{n(k)}, x_{n(k)}), T_{\alpha_0}(y_{m(k)}, x_{m(k)})))\\ &\leq \psi(M(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}))\\ &\quad -\phi(M(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}))\\ &\quad +L\theta(N(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})).\end{aligned}$$

Since ψ is a nondecreasing function, we have

$$\begin{aligned}&\max\{\psi(d(gx_{n(k)+1}, gx_{m(k)+1})), \psi(d(gy_{n(k)+1}, gy_{m(k)+1}))\}\\ &\quad = \psi(\max\{d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1})\}).\end{aligned}$$

Taking the upper limit as $k \to \infty$, and using (2.25) and (2.26), we get

$$\begin{aligned}\psi(\epsilon) &= \psi(\max\{\limsup_{k\longrightarrow\infty} d(gx_{n(k)+1}, gx_{m(k)+1}),\\ &\qquad \limsup_{k\longrightarrow\infty} d(gy_{n(k)+1}, gy_{m(k)+1})\})\\ &\leq \psi(\limsup_{k\longrightarrow\infty} M(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}))\\ &\quad -\phi(\liminf_{k\longrightarrow\infty} M(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}))\\ &\quad +L\theta(\limsup_{k\longrightarrow\infty} N(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}))\\ &\leq \psi(\epsilon) - \phi(\epsilon)\\ &< \psi(\epsilon).\end{aligned}$$

It is a contradiction. Therefore, (2.15) holds and we have

$$\lim_{n,m\to\infty} d(gx_n, gx_m) = 0 \ \textit{and} \ \lim_{n,m\to\infty} d(gy_n, gy_m) = 0.$$

Since X is a complete metric space, there exist $x, y \in X$ such that

$$\begin{aligned}\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} T_{\alpha_0}(x_n, y_n) = x \text{ and } \lim_{n\to\infty} gy_{n+1}\\ = \lim_{n\to\infty} T_{\alpha_0}(y_n, x_n) = y.\end{aligned} \tag{2.29}$$

From the commutativity of T_{α_0} and g, we have

$$\begin{aligned}g(gx_{n+1}) = g(T_{\alpha_0}(x_n, y_n)) = T_{\alpha_0}(gx_n, gy_n),\\ g(gy_{n+1}) = g(T_{\alpha_0}(y_n, x_n)) = T_{\alpha_0}(gy_n, gx_n).\end{aligned} \tag{2.30}$$

We now show that $gx = T_{\alpha_0}(x, y)$ and $gy = T_{\alpha_0}(y, x)$. Suppose that the assumption (i) holds. For all n from the continuity of T_{α_0} and g, and letting $n \to \infty$ in (2.30), we get

$$gx = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} g(T_{\alpha_0}(x_n, y_n)) = \lim_{n\to\infty} T_{\alpha_0}(gx_n, gy_n) = T_{\alpha_0}(x, y),$$

$$gy = \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} g(T_{\alpha_0}(y_n, x_n)) = \lim_{n\to\infty} T_{\alpha_0}(gy_n, gx_n) = T_{\alpha_0}(y, x).$$

Finally, suppose that (ii) holds. Since $\{gx_n\}$ is non-decreasing sequence and $gx_n \to x$ and as $\{gy_n\}$ is non-increasing sequence and $gy_n \longrightarrow y$, by assumption (ii), we have $gx_n \leq x$ and $gy_n \geq y$ for all n. By (2.30) and g is continuous, we have

$$gx = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} g(T_{\alpha_0}(x_n, y_n)) = \lim_{n\to\infty} T_{\alpha_0}(gx_n, gy_n),$$
$$gy = \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} g(T_{\alpha_0}(y_n, x_n)) = \lim_{n\to\infty} T_{\alpha_0}(gy_n, gx_n).$$

Now we have

$$d(gx, T_{\alpha_0}(x, y)) \leq d(gx, g(gx_{n+1})) + d(g(gx_{n+1}), T_{\alpha_0}(x, y)).$$

Taking $n \to \infty$ in the above inequality, using (2.2) we have,

$$\begin{aligned} d(gx, T_{\alpha_0}(x, y)) &\leq \lim_{n\to\infty} d(gx, g(gx_{n+1})) + \lim_{n\to\infty} d(g(T_{\alpha_0}(x_n, y_n)), T_{\alpha_0}(x, y)) \\ &\leq \lim_{n\to\infty} d(T_{\alpha_0}(gx_n, gy_n), T_{\alpha_0}(x, y)). \end{aligned}$$

Similarly, we can show that

$$d(gy, T_{\alpha_0}(y, x)) \leq \lim_{n\to\infty} d(T_{\alpha_0}(gy_n, gx_n), T_{\alpha_0}(y, x)).$$

Therefore,

$$\begin{aligned} &\psi(\max\{d(gx, T_{\alpha_0}(x, y)), d(gy, T_{\alpha_0}(y, x))\}) \\ &\quad = \max\{\psi(d(gx, T_{\alpha_0}(x, y))), \psi(d(gy, T_{\alpha_0}(y, x)))\} \\ &\quad \leq \limsup_{n\longrightarrow\infty} \max\{\psi(d(T_{\alpha_0}(gx_n, gy_n), T_{\alpha_0}(x, y))), \\ &\qquad \psi(d(T_{\alpha_0}(gy_n, gx_n), T_{\alpha_0}(y, x)))\} \\ &\quad \leq \limsup_{n\longrightarrow\infty} \psi(M(gx_n, gy_n, x, y)) - \liminf_{n\longrightarrow\infty} \phi(M(gx_n, gy_n, x, y)) \\ &\qquad + L \limsup_{n\longrightarrow\infty} \theta(N(gx_n, gy_n, x, y)) \\ &\quad \leq \psi(\max\{d(gx, T_{\alpha_0}(x, y)), d(gy, T_{\alpha_0}(y, x))\}) \\ &\qquad -\phi(\max\{d(gx, T_{\alpha_0}(x, y)), d(gy, T_{\alpha_0}(y, x))\}), \end{aligned}$$

which implies that $d(gx, T_{\alpha_0}(x, y)) = 0$ and $d(gy, T_{\alpha_0}(y, x)) = 0$, that is,

$$gx = T_{\alpha_0}(x, y) \text{ and } gy = T_{\alpha_0}(y, x).$$

Now, we will prove that for any $\alpha \in \Lambda$, $gx = T_\alpha(x, y)$ and $gy = T_\alpha(y, x)$. Suppose, to the contrary, that at least one of $gx = T_\alpha(x, y)$ or $gy = T_\alpha(y, x)$ is not equal. Then there exists an $\alpha_1 \in \Lambda$ such that

$$r = \max\{d(gx, T_{\alpha_1}(x, y)), d(gy, T_{\alpha_1}(y, x)))\} > 0.$$

Using the property of ψ and (2.1), we have

$$\begin{aligned} \psi(r) &= \psi(\max\{d(gx, T_{\alpha_1}(x, y)), d(gy, T_{\alpha_1}(y, x)))\}) \\ &\quad = \max\{\psi(d(gx, T_{\alpha_1}(x, y))), \psi(d(gy, T_{\alpha_1}(y, x)))\} \\ &\quad \leq \psi(M(x, y, x, y)) - \phi(M(x, y, x, y)) + L\theta(N(x, y, x, y)), \end{aligned}$$

where

$$\begin{aligned} M&(x,y,x,y)\\ &= M(y,x,y,x) = \max\Big\{d(gx,gx), d(gy,gy), d(gx,T_{\alpha_1}(x,y)), d(gx,T_{\alpha_0}(x,y)),\\ &\quad (gy,T_{\alpha_1}(y,x)), d(gy,T_{\alpha_0}(y,x)),\\ &\quad \frac{d(gx,T_{\alpha_1}(x,y))+d(gx,T_{\alpha_0}(x,y))}{2}, \frac{d(gy,T_{\alpha_1}(y,x))+d(gy,T_{\alpha_0}(y,x))}{2}\Big\}\\ &= \max\{d(gx,T_{\alpha_1}(x,y)), d(gy,T_{\alpha_1}(y,x)))\} = r, \end{aligned}$$

and

$$\begin{aligned} N(x,y,x,y) = N(y,x,y,x) = \min\{&d(gx,T_{\alpha_0}(x,y)), d(gu,T_{\alpha_1}(u,v)),\\ &d(gu,T_{\alpha_0}(x,y)), d(gx,T_{\alpha_1}(u,v))\} = 0. \end{aligned}$$

This implies

$$\begin{aligned} \psi(r) &\le \psi(M(x,y,x,y)) - \phi(M(x,y,x,y)) + L\theta(N(x,y,x,y))\\ &\le \psi(r) - \phi(r)\\ &< \psi(r), \end{aligned}$$

which is a contradiction. Thus (x,y) is a coupled coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$. $\square$

Corollary 2.2. *Suppose that $(X,d,\le)$ is a partially ordered complete metric space. Suppose $g : X \longrightarrow X$ and $\{T_\alpha : X \times X \longrightarrow X : \alpha \in \Lambda\}$ are such that T_{α_0} has the mixed g-monotone property and commutes with g on X such that there exist two elements $x_0, y_0 \in X$ with $gx_0 \le T_{\alpha_0}(x_0,y_0)$ and $gy_0 \ge T_{\alpha_0}(y_0,x_0)$. Suppose there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi(d(T_{\alpha_0}(x,y), T_\alpha(u,v))) \le \psi(M(x,y,u,v)) - \phi(M(x,y,u,v)), \tag{2.31}$$

where

$$\begin{aligned} M(x,y,u,v) = \max\{&d(gx,gu), d(gy,gv), d(gx,T_\alpha(x,y)), d(gu,T_{\alpha_0}(u,v)),\\ &d(gy,T_\alpha(y,x)), d(gv,T_{\alpha_0}(v,u)),\\ &\tfrac{d(gx,T_\alpha(u,v))+d(gu,T_{\alpha_0}(x,y))}{2},\\ &\tfrac{d(gy,T_\alpha(v,u))+d(gv,T_{\alpha_0}(y,x))}{2}\Big\} \end{aligned}$$

for all $x,y,u,v \in X$, $\alpha \in \Lambda$ for which $gx \le gu$ and $gy \ge gv$. Suppose $T_{\alpha_0}(X \times X) \subseteq g(X)$, g is continuous and also suppose either

(i) *T_{α_0} is continuous or*
(ii) *X has the following property:*
  (a) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all n,*
  (b) *if a non-increasing sequence $\{y_n\} \to y$, then $y \le y_n$ for all n.*

Then $\{T_\alpha : \alpha \in \Lambda\}$ and g have coupled coincidence point in X.

Corollary 2.3. *Suppose that $(X,d,\le)$ is a partially ordered complete metric space. Suppose $\{T_\alpha : X \times X \longrightarrow X : \alpha \in \Lambda\}$ are such that T_{α_0} has the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$x_0 \leq T_{\alpha_0}(x_0, y_0)$ and $y_0 \geq T_{\alpha_0}(y_0, x_0)$. Suppose there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\psi(d(T_{\alpha_0}(x, y), T_\alpha(u, v))) \leq \psi(M(x, y, u, v)) - \phi(M(x, y, u, v)), \tag{2.32}$$

where

$$\begin{aligned} M(x, y, u, v) = \max\Big\{ & d(x, u), d(y, v), d(x, T_\alpha(x, y)), d(u, T_{\alpha_0}(u, v)), \\ & d(y, T_\alpha(y, x)), d(v, T_{\alpha_0}(v, u)), \\ & \tfrac{d(x, T_\alpha(u, v)) + d(u, T_{\alpha_0}(x, y))}{2}, \\ & \tfrac{d(y, T_\alpha(v, u)) + d(v, T_{\alpha_0}(y, x))}{2} \Big\}, \end{aligned}$$

for all $x, y, u, v \in X$, $\alpha \in \Lambda$ for which $x \leq u$ and $y \geq v$. Also suppose either

(i) *T_{α_0} is continuous or*
(ii) *X has the following property:*
  (a) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n;*
  (b) *if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all n.*

Then $\{T_\alpha : \alpha \in \Lambda\}$ have coupled fixed point in X.

3. Uniqueness of common fixed point

In this section we shall provide some sufficient conditions under which $\{T_\alpha : \alpha \in \Lambda\}$ and g have a unique common fixed point. Note that if $(X, \leq)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order relation, for all $(x, y), (z, t) \in X \times X$:

$$(x, y) \leq (z, t) \iff x \leq z, \; y \geq t.$$

From Theorem 2.1, it follows that the set $C(T_\alpha, g)$ of coupled coincidences is nonempty.

Theorem 3.1. *By adding to the hypotheses of Theorem 2.1, the condition for every (x, y) and (z, t) in $X \times X$, there exists a $(u, v) \in X \times X$ such that $(T_{\alpha_0}(u, v), T_{\alpha_0}(v, u))$ is comparable to $(T_{\alpha_0}(x, y), T_{\alpha_0}(y, x))$ and to $(T_{\alpha_0}(z, t), T_{\alpha_0}(t, z))$, and $\{T_\alpha\}$ and g are w-compatible. Then $\{T_\alpha\}$ and g have a unique coupled common fixed point.*

Proof. We know, from Theorem 2.1, that there exists at least a coupled coincidence point. Suppose that (x, y) and (z, t) are coupled coincidence points of T_α and g, that is, $T_\alpha(x, y) = gx$, $T_\alpha(y, x) = gy$, $T_\alpha(z, t) = gz$ and $T_\alpha(t, z) = gt$. We shall show that $gx = gz$ and $gy = gt$. By the assumptions, there exists $(u, v) \in X \times X$ such that $(T_{\alpha_0}(u, v), T_{\alpha_0}(v, u))$ is comparable to $(T_{\alpha_0}(x, y), T_{\alpha_0}(y, x))$ and to $(T_{\alpha_0}(z, t), T_{\alpha_0}(t, z))$. Without any restriction of the generality, we can assume that

$$(T_{\alpha_0}(x, y), T_{\alpha_0}(y, x)) \leq (T_{\alpha_0}(u, v), T_{\alpha_0}(v, u)) \textit{ and}$$
$$(T_{\alpha_0}(z, t), T_{\alpha_0}(t, z)) \leq (T_{\alpha_0}(u, v), T_{\alpha_0}(v, u)).$$

Put $u_0 = u, \; v_0 = v$ and choose $(u_1, \; v_1) \in X \times X$ such that

$$gu_1 = T_{\alpha_0}(u_0, v_0), \; gv_1 = T_{\alpha_0}(v_0, u_0).$$

For $n \geq 1$, continuing this process we can construct sequences $\{gu_n\}$ and $\{gv_n\}$ such that

$$gu_{n+1} = T_{\alpha_0}(u_n, v_n),\ gv_{n+1} = T_{\alpha_0}(v_n, u_n) \text{ for all } n.$$

Further, set $x_0 = x$, $y_0 = y$ and $z_0 = z$, $t_0 = t$ and on the same way define sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$, $\{gt_n\}$. Since $(gx, gy) = (T_{\alpha_0}(x, y), T_{\alpha_0}(y, x)) = (gx_1, gy_1)$ and $(T_{\alpha_0}(u, v), T_{\alpha_0}(v, u)) = (gu_1, gv_1)$ are comparable, $(gx, gy) \leq (gu, gv)$. One can show, by induction, that

$$(gx_n, gy_n) \leq (gu_n, gv_n) \text{ for all } n. \tag{3.1}$$

Thus from (2.1), we have

$$\begin{aligned}\psi(d(gx, gu_{n+1})) &= \psi(d(T_{\alpha_0}(x, y), T_{\alpha_0}(u_n, v_n))) \\ &\leq \psi(M(x, y, u_n, v_n)) - \phi(M(x, y, u_n, v_n)) \\ &\quad + L\theta(N(x, y, u_n, v_n)),\end{aligned}$$

where

$$\begin{aligned}M(x, y, u_n, v_n) = \max\Big\{& d(gx, gu_n), d(gy, gv_n), d(gx, T_{\alpha_0}(x, y)), \\ & d(gu_n, T_{\alpha_0}(u_n, v_n)), d(gy, T_{\alpha_0}(y, x)), \\ & d(gv_n, T_{\alpha_0}(v_n, u_n)), \frac{d(gx, T_{\alpha_0}(u_n, v_n)) + d(gu_n, T_{\alpha_0}(x, y))}{2}, \\ & \frac{d(gy, T_{\alpha_0}(v_n, u_n)) + d(gv_n, T_{\alpha_0}(y, x))}{2}\Big\} \\ = \max\{& d(gx, gu_n), d(gy, gv_n), d(gy, gv_{n+1}), d(gx, gu_{n+1})\}.\end{aligned}$$

It is easy to show that

$$M(x, y, u_n, v_n) = \max\{d(gx, gu_n), d(gy, gv_n)\},$$

and

$$N(x, y, u_n, v_n) = 0.$$

Hence

$$\begin{aligned}\psi(d(gx, gu_{n+1})) \leq\ & \psi(\max\{d(gx, gu_n), d(gy, gv_n)\}) \\ & - \phi(\max\{d(gx, gu_n), d(gy, gv_n)\}).\end{aligned} \tag{3.2}$$

Similarly, one can prove that

$$\begin{aligned}\psi(d(gy, gv_{n+1})) \leq\ & \psi(\max\{d(gx, gu_n), d(gy, gv_n)\}) \\ & - \phi(\max\{d(gx, gu_n), d(gy, gv_n)\}).\end{aligned} \tag{3.3}$$

Combining (3.2),(3.3) and the fact that $\max\{\psi(a), \psi(b)\} = \psi(\max\{a, b\})$ for $a, b \in [0, \infty)$, we have

$$\begin{aligned}&\psi(\max\{d(gx, gu_{n+1}), d(gy, gv_{n+1})\}) \\ &\quad = \max\{\psi(d(gx, gu_{n+1})), \psi(d(gy, gv_{n+1}))\} \\ &\quad \leq \psi(\max\{d(gx, gu_n), d(gy, gv_n)\}) \\ &\qquad - \phi(\max\{d(gx, gu_n), d(gy, gv_n)\}) \\ &\quad \leq \psi(\max\{d(gx, gu_n), d(gy, gv_n)\}).\end{aligned} \tag{3.4}$$

Using the non-decreasing property of ψ, we get

$$\max\{d(gx, gu_{n+1}), d(gy, gv_{n+1})\} \leq \max\{d(gx, gu_n), d(gy, gv_n)\},$$

which implies that $\max\{d(gx, gu_n), d(gy, gv_n)\}$ is a non-increasing sequence. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \max\{d(gx, gu_n), d(gy, gv_n)\} = r.$$

Passing the upper limit in (3.4) as $n \to \infty$, we obtain

$$\psi(r) \leq \psi(r) - \phi(r),$$

which implies that $\phi(r) = 0$ and then $r = 0$. We deduce that

$$\lim_{n\to\infty} \max\{d(gx, gu_n), d(gy, gv_n)\} = 0,$$

which concludes

$$\lim_{n\to\infty} d(gx, gu_n) = \lim_{n\to\infty} d(gy, gv_n) = 0. \tag{3.5}$$

Similarly, one can prove that

$$\lim_{n\to\infty} d(gz, gu_n) = \lim_{n\to\infty} d(gt, gv_n) = 0. \tag{3.6}$$

From (3.5) and (3.6), we have $gx = gz$ and $gy = gt$. Since $gx = T_\alpha(x, y)$ and $gy = T_\alpha(y, x)$, by w-compatible of $\{T_\alpha\}$ and g, we have

$$g(gx) = g(T_\alpha(x, y)) = T_\alpha(gx, gy),\ g(gy) = g(T_\alpha(y, x)) = T_\alpha(gy, gx). \tag{3.7}$$

Denote $gx = a$ and $gy = b$; then from (3.7),

$$g(a) = T_\alpha(a, b),\ g(b) = T_\alpha(b, a). \tag{3.8}$$

Thus (a, b) is a coupled coincidence point; it follows that $ga = gz$ and $gb = gy$, that is,

$$g(a) = a,\ g(b) = b. \tag{3.9}$$

From (3.8) and (3.9),

$$a = g(a) = T_\alpha(a, b),\ b = g(b) = T_\alpha(b, a). \tag{3.10}$$

Therefore, (a, b) is a coupled common fixed point of $\{T_\alpha\}$ and g. To prove the uniqueness of the point (a, b), assume that (c, d) is another coupled common fixed point of T_α and g. Then we have

$$c = gc = T_\alpha(c, d),\ d = gd = T_\alpha(d, c).$$

Since (c, d) is a coupled coincidence point of $\{T_\alpha\}$ and g, we have $gc = gx = a$ and $gd = gy = b$. Thus $c = gc = ga = a$ and $d = gd = gb = b$, which is the desired result. □

Since every commuting pair of functions is a w-compatible, we have the following corollary:

Corollary 3.2. *By adding to the hypotheses of Theorem 2.1, the condition for every (x, y) and (z, t) in $X \times X$, there exists a $(u, v) \in X \times X$ such that $(T_{\alpha_0}(u, v), T_{\alpha_0}(v, u))$ is comparable to $(T_{\alpha_0}(x, y), T_{\alpha_0}(y, x))$ and to $(T_{\alpha_0}(z, t), T_{\alpha_0}(t, z))$, and $\{T_\alpha\}$ and g are commuting. Then $\{T_\alpha\}$ and g have a unique coupled common fixed point.*

Theorem 3.3. *In addition to the hypotheses of Theorem 3.1, if gx_0 and gy_0 are comparable, then $\{T_\alpha\}$ and g have a unique common fixed point, that is, there exists $x \in X$ such that $x = gx = T_\alpha(x, x)$.*

Proof. Following the proof of Theorem 3.1, $\{T_\alpha\}$ and g have a unique coupled common fixed point (x, y). We only have to show that $x = y$. Since gx_0 and gy_0 are comparable, we may assume that $gx_0 \leq gy_0$. By using the mathematical induction, one can show that

$$gx_n \leq gy_n \text{ for all } n \geq 0, \tag{3.11}$$

where $\{gx_n\}$ and $\{gy_n\}$ are defined by (2.2). From (2.29), we have

$$\begin{aligned}
\psi(d(x,y)) &= \limsup_{n \longrightarrow \infty} \psi(d(gx_{n+1}, gy_{n+1})) = \limsup_{n \longrightarrow \infty} \psi(d(T_{\alpha_0}(x_n, y_n), T_{\alpha_0}(y_n, x_n))) \\
&\leq \limsup_{n \longrightarrow \infty} \psi(M(x_n, y_n, y_n, x_n)) - \liminf_{n \longrightarrow \infty} \phi(M(x_n, y_n, y_n, x_n)) \\
&\quad + \limsup_{n \longrightarrow \infty} L\theta(N(x_n, y_n, y_n, x_n)) \\
&\leq \psi(d(x,y)) - \liminf_{n \longrightarrow \infty} \phi(M_s(x_n, y_n, y_n, x_n)) \\
&< \psi(d(x,y)),
\end{aligned}$$

a contradiction. Therefore $x = y$, that is, $\{T_\alpha\}$ and g have a common fixed point. □

4. Application of Darbo type coupled fixed point theorem

Suppose $\mathcal{E}$ is a real Banach space with the norm $\| \, . \, \|$. Let $B[y, d]$ be a closed ball in $\mathcal{E}$ centered at y and radius d. If X is a nonempty subset of $\mathcal{E}$, then we denote $\bar{X}$ and Conv Conv$X$ the closure and convex closure of X. Moreover, let $\mathcal{M}_{\mathcal{E}}$ denote the family of all nonempty and bounded subsets of $\mathcal{E}$ and $\mathcal{N}_{\mathcal{E}}$ its subfamily consisting of all relatively compact sets. We denote by $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+ = [0, \infty)$.

Now we recall the definition of a measure of noncompactness.

Definition 4.1 [7]. A function $\mu : \mathcal{M}_{\mathcal{E}} \to \mathbb{R}_+$ is called a measure of noncompactness in $\mathcal{E}$ if it satisfies the following conditions:

(i) the family $ker\mu = \{X \in \mathcal{M}_{\mathcal{E}} : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathcal{N}_{\mathcal{E}}$.
(ii) $X \subseteq Y \implies \mu(X) \leq \mu(Y)$.
(iii) $\mu(\bar{X}) = \mu(X)$.
(iv) $\mu(\text{Conv}X) = \mu(X)$.
(v) $\mu(\lambda X + (1 - \lambda) Y) \leq \lambda \mu(X) + (1 - \lambda) \mu(Y)$ for $\lambda \in [0, 1]$.
(vi) if $X_n \in \mathcal{M}_{\mathcal{E}}, X_n = \bar{X}_n, X_{n+1} \subset X_n$ for $n = 1, 2, 3, ...$ and $\lim_{n \to \infty} \mu(X_n) = 0$, then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \phi$.

The family kerμ is said to be *kernel of measure* μ. Observe that the intersection set X_∞ from (vi) is a member of the family kerμ. Since $\mu(X_\infty) \leq \mu(X_n)$ for any n, we have $\mu(X_\infty) = 0$. This gives $X_\infty \in$ kerμ.

A measure μ is said to be sublinear if it satisfies the following conditions:

(1) $\mu(\lambda X) = |\lambda| \mu(X)$ for $\lambda \in \mathbb{R}$.
(2) $\mu(X + Y) \leq \mu(Y) + \mu(Y)$.

A sublinear measure of noncompactness μ satisfying the condition:

$$\mu(X \cup Y) = \max\{\mu(\lambda X), \mu(\lambda Y)\}$$

and such that $\ker\mu = \mathcal{N}_{\mathcal{E}}$ is said to be regular.

For a bounded subset S of a metric space X, the Kuratowski measure of noncompactness is defined as

$$\alpha(S) = \inf\left\{\delta > 0 : S = \bigcup_{i=1}^{n} S_i,\ \operatorname{diam}(S_i) \le \delta\ for\ 1 \le i \le n, n \in \mathbb{N}\right\},$$

where $\operatorname{diam}(S_i)$ denotes the diameter of the set S_i, that is,

$$\operatorname{diam}(S_i) = \sup\{d(x,y) : x, y \in S_i\}.$$

We consider the space $\mathcal{B} = BC(\mathbb{R}_+ \times \mathbb{R}_+)$ of real-valued continuous and bounded functions defined on $\mathbb{R}_+ \times \mathbb{R}_+$. It is clear that $\mathcal{B}$ is a Banach space with respect to the norm

$$\| x \| = \sup\{|x(t,s)| : t, s \ge 0\},\ x \in \mathcal{B}.$$

Let X be a fixed nonempty and bounded subset of the space $\mathcal{B} = BC(\mathbb{R}_+ \times \mathbb{R}_+)$ and τ be a fixed positive number. For $x \in X$ and $\epsilon > 0$, denote by $\omega^\tau(x, \epsilon)$ the modulus of the continuity function x on the interval $[0, \tau]$, i.e.

$$\omega^\tau(x,\epsilon) = \sup\{|x(t,s) - x(u,v)| : t, s, u, v \in [0,\tau], |t-u| \le \epsilon, |s-v| \le \epsilon\}.$$

Further we define

$$\omega^\tau(X,\epsilon) = \sup\{\omega^\tau(x,\epsilon) : x \in X\}.$$
$$\omega_0^\tau(X) = \lim_{\epsilon \to 0} \omega^\tau(x,\epsilon)$$

and

$$\omega_0(X) = \lim_{\tau \to \infty} \omega_0^\tau(X).$$

Moreover, for two fixed numbers $t, s \in \mathbb{R}_+$, let us define the function μ on the family $\mathcal{M}_{\mathcal{B}}$ by the following formulae:

$$\mu(X) = \omega_0(X) + \lim_{t,s \to \infty} \sup \operatorname{diam} X(t,s),$$

where $X(t,s) = \{x(t,s) : t, s \in \mathbb{R}_+\}$ and

$$\operatorname{diam} X(t,s) = \sup\{|x(t,s) - y(t,s)| : x, y \in X\}.$$

The function μ is a measure of noncompactness in the space $\mathcal{B} = BC(\mathbb{R}_+ \times \mathbb{R}_+)$ (see [6]).

By using Darbo type coupled fixed point theorem discussed in Theorems 2.2 and 2.3 in [12] and Theorem 3.1 of Das et al. [12], we study the existence of a solution to the system of following nonlinear functional integral equations:

$$\begin{cases} x(t,s) = e^{-ts} + \frac{\ln(1+|x(t,s)|)}{12+ts} + \frac{e^{-ts}\ln(1+|y(t,s)|)}{8} \\ \qquad + \ln\left(1 + \frac{1}{4}\left|\int\limits_0^s\int\limits_0^t \frac{\cos(1+vwy(v,w))}{e^{ts}}\right| dvdw\right) \\ y(t,s) = e^{-ts} + \frac{\ln(1+|y(t,s)|)}{12+ts} + \frac{e^{-ts}\ln(1+|x(t,s)|)}{8} \\ \qquad + \ln\left(1 + \frac{1}{4}\left|\int\limits_0^s\int\limits_0^t \frac{\cos(1+vwx(v,w))}{e^{ts}}\right| dvdw\right) \end{cases} \tag{4.1}$$

Here

$$f(t,s,x,y,z) = e^{-ts} + \frac{\ln(1+|x(t,s)|)}{12+ts} + \frac{e^{-ts}}{8}\ln(1+|y(t,s)|) + \ln\left(1+\frac{|z|}{2}\right),$$

$$\bar{g}(t,s,v,w,x(v,w),y(v,w)) = \frac{\cos(1+vwy(v,w))}{e^{ts}},$$

$$\phi_1(t,s) = \ln\left(1+\frac{t+s}{2}\right), \alpha(t) = t = \beta(t), \psi_2(t) = \frac{t}{2},$$

where α, β and f are continuous.

We have

$$|f(t,s,0,0,0)| = e^{-ts}$$

is bounded and $M = 1$.

Let $t, s \in \mathbb{R}_+$, $x, y, z, \bar{u}, \bar{v}, \bar{w} \in \mathbb{R}$ with $|x| \geq |\bar{u}|, |y| \geq |\bar{v}|$. By Mean Value Theorem on $\ln\left(1+\frac{|z|}{2}\right)$ and $\ln\left(1+\frac{t+s}{2}\right) \in \hat{\phi}$ (for details of $\hat{\phi}$, see Sect. 2 of [12]), we get

$$\begin{aligned} &|f(t,s,x,y,z) - f(t,s,\bar{u},\bar{v},\bar{w})| \\ &\leq \frac{1}{12+ts}\left|\ln(1+|x| - \ln(1+|\bar{u}|))\right| + \frac{e^{-ts}}{8}\left|\ln(1+|y|) - \ln(1+|\bar{v}|)\right| \\ &\quad + \left|\ln\left(1+\frac{|z|}{2}\right) - \ln\left(1+\frac{|\bar{w}|}{2}\right)\right| \\ &\leq \frac{1}{12+ts}\left|\ln\left(\frac{1+|x|}{1+|\bar{u}|}\right)\right| + \frac{e^{-ts}}{8}\left|\ln\left(\frac{1+|y|}{1+|\bar{v}|}\right)\right| + \frac{1}{2}\left||z| - |\bar{w}|\right| \\ &\leq \frac{1}{12}\left|\ln\left(\frac{1+|x|}{1+|\bar{u}|}\right)\right| + \frac{1}{8}\left|\ln\left(\frac{1+|y|}{1+|\bar{v}|}\right)\right| + \frac{1}{2}\left||z| - |\bar{w}|\right| \\ &= \frac{1}{12}\left|\ln\left(1+\frac{|x|-|\bar{u}|}{1+|\bar{u}|}\right)\right| + \frac{1}{8}\left|\ln\left(1+\frac{|y|-|\bar{v}|}{1+|\bar{v}|}\right)\right| + \frac{1}{2}\left||z| - |\bar{w}|\right| \\ &\leq \frac{1}{8}\ln(1+|x-\bar{u}|) + \frac{1}{8}\ln(1+|y-\bar{v}|) + \frac{1}{2}\left||z| - |\bar{w}|\right| \\ &\leq \frac{1}{4}\ln\left(1+\frac{|x-\bar{u}| + |y-\bar{v}|}{2}\right) + \frac{1}{2}|z-\bar{w}| \\ &= \frac{1}{4}\phi_1(|x-\bar{u}|, |y-\bar{v}|) + \psi_2(|z-\bar{w}|). \end{aligned}$$

Clearly $\bar{g}$ is continuous. Moreover, for each $t, s, u, v \in \mathbb{R}_+$, and $x, y, x_1, y_1 \in \mathbb{R}$,

$$|\bar{g}(t,s,v,w,x(v,w),y(v,w)) - \bar{g}(t,s,v,w,x_1(v,w),y_1(v,w))| \leq \frac{2}{e^{ts}}.$$

Therefore,

$$\int_0^s \int_0^t |\bar{g}(t,s,v,w,x(v,w),y(v,w)) - \bar{g}(t,s,v,w,x_1(v,w),y_1(v,w))|\, dvdw \leq \frac{2}{e^{ts}}.$$

and

$$\lim_{t,s\to\infty} \int_0^s \int_0^t |\bar{g}(t,s,v,w,x(v,w),y(v,w)) - \bar{g}(t,s,v,w,x_1(v,w),y_1(v,w))|\, dvdw = 0.$$

Therefore,

$$\left| \int_0^s \int_0^t \bar{g}(t,s,v,w,x(v,w),y(v,w)) dvdw \right| \leq \frac{2ts}{e^{ts}}$$

for any $t, s, v, w \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$. Thus

$$G \leq \sup \left\{ \frac{2ts}{e^{ts}} : t, s \geq 0 \right\} = \frac{2}{e}.$$

Now substituting the values of M, G, ϕ_1 and ψ_2 in assumption (5) of the Theorem 3.1 [12], we get the following inequality:

$$1 + \ln(1+r) + \frac{1}{e} \leq r$$

Consequently, all the conditions of the Theorem 3.1 [12] are satisfied and hence the system of Eq. (4.1) has at least one solution in $\mathcal{B} \times \mathcal{B}$.

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References

[1] Abbas, M., Ali Khan, M., Radenović, S.: Common coupled fixed point theorems in cone metric spaces for w-compatible mappings. Appl. Math. Comput. **217**, 195–202 (2010)

[2] Agarwal, R.P., Sintunavarat, W., Kumam, P.: Coupled coincidence point and common coupled fixed point theorems lacking the mixed monotone property. Fixed Point Theory Appl. **2013**, 22 (2013)

[3] Arab, R., Rabbani, M.: Coupled coincidence and common fixed point theorems for mappings in partially ordered metric spaces. Math. Sci. Lett. **3**(2), 81–87 (2014)

[4] Arab, R.: A common coupled fixed point theorem for two pairs of w-compatible mappings in G-metric spaces. Sohag J. Math. **1**(1), 37–43 (2014)

[5] Arab, R.: Coupled coincidence point results on partial metric spaces. Sohag J. Math. **2**(1), 23–28 (2015)

[6] Arab, R., Allahyari, R., Haghighi, A.S.: Existence of solutions of infinite systems of integral equations in two variables via measure of noncompactness. Appl. Math. Comput. **246**, 283–291 (2014)

[7] Banaś, J., Goebel, K.: Measure of Noncompactness in Banach Spaces. Lecture Notes in Pure and Applied Mathematics, vol. 60. Marcel Dekker, New York (1980)

[8] Berinde, V.: Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces. Nonlinear Anal. TMA **74**(18), 7347–7355 (2011)

[9] Bhaskar, T.G., Lakshmikantham, V.: Fixed point theory in partially ordered metric spaces and applications. Nonlinear Anal. **65**, 1379–1393 (2006)

[10] Choudhury, B.S., Kundu, A.: A coupled coincidence point result in partially ordered metric spaces for compatible mappings. Nonlinear Anal. **73**, 2524–2531 (2010)

[11] Choudhury, B.S., Metiya, N., Kundu, A.: Coupled coincidence point theorems in ordered metric spaces. Ann. Univ. Ferrara **57**, 1–16 (2011)

[12] A.Das, B. Hazarika, R. Arab, M. Mursaleen, Applications of a fixed point theorem to the existence of solutions to the nonlinear functional integral equations in two variables, Rend. Circ. Mat. Palermo, II. Ser, https://doi.org/10.1007/s12215-018-0347-9

[13] Jungck, G.: Commuting mappings and fixed points. Am. Math. Mon. **83**, 261–263 (1976)

[14] Jungck, G.: Compatible mappings and common fixed points. Int. J. Math. Math. Sci. **9**, 771–779 (1986)

[15] Lakshmikantham, V., Ćirić, L.: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. TMA **70**(12), 4341–4349 (2009)

[16] Luong, N.V., Thuan, N.X.: Coupled fixed point theorems in partially ordered metric spaces. Bul. Math. Anal. Appl. **4**, 16–24 (2010)

[17] Shatanawi, W., Samet, B., Abbas, M.: Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces. Math. Comput. Model. **55**(34), 680–687 (2012)

[18] Tasković, M.R.: Axiom of Infinite choice, transversal ordered spring spaces and fixed points. Fixed Point Theory Appl. **2018**, 10 (2018). (**32 pages**)

[19] Petrusel, A., Petruśel, G., Xiao, Yi-bin, Yao, Jen-Chih: Fixed point theorems for generalized contractions with applications to coupled fixed point theory. J. Nonlinear Convex Anal. **19**(1), 71–81 (2018)

[20] Ran, A.C.M., Reurings, M.C.B.: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Amer. Math. Soc. **132**, 1435–1443 (2004)

[21] Nieto, J.J., Rodrguez-López, R.: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223–239 (2005)

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